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Self-similar fragmentations derived from the stable tree I

Splitting at heights

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Abstract. The basic object we consider is a certain model of continuum random tree, called the stable tree. We construct a fragmentation process $(F^{-}(t), t \geq 0)$ out of this tree by removing the vertices located under height t. Thanks to a self-similarity property of the stable tree, we show that the fragmentation process is also self-similar. The semigroup and other features of the fragmentation are given explicitly. Asymptotic results are given, as well as a couple of related results on continuous-state branching processes.

1. Introduction

The recent advances in the study of coalescence and fragmentation processes pointed at the key role played by tree structures in this topic, both at the discrete and continuous level [15, 3, 4]. Our goal here is to push further the investigation, begun in [3, 9], of a category of fragmentations obtained by cutting a certain continuum random tree. The tree that was fragmented in the latter articles is the Brownian Continuum Random Tree of Aldous, and the fragmentation is related to the so-called *standard additive coalescent*. The family of trees we consider is a natural but technically involved "Lévy generalization" of the Brownian tree. It has been introduced in Duquesne and Le Gall [14], and implicitly considered in the previous work of Kersting [18]. Some of these trees, which we call the *stable trees*, enjoy certain self-similar properties as their Brownian companion. In the present work the crucial property is that when removing the vertices of the stable tree located under a fixed *height* (or distance to the root), the remaining object is a forest of smaller trees that have the same law as the original one up to rescaling. This is formalized in Lemma 3 below. This way of logging the stable tree induces a *fragmentation process* which by the property explained above turns out to be a *self-similar fragmentation*, the theory of such processes being extensively studied by Bertoin [8–10]. The goal of this paper is to describe the characteristics and

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give some properties of this fragmentation process. We will have to use stochastic processes and combinatorial approaches in the same time; in particular, we will encounter σ-finite generalizations of the (α, θ)-partitions of [26], which are distributions on the set of partitions of $\mathbb{N} = \{1, 2, \dots\}$, as well as we will need the construction of the stable tree out of stable Lévy processes and its connection to continuous-state branching processes (CSBP) explained in [14].

In a companion paper [23] we will consider another way of obtaining a self-similar fragmentation by another cutting device on the stable tree, using the heuristic fact that when cutting at random one node (or "hub") in the the stable tree, the trunk and branches that have been separated are scaled versions of the initial tree. Surprisingly, although this other device looks quite different from the first (no mass is lost when cutting hubs, whereas there is a loss of mass when we throw everything that is located under the height h), it turns out that the only difference between these two fragmentations is the speed at which fragments decay, hence generalizing a "duality" relation stressed by Bertoin in [9] between two different fragmentations of the Brownian tree (one of these fragmentations being a direct analog of the fragmentation F^- considered here).

To state our main results, let us introduce quickly the already mentioned tree structures and fragmentation processes, postponing the details to a further section.

Let $S = \{\mathbf{s} = (s_1, s_2, \dots) : s_1 \geq s_2 \geq \dots \geq 0, \sum_{i \geq 1} s_i \leq 1\}$, endowed with the topology of pointwise convergence. A ranked self-similar fragmentation process $(F(t), t \geq 0)$ with index $\beta \in \mathbb{R}$ is a S-valued Markov process that is continuous in probability, such that $F(0) = (1, 0, 0, \dots)$ and such that conditionally on $F(t) = (x_1, x_2, \dots)$, $F(t + t')$ has the law of the decreasing arrangement of the sequences $x_i F^{(i)}(x_i^\beta t')$, where the $F^{(i)}$ are independent with the same law as F. That is, after time t, the different fragments evolve independently with a speed that depends on their size. It has been shown in [9] that such fragmentations are characterized by a 3-tuple (β, c, ν), where β is the index, $c \geq 0$ is an "erosion" real constant saying that the fragments may melt continuously at some rate depending on c, and ν is a σ-finite measure on S that attributes mass 0 to $(1, 0, \dots)$ and that integrates $\mathbf{s} \mapsto (1 - s_1)$. This measure governs the sudden dislocations in the fragmentation process, and the integrability assumption ensures that these dislocations do not occur too quickly, although the fragmentation epochs may form a dense subset of $\mathbb{R}_+$ as soon as $\nu(S) = +\infty$. When $\beta < 0$, a positive fraction of the mass can disappear within a finite time, even though there is no loss of mass due to erosion nor to sudden dislocations. This phenomenon will be crucial in the fragmentation F^- below.

The trees we are considering are continuum random trees. Intuitively, they are metric spaces with an "infinitely ramified" tree structure, which can be considered as genealogical structures combined with two measures: a σ-finite *length measure* supported by the "skeleton" of the tree and a finite *mass measure* supported by its leaves, which are everywhere dense in the tree. These trees can be defined in several equivalent ways:

- as a weak limit of Galton-Watson trees
- through its *height process* H, which is a positive continuous process on $[0, 1]$. To a point $u \in [0, 1]$ corresponds a vertex of the tree with height (distance to the

root) equal to H_u, and the mass measure on the tree is represented by Lebesgue's measure on [0, 1]
- through its explicit "marginals", that is, the laws of subtrees spanned by a random sample of leaves.

We will have to use the second (stochastic process) and third (combinatorial) points of view. We know from the works of Duquesne and Le Gall [14] and Duquesne [13] that one may define a particular instance of tree, called the *stable tree with index α* (for some $\alpha \in (1, 2]$). When $\alpha = 2$, the stable tree is equal to the Brownian CRT of Aldous [2], in which case the height process is a Brownian excursion conditioned to have duration 1. We will recall the rigorous construction of the height process of the stable tree in Sect. 2.2, but let us state our results now. Fix $\alpha \in (1, 2)$ and let $(H_s, 0 \le s \le 1)$ be the height process of the stable tree with index α.

The fragmentation process, that we call F^-, is defined as follows. For each $t \ge 0$, let $I_-(t)$ be the open subset of (0, 1) defined by

$$I_-(t) = \{s \in (0, 1) : H_s > t\}.$$

With our intuitive interpretation of the height process, $I_-(t)$ is the set of vertices of the tree with height $> t$. We denote by $F^-(t)$ the decreasing sequence of the lengths of the connected components of $I_-(t)$. Hence, $F^-(t)$ is the sequence of the masses of the tree components obtained by cutting the stable tree below height t. The boundedness of H implies that $F^-(t) = (0, 0, \dots)$ as soon as $t \ge \max_{0\le s\le 1} H_s$. As mentioned above, F^- is a direct generalization of the fragmentation F in [9, Section 4]. However, the nature of F^- strongly differs from that of F, because the latter is *binary* (a fragment breaks into exactly two fragments when a sudden dislocation occurs, which one expresses by $\nu\{\mathbf{s} : s_1 + s_2 < 1\} = 0$ where ν is the dislocation measure of F), while F^- is *infinitary* (the dislocation measure ν_- satisfies $\nu_-\{\mathbf{s} : s_N = 0\} = 0$ for every $N \ge 1$). This difference is due to the fact that the Brownian tree is itself binary, a property one can deduce from the well-known fact that local infima of the Brownian motion are pairwise distinct. By contrast, as explained below, the local infima of the height process of (non-Brownian) stable trees are attained at an infinite number of locations, so the stable trees are infinitary (see the construction of the tree out of its height process in Sect. 2.2).

Proposition 1. *The process F^- is a ranked self-similar fragmentation with index $1/\alpha - 1 \in (-1/2, 0)$ and erosion coefficient* 0.

Notice that, as mentioned before, F^- loses some mass, and eventually disappears completely in finite time even though the erosion is 0. This is due, of course, to the fact that the self-similarity index is negative.

Our main result is a description of the dislocation measure $\nu_-(\mathrm{d}\mathbf{s})$ of F^-. Let us introduce some notation. For $\alpha \in (1, 2)$, let $(T_x, x \ge 0)$ be a stable subordinator with Laplace exponent $\lambda^{1/\alpha}$, that is, $(T_x, x \ge 0)$ has the same law as $(\sum_{y_i\le x} r_i, x \ge 0)$, where $(y_i, r_i, i \ge 1)$ are the atoms of a Poisson point measure on $\mathbb{R}_+ \times (0, \infty)$ with intensity $c_\alpha \mathrm{d}y\mathrm{d}r/r^{1+1/\alpha}$, where $c_\alpha = (\alpha\Gamma(1 - 1/\alpha))^{-1}$. We denote by $\Delta T_x = T_x - T_{x-}$ the jump at level x and by $\Delta T_{[0,x]}$ the sequence of the jumps of T before time x, and ranked in decreasing order. Define the measure ν_α on S by

$$\nu_\alpha(d\mathbf{s}) = E\left[T_1\,;\ \frac{\Delta T_{[0,1]}}{T_1} \in d\mathbf{s}\right] \tag{1}$$

where the last expression means that for any positive measurable function G, the quantity $\nu_\alpha(G)$ is equal to $E[T_1\, G(T_1^{-1}\Delta T_{[0,1]})]$.

Theorem 1. *The dislocation measure of F^- is $\nu_- = D_\alpha \nu_\alpha$, where*

$$D_\alpha = \frac{\alpha(\alpha-1)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(2-\alpha)} = \frac{\alpha^2\Gamma\left(2-\frac{1}{\alpha}\right)}{\Gamma(2-\alpha)}.$$

Some comments about this. First, the dislocation measure charges only the sequences $\mathbf{s}$ for which $\sum_{i\geq 1} s_i = 1$, that is, no mass can be lost within a sudden dislocation. Second, we recognize an expression close to [27], of a Poisson-Dirichlet type distribution. Recall from [29, 27] that the (β, θ) Poisson-Dirichlet distribution is the law on S given by

$$\mathrm{PD}(\beta,\theta)(d\mathbf{s}) = \frac{\Gamma(\theta+1)}{\Gamma(\theta/\beta+1)} E\left[(T_1^\beta)^{-\theta};\ \frac{\Delta T^\beta_{[0,1]}}{T^\beta_1} \in d\mathbf{s}\right], \tag{2}$$

where T^β is a β-stable subordinator with Laplace exponent λ^β, and the definition makes sense if $\beta \in (0,1)$ and $\theta > -\beta$. With this notation, $\nu_\alpha(d\mathbf{s})$ looks like a "renormalized Poisson-Dirichlet $(1/\alpha, -1)$ distribution". However, it has to be noticed that this corresponds to a forbidden parametrization $\theta = -1$, and indeed, the measure that we obtain is infinite since $E[T_1] = \infty$. This measure integrates $\mathbf{s} \mapsto 1 - s_1$ though, just as it has to. Indeed, $E[T_1 - \Delta_1]$ is finite if Δ_1 denotes the largest jump of T before time 1. To see this, notice that $\Delta_1 \geq \Delta_1^*$ where Δ_1^* is a size-biased pick from the jumps of T before time 1, and it follows from Lemma 1 in Sect. 2.1 below and scaling arguments that $T_1 - \Delta_1^*$ has finite expectation.

The rest of the paper is organized as follows. In Sect. 2 we first recall some facts about Lévy processes, excursions, and conditioned subordinators. Then we give the rigorous description of the stable tree, and state some properties of the height process that we will need. Last we recall some facts about self-similar fragmentations. We then obtain the characteristics of F^- in Sect. 3 and derive its semigroup. We insist on the fact that knowing explicitly the semigroup of a fragmentation process is in general a very complicated problem, see [24] for somehow surprising negative results in this vein. However, most of the fragmentation processes that have been extensively studied in recent years [3, 7, 22, 9] do have known, and sometimes strange-looking semigroups involving conditioned Poisson clouds. And as a matter of fact, the fragmentation F^+ considered in the companion paper [23] has also an explicit semigroup. We end the study of F^- by giving asymptotic results for small times in Sect. 4. These results need some properties of conditioned continuous-state branching processes, which are in the vein of Jeulin's results for the rescaled Brownian excursion and its local times. We prove these properties in Sect. 5, where we give the rigorous definition of some processes that are used heuristically in Sect. 3 to conjecture the form of the dislocation measure.

2. Preliminaries

2.1. Stable processes, excursions, conditioned inverse subordinator

Throughout the paper, we let $(X_s, s \geq 0)$ be the canonical process in the Skorokhod space $\mathbb{D}([0,\infty))$ of càdlàg paths on $[0,\infty)$. Recall that a Lévy process is a real-valued càdlàg process with independent and stationary increments. We fix $\alpha \in (1,2)$. Let P be the law that makes X a stable Lévy process with no negative jumps and Laplace exponent $E[\exp(-\lambda X_s)] = \exp(\lambda^\alpha)$ for $s, \lambda \geq 0$, where E is the expectation associated with P. Such a process has infinite variation and satisfies $E[X_1] = 0$. When there is no ambiguity, we may sometimes speak of X as being itself the Lévy process with law P. Writing this in the form of the Lévy-Khintchine formula, we have :

$$E[\exp(-\lambda X_s)] = \exp\left(s\int_0^\infty \frac{C_\alpha \mathrm{d}x}{x^{1+\alpha}}(e^{-\lambda x} - 1 + \lambda x)\right), \quad s, \lambda \geq 0, \tag{3}$$

where $C_\alpha = \alpha(\alpha-1)/\Gamma(2-\alpha)$. That is, the Lévy measure of X under P is $C_\alpha x^{-1-\alpha}\mathrm{d}x\mathbf{1}_{\{x>0\}}$. An important property of X is then the scaling property: under P,

$$\left(\frac{1}{\lambda^{1/\alpha}}X_{\lambda s}, s \geq 0\right) \stackrel{d}{=} (X_s, s \geq 0) \qquad \text{for all } \lambda > 0.$$

It is known [31] that under P, X_s has a density $(p_s(x), x \in \mathbb{R})$ for every $s > 0$, such that $p_s(x)$ is jointly continuous in x and s.

Excursions Let $\underline{X}$ be the infimum process of X, defined for $s \geq 0$ by

$$\underline{X}_s = \inf\{X_u, 0 \leq u \leq s\}.$$

By Itô's excursion theory for Markov processes, the excursions away from 0 of the process $X - \underline{X}$ under P are distributed according to a Poisson point process that can be described by the Itô excursion measure, which we call N. We now either consider the process X under the law P that makes it a Lévy process starting at 0, or under the σ-finite measure N under which the sample paths are excursions with finite lifetime ζ (since $E[X_1] = 0$). Let $N^{(v)}$ be a regular version of the probability law $N(\cdot|\zeta = v)$, which is weakly continuous in v. That is, for any positive continuous functional G,

$$N(G) = \int_{(0,\infty)} N(\zeta \in \mathrm{d}v)N^{(v)}(G)$$

and $\lim N^{(w)}(G) = N^{(v)}(G)$ as $w \to v$. Such a version can be obtained by scaling: for any fixed $\eta > 0$, the process

$$\left((v/\zeta)^{1/\alpha}X_{\zeta s/v}, 0 \leq s \leq v\right) \qquad \text{under } N(\cdot|\zeta > \eta) = \frac{N(\cdot, \zeta > \eta)}{N(\zeta > \eta)}$$

is $N^{(v)}$. See [12] for this and other interesting ways to obtain processes with law $N^{(v)}$ by path transformations. In particular, one has the scaling property at the level of conditioned excursions: under $N^{(v)}$, $\left(v^{-1/\alpha}X_{vs}, 0 \leq s \leq 1\right)$ has law $N^{(1)}$.

First-passage subordinator Let T be the right-continuous inverse of the increasing process $-\underline{X}$, that is,

$$T_x = \inf\{s \geq 0 : \underline{X}_s < -x\}.$$

Then it is known that under P, T is a subordinator, that is, an increasing Lévy process. According to [6, Theorem VII.1.1], its Laplace exponent ϕ is the inverse function of the restriction of the Laplace exponent of X to $\mathbb{R}_+$. Thus $\phi(\lambda) = \lambda^{1/\alpha}$, and T is a stable subordinator with index $1/\alpha$, as defined above. The Lévy-Khintchine formula gives, for $\lambda, x \geq 0$,

$$E[\exp(-\lambda T_x)] = \exp(-x\lambda^{1/\alpha}) = \exp\left(-x\int_0^\infty \frac{c_\alpha \mathrm{d}y}{y^{1+1/\alpha}}(1-e^{-\lambda y})\right).$$

where c_α has been defined in the introduction. Recall that X has a marginal density $p_s(\cdot)$ at time s under P. Then under P, the inverse subordinator T has also jointly continuous densities, given by (see e.g. [6, Corollary VII.1.3])

$$q_x(s) = \frac{P(T_x \in \mathrm{d}s)}{\mathrm{d}s} = \frac{x}{s}p_s(x). \tag{4}$$

This equation can be derived from the ballot theorem of Takács [32].

Let us now discuss the conditioned forms of distributions of the sequence $\Delta T_{[0,x]}$ given T_x. An easy way to obtain nice regular versions for these conditional laws is developed in [25, 27], and uses the notion of size-biased fragment. Precisely, the range of any subordinator, with drift 0 say (which we will assume in the sequel), between times 0 and x, induces a partition of $[0, T_x]$ into subintervals with sum T_x. Consider a sequence $(U_i, i \geq 1)$ of independent uniform $(0, 1)$ variables, independent of T, and let $\Delta_1^*(x), \Delta_2^*(x), \ldots$ be the sequence of the lengths of these intervals in the order in which they are discovered by the U_i's. That is, $\Delta_1^*(x)$ is the length of the interval in which $T_x U_1$ falls, $\Delta_2^*(x)$ is the length of the first interval different from the one containing $T_x U_1$ in which $T_x U_i$ falls, and so on. Then Palm measure results for Poisson clouds give the following result (specialized to the case of stable subordinators).

Lemma 1. *The joint law under P of $(\Delta_1^*(x), T_x)$ is*

$$P(\Delta_1^*(x) \in \mathrm{d}y, T_x \in \mathrm{d}s) = \frac{c_\alpha x q_x(s-y)}{s y^{1/\alpha}}\mathrm{d}y\mathrm{d}s, \tag{5}$$

and more generally for $j \geq 1$,

$$P\left(\Delta_j^*(x) \in \mathrm{d}y \,\middle|\, T_x = s_0, \Delta_k^*(x) = s_k, 1 \leq k \leq j-1\right) = \frac{c_\alpha x q_x(s-y)}{s y^{1/\alpha} q_x(s)}\mathrm{d}y,$$

where $s = s_0 - s_1 - \ldots - s_{j-1}$.

This gives a nice regular conditional version for $(\Delta_i^*(x), i \geq 1)$ given T_x, and thus induces a conditional version for $\Delta T_{[0,x]}$ given T_x, by ranking, where $\Delta T_{[0,x]}$ is the sequence of jumps of T before x, ranked in decreasing order of magnitude.

2.2. *The stable tree*

We now introduce the models of trees we will consider. This section is mainly inspired by [14, 13]. With the notations of Sect. 2.1, for $u \geq 0$, let $R^{(u)}$ be the time-reversed process of X at time u:

$$R_s^{(u)} = X_u - X_{(u-s)-} \quad , \quad 0 \leq s \leq u.$$

It is standard that this process has the same law as X killed at time u under P. Let also

$$\overline{R}_s^{(u)} = \sup_{0 \leq v \leq s} R_v^{(u)} \quad , \quad 0 \leq s \leq u$$

be its supremum process. We let H_u be the local time at 0 of the process $R^{(u)}$ reflected under its supremum $\overline{R}^{(u)}$ up to time u. The normalization can be chosen so that we have the limit in probability

$$H_u = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^u \mathbf{1}_{\{\overline{R}_s^{(u)} - R_s^{(u)} \leq \varepsilon\}} \mathrm{d}s.$$

It is known by [14, Theorem 1.4.3] that H admits a continuous version, with which we shall work in the sequel. It has to be noticed that H is not a Markov process (the only exception in the theory of Lévy trees is the Brownian tree obtained when P is the law of Brownian motion with drift, which has been excluded in our discussion). As a matter of fact, it can be checked that under P, H admits local minima that are attained an infinite number of times, a property that strongly contrasts with Brownian motion or Lévy processes with infinite variation. To see this, consider a jump time t of X, and let $t_1, t_2 > t$ so that $\inf_{t \leq u \leq t_i} X_u = X_{t_i}$ and $X_{t-} < X_{t_i} < X_t$, $i \in \{1, 2\}$. Then it is easy to see that $H_t = H_{t_1} = H_{t_2}$ and that one may in fact find an infinite number of distinct t_i's satisfying the properties of t_1, t_2. On the other hand, it is not difficult to see that H_t is a local minimum of H. One can in fact deduce from the fact that F^- is infinitary that *every* local minimum is attained an infinite number of times, as mentioned in the introduction.

It is shown in [14] that the definition of H still makes sense under the σ-finite measure N rather than the probability law P. The process H is then defined only on $[0, \zeta]$, and we call it the *excursion of the height process*. One can define without difficulty, using the scaling property, the height process under the laws $N^{(v)}$: this is simply the law of

$$\left((v/\zeta)^{1-1/\alpha} H_{\zeta t/v}, 0 \leq t \leq v\right) \qquad \text{under } N(\cdot|\zeta > \eta).$$

Call it the law of the excursion of the height process with duration v. The following scaling property is the key for the self-similarity of F^-: for every $x > 0$,

$$(v^{1/\alpha-1} H_{sv}, 0 \leq s \leq 1) \text{ under } N^{(v)} \stackrel{d}{=} (H_s, 0 \leq s \leq 1) \text{ under } N^{(1)}. \qquad (6)$$

This property is inherited from the scaling property of X, and it is easily obtained e.g. by the above definition of H as an approximation.

An important tool for studying the height process is its *local time process*, or *width process*, which we will denote by $(L^t_s, t \geq 0, s \geq 0)$. It can be obtained for every fixed s, t by the limit in probability

$$L^t_s = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t < H_u \leq t+\varepsilon\}} \mathrm{d}u.$$

L^t_s is then the density of the occupation measure of H at level t and time s. For $t = 0$, one has that $(L^0_s, s \geq 0)$ is the inverse of the subordinator T, which is a reminiscent of the fact that the excursions of the height process are in one-to-one correspondence with excursions of X with the same lengths. According to the Ray-Knight theorem [14, Theorem 1.4.1], for every $x > 0$, the process $(L^t_{T_x}, t \geq 0)$ is a continuous-time branching process with branching mechanism λ^α, in short α-CSBP. We will recall basic and less basic features about this processes in Sect. 5, where in particular an interpretation for the law of the process $(L^t_1, t \geq 0)$ under $N^{(1)}$ will be given. For now we just note that for every x the process $(L^t_{T_x}, t \geq 0)$ is a process with no negative jumps, and a jump of this process at time t corresponds precisely to one of the infinitely often attained local infima of the height process. With the forthcoming interpretation of the tree encoded within excursions of the height process, this means that there is a branchpoint with infinite degree at level t. It is again possible to define the local time process under the excursion measure N, and by scaling it is also possible to define the local time process under $N^{(v)}$.

Let us now motivate the term of "height process" for H. Under the σ-finite "law" N, we define a tree structure following [2, 21].

First we introduce some extra vocabulary. Let $\mathbf{T}$ be the set of finite rooted plane trees, that is, for any $\mathcal{T} \in \mathbf{T}$, each set of children of a vertex $v \in \mathcal{T}$ is ordered as first, second, ..., last child. Let $\mathbf{T}^* \subset \mathbf{T}$ be those rooted plane trees for which the out-degree (number of children) of vertices is never 1. Let $\mathbf{T}_n$ and $\mathbf{T}^*_n$ be the corresponding sets of trees that have exactly n leaves (vertices with out-degree 0). A marked tree ϑ is a pair $(\mathcal{T}, \{h_v, v \in \mathcal{T}\})$ where $\mathcal{T} \in \mathbf{T}$ and $h_v \geq 0$ for every vertex v of $\mathcal{T}$ (which we denote by $v \in \mathcal{T}$). The tree $\mathcal{T}$ is called the skeleton of ϑ, and the h_v's are the marks. These marks induce a distance on the tree, given by $d_\vartheta(v, v') = \sum_{w \in [[v, v']]} h_w$ if $v, v' \in \mathcal{T}$ are two vertices of the marked tree, where $[[v, v']]$ is the set of vertices of the path from v to v' in the skeleton. The distance of a vertex to the root will be called its *height*. Let $\mathbb{T}^*_n$ be the set of marked trees with n leaves and no out-degree equal to 1.

Let $(U_i, i \geq 1)$ be independent random variables with uniform law on $(0, 1)$ and independent of the excursion H of the height process. One may define a random marked tree $\vartheta(U_1, \dots, U_k) = \vartheta_k \in \mathbb{T}^*_k$, as follows. For $u, v \in [0, \zeta]$ let $m(u, v) = \inf_{s \in [u,v]} H_s$. Roughly, the key fact about ϑ_k is that the height of the i-th leaf to the root is $H_{U_{(i)}}$, where $(U_{(i)}, 1 \leq i \leq k)$ are the order statistics of $(U_i, 1 \leq i \leq k)$, and the ancestor of the i-th and j-th leaves has height $m(\zeta U_{(i)}, \zeta U_{(j)})$ for every i, j. This allows to build recursively a tree by first putting the mark $h_{\mathrm{root}} = \inf_{1 \leq i \leq j \leq k} m(U_i, U_j)$ on a root vertex. Let c_{root} be the number of excursions of H above level h_{root} in which at least one ζU_i falls. Attach c_{root} vertices to the root, and let the i-th of these vertices be the root of the tree embedded in the

i-th of these excursions above level h_{root}. Go on until the excursions separate the variables U_i. By construction $\vartheta_k \in \mathbb{T}_k^*$. Adding a $(k+1)$-th variable U_{k+1} to the first k just adds a new branch to the tree in a consistent way as k varies.

As noted above, we may as well define the trees $(\vartheta_k, k \geq 0)$ under the law $N^{(1)}$ by means of scaling.

Definition 1. *The family of marked trees* $(\vartheta_k, k \geq 1)$ *associated with the height process under the law* $N^{(1)}$ *is called the stable tree.*

Remark 1. The previous definition is not the only way to characterize the same object. Alternatively, one easily sees that the marked tree ϑ_k can be interpreted as a subset of l^1, each new branch going in a direction orthogonal to the preceding branches, in a consistent way as k varies. Then it makes sense to take the metric completion of $\cup_{k\geq 1}\vartheta_k$, which we could also call the stable tree, and one can check that the branchpoints of this tree all have infinite degree because the local minima of H are attained an infinite number of times. This object is also isometric to the space obtained by taking the quotient of $[0, 1]$ endowed with the pseudo-metric

$$d(u, v) = H_u + H_v - 2m(u, v), \quad u, v \in [0, 1],$$

with respect to the equivalence relation $u \equiv v \iff d(u, v) = 0$. With this way of looking at things, the leaves of the tree are uncountable and everywhere dense in the tree, and the empirical distribution on the leaves of ϑ_k converges weakly to a probability measure on the stable tree, called the mass measure. Then it turns out that ϑ_k is equal in law to the subtree of the stable tree that is spanned by the root and k independent leaves distributed according to the mass measure. Hence, the mass measure is represented by Lebesgue measure on $[0, 1]$ in the coding of the stable tree through its height process. This is coherent with the definition of $F^-(t)$ as the "masses of the tree components located above height t". The equivalence between these possible definitions is discussed in [2].

The key property for obtaining the dislocation measure of F^- is the following description of the law of the skeleton of ϑ_n, and the mark of the root of ϑ_1. For $\mathcal{T} \in \mathbf{T}$ let $\mathcal{N}_\mathcal{T}$ be the set of non-leaf vertices of $\mathcal{T}$ and for $v \in \mathcal{N}_\mathcal{T}$ let $c_v(\mathcal{T})$ be the number of children of v. From the more complete description of the marked trees in [14, Theorem 3.3.3], we recall that

Proposition 2. *The probability that the skeleton of* ϑ_n *is* $\mathcal{T} \in \mathbf{T}_k^*$ *is*

$$\frac{n!}{(\alpha-1)(2\alpha-1)\ldots((n-1)\alpha-1)} \prod_{v\in\mathcal{N}_\mathcal{T}} \frac{|(\alpha-1)(\alpha-2)\ldots(\alpha-c_v(\mathcal{T})+1)|}{c_v(\mathcal{T})!}.$$

Moreover, the law of the mark of the root in ϑ_1 *is*

$$N^{(1)}(H_{U_1} \in \mathrm{d}h) = \alpha\Gamma\left(1-\frac{1}{\alpha}\right)\chi_{\alpha h}(1)\mathrm{d}h,$$

where $(\chi_x(s), s \geq 0)$ *is the density of the stable* $1-1/\alpha$ *subordinator (with Laplace exponent equal to* $\lambda^{1-1/\alpha}$*) at time* x.

2.3. Some results on self-similar fragmentations

In this section we are going to recall some basic facts about the theory of self-similar fragmentations, and also introduce some useful ways to recover the characteristics of these fragmentations. We will suppose that the fragmentations we consider are not trivial, that is, they are not equal to their initial state for every time. It will be useful to consider not only S-valued (or *ranked*) fragmentations, but also fragmentations with values in the set of open subsets of (0, 1) and in the set of partitions of $\mathbb{N} = \{1, 2, \dots\}$, respectively called *interval* and *partition-valued* fragmentations. As established in [9, 5], there is a one-to-one mapping between the laws of the three kinds of fragmentation when they satisfy a self-similarity property that is similar to that of the ranked fragmentations. That is, each of them is characterized by the same 3-tuple (β, c, ν) introduced above. To be completely accurate, we should stress that there actually exist several versions of interval partitions that give the same ranked or partition-valued fragmentation, but all these versions have the same characteristics (β, c, ν). Let us make the terms precise.

Let $\mathcal{P}$ be the set of unordered partitions of $\mathbb{N}$. An exchangeable partition Π is a $\mathcal{P}$-valued random variable whose restriction Π_n to $[n] = \{1, \dots, n\}$ has an invariant law under the action of the permutations of $[n]$, for every n. By Kingman's representation theorem [19, 1], the blocks of exchangeable partitions of $\mathbb{N}$ admit almost-sure asymptotic frequencies, that is, if $\Pi = \{B_1, B_2, \dots\}$ where the B_i's are listed by order of their least element, then

$$\Lambda(B_i) = \lim_{n\to\infty} \frac{\mathrm{Card}\,(B_i \cap [n])}{n}$$

exists a.s. for every $i \geq 0$. Denoting by $\Lambda(\Pi)$ the ranked sequence of these asymptotic frequencies, $\Lambda(\Pi)$ is then a S-valued random variable, whose law characterizes that of Π.

A self-similar partition-valued fragmentation $(\Pi(t), t \geq 0)$ with index β is a $\mathcal{P}$-valued càdlàg process that is continuous in probability, exchangeable, meaning that for every permutation σ of $\mathbb{N}$, $(\sigma\Pi(t), t \geq 0)$ and $(\Pi(t), t \geq 0)$ have the same law, and such that given $\Pi(t) = \{B_1, B_2, \dots\}$, the variable $\Pi(t + t')$ has the law of the partition with blocks $\Pi^{(i)}(\Lambda(B_i)^\beta t') \circ B_i$ where the $\Pi^{(i)}$ are independent copies of Π. Here, the operation $\circ$ is the natural "fragmentation" operation of a set by a partition: if $\Pi = \{B_1, B_2, \dots\}$ and $C \subset \mathbb{N}$, then $\Pi \circ C$ is the partition of C with blocks $B_i \cap C$.

A self-similar interval partition $(I(t), t \geq 0)$ with index β is a process with values in the open subsets $\mathcal{O}$ of (0, 1) which is right-continuous and continuous in probability for the usual Hausdorff distance between the complementary sets $[0, 1]\backslash\mathcal{O}$, with the property that given $I(t) = \cup_{i\geq 1} I_i$ say, where the I_i are the disjoint connected components of $I(t)$, the set $I(t+t')$ has the law of $\cup_{i\geq 1} g_i(I^{(i)}(t'|I_i|^\beta))$, where $|I_i|$ is the length of I_i, g_i is the affine transformation that maps (0, 1) to I_i and conserves orientation and the $I^{(i)}$ are independent copies of I.

Consider an interval self-similar fragmentation $(I(t), t \geq 0)$, with characteristics $(\beta, 0, \nu)$ (the case when $c > 0$ would be similar, but we do not need it in the sequel). Let $U_i, i \geq 1$ be independent uniform random variables on (0, 1). These induce a partition-valued fragmentation $(\Pi(t), t \geq 0)$ by letting $i \overset{\Pi(t)}{\sim} j$ iff U_i and U_j are in the same connected component of $I(t)$. It is known [9] that Π is a self-similar fragmentation with values in the set of partitions of $\mathbb{N}$ and characteristics $(\beta, 0, \nu)$. For $n \geq 2$ let $\mathcal{P}_n^*$ be the set of partitions of $\mathbb{N}$ whose restriction to $[n]$ is non-trivial, i.e. different from $\{[n]\}$. Then there is some random time $t_n > 0$ such that the restriction of $\Pi(t)$ to $[n]$ jumps from the trivial state $\{[n]\}$ to some non-trivial state at time t_n. Let $\rho(n)$ be the law of the restriction of $\Pi(t_n)$ to $[n]$. The next lemma states that the knowledge of the family $(\rho(n), n \geq 2)$ almost determines the dislocation measure ν of the fragmentation. Precisely, we introduce from [8] the notion of characteristic measure of the fragmentation. This measure, denoted by κ, is a σ-finite measure supported by the non-trivial partitions of $\mathbb{N}$, which is determined by the dislocation measure of the fragmentation. This measure may be written as

$$\kappa(\mathrm{d}\pi) = \int_S \nu(\mathrm{d}\mathbf{s})\kappa_{\mathbf{s}}(\mathrm{d}\pi),$$

where $\kappa_{\mathbf{s}}$ is the law of the exchangeable partition of $\mathbb{N}$ with ranked asymptotic frequencies given by $\mathbf{s}$. Conversely, this measure characterizes the dislocation measure ν (simply by taking the asymptotic frequencies of the generic partition under κ).

Lemma 2. *The restriction of κ to the non-trivial partitions of $[n]$, for $n \geq 2$, equals $q(n)\rho(n)$, for some sequence $(q(n), n \geq 2)$ of strictly positive numbers. As a consequence, the dislocation measure of the fragmentation I is characterized by the sequence of laws $(\rho(n), n \geq 2)$, up to a multiplicative constant.*

Otherwise said, and using the correspondence between self-similar fragmentations with same dislocation measure and different indices established by Bertoin [9] by introducing the appropriate time-changes, if we have two interval-valued self-similar fragmentations I and I' with the same index and no erosion, and with the same associated probabilities $\rho(n)$ and $\rho'(n)$, $n \geq 1$, then there exists $K > 0$ such that $(I(Kt), t \geq 0)$ has the same dislocation measure as I'.

Proof. Suppose $\beta = 0$, then the result is almost immediate by the results of [8] on homogeneous fragmentation processes. In this case $q(n)$ is the inverse of the expected jump time of Π in $\mathcal{P}_n^*$, and the restriction of the measure $q(n+1)\rho(n+1)$ to the set of non-trivial partitions of $[n]$ is $q(n)\rho(n)$, for every $n \geq 1$. Hence, it is easy to see that the knowledge on $\rho(n)$ determines uniquely the sequence $(q(n), n \geq 1)$, up to a multiplicative positive constant: one simply has $q(n)/q(n+1) = \rho(n+1)(\pi|_{[n]} : \pi \in \mathcal{P}_n^*)$, where $\pi|_{[n]}$ denotes the restriction of π to $[n]$. It remains to notice that the sequence of restrictions $(q(n)\rho(n), n \geq 2)$ characterizes κ.

When $\beta \neq 0$, we obtain the same results by noticing that the law $\rho(n)$ still equals the law of the restriction to $[n]$ of the exchangeable partition with limiting frequencies having the "law" ν and restricted to $\mathcal{P}_n^*$, up to a multiplicative constant. Indeed, let $I^*(t)$ be the subinterval of $I(t)$ containing U_1 at time t, and recall [9] that if

$$a(t) = \inf\left\{u \geq 0 : \int_0^u |I^*(v)|^\beta \mathrm{d}v > t\right\},$$

then $(|I^*(a(t))|, t \geq 0)$ evolves as the fragment containing U_1 in an interval fragmentation with characteristics $(0, 0, \nu)$. Now, before time t_n, the fragment containing U_1 is the same as that containing all the $(U_i, 1 \leq i \leq n)$. Hence, $a(t_n)$ is the first time when Π' jumps in $\mathcal{P}_n^*$ for some homogeneous partition-valued fragmentation process Π' with characteristics $(0, 0, \nu)$, and the law of $\Pi'(a(t_n))$ restricted to $[n]$ is $\rho(n)$. Hence the result. □

We also cite the following result [24, Proposition 3] which allows to recover the dislocation measure of a self-similar fragmentation with positive index out of its semigroup. We will not use this proposition in a proof, but it is useful to keep it in mind to conjecture the form of the dislocation measure of F^-, as it will be done below.

Proposition 3. *Let* $(F(t), t \geq 0)$ *be a ranked self-similar fragmentation with characteristics* $(\beta, 0, \nu)$*,* $\beta \geq 0$*. Then for every continuous bounded function G on S which is null on an open neighborhood of* $(1, 0, \dots)$*, one has*

$$\frac{1}{t} E[G(F(t))] \underset{t \downarrow 0}{\rightarrow} \nu(G).$$

3. Study of F^-

We now specifically turn to the study of F^- defined in the introduction. Although some of the results below may be easily generalized to a broader "Lévy context", we will suppose in this section that X is a stable process with index $\alpha \in (1, 2)$, with first-passage subordinator T. The references to height processes, excursion measures and so on, will always be with respect to this process, unless otherwise specified. Also, for the needs of the proofs below, we define the process $(F^-(t), t \geq 0)$ not only under the law $N^{(1)}$ used to define the stable tree, but also for all the excursion measures $N^{(v)}$ and N. Under $N^{(v)}$, let $F^-(t)$ be the decreasing sequence of lengths of the constancy intervals of $I_-(t) = \{s \in (0, v) : H_s > t\}$ (v is replaced by ζ under N). To avoid confusions, we will always mention in Sect. 3.1 the measure we are working with, but this formalism will be abandoned in the following sections where no more use of $N^{(v)}$ is made with $v \neq 1$.

The study contains four steps. First we prove the self-similarity property for F^- and make its semigroup explicit. Heuristic arguments based on generators of conditioned CSBP's allow us to conjecture the rough shape of the dislocation measure. Then we prove that the erosion coefficient is 0 by studying the evolution of a tagged fragment. We then apply Lemma 2, giving us the dislocation measure up to a constant, and we finally recover the constant by re-obtaining the results needed in the second step by another computation.

3.1. Self-similarity and semigroup

The self-similarity and the description of the semigroup rely strongly on the following result, which is a variant of [14, Proposition 1.3.1]. For $t, s \geq 0$ let

$$\gamma_s^t = \inf\left\{u \geq 0 : \int_0^u \mathbf{1}_{\{H_v > t\}} \mathrm{d}v > s\right\}$$

and

$$\widetilde{\gamma}_s^t = \inf\{u \geq 0 : \int_0^u \mathbf{1}_{\{H_v \leq t\}} \mathrm{d}v > s\}.$$

Denote by $\mathcal{H}_t$ the sigma-field generated by the process $(H_{\widetilde{\gamma}_s^t}, s \geq 0)$ and the P-negligible sets. Let also $(H_s^t, s \geq 0)$ be the process $(H_{\gamma_s^t} - t, s \geq 0)$. Then under P, H^t is independent of $\mathcal{H}_t$, and its law is the same as that of H under P.

As a first consequence, we obtain that the excursions of H above level t, that is, the excursions of H^t above level 0, are, conditionally on their durations, independent excursions of H. This simple result allows us to state the Markov property and self-similarity of F^-. In the following statement, it has to be understood that we work under the probability $N^{(1)}$ and that the process H that is considered is the same that is used to construct F^-.

Lemma 3. *Conditionally on $F^-(t) = (x_1, x_2, \dots)$, the excursions of H above level t, that is, of H^t away from* 0*, are independent excursions with respective laws $N^{(x_1)}, N^{(x_2)}, \dots$.*

As a consequence, the process F^- is a self-similar fragmentation process with index $1/\alpha - 1$.

Proof. By the previous considerations on H^t, we have that under P, given that the lengths of interval components of the set $\{s \in [0, T_1] : H_s > t\}$ ranked in decreasing order are equal to $(x_1, x_2, \dots)$, the excursions of the killed process $(H(t), 0 \leq t \leq T_1)$ above level t are independent excursions of H with durations $x_1, x_2, \dots$. The first part of the statement follows by considering the first excursion of H (or of X) that has duration greater than some $v > 0$, which gives the result under the measure $N(\cdot, \zeta > v)$, hence for N, hence for $N^{(v)}$ for almost all v, and then for $v = 1$ by continuity of the measures $N^{(v)}$.

Thus, conditionally on $F^-(t) = (x_1, x_2, \dots)$, the process $(F^-(t + t'), t \geq 0)$ has the same law as the random sequence obtained by taking independent excursions $H^{(x_1)}, H^{(x_2)}, \dots$ with durations $x_1, x_2, \dots$ of the height process, and then arranging in decreasing order the lengths of constancy intervals of the sets

$$\{s \in [0, x_i] : H_s^{(x_i)} > t'\}.$$

It thus follows from the scaling property (6) of the excursions of H that given $F^-(t) = (x_1, x_2, \dots)$, the process $(F^-(t + t'), t' \geq 0)$ has the same law as the decreasing rearrangement of the processes $(x_i F_{(i)}^-(x_i^{1/\alpha - 1} t'), t' \geq 0)$, where the $F_{(i)}^-$'s are independent copies of F^-. The fact that F^- is a Markov process that is continuous in probability easily follows, as does the self-similar fragmentation property with the index $1/\alpha - 1$. □

We now turn our attention to the semigroup of F^-.

Proposition 4. *For every $t \geq 0$ one has*

$$N^{(1)}(F^-(t) \in \mathrm{d}\mathbf{s}) = \int_{\mathbb{R}_+ \times [0,1]} N^{(1)} \left(L_1^t \in \mathrm{d}\ell, \int_t^\infty \mathrm{d}b \, L_1^b \in \mathrm{d}z \right) P\left(\Delta T_{[0,\ell]} \in \mathrm{d}\mathbf{s} \,|\, T_\ell = z \right), \quad (7)$$

with the convention that the law $P(\Delta T_{[0,0]} \in \mathrm{d}\mathbf{s} | T_0 = z)$ *is the Dirac mass at the sequence* $(z, 0, 0 \dots)$ *for every* $z \geq 0$.

Proof. It suffices to prove the result for some fixed $t > 0$. Let $\omega(t) = \inf\{s \geq 0 : H_s > t\}$, $d_{\omega(t)} = \inf\{s \geq \omega(t) : X_s = \underline{X}_s\}$ and $g_{\omega(t)} = \sup\{s \leq \omega(t) : X_s = \underline{X}_s\}$. Call $\mathcal{F}^-(t)$ the ranked sequence of the lengths of the interval components of the set $\{s \in [\omega(t), d_{\omega(t)}] : H_s > t\}$. Notice that under the law $N^{(1)}$, $\mathcal{F}^-$ would be F^-, but we will first define $\mathcal{F}^-$ under P. By the definition of H, $\omega(t)$ and $d_{\omega(t)}$ are stopping times with respect to the natural filtration generated by X. In fact, it also holds that $\omega(t)$ is a *terminal time*, that is,

$$\omega(t) = s + \inf\{u \geq 0 : H_{s+u} > t\} \quad \text{on } \{\omega(t) > s\}.$$

Moreover, $0 < \omega(t) < \infty$ P-a.s., because of the continuity of H and the fact that excursions of H have a positive probability to hit level t (which follows e.g. by scaling). Recall the notations at the beginning of the section, and denote by A^t and $\widetilde{A}^t$ the right-continuous inverses of γ^t and $\widetilde{\gamma}^t$. Then the local time $L^t_{d_{\omega(t)}}$ is the local time at level 0 and time $A^t_{d_{\omega(t)}}$ of the process H^t. This is also equal to the local time of $(H_{\widetilde{\gamma}^t_s}, s \geq 0)$ at level t and time $\widetilde{A}^t_{d_{\omega(t)}}$. This last time is $\mathcal{H}_t$-measurable, as it is the first time the process $(H_{\widetilde{\gamma}^t_s}, s \geq 0)$ hits back 0 after first hitting t. Hence $L^t_{d_{\omega(t)}}$ is $\mathcal{H}_t$-measurable, hence independent of H^t. Let T^t be the inverse local time of H^t at level 0, which is $\sigma(H^t)$-measurable, hence independent of $\mathcal{H}_t$, and has same law as T since H^t has same law as H under P. Notice that $\mathcal{F}^-(t)$ equals the sequence $\Delta T^t_{[0, L^t_{d_{\omega(t)}}]}$, and that the $\sigma(H^t)$-measurable random variable $\int_t^\infty \mathrm{d}b\, L^b_{d_{\omega(t)}} = T^t(L^t_{d_{\omega(t)}})$. Thus, conditionally on $L^t_{d_{\omega(t)}} = \ell$ and $\int_t^\infty \mathrm{d}b\, L^b_{d_{\omega(t)}} = z$, $\mathcal{F}^-(t)$ has law $P(\Delta T_{[0,\ell]} \in \mathrm{d}\mathbf{s} | T_\ell = z)$. Hence

$$\begin{aligned} P(\mathcal{F}^-(t) \in \mathrm{d}\mathbf{s}) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} & P\left(L^t_{d_{\omega(t)}} \in \mathrm{d}\ell\,,\ \int_t^\infty \mathrm{d}b\, L^b_{d_{\omega(t)}} \in \mathrm{d}z \right) \\ & \times P(\Delta T_{[0,\ell]} \in \mathrm{d}\mathbf{s} | T_\ell = z), \end{aligned}$$

and also, since $d_{\omega(t)} - g_{\omega(t)} = \int_0^\infty \mathrm{d}b\, (L^b_{d_{\omega(t)}} - L^b_{g_{\omega(t)}})$ and since $\int_0^t \mathrm{d}b\, (L^b_{d_{\omega(t)}} - L^b_{g_{\omega(t)}})$ is independent of $\sigma(H^t)$, the result also holds conditionally on $d_{\omega(t)} - g_{\omega(t)}$, namely

$$\begin{aligned} & P(\mathcal{F}^-(t) \in \mathrm{d}\mathbf{s} | d_{\omega(t)} - g_{\omega(t)}) \\ & = \int_{\mathbb{R}_+ \times \mathbb{R}_+} P\left(L^t_{d_{\omega(t)}} \in \mathrm{d}\ell\,,\ \int_t^\infty \mathrm{d}b\, L^b_{d_{\omega(t)}} \in \mathrm{d}z \,\middle|\, d_{\omega(t)} - g_{\omega(t)} \right) \\ & \quad \times P(\Delta T_{[0,\ell]} \in \mathrm{d}\mathbf{s} | T_\ell = z). \end{aligned}$$

Now notice that the excursion of H straddling time $\omega(t)$ is the first excursion of H that attains level t, and apply [30, Proposition XII.3.5] to obtain that

$$\begin{aligned} & P(\mathcal{F}^-(t) \in \mathrm{d}\mathbf{s} | d_{\omega(t)} - g_{\omega(t)} = v) \\ & = N^{(v)}(\zeta > \omega(t))^{-1}\, N^{(v)}(F_1^-(t) \in \mathrm{d}\mathbf{s},\ v > \omega(t)), \end{aligned}$$

and similarly

$$P\left(L^t_{d_{\omega(t)}} \in \mathrm{d}\ell\,,\ \int_t^\infty \mathrm{d}b\, L^b_{d_{\omega(t)}} \in \mathrm{d}z \,\middle|\, d_{\omega(t)} - g_{\omega(t)} = v\right)$$
$$= N^{(v)}(\zeta > \omega(t))^{-1}\, N^{(v)}\left(L^t_v \in \mathrm{d}\ell\,,\ \int_t^\infty \mathrm{d}b\, L^b_v \in \mathrm{d}z\,,\ v > \omega(t)\right),$$

for almost every v. Finally, notice that $\mathcal{F}^-(t) = F^-(t)$ under N and the $N^{(v)}$'s and that we may remove the indicator of $v > \omega(t)$ since a.s. under $N^{(v)}$, $L^t_v = 0$ if and only if $\max H \le t$, to obtain

$$N^{(v)}(F^-(t) \in \mathrm{d}\mathbf{s})$$
$$= \int_{\mathbb{R}+\times\mathbb{R}_+} N^{(v)}\left(L^t_v \in \mathrm{d}\ell,\ \int_t^\infty \mathrm{d}b\, L^b_v \in \mathrm{d}z\right) P(\Delta T_{[0,\ell]} \in \mathrm{d}\mathbf{s} | T_\ell = z).$$

Using scaling allows to take $v = 1$, entailing the claim. □

As a consequence of this result we may conjecture the shape of the dislocation measure of F^-. The next subsections will give essentially the rigorous proof of this conjecture, but finding ν_- directly from the forthcoming computations would certainly have been tricky without any former intuition. Roughly, suppose that the statement of Proposition 3 remains true for negative self-similarity indices (which is probably true, but we will not need it anyway). Then take G a bounded continuous function that is null on a neighborhood of $(1, 0, \dots)$ and write

$$N^{(1)}(G(F^-(t))) = \int_{\mathbb{R}_+\times[0,1]} N^{(1)}\left(L^t_1 \in \mathrm{d}x,\ \int_t^\infty \mathrm{d}b\, L^b_1 \in \mathrm{d}z\right)$$
$$\times E[G(\Delta T_{[0,x]}) | T_x = z].$$

Call $J(x, z)$ the expectation in the integral on the right hand side. Dividing by t and letting $t \downarrow 0$ should yield the generator of the $\mathbb{R}^2_+$-valued process $((L^t_1, \int_t^\infty \mathrm{d}b L^b_1), t \ge 0)$, evaluated at the function J and at the starting point $(0, 1)$. Now, we interpret (see Sect. 5 for definitions) the process $(L^t_1, t \ge 0)$ under $N^{(1)}$ as the α-CSBP conditioned both to start at 0 and stay positive, and to have a total progeny equal to 1. It is thus heuristically a Doob h-transform of the initial CSBP with harmonic function $h(x) = x$, and conditioned to come back near 0 when its integral comes near 1. Now as a consequence of Lamperti's time-change between CSBP's and Lévy processes, the generator of the CSBP started at x is $x\mathcal{L}(x, \mathrm{d}y)$ where $\mathcal{L}$ is the generator of the stable Lévy process with index α:

$$\mathcal{L}f(x) = \int_0^\infty \frac{C_\alpha \mathrm{d}y}{y^{\alpha+1}}(f(x+y) - f(x) - yf'(x)),$$

where f stands for a generic function in the Schwartz space. This, together with well-known properties for generators of h-transforms allows to conjecture that the generator $\mathcal{L}'$ of the CSBP conditioned to stay positive and started at 0 is given by

$$\mathcal{L}'f(0) = \int_0^\infty \frac{C_\alpha \mathrm{d}y}{y^\alpha}(f(y) - f(0)),$$

for a certain class of nice functions f, so roughly, the conditioned CSBP jumps at time 0+ to level y at rate $C_\alpha y^{-\alpha}\mathrm{d}y$. On the other hand, conditioning to come back to 0 when the progeny reaches 1 should introduce the extra term $q_y(1)$ (recall its definition (4)) in the integral with a certain coefficient, since the total progeny of a CSBP started at y is equal in law to T_y, as a consequence of Ray-Knight's theorem. To be a bit more accurate, the CSBP starting at y and conditioned to stay positive should be in $[0, \varepsilon]$ when its integral equals 1 with probability close to $g(\varepsilon)y^{-1}q_y(1)$ for some positive g with $g(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. Indeed, by the conditioned form of Lamperti's theorem of [20] to be recalled in Sect. 5, this is the same as the probability that the Lévy process started at y and conditioned to stay positive is in $[0, \varepsilon]$ at time 1. With the notations of Sect. 5, this is

$$P_y^{\uparrow}(X_1 \leq \varepsilon) = \int_0^\varepsilon xy^{-1}P_y(X_1 \in \mathrm{d}x, T_0 > 1).$$

We may expect that the quantity $P_y(X_1 \in \mathrm{d}x, T_0 > 1)$ can be expressed as $r(y,x)\mathrm{d}x$ with $r(y,x) \sim g'(x)q_y(1)$ as $x \downarrow 0$ for some g' vanishing at 0. Consequently, we expect that under $N^{(1)}$, the process $(L_1^t, t \geq 0)$ jumps at time 0+ to level $y > 0$ at rate $Cy^{-\alpha-1}q_y(1)\mathrm{d}y$ for some $C > 0$. This, thanks to Lemma 3, allows to conjecture the form of the dislocation measure as

$$\nu_-(G) = C\int_0^\infty \frac{\mathrm{d}y\, q_y(1)}{y^{\alpha+1}} E[G(\Delta T_{[0,y]})|T_y = 1]$$

for some $C > 0$, that can be shown to be equal to αD_α with some extra care, but we do not need it at this point. It is then easy to reduce this to the form of Theorem 1: by using the scaling identities and changing variables $u = y^{-\alpha}$, we have

$$\begin{aligned}
&\int_0^\infty \frac{\mathrm{d}y\, q_y(1)}{y^{\alpha+1}} E[G(\Delta T_{[0,y]})|T_y = 1] \\
&\quad = \int_0^\infty \frac{\mathrm{d}y\, q_1(y^{-\alpha})}{y^{2\alpha-1}} E[G(y^\alpha \Delta T_{[0,1]})|y^\alpha T_1 = 1] \\
&\quad = \int_0^\infty \alpha^{-1}\mathrm{d}u\, u\, q_1(u)E[G(u^{-1}\Delta T_{[0,1]})|T_1 = u] \\
&\quad = \alpha^{-1}E[T_1 G(T_1^{-1}\Delta T_{[0,1]})],
\end{aligned}$$

as wanted.

This very rough program of proof could probably be "upgraded" to a real rigorous proof, but the technical difficulties on generators of processes would undoubtedly make it quite involved. We are going to use a path that uses more the structure of the stable tree.

3.2. Erosion and first properties of the dislocation measure

From this section on, F^- is exclusively defined under $N^{(1)}$, so that we may use the nicer notations $P(F^-(t) \in \mathrm{d}\mathbf{s})$ or $E[G(F^-(t))]$ instead of $N^{(1)}(F^-(t) \in \mathrm{d}\mathbf{s})$ or $N^{(1)}(G(F^-(t)))$ if there is no ambiguity.

Lemma 4. *The erosion coefficient c of F^- is 0, and the dislocation measure $\nu_-(\mathrm{d}\mathbf{s})$ charges only $\{\mathbf{s} \in S : \sum_{i=1}^{+\infty} s_i = 1\}$.*

Proof. We will follow the analysis of Bertoin [9], using the law of the time at which a tagged fragment vanishes. Let U be uniform on $(0, 1)$ and independent of the height process of the stable tree. Recall the definition of $F^-(t)$ out of the open set $I_-(t)$ and let $\lambda(t) = |I^*(t)|$ be the size of the interval $I^*_-(t)$ of $I_-(t)$ that contains U. As in Sect. 2.3, if we define

$$a(t) = \inf\left\{u \geq 0 \ : \ \int_0^u \lambda(v)^{1/\alpha-1}\mathrm{d}v > t\right\} \quad , \quad t \geq 0,$$

then $(-\log(\lambda(a(t))), t \geq 0)$ is a subordinator with Laplace exponent

$$\Phi(r) = -\log E[\lambda(a(t))^r] = c(r+1) + \int_S \left(1 - \sum_{n=1}^{+\infty} s_n^{r+1}\right) \nu_-(\mathrm{d}\mathbf{s}). \tag{8}$$

Moreover, if $\xi = H_U$ is the lifetime of the tagged fragment, then

$$E[\xi^k] = \frac{k!}{\prod_{i=1}^k \Phi\left(i\left(1 - \frac{1}{\alpha}\right)\right)}. \tag{9}$$

For the computation, recall that the density $(\chi_x(s), s \geq 0)$ introduced in Proposition 2 is characterized by its Laplace transform

$$\int_0^{+\infty} e^{-\mu s}\chi_x(s)\mathrm{d}s = \exp(-x\mu^{1-1/\alpha}). \tag{10}$$

We may now compute the moments of ξ. By Proposition 2,

$$E[\xi^k] = \int_0^{+\infty} h^k \alpha\Gamma\left(1 - \frac{1}{\alpha}\right)\chi_{\alpha h}(1)\mathrm{d}h = \frac{\Gamma\left(1 - \frac{1}{\alpha}\right)}{\alpha^k}\int_0^{+\infty} x^k \chi_x(1)\mathrm{d}x.$$

To compute this we use (10) and Fubini's theorem to get

$$\int_0^{+\infty} \mathrm{d}s\, e^{-\mu s} \int_0^{+\infty} \mathrm{d}x\ \chi_x(s)x^k = \int_0^{+\infty} x^k \exp(-x\mu^{1-1/\alpha})\mathrm{d}x = \frac{k!}{\mu^{(k+1)(1-1/\alpha)}},$$

and then the last term above is equal to

$$\frac{k!}{\Gamma\left((k+1)\left(1 - \frac{1}{\alpha}\right)\right)}\int_0^{+\infty} \mathrm{d}u\, e^{-\mu u} u^{(k+1)(1-1/\alpha)-1}.$$

Inverting Laplace transforms and taking $u = 1$ thus give

$$\int_0^{+\infty} x^k \chi_x(1)\mathrm{d}x = \frac{k!}{\Gamma\left((k+1)\left(1 - \frac{1}{\alpha}\right)\right)},$$

hence we finally get

$$E[\xi^k] = \frac{k!\Gamma\left(1 - \frac{1}{\alpha}\right)}{\alpha^k\Gamma\left((k+1)\left(1 - \frac{1}{\alpha}\right)\right)}.$$

Using (9) we now obtain that

$$\Phi\left(k\left(1-\frac{1}{\alpha}\right)\right)=\alpha\frac{\Gamma\left((k+1)\left(1-\frac{1}{\alpha}\right)\right)}{\Gamma\left(k\left(1-\frac{1}{\alpha}\right)\right)},\quad k=1,2,\dots$$

Thus, for r of the form $k(1-1/\alpha)$,

$$\Phi(r)=\alpha\frac{\Gamma\left(r+1-\frac{1}{\alpha}\right)}{\Gamma(r)}=\frac{r}{\Gamma\left(1+\frac{1}{\alpha}\right)}B\left(r+1-\frac{1}{\alpha},\frac{1}{\alpha}\right). \tag{11}$$

It is not difficult, using the integral representation of the function B, then changing variables and integrating by parts, to write this in Lévy-Khintchine form, that is, for every $r\geq 0$,

$$\frac{r}{\Gamma\left(1+\frac{1}{\alpha}\right)}B\left(r+1-\frac{1}{\alpha},\frac{1}{\alpha}\right)=\int_0^\infty \mathrm{d}x\frac{\left(1-\frac{1}{\alpha}\right)e^x}{\Gamma\left(1+\frac{1}{\alpha}\right)(e^x-1)^{2-1/\alpha}}\left(1-e^{-xr}\right), \tag{12}$$

and it follows that (11) remains true for every $r\geq 0$, because $\lambda(a(t))^{1-1/\alpha}$ is characterized by its moments, hence generalizing Equation (12) in [9] in the Brownian case. It also gives the formula

$$L(\mathrm{d}x)=\frac{\left(1-\frac{1}{\alpha}\right)e^x\mathrm{d}x}{\Gamma\left(1+\frac{1}{\alpha}\right)(e^x-1)^{2-1/\alpha}}$$

for the Lévy measure $L(\mathrm{d}x)$ of Φ, hence generalizing Equation (11) in [9].

To conclude, we just notice that $\Phi(0)=0$, which by (8) gives both $c=0$ and $\int_S \nu_-(\mathrm{d}\mathbf{s})(1-\sum_{i=1}^\infty s_i)=0$, implying the result. □

3.3. Dislocation measure

The dislocation measure of F^- will now be obtained by explicitly computing the law of the first fragmentation of the fragments marked by n independent uniform variables $U_1,\dots,U_n$ on $(0,1)$, as explained in Sect. 2.3. This is going to be a purely combinatorial computation based on the first formula of Proposition 2. What we want to compute is the law of the partition of $[n]$ induced by the partition $I_-(t_n)$ and the variables $U_1,\dots,U_n$ at the time t_n when they are first separated. We want to evaluate the probability $\rho_-(n)(\{\pi_n\})$ that the partition induced by $I_-(t_n)$ and the variables $(U_1,\dots,U_n)$ equals some non-trivial partition π_n of $[n]$ with blocks $A_1,\dots,A_k$ having sizes $n_1,\dots,n_k$ with sum n ($n,k\geq 2$). In terms of the stable tree described in Sect. 2.2, this is simply the probability that the skeleton of the marked tree ϑ_n is such that the root has out-degree k, and the k trees that are rooted at the children of the root have $n_1,n_2,\dots,n_k$ leaves, times $n_1!\dots n_k!/n!$, which is the probability that labeling by i the leaf associated to the variable U_i, for $1\leq i\leq n$, induces the partition π_n (where i and j are in the same block if the leaves labeled i,j share the same child of the root as a common ancestor). Let $\mathbf{T}^*_{n_1,\dots,n_k}$ be the set of trees of $\mathbf{T}^*_n$ that have this last property. For $x\geq 0$ and $n\geq 0$ we denote by $[x]_n$ the quantity $\prod_{i=0}^{n-1}(x+i)=\Gamma(x+n)/\Gamma(x)$.

Lemma 5. *Let π_n be a partition of $[n]$ with $k \geq 2$ blocks having sizes $n_1, n_2, \ldots,$ n_k. Then*

$$\rho_-(n)(\{\pi_n\}) = \frac{D_\alpha \Gamma(k-\alpha)}{\alpha^k \Gamma\left(n - \frac{1}{\alpha}\right)} \prod_{i=1}^{k} \left[1 - \frac{1}{\alpha}\right]_{n_i - 1}.$$

Proof. Recall that we want to compute the probability that the skeleton of the marked tree ϑ_n has a root with k children, and the fringe subtrees spanned by these children are trees of $\mathbf{T}^*_{n_i}$ for $1 \leq i \leq k$. The fact that the first displayed quantity in Proposition 2 defines a probability on $\mathbf{T}^*_n$ implies

$$\begin{aligned}
&\sum_{\mathcal{T} \in \mathbf{T}^*_n} \prod_{v \in \mathcal{N}_\mathcal{T}} \frac{|(\alpha-1)(\alpha-2)\ldots(\alpha - c_v(\mathcal{T}) + 1)|}{c_v(\mathcal{T})!} \\
&\quad = \frac{(\alpha-1)(2\alpha-1)\ldots((n-1)\alpha - 1)}{n!} \\
&\quad = \frac{\alpha^{n-1}}{n!} \left[1 - \frac{1}{\alpha}\right]_{n-1}.
\end{aligned}$$

Now we compute, using Proposition 2,

$$\begin{aligned}
&\rho_-(n)(\{\pi_n\}) \\
&\quad = \sum_{\mathcal{T} \in \mathbf{T}^*_{n_1,\ldots,n_k}} \frac{n! n_1! \ldots n_k!}{\alpha^{n-1}\left[1 - \frac{1}{\alpha}\right]_{n-1} n!} \prod_{v \in \mathcal{N}_\mathcal{T}} \frac{|(\alpha-1)(\alpha-2)\ldots(\alpha - c_v(\mathcal{T}) + 1)|}{c_v(\mathcal{T})!} \\
&\quad = \frac{n_1! \ldots n_k! |(\alpha-1)(\alpha-2)\ldots(\alpha - k + 1)|}{\alpha^{n-1} k! \left[1 - \frac{1}{\alpha}\right]_{n-1}} \\
&\qquad \times \sum_{\mathcal{T} \in \mathbf{T}^*_{n_1,\ldots,n_k}} \prod_{v \in \mathcal{N}_\mathcal{T} \setminus \{\text{root}\}} \frac{|(\alpha-1)(\alpha-2)\ldots(\alpha - c_v(\mathcal{T}) + 1)|}{c_v(\mathcal{T})!} \\
&\quad = \frac{(\alpha-1)\Gamma(k-\alpha)\Gamma\left(1 - \frac{1}{\alpha}\right)}{k! \alpha^{n-1} \Gamma(2-\alpha) \Gamma\left(n - \frac{1}{\alpha}\right)} \\
&\qquad \times k! n_1! \ldots n_k! \prod_{i=1}^{k} \sum_{\mathcal{T} \in \mathbf{T}^*_{n_i}} \prod_{v \in \mathcal{N}_\mathcal{T}} \frac{|(\alpha-1)(\alpha-2)\ldots(\alpha - c_v(\mathcal{T}) + 1)|}{c_v(\mathcal{T})!},
\end{aligned}$$

where the last equality stems from the definition of $\mathbf{T}^*_{n_1,\ldots,n_k}$, and where the factor $k!$ appears because the k fringe subtrees spanned by the sons of the root may appear in any order. By the first formula of the proof this now reduces to

$$\rho_-(n)(\{\pi_n\}) = \frac{D_\alpha \Gamma(k-\alpha) \prod_{i=1}^{k} n_i!}{\alpha^n \Gamma\left(n - \frac{1}{\alpha}\right)} \prod_{i=1}^{k} \frac{\alpha^{n_i - 1}}{n_i!} \left[1 - \frac{1}{\alpha}\right]_{n_i - 1},$$

giving the result. □

Comparing with Lemma 2 implies, since $c = 0$, that the dislocation measure ν_- of F^- is thus determined up to a multiplicative constant. Since we have a conjectured form $D_\alpha \nu_\alpha$ for the dislocation measure ν_- of F^-, we just have to compute the quantity $\kappa_-(\pi)$ for κ_- the exchangeable measure on $\mathcal{P}$ with frequencies given by the conjectured ν_-. Precisely, we have

Lemma 6. *Let π_n be a partition of $[n]$ with $k \geq 2$ blocks and block sizes $n_1, \dots, n_k$. Then*

$$\kappa_-^n(\{\pi_n\}) := \kappa_-(\{\pi \in \mathcal{P} : \pi|_{[n]} = \pi_n\}) = \frac{D_\alpha \Gamma(k-\alpha)}{\alpha^{k-1}\Gamma(n-1)} \prod_{i=1}^k \left[1 - \frac{1}{\alpha}\right]_{n_i - 1}$$

Before proving this we state from (74) in section 6 of [27] (notice that the α there is our $1/\alpha$):

Proposition 5. *Let $\theta > -1/\alpha$ and recall (2) the definition of the Poisson-Dirichlet* $\mathrm{PD}(1/\alpha, \theta)$ *distribution. Let π_n be a partition of $[n]$ with non-void block sizes $n_1, \dots, n_k$. Then the probability that the restriction to $[n]$ of the exchangeable partition of $\mathcal{P}$ with frequencies having law* $\mathrm{PD}(1/\alpha, \theta)(\mathrm{d}\mathbf{s})$ *is π_n is given by*

$$p_\theta(n_1, \dots, n_k) = \frac{[\alpha\theta + 1]_{k-1}}{\alpha^{k-1}[\theta + 1]_{n-1}} \prod_{i=1}^k \left[1 - \frac{1}{\alpha}\right]_{n_i - 1}$$

Proof of Lemma 6. The computation of the κ_-^n associated with the conjectured dislocation measure ν_- can go through the same lines as the proof of Proposition 5 given in [27], using the explicit densities for size-biased picks among the jumps of the subordinator T. However, we use the following more direct proof. For $\theta \geq -1$ write

$$\nu_\theta = D_\alpha E\left[T_1^{-\theta}; \frac{\Delta T_{[0,1]}}{T_1} \in \mathrm{d}\mathbf{s}\right],$$

so $\nu_\theta = D_\alpha(\Gamma(\alpha\theta + 1)/\Gamma(\theta + 1))\mathrm{PD}(1/\alpha, \theta)$ for $\theta > -1/\alpha$. Recall from the above the notation $\kappa_{\mathbf{s}}(\mathrm{d}\pi)$ for the law of the exchangeable partition of $\mathbb{N}$ with ranked asymptotic frequencies given by $\mathbf{s}$. Define

$$\kappa_\theta(\mathrm{d}\pi) = \int_S \nu_\theta(\mathrm{d}\mathbf{s})\kappa_{\mathbf{s}}(\mathrm{d}\pi) = D_\alpha E\left[T_1^{-\theta}\kappa_{\Delta T_{[0,1]}/T_1}(\mathrm{d}\pi)\right], \tag{13}$$

and for π_n a partition of $[n]$ with block sizes $n_1, \dots, n_k$ write $\kappa_\theta^n(\{\pi_n\}) = \kappa_\theta(\{\pi \in \mathcal{P} : \pi|_{[n]} = \pi_n\})$. Notice that when $n, k \geq 2$ and $\mathbf{s} \in S$, we have $\kappa_{\mathbf{s}}(\{\pi \in \mathcal{P} : \pi|_{[n]} = \pi_n\}) \leq n(1 - s_1)$ (this is easy by Kingman's exchangeable partitions representation theorem, see e.g. [8, p. 310]). Moreover, the fact that ν_- integrates $\mathbf{s} \mapsto 1 - s_1$ is easily generalized to ν_θ for $\theta > -1$. We deduce that the map $\theta \mapsto \kappa_\theta^n(\{\pi_n\})$ is analytic on $\{\theta \in \mathbb{C} : \mathrm{Re}(\theta) > -1\}$. The same holds for

$$D_\alpha \frac{\Gamma(\alpha\theta + 1)}{\Gamma(\theta + 1)} p_\theta(n_1, \dots, n_k) = \frac{D_\alpha \Gamma(\alpha\theta + k)}{\alpha^{k-1}\Gamma(\theta + n)} \prod_{i=1}^k \left[1 - \frac{1}{\alpha}\right]_{n_i - 1} \tag{14}$$

provided $k \geq 2$, and by Proposition 5 they are equal on $\theta \in (-1/\alpha, \infty)$. Thus they are equal on $\{\theta \in \mathbb{C} : \mathrm{Re}(\theta) > -1\}$, so the limits as $\theta \in \mathbb{R} \downarrow -1$ of $\kappa_\theta^n(\{\pi_n\})$ and of (14) coincide. Using (13) and a dominated convergence argument we have $\kappa_\theta^n(\{\pi_n\}) \to \kappa_-^n(\{\pi_n\})$ as $\theta \downarrow -1$, so

$$\kappa_-^n(\{\pi_n\}) = \frac{D_\alpha \Gamma(k-\alpha)}{\alpha^{k-1}\Gamma(n-1)} \prod_{i=1}^k \left[1 - \frac{1}{\alpha}\right]_{n_i - 1},$$

as wanted. □

Remark 2. By analogy with the EPPF (exchangeable partition probability function) that allows to characterize the law of exchangeable partitions, expressions such as in Lemma 6 could be called "exchangeable partition distribution functions", as they characterize σ-finite exchangeable measures on the set of partitions of $\mathbb{N}$. The expression in Lemma 6 should be interpreted as an EPDF for a generalized $(1/\alpha, \theta)$ partition (see [26]), for $\theta = -1$. One certainly could imagine more general exchangeable partitions as θ goes further in the negative axis: this would impose more and more stringent constraints on the number of blocks of the partitions.

Therefore, we obtain that

$$\kappa_-^n = \alpha(\Gamma(n - 1/\alpha)/\Gamma(n-1))\rho_-(n)$$

on the set of non-trivial partitions of $[n]$. Lemma 2 implies that the dislocation measure of F^- is equal to the conjectured ν_- up to a multiplicative constant. We are going to recover the missing information with the help of the computation of Φ above.

3.4. *The missing constant*

In this section, we compute the Laplace exponent Φ of the subordinator $-\log(\lambda(a(\cdot)))$ of Sect. 3.2, whose value is indicated in (11), directly from formulas (8) and (1). Let

$$\Phi_0(r) = \int_S \left(1 - \sum_{n=1}^{\infty} s_n^{r+1}\right) \nu_-(\mathrm{d}\mathbf{s}),$$

where ν_- is the measure given in Theorem 1. If we can prove that $\Phi_0(r) = \Phi(r)$ for every $r \geq 0$, we will therefore have established that the normalization of ν_- is the appropriate one. By (1),

$$\begin{aligned}
\Phi_0(r) &= D_\alpha E\left[T_1 \left(1 - \sum_{0 \leq x \leq 1} \left(\frac{\Delta T_x}{T_1}\right)^{r+1}\right)\right] \\
&= D_\alpha \int_0^\infty \mathrm{d}u\, u\, q_1(u) E\left[1 - \sum_{0 \leq x \leq 1} \left(\frac{\Delta T_x}{u}\right)^{r+1} \middle| T_1 = u\right] \\
&= D_\alpha \int_0^\infty \mathrm{d}u\, u\, q_1(u) E\left[1 - \left(\frac{\Delta_1^*}{u}\right)^r\right]
\end{aligned}$$

where Δ_1^* is a size-biased pick from the jumps of T_x, for $0 \le x \le 1$, conditionally on $T_1 = u$. Using formula (5) and recalling that T has Lévy measure $c_\alpha x^{-1-1/\alpha}\mathrm{d}x$, we can write

$$\begin{aligned}\Phi_0(r) &= D_\alpha \int_0^\infty \mathrm{d}u\, u\, q_1(u) \int_0^u \mathrm{d}x (1-(x/u)^r) \frac{c_\alpha q_1(u-x)}{u x^{1/\alpha} q_1(u)} \\ &= D_\alpha \int_0^\infty \mathrm{d}u \int_0^1 \mathrm{d}y\, c_\alpha u^{1-1/\alpha} q_1(u(1-y)) \frac{1-y^r}{y^{1/\alpha}} \\ &= D_\alpha \int_0^1 \mathrm{d}y \frac{c_\alpha (1-y^r)}{y^{1/\alpha}(1-y)^{2-1/\alpha}} \int_0^\infty \mathrm{d}u\, u^{1-1/\alpha} q_1(u)\end{aligned}$$

as obtained by Fubini's theorem, and linear changes of variables. The integral in $\mathrm{d}u$ equals $\mathbb{E}[T_1^{1-1/\alpha}]$, which is $\Gamma(2-\alpha)/\Gamma(1/\alpha)$ (see e.g. (43) in [26]). Using the expressions for D_α, c_α and the identity $\alpha^{-1}\Gamma(1/\alpha) = \Gamma(1+1/\alpha)$, it remains to compute the quantity

$$\frac{1-\frac{1}{\alpha}}{\Gamma\left(1+\frac{1}{\alpha}\right)} \int_0^1 \mathrm{d}y \frac{y^{-1/\alpha}(1-y^r)}{(1-y)^{2-1/\alpha}}.$$

But this is exactly the expression (12) after changing variables $y = e^{-x}$, and it is thus equal to $rB(r+1-1/\alpha, 1/\alpha)/\Gamma(1+1/\alpha)$, which is (11) as wanted, thus completing the proof of Theorem 1.

4. Small-time asymptotics

In this section we study the asymptotic behavior of F^- for small times. Precisely, let $M(t) = \sum_{i\ge 1} F_i^-(t)$ denote the total mass of F^- at time t. Let $(Y_x, x \ge 0)$ denote an α-CSBP, started at 0 and conditioned to stay positive. See the following section for definitions. We have the following result, that generalizes and mimics somehow results from [3, 5, 24]. However, these results dealt with self-similar fragmentations with positive indices, and also, the occurrence of the randomization introduced by Y_1 below is somehow unusual.

Proposition 6. *The following convergence in law holds:*

$$t^{\alpha/(1-\alpha)}(M(t) - F_1^-(t), F_2^-(t), F_3^-(t), \dots) \xrightarrow[t\downarrow 0]{d} (T_{Y_1}, \Delta_1, \Delta_2, \dots)$$

where T is the stable $1/\alpha$ subordinator as above, independent of Y, and $\Delta_1, \Delta_2, \dots$ are the jumps of $(T_x, 0 \le x \le Y_1)$ ranked in decreasing order of magnitude.

For this we are going to use the following lemma, which resembles the result of Jeulin in [16] relating a scaled normalized Brownian excursion and a 3-dimensional Bessel process. The proof is postponed to the following section. Recall that $(L_1^t, t \ge 0)$ stands for the local time of the height process up to time 1.

Lemma 7. *The following convergence in law holds:*

$$\text{Under } N^{(1)}, \quad (t^{1/(1-\alpha)} L_1^{tx}, x \geq 0) \underset{t \downarrow 0}{\overset{d}{\to}} (Y_x, x \geq 0),$$

and this last limit is independent of the initial process $(L_1^t, t \geq 0)$*. In particular,* $t^{1/(1-\alpha)} L_1^t$ *converges in distribution to* Y_1 *as* $t \downarrow 0$*.*

In the sequel let $(y_t, \overline{y}_t)$ have the law of $(L_1^t, \int_t^\infty \mathrm{d}b L_1^b)$ under $N^{(1)}$.

Proof of Proposition 6. Following the method of Aldous and Pitman [3], we are actually going to prove that for every k,

$$t^{\alpha/(1-\alpha)}(M(t) - F_1^*(t), F_2^*(t), F_3^*(t), \ldots, F_k^*(t)) \underset{t \downarrow 0}{\overset{d}{\to}} (T_{Y_1}, \Delta_1^*, \Delta_2^*, \ldots, \Delta_{k-1}^*), \tag{15}$$

for every $k \geq 1$, where the quantities with the stars are the size-biased quantities associated with the ones of the statement, and this is sufficient. We are going to proceed by induction on k. To start the induction, let g be a continuous function with compact support and write, using Lemma 1, Proposition 4, then changing variables and using scaling identities,

$$\begin{aligned}
&E[g(t^{\alpha/(1-\alpha)}(M(t) - F_1^*(t)))] \\
&\quad = E\left(\int_0^{\overline{y}_t} \mathrm{d}u \frac{c_\alpha\, y_t\, q_{y_t}(\overline{y}_t - u)}{\overline{y}_t\, u^{1/\alpha}\, q_{y_t}(\overline{y}_t)} g(t^{\alpha/(1-\alpha)}(\overline{y}_t - u))\right) \\
&\quad = E\left(\int_0^{t^{\alpha/(1-\alpha)}\overline{y}_t} \mathrm{d}v \frac{t^{\alpha/(\alpha-1)}\, c_\alpha\, y_t\, q_1\left(\frac{v}{t^{\alpha/(1-\alpha)} y_t^\alpha}\right)}{(\overline{y}_t - t^{\alpha/(\alpha-1)} v)^{1/\alpha}\, \overline{y}_t\, q_1\left(\frac{\overline{y}_t}{y_t^\alpha}\right)} g(v)\right).
\end{aligned} \tag{16}$$

By making use of Skorokhod's representation theorem, we may suppose that the convergence of $(t^{1/(1-\alpha)} y_t, t^{\alpha/(1-\alpha)} \overline{y}_t)$ to (Y_1, ∞) is almost-sure. Now the integral inside the expectation is the integral according to a probability law, hence it is dominated by the supremum of $|g|$, so it suffices to show that the integral converges a.s. to apply dominated convergence. For almost every ω, there exists ε such that if $t < \varepsilon$, $t^{\alpha/(1-\alpha)} \overline{y}_t(\omega) > K$ where K is the right-end of the support of g. For such an ω and t, the integral is thus

$$\begin{aligned}
&\int_0^K \mathrm{d}v\, g(v) \frac{c_\alpha t^{\alpha/(\alpha-1)} y_t q_1(v(t^{1/(1-\alpha)} y_t)^{-\alpha})}{\overline{y}_t^{1+1/\alpha}(1 - t^{\alpha/(\alpha-1)} v/\overline{y}_t)^{1/\alpha} q_1(\overline{y}_t y_t^{-\alpha})} \\
&\quad \leq M \frac{t^{\alpha/(\alpha-1)} y_t}{\overline{y}_t^{1+1/\alpha} q_1(\overline{y}_t y_t^{-\alpha})} \int_0^K \mathrm{d}v\, q_1\left(\frac{v}{t^{\alpha/(1-\alpha)} y_t^\alpha}\right)
\end{aligned}$$

for some constant M not depending on t. Now we use the fact from [31] that q_1 is bounded and

$$q_1(x) \underset{x \to \infty}{=} c_\alpha x^{-1-1/\alpha} + O(x^{-1-2/\alpha}).$$

This allows to conclude by dominated convergence that the integral in (16) a.s. goes to

$$\int_0^K \mathrm{d}v\, g(v) \frac{q_1(v/Y_1^\alpha)}{Y_1^\alpha} = \int_0^K \mathrm{d}v\, g(v) q_{Y_1}(v),$$

and by dominated convergence its expectation converges to the expectation of the above limit, that is $E[g(T_{Y_1})]$.

To implement the recursive argument, suppose that (15) holds for some $k \geq 1$. Let g and h be continuous bounded functions on $\mathbb{R}_+$ and $\mathbb{R}_+^k$ respectively. Write $(y_t, \overline{y}_t, \Delta_1(t), \Delta_2(t) \dots)$ for a sequence with the same law as $(L_1^t, \int_t^\infty \mathrm{d}s L_1^s, \Delta T'_{[0, L_1^t]})$ given $T'_{L_1^t} = \int_t^\infty \mathrm{d}s L_1^s$, where L_1 is taken under $N^{(1)}$ and T' is a stable $1/\alpha$ subordinator, taken independent of L. Last, let $\Delta_1^*(t), \Delta_2^*(t), \dots$ be the size-biased permutation associated with $\Delta_1(t), \Delta_2(t), \dots$. By Proposition 4, conditioning, and using Lemma 1, we have

$$\begin{aligned}
&E[g(t^{\alpha/(1-\alpha)} F_{k+1}^*(t)) h(t^{\alpha/(1-\alpha)}(M(t) - F_1^*(t), F_2^*(t), \dots, F_k^*(t)))] \\
&= E\Bigg[h(t^{\alpha/(1-\alpha)}(\overline{y}_t - \Delta_1^*(t), \Delta_2^*(t), \dots, \Delta_k^*(t))) \int_0^{\overline{y}_t - \sum_{i=1}^k \Delta_i^*(t)} \mathrm{d}u\, g(t^{\alpha/(1-\alpha)} u) \\
&\quad \times \frac{c_\alpha\, y_t\, q_{y_t}\left(\overline{y}_t - \sum_{i=1}^k \Delta_i^*(t) - u\right)}{u^{1/\alpha}\left(\overline{y}_t - \sum_{i=1}^k \Delta_i^*(t)\right) q_{y_t}\left(\overline{y}_t - \sum_{i=1}^k \Delta_i^*(t)\right)}\Bigg]
\end{aligned}$$

Similarly as above, we show by changing variables and then using the scaling identities and the asymptotic behavior of q_1 that this converges to

$$\begin{aligned}
&E\Bigg[h(T_{Y_1}, \Delta_1^*, \dots, \Delta_{k-1}^*) \int_0^{T_{Y_1} - \sum_{i=1}^k \Delta_i^*} \mathrm{d}v\, g(v) \\
&\quad \times \frac{c_\alpha\, Y_1\, q_{Y_1}\left(T_{Y_1} - \sum_{i=1}^{k-1} \Delta_i^* - v\right)}{v^{1/\alpha}(T_{Y_1} - \sum_{i=1}^{k-1} \Delta_i^*) q_{Y_1}\left(T_{Y_1} - \sum_{i=1}^{k-1} \Delta_i^*\right)}\Bigg]
\end{aligned}$$

and by Lemma 1 this is $E[h(T_{Y_1}, \Delta_1^*, \dots, \Delta_{k-1}^*) g(\Delta_k^*)]$. This finishes the proof. □

The method used in this proof can also show that the rescaled remaining mass $t^{\alpha/(1-\alpha)}(1 - M(t))$ converges in distribution to $\int_0^1 Y_v\, \mathrm{d}v$ jointly with the vector of the proposition.

5. Some results on continuous-state branching processes

In this section we develop the material needed to prove Lemma 7. In the course, we will give an analog of Jeulin's theorem [17] linking the local time process of a Brownian excursion to another time-changed Brownian excursion. To stay in the line of the present paper, we will suppose that the laws we consider are associated to stable processes, but all of the results (except the proof of Lemma 7 which strongly uses scaling) can be extended to more general Lévy processes and their associated CSBP's. To avoid confusions, we will denote by $(Z_t, t \geq 0)$ the different CSBP's we will consider, or to be more precise, we let $(Z_t, t \geq 0)$ instead of $(X_s, s \geq 0)$ be the canonical process on $\mathbb{D}([0,\infty))$ when dealing with the laws $\mathbb{P}_x, \mathbb{P}_x^{\uparrow}, \dots$ associated to CSBP's.

Definition 2. *For any $x > 0$, let $\mathbb{P}_x$ be the unique law on $\mathbb{D}([0,\infty))$ that makes the canonical process $(Z_t, t \geq 0)$ a right-continuous Markov process starting at x with transition probabilities characterized by*

$$\mathbb{E}[\exp(-\lambda Z_{t+r})|Z_t = y] = \exp(-y u_r(\lambda)),$$

where $u_r(\lambda) = (\lambda^{1-\alpha} + (\alpha - 1)r)^{1/(1-\alpha)}$ is determined by the equation

$$\int_{u_r(\lambda)}^{\lambda} \frac{\mathrm{d}v}{v^\alpha} = r.$$

Then $\mathbb{P}_x$ is called the law of of the α-CSBP started at x.

Remark 3. For more general branching mechanisms, the definition of $u_r(\lambda)$ is modified by replacing v^α by $\psi(v)$, where ψ is the Laplace exponent of a spectrally positive Lévy process with infinite variation that oscillates or drifts to $-\infty$.

Recall the setting of Sect. 2.1, and let P_x be law under which X is the spectrally positive stable process with Laplace exponent λ^α and started at $x > 0$, that is, the law of $x + X$ under P. Let E_x be the corresponding expectation. Define the time-change $(\tau_t, t \geq 0)$ by

$$\tau_t = \inf\left\{u \geq 0 : \int_0^u \frac{\mathrm{d}v}{X_{v\wedge h_0}} > t\right\},$$

where $h_0 = \inf\{s > 0 : X_s = 0\}$ is the first hitting time of 0. This definition makes sense either under the law P_x, for $x > 0$, or the σ-finite excursion measure N (we will see below that under N, τ is not the trivial process identical to 0).

Theorem 2. *We have the following identities in law: for every $x > 0$,*

$$(L^t_{T_x}, t \geq 0) \text{ under } P \overset{d}{=} (X_{\tau_t}, t \geq 0) \text{ under } P_x,$$

and both have law $\mathbb{P}_x$. Moreover,

$$(L^t_\zeta, t \geq 0) \text{ under } N \overset{d}{=} (X_{\tau_t}, t \geq 0) \text{ under } N.$$

The first part is already known and is a conjunction of Lamperti's theorem (stating that $(X_{\tau_t}, t \geq 0)$ under P_x has law $\mathbb{P}_x$) and the Ray-Knight theorem mentioned in Sect. 2.2. We will use it to prove the second part. First we introduce some notations, which were already used in a heuristic way above.

For $x > 0$ one can define the law $P_x^\uparrow$ of the stable process started at x and conditioned to stay positive by means of Doob's theory of harmonic h-transforms. It is characterized by the property

$$E_x^\uparrow[F(X_s, 0 \le s \le K)] = E_x\left[\frac{X_K}{x} F(X_s, 0 \le s \le K), K < T_0\right]$$

for any positive measurable functional F. Here T_0 denotes as above the first hitting time of 0 by X. It can be shown (see e.g. [12]) that $P_x^\uparrow$ has a weak limit as $x \to 0$, which we call $P^\uparrow$, the law of the stable process conditioned to stay positive.

Similarly, we define the CSBP conditioned to stay positive according to [20], by letting $\mathbb{P}_x$ be the law of the CSBP started at $x > 0$, then setting

$$\mathbb{E}_x^\uparrow[F(Z_t, 0 \le t \le K)] = \mathbb{E}_x\left[\frac{Z_K}{x} F(Z_s, 0 \le s \le K)\right].$$

We want to show that a $x \downarrow 0$ limit also exists in this case. This is made possible by the interpretation of [20] of the law $\mathbb{P}_x^\uparrow$ in terms of a CSBP with *immigration*. To be concise, we have

Lemma 8. *For $x > 0$, the law $\mathbb{P}_x^\uparrow$ is the law of the α-CSBP with immigration function $\alpha\lambda^{\alpha-1}$ and started at x. That is, under $\mathbb{P}_x^\uparrow$, $(Z_t, t \ge 0)$ is a Markov process starting at x and with transition probabilities*

$$\mathbb{E}_x^\uparrow[\exp(-\lambda Z_{t+r})|Z_t = y] = \exp\left(-y u_r(\lambda) - \int_0^r \alpha u_v(\lambda)^{\alpha-1} \mathrm{d}v\right).$$

As a consequence, the laws $\mathbb{P}_x^\uparrow$ converge weakly as $x \downarrow 0$ to a law $\mathbb{P}_0^\uparrow = \mathbb{P}^\uparrow$, which is the law of a Markov process with same transition probabilities and whose entrance law is given by the above formula, taking $t = y = x = 0$. It is also easy that the law $\mathbb{P}^\uparrow$ is that of a Feller process according to the definition for $u_r(\lambda)$.

It is shown in [20] that Lamperti's correspondence is still valid between conditioned processes started at $x > 0$: the process $(X_{\tau_t}, t \ge 0)$ under the law $P_x^\uparrow$ has law $\mathbb{P}_x^\uparrow$. To be more accurate, the exact statement is that if the process $(Z_t, t \ge 0)$ has law $\mathbb{P}_x^\uparrow$, then the process $(Z_{C_s}, s \ge 0)$ has law $P_x^\uparrow$ where

$$C_s = \inf\left\{u \ge 0 : \int_0^u \mathrm{d}v Z_v > s\right\},$$

but this is the second part of Lamperti's transformation, which is easily inverted (see also the comment at the end of the section). We generalize this to

Lemma 9. *The process $(X_{\tau_t}, t \ge 0)$ under the law $P^\uparrow$ has law $\mathbb{P}^\uparrow$.*

Part of this lemma is that $\tau_t > 0$ for every t.

Proof. For fixed $\eta > 0$, let

$$\tau_t^\eta = \inf\left\{u : \int_\eta^{u\vee\eta} \frac{\mathrm{d}v}{X_v} > t\right\}.$$

This is well defined under $P^\uparrow$ since $X_t > 0$ for all $t > 0$ a.s. under this law. Then since $\int_\eta^{u\vee\eta} \mathrm{d}v/X_v = \int_0^{u-\eta} \mathrm{d}v/X_{\eta+v}$ whenever $u \geq \eta$ and is null else, we have that

$$\tau_t^\eta = \eta + \inf\left\{u \geq 0 : \int_0^u \frac{\mathrm{d}v}{X_{\eta+v}} > t\right\}.$$

That is, $\tau^\eta - \eta$ equals the time-change τ defined above, but associated to the process $(X_{\eta+t}, t \geq 0)$ (notice that h_0 plays no role here since we are dealing with processes that are strictly positive on $(0, \infty)$). Under $P^\uparrow$, this process is independent of $(X_s, 0 \leq s \leq \eta)$ conditionally on X_η and has law $P^\uparrow_{X_\eta}$. Hence, by Lamperti's identity, conditionally on $(X_s, 0 \leq s \leq \eta)$ under $P^\uparrow$, the process $(X_{\tau_t^\eta}, t \geq 0)$ has law $\mathbb{P}^\uparrow_{X_\eta}$. Hence, for any continuous bounded functional G on the paths defined on $[0, K]$ for some $K > 0$,

$$E^\uparrow[G(X_{\tau_t^\eta}, 0 \leq t \leq K)] = E^\uparrow[\mathbb{E}^\uparrow_{X_\eta}[G(Z_t, 0 \leq t \leq K)]].$$

Now, it is not difficult to see that τ^η decreases to the limit τ uniformly on compact sets. Thus, using the right-continuity of X on the one hand, and the Feller property on the other (in fact, less than the Feller property is needed here), we obtain by letting $\eta \downarrow 0$ in the above identity

$$E^\uparrow[G(X_{\tau_t}, 0 \leq t \leq K)] = \mathbb{E}^\uparrow[G(Z_t, 0 \leq t \leq K)],$$

which is the desired identity. In particular, τ cannot be identically 0. □

Remark 4. Notice that the fact that the time-change τ_t is still well-defined under the law $P^\uparrow$ can be double-checked by a law of the iterated logarithm for the law $P^\uparrow$. See also the end of the section.

Motivated by the definition in Pitman-Yor [28] for the excursion measure away from 0 of continuous diffusions for which 0 is an exit point (and initially by Itô's description of the Brownian excursion measure linking the three-dimensional Bessel process semigroup to the entrance law of Brownian excursions), we now state the following

Proposition 7. *The process $(L_\zeta^t, t \geq 0)$ under the measure N is governed by the excursion measure of the CSBP with characteristic λ^α. That is, its entrance law $N(L_\zeta^t \in \mathrm{d}y)$ for $t > 0$ is equal to $y^{-1}\mathbb{P}^\uparrow(Z_t \in \mathrm{d}y)$ for $y > 0$ (and it puts mass ∞ on $\{0\}$), and given $(L_\zeta^u, 0 \leq u \leq t)$, the process $(L_\zeta^{t+t'}, t' \geq 0)$ has law $\mathbb{P}_{L_\zeta^t}$.*

The use of the height process and its local time under N, and hence of an "excursion measure" associated to the genealogy of CSBP's, snakes and superprocesses, is a very natural tool, however it does not seem that the above proposition, which states that this notion of "excursion measure" is the most natural one, has been checked somewhere. However, as noticed in [28], since the point 0 is not an entrance point for the initial CSBP, one cannot define a reentering diffusion by sticking the atoms of a Poisson measure with intensity given by this excursion measure, because the durations are almost never summable.

Proof. The law $\mathbb{P}^{\uparrow}(Z_t \in \mathrm{d}y)$ is the weak limit of $\mathbb{P}^{\uparrow}_x(Z_t \in \mathrm{d}y) = x^{-1}y\mathbb{P}(Z_t \in \mathrm{d}y)$ as $x \to 0$. Since by the properties of the CSBP mentioned in Sect. 2.2, we have $\mathbb{E}_x[\exp(-\lambda Z_t)] = \exp(-xu_t(\lambda))$, we obtain

$$\int_0^\infty \frac{\mathbb{P}^{\uparrow}_x(Z_t \in \mathrm{d}y)}{y}(1 - e^{-\lambda y}) = \int_0^\infty \frac{\mathbb{P}_x(Z_t \in \mathrm{d}y)}{x}(1 - e^{-\lambda y}) = \frac{1 - e^{-xu_t(\lambda)}}{x}.$$

This converges to $u_t(\lambda)$ as $x \to 0$, and thanks to the proof of [14, Theorem 1.4.1], this equals $N(1 - \exp(-\lambda L^t_\zeta))$. This gives the identity of the entrance laws. For the Markov property we use excursion theory and Ray-Knight's theorem. Let $0 < t_1 < \ldots < t_n < t$, then Markov's property for $(L^t_{T_1}, t \geq 0)$ entails that for every $\lambda_1, \ldots, \lambda_n, \lambda \geq 0$,

$$\begin{aligned} &E\left[\exp\left(-\sum_{i=1}^n \lambda_i L^{t_i}_{T_1} - \lambda L^t_{T_1}\right)\right] \\ &\quad = E\left[\exp\left(-\sum_{i=1}^{n-1} \lambda_i L^{t_i}_{T_1} - (\lambda_n + u_{t-t_n}(\lambda))L^{t_n}_{T_1}\right)\right]. \end{aligned}$$

On the other hand, we may write $L^t_{T_1} = \sum_{0<s\leq 1}(L^t_{T_s} - L^t_{T_s-})$ so that the Laplace exponent identity for Poisson point processes applied to both sides of the above displayed expression gives after taking logarithms:

$$\begin{aligned} &N\left(1 - \exp\left(-\sum_{i=1}^n \lambda_i L^{t_i}_\zeta - \lambda L^t_\zeta\right)\right) \\ &\quad = N\left(1 - \exp\left(-\sum_{i=1}^{n-1} \lambda_i L^{t_i}_\zeta - (\lambda_n + u_{t-t_n}(\lambda))L^{t_n}_\zeta\right)\right), \end{aligned}$$

so that a substraction gives

$$\begin{aligned} &N\left(\exp\left(-\sum_{i=1}^n \lambda_i L^{t_i}_\zeta\right)(1 - \exp(-\lambda L^t_\zeta))\right) \\ &\quad = N\left(\exp\left(-\sum_{i=1}^n \lambda_i L^{t_i}_\zeta\right)(1 - \exp(-u_{t-t_n}(\lambda)L^{t_n}_\zeta))\right) \\ &\quad = N\left(\exp\left(-\sum_{i=1}^n \lambda_i L^{t_i}_\zeta\right)\mathbb{E}_{L^{t_n}_\zeta}[1 - \exp(-\lambda Z_{t-t_n})]\right). \end{aligned}$$

Hence the Markov property. $\square$

Proof of Theorem 2. It just remains to prove the second statement. For this we let $\eta > 0$ and we define as above the time change τ^η_t. Using the Markov property under the measure N, we again have that under N, $(X_{\eta+s}, s \geq 0)$ is independent of $(X_s, 0 \leq s \leq \eta)$ conditionally on X_η and has the law $P^{h_0}_{X_\eta}$ of the stable process started at X_η and killed at time h_0. Hence, by Lamperti's identity, under N and conditionally on $(X_s, 0 \le s \le \eta)$, the process $(X_{\tau_t^\eta}, t \ge 0)$ has law $\mathbb{P}_{X_\eta}$. Thus if $\eta < t_1 < \ldots < t_n < t$ and if $g_1, \ldots, g_n, g$ are positive continuous functions with compact support that does not contain 0, then

$$\begin{aligned}
&N\left(\prod_{i=1}^n g_i(X_{\tau_{t_i}^\eta})\ g(X_{\tau_t^\eta})\right) \\
&\quad = \int_0^\infty N(X_\eta \in \mathrm{d}x)\mathbb{E}_x\left[\prod_{i=1}^n g_i(Z_{t_i-\eta})\ g(Z_{t-\eta})\right] \\
&\quad = \int_0^\infty N(X_\eta \in \mathrm{d}x)\mathbb{E}_x\left[\prod_{i=1}^n g_i(Z_{t_i-\eta})\ \mathbb{E}_{Z_{t_n-\eta}}[g(Z_{t-t_n})]\right].
\end{aligned}$$

As for the CSBP, the entrance law $N(X_\eta \in \mathrm{d}x)$ equals $x^{-1}P^\uparrow(X_\eta \in \mathrm{d}x)$ for $x > 0$. So we recast the last expression as

$$\begin{aligned}
&\int_0^\infty P^\uparrow(X_\eta \in \mathrm{d}x)\mathbb{E}_x\left[\frac{\prod_{i=1}^n g_i(Z_{t_i-\eta})}{x}\mathbb{E}_{Z_{t_n-\eta}}[g(Z_{t-t_n})]\right] \\
&\quad = \int_0^\infty P^\uparrow(X_\eta \in \mathrm{d}x)\mathbb{E}_x^\uparrow\left[\frac{\prod_{i=1}^n g_i(Z_{t_i-\eta})}{Z_{t_n-\eta}}\mathbb{E}_{Z_{t_n-\eta}}[g(Z_{t-t_n})]\right].
\end{aligned}$$

Now we let $\eta \downarrow 0$, using the right continuity and the Feller property of the CSBP, to obtain

$$N\left(\prod_{i=1}^n g_i(X_{\tau_{t_i}})\ g(X_{\tau_t})\right) = \mathbb{E}^\uparrow\left[\frac{\prod_{i=1}^n g_i(Z_{t_i})}{Z_{t_n}}\mathbb{E}_{Z_{t_n}}[g(Z_{t-t_n})]\right].$$

Hence, thanks to Proposition 7 we obtain that under N the process $(X_{\tau_t}, t \ge 0)$ has the same entrance law and Markov property as $(L_\zeta^t, t \ge 0)$, hence the same law. □

Proof of Lemma 7. Let G be a continuous bounded functional on the paths with lifetime K. We want to show that $N^{(1)}[G(t^{1/(1-\alpha)}L_1^{tx}, 0 \le x \le K)]$ goes to $E^\uparrow[G(X_{\tau_x}, 0 \le x \le K)]$. By Theorem 2, the process $(L_v^x, x \ge 0)$ under $N^{(v)}$ is equal to the process $(X_{\tau_x}, x \ge 0)$ under the law $N^{(v)}$ for almost every v, and we can take $v = 1$ by the usual scaling argument. By [12], the law $N^{(1)}$ can be obtained as the bridge with length 1 of the stable process conditioned to stay positive, and there exists a positive measurable space-time harmonic function $(h_r(x), 0 < r < 1, x \ge 0)$ such that for every functional J and every $r < 1$,

$$N^{(1)}[J(X_s, 0 \le s \le r)] = E^\uparrow[h_r(X_r)J(X_s, 0 \le s \le r)].$$

We now use essentially the same proof as in [11, Lemma 6]. Let $\varepsilon > 0$. Since $\tau_{tK} \wedge \varepsilon$ is a stopping time for the natural filtration of X,

$$\begin{aligned}
&N^{(1)}[G(t^{1/(1-\alpha)}X_{\tau_{tx}\wedge\varepsilon}, 0 \le x \le K)] \\
&\quad = E^\uparrow[h_\varepsilon(X_\varepsilon)G(t^{1/(1-\alpha)}X_{\tau_{tx}\wedge\varepsilon}, 0 \le x \le K)] \\
&\quad = E^\uparrow[E^\uparrow[h_\varepsilon(X_\varepsilon)|X_{\tau_{tK}\wedge\varepsilon}]G(t^{1/(1-\alpha)}X_{\tau_{tx}\wedge\varepsilon}, 0 \le x \le K)].
\end{aligned}$$

Since $\tau_{tK} \to 0$ a.s. as $t \downarrow 0$, we obtain the same limit if we remove the ε in the left-hand side, hence giving $\lim N^{(1)}[G(t^{1/(1-\alpha)}L_1^{tx}, 0 \leq x \leq K)]$ by Theorem 2. Using the backwards martingale convergence theorem we obtain that the conditional expectation on the right-hand side converges to $E^{\uparrow}[h_\varepsilon(X_\varepsilon)] = 1$. So

$$\lim_{t \downarrow 0} N^{(1)}[G(t^{1/(1-\alpha)}L_1^{tx}, 0 \leq x \leq K)] = \lim_{t \downarrow 0} E^{\uparrow}[G(t^{1/(1-\alpha)}X_{\tau_{tx}}, 0 \leq x \leq K)]$$

and the last expression is constant, equal to $E^{\uparrow}[G(X_{\tau_x}, 0 \leq x \leq K)]$ by scaling, hence the result by Lamperti's transform. The independence with the initial process is a refinement of the argument above, using the Markov property at the time $\tau_{tK} \wedge \varepsilon$. $\square$

One final comment. It may look quite strange in the proofs above that the *a priori* ill-defined time τ_t under the laws $P^{\uparrow}$ or N somehow has to be non-degenerate by the proofs we used, even though no argument on the path behavior near 0 has been given for these laws. As a matter of fact, things are maybe clearer when considering also the inverse Lamperti transform. As above, for some process Z that is strictly positive on a set of the form $(0, K)$, $K > 0$, we let

$$C_s = \inf\left\{u \geq 0 : \int_0^u \mathrm{d}v\, Z_v > s\right\}.$$

Define the process X by $X_s = Z_{C_s}$. Then we claim that the map $s \mapsto 1/X_s$ is integrable on a neighborhood of 0 and that $X_{\tau_t} = Z_t$. Indeed, by a change of variables $w = C_v$, one has:

$$\int_0^u \frac{\mathrm{d}v}{X_v} = \int_0^u \frac{\mathrm{d}v}{Z_{C_v}} = \int_0^{C_u} \frac{Z_w \mathrm{d}w}{Z_w} = C_u < \infty,$$

as long as $u < C^{-1}(\infty) = \inf\{s : X_s = 0\}$, which is strictly positive by the hypothesis made on Z. This kind of arguments also shows that as soon as we have one side of Lamperti's theorem, i.e. $X_s = Z_{C_s}$ or $Z_t = X_{\tau_t}$, with non-degenerate C or τ, then the other side is true. In particular, Theorem 2 and Lemma 9 could be restated with the inverse statement giving the Lévy process by time-changing the CSBP with C.

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