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Exact packing dimension in random recursive constructions

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Abstract. We explore the exact packing dimension of certain random recursive constructions. In case of polynomial decay at 0 of the distribution function of random variable X, associated with the construction, we prove that it does not exist, and in case of exponential decay it is $t^\alpha |\log|\log t||^\beta$, where α is the fractal dimension of the limit set and $1/\beta$ is the rate of exponential decay.

1. Introduction

Let $n \in \mathbb{N}$, $n \geq 2$, $\Delta = \{1, 2, \dots, n\}$, $\Delta^* = \bigcup_{j=0}^{\infty} \Delta^j$ is the set of all finite sequences of numbers $1, \dots, n$, and $\Delta^{\mathbb{N}}$ is the set of all infinite sequences of such numbers. The result of concatenation of two finite sequences σ and τ from Δ^* is denoted by $\sigma * \tau$. For a finite sequence σ its length will be denoted by $|\sigma|$. For $k \in \mathbb{N}$ and $\sigma \in \Delta^*$ such that $|\sigma| \geq k$, $\sigma|_k$ is a sequence consisting of first k numbers in σ. There is a natural partial order on the n-ary tree Δ^* : $\sigma \prec \tau$ if and only if the sequence τ starts with σ.

The random recursive construction was first defined by Mauldin and Williams in [15] with n being not necessarily finite. Suppose that J is a compact subset of $\mathbb{R}^d$ such that $J = \mathrm{Cl}(\mathrm{Int}(J))$, without loss of generality its diameter is 1. A random recursive construction is a probability space (Ω, Σ, P) and a collection of random subsets of $\mathbb{R}^d$ $\{J_\sigma(w) | w \in \Omega, \sigma \in \Delta^*\}$ such that

(i) $J_\emptyset = J$ a.s.,
(ii) The maps $w \to J_\sigma(w)$ are measurable with respect to Σ,
(iii) The sets J_σ, if not empty, are geometrically similar to J,
(iv) $J_{\sigma * i}$ is a proper subset of J_σ for all $\sigma \in \Delta^*$ and $i \in \Delta$ provided $J_\sigma \neq \emptyset$,

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(v) The construction satisfies the random *open set condition*: if σ and τ are two sequences of the same length, then $\mathrm{Int}(J_\sigma) \cap \mathrm{Int}(J_\tau) = \emptyset$, and finally
(vi) There is a collection of random i.i.d. vectors $\hat{T}_\sigma = (T_{\sigma*1}, \ldots, T_{\sigma*n}) : \Omega \to [0, 1]^n$, $\sigma \in \Delta^*$ such that $\mathrm{diam}(J_{\sigma*i}) = \mathrm{diam}(J_\sigma) T_{\sigma*i}$ provided $J_\sigma \neq \emptyset$.

The object of study is the random limit set, or fractal, $K(w) = \bigcap_{k=1}^{\infty} \bigcup_{\sigma\in\Delta^k} J_\sigma(w)$.

Note that this setting does not account for placement of the sets $J_{\sigma*i}$ within J_σ. Thus these constructions include as a special case the random self-similar sets defined independently by Graf in [6] and by Mauldin and Williams in [15]. Random self-similar sets can be obtained by choosing the similarity mappings according to some probability distribution and thus may be regarded as random iterated function systems.

If $\mu = E\left[\sum_{i=1}^{n} T_i^0\right] \leq 1$ (by convention, $0^0 = 0$), then $K(w)$ is almost surely an empty set or a point, and we exclude that case from further consideration. Mauldin and Williams in [15] have found the Hausdorff dimension, α, of the limit set $K(w)$, provided $K(w) \neq \emptyset$, $\alpha = \inf\{\beta | E\left[\sum_{i=1}^{n} T_i^\beta\right] \leq 1\}$ a.s. In case $n < \infty$, α is the solution of equation $E\left[\sum_{i=1}^{n} T_i^\alpha\right] = 1$. Berlinkov and Mauldin in [3] proved that the packing, upper and lower Minkowski (box-counting) dimensions of the limit set almost surely equal the Hausdorff dimension. For the definitions of Hausdorff and packing measures and dimensions, as well as definitions of upper and lower Minkowski dimensions, the reader is referred to the book of Mattila ([14]).

Graf et al. in [7] have found under certain conditions the exact Hausdorff dimension of the limit set, that is a gauge function $\varphi(t)$ (a non-decreasing function such that $\varphi(0+) = 0$), so that the φ-Hausdorff measure of $K(w)$ is positive and finite almost surely given $K(w) \neq \emptyset$.

In [3] Berlinkov and Mauldin have found an upper bound on exact packing dimension. In this paper we prove that this upper bound is the best under the *random strong open set condition* and certain other conditions. Let $K_\sigma(w) = \bigcup_{\substack{\eta\in\Delta^{\mathbb{N}} \\ \eta|_{|\sigma|}=\sigma}} \bigcap_{i=1}^{\infty} J_{\eta|_i}(w) \subset J_\sigma(w) \cap K(w)$.

Definition. *The construction satisfies the random strong open set condition if there exist* $\rho_0, \hat{p}_0 > 0$ *and* $s_0 \in \mathbb{N} \cup \{0\}$ *such that for every* $\sigma \in \Delta^*$, *there is an event* $\hat{R}_\sigma$ *in the* σ*-algebra generated by the maps* $w \to T_{\sigma*\tau}(w)$ *with* $0 < |\tau| \leq s_0$, $P(\hat{R}_\sigma | K_\sigma \neq \emptyset) \geq \hat{p}_0$, *such that for every* $w \in \hat{R}_\sigma \cap \{K_\sigma \neq \emptyset\}$ *there exists* $x \in K_\sigma$ *with* $\mathrm{dist}(x, \partial J_\sigma) \geq \rho_0 l_\sigma$.

This condition for random recursive constructions was introduced in [3]. Though it did not mention that the event of obtaining a point of the limit set far enough from the boundary of a cell must be determined by the first few generations of reduction ratios, this was silently assumed in the proof of theorem 5. The connection between random open set condition and random strong open set condition for random self-similar sets was studied by Patzschke in [17].

Before proceeding with the proofs, let us introduce more notation. For $\sigma \in \Delta^*$, let $l_\sigma(w) = \mathrm{diam}(J_\sigma(w)) = \prod_{i=1}^{|\sigma|} T_{\sigma|_i}$, and consider the sequence $\left\{ \sum_{\tau\in\Delta^k} \prod_{i=1}^{|\tau|} T^\alpha_{\sigma*\tau|_i} \right\}$, $k \in \mathbb{N}$. It has been proved in [6], [7], [15], that this sequence forms an L^p-bounded martingale for all $p \geq 1$, and if we denote the limit of this sequence by X_σ, or X in case $\sigma = \emptyset$, then all X_σ's will have the same distribution with expectation 1, finite moments of all orders and moreover, for σ and $\tau \in \Delta^*$ such that $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$, X_σ and X_τ are independent. In [15] it has been proved that $X(w) = 0$ if and only if $K(w) = \emptyset$ for a.e. w.

We call $\Gamma \subset \Delta^*$ an antichain if for all $\tau, \sigma \in \Gamma$ $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$. An antichain Γ is maximal, if for all $\eta \in \Delta^{\mathbb{N}}$ there exists a unique $k \in \mathbb{N}$ such that $\eta|_k \in \Gamma$. By equation (1.9) in [7], with probability 1, for every maximal antichain Γ and every $\sigma \in \Delta^*$, $X_\sigma = \sum_{\tau\in\Gamma} \prod_{i=1}^{|\tau|} T^\alpha_{\sigma*\tau|_i} X_{\sigma*\tau} = \sum_{\tau\in\Gamma} X_{\sigma*\tau} l^\alpha_{\sigma*\tau}/l^\alpha_\sigma$.

Graf et al. in [7] have demonstrated that with each construction one can associate 3 measures, denoted ν_w (the construction measure), μ_w and Q as follows. First, ν_w is determined by setting for a compact set $K \subset \mathbb{R}^d$

$$\nu_w(K) = \lim_{k\to\infty} \sum_{\substack{\sigma\in\Delta^k \\ J_\sigma\cap K\neq\emptyset}} l^\alpha_\sigma(w) X_\sigma(w).$$

Second, μ_w, a measure on $\Delta^{\mathbb{N}}$, is determined from each set $A(\sigma) = \{\eta \in \Delta^{\mathbb{N}} \mid \sigma \prec \eta\}$, a clopen subset of $\Delta^{\mathbb{N}}$, by

$$\mu_w(A(\sigma)) = l^\alpha_\sigma(w) X_\sigma(w)$$

and μ_w is extended to a Borel measure on $\Delta^{\mathbb{N}}$. Finally, Q is a measure on the product space $\Delta^{\mathbb{N}} \times \Omega$. If for a Borel set B, we let $B_w = \{\eta \in \Delta^{\mathbb{N}} \mid (\eta, w) \in B\}$, then

$$Q(B) = \int \mu_w(B_w) dP(w).$$

Expectations with respect to measures P and Q are connected in the following way: if Γ is a map from Ω into the countable set of all maximal antichains in Δ^* such that for each maximal antichain Υ, $\Gamma^{-1}(\Upsilon)$ is in the σ-algebra generated by $\{J_\sigma | \sigma \preceq \Upsilon\}$ and $Y : \Delta^{\mathbb{N}} \times \Omega \to \mathbb{R}$ is a random variable such that $Y(\eta, w) = Y(\eta', w)$ provided $\eta|_{\Gamma(w)} = \eta'|_{\Gamma(w)}$, then

$$E_Q[Y] = E\left[\sum_{\sigma\in\Gamma} l^\alpha_\sigma X_\sigma Y(\sigma, \cdot) \right].$$

In particular, if $A \subset \Delta^{\mathbb{N}} \times \Omega$ and there exists $k \in \mathbb{N}$ such that $\mathbf{1}_A(\eta, w) = \mathbf{1}_A(\eta', w)$ whenever $\eta|_k = \eta'|_k$, then

$$Q(A) = E_Q[\mathbf{1}_A] = E\left[\sum_{|\sigma|=k} l^\alpha_\sigma X_\sigma \mathbf{1}_A(\sigma, \cdot) \right].$$

For $k \in \mathbb{N}$ and $(\eta, w) \in \Delta^{\mathbb{N}} \times \Omega$ denote $X_k(\eta, w) = X_{\eta|_k}(w)$, $T_k(\eta, w) = T_{\eta|_k}(w)$, $l_k(\eta, w) = l_{\eta|_k}(w)$. Thus for all $p > 0$ and $k \in \mathbb{N}$, $E_Q[X_k^p] = E[X^{p+1}] < \infty$. We denote $\hat{R}_k = \{(\eta, w) \in \Delta^{\mathbb{N}} \times \Omega | w \in \hat{R}_{\eta|_k}\} = \bigcup_{|\sigma|=k} A(\sigma) \times \hat{R}_\sigma$.

The exact Hausdorff dimension was determined in [7] by considering the behaviour of the distribution function of the random variable X at infinity. As it turns out, the exact packing dimension is determined by the behaviour of the same distribution function at 0. Berlinkov and Mauldin in [3] proved (under certain conditions) that if $P(0 < X \le a) \le C_2 a^\beta$, $a \in (0, 1)$, then for the function $\varphi(t) = t^\alpha g(t)$, $\int_{0+} \frac{g^{\beta+1}(s)}{s} ds < \infty$ implies $\mathcal{P}^\varphi(K) = 0$ a.s., where $\mathcal{P}^\varphi$ denotes the packing measure with respect to the gauge function φ. Later in the text we refer to this situation as the case of polynomial decay (with parameter β). Assuming that for some $C_1, C_2 > 0$, $C_1 a^\beta \le P(0 < X \le a) \le C_2 a^\beta$ for all $a \in (0, 1)$, we prove in theorem 2 the conjecture about the lower bound, namely, if for the function $\varphi(t) = t^\alpha g(t)$, $\int_{0+} \frac{g^{\beta+1}(s)}{s} ds = \infty$, then $\mathcal{P}^\varphi(K) = \infty$ a.s. provided $K \neq \emptyset$. Thus, the exact packing dimension does not exist in this case.

If $t_0 = \underline{\lim}_{x\to 0} -x^{-1/\beta} \log P(0 < X \le x) = \sup\{t \ge 0 | E\left[e^{tX^{1/\beta}}\right] < \infty\} > 0$ for some $\beta \in \mathbb{R}$, it has been proved in [3], that for $\varphi(t) = t^\alpha |\log|\log t||^\beta$, $\mathcal{P}^\varphi(K) < \infty$ a.s. We call this situation the case of exponential decay (with parameter β). In this case $\beta < 0$, and the "add-on function", $g(t) = |\log|\log t||^\beta$. Assuming that for some $C_1, C_2 > 0$, $C_1 a^{1/\beta} \le -\log P(0 < X \le a) \le C_2 a^{1/\beta}$ for all $a \in (0, 1)$, we prove in theorem 2 that $\mathcal{P}^\varphi(K) > 0$ a.s. provided $K \neq \emptyset$. When this paper was being refereed, the author was informed that in the case of Galton-Watson tree (example 4) the problem has been solved independently by Watanabe in [21].

2. Results

In many previous papers (see, e.g. [5],[11], [22]) concerning the exact packing dimension of stochastic processes, the authors proved its non-existence only for the gauge functions of the type $\varphi(t) = t^\alpha g(t)$ where $g(t)$ is monotone, right-continuous and satisfies the *doubling condition*, $\overline{\lim}_{t\to 0} g(2t)/g(t) < \infty$. Whether the packing measure with gauge function $\varphi(t)$ was infinite or zero was determined first by looking at an integral, and then deciding from that whether the series $\sum_{i=1}^{\infty} g(2^{-i})$ converges or diverges. We show in lemma 1 that these restrictions on $g(t)$ are unnecessary.

Lemma 1. *If $\varphi(t) = t^\alpha g(t)$ is a gauge function, $\beta \ge 0$, then $\int_{0+} \frac{g^{\beta+1}(s)}{s} ds = +\infty$ if and only if for every $N > 1$ and $0 < \rho < 1$, $\sum_{k=1}^{\infty} g^{\beta+1}(\rho N^{-k}) = +\infty$.*

Proof. Fix $0<\rho<1$, $N>1$. Let $g_1(x)=g^{\beta+1}(\rho N^{-x})$, then $\int\limits_{0^+}\frac{g^{\beta+1}(s)}{s}ds=+\infty$ if and only if $\int\limits^{\infty} g_1(x)dx=+\infty$. Using that $\varphi(t)$ is non-decreasing, we obtain

$$g_1(k+y)=g^{\beta+1}(\rho N^{-k}N^{-y})\le g^{\beta+1}(\rho N^{-k})/N^{-y\alpha(\beta+1)}=g_1(k)N^{y\alpha(\beta+1)}$$

for all $0\le y\le 1$, and if $\int\limits^{\infty} g_1(x)dx=\infty$, then

$$\sum_{k=1}^{\infty} g_1(k)\ge N^{-\alpha(\beta+1)}\sum_{k=1}^{\infty}\sup\{g_1(x)|k\le x\le k+1\}=\infty.$$

In the opposite direction, suppose that for some $0<\rho<1$, $N>1$, $\sum\limits_{k=1}^{\infty} g_1(k)=\infty$. Since $\varphi(t)$ is non-decreasing, for every $0\le y\le 1$ and $k\in\mathbb{N}$, we have

$$\begin{aligned} g_1(k+1-y)&=g^{\beta+1}(\rho N^{-k-1}N^{y})\ge N^{-y\alpha(\beta+1)}g^{\beta+1}(\rho N^{-k-1})\\ &=N^{-y\alpha(\beta+1)}g_1(k+1),\end{aligned}$$

and therefore

$$\int^{\infty} g_1(x)dx\ge\sum^{\infty}\inf\{g_1(x)|k\le x\le k+1\}\ge N^{-\alpha(\beta+1)}\sum^{\infty} g_1(k+1)=\infty.\square$$

The following two lemmas interpret the conditions given to us with respect to probability measure P, in terms of probability measure Q, which is the main tool we work with. In exponential case denote $t_0=\varliminf\limits_{x\to 0} -x^{-1/\beta}\log P(0<X\le x)<\infty$.

Lemma 2. *In case of exponential decay with parameter β, for all $t>t_0$ and all $k\in\mathbb{N}$, $E_Q[e^{tX_k^{1/\beta}}]=E[Xe^{tX^{1/\beta}}]=\infty$. For all $0<\rho<1$, $N>1$ and $C>t_0^{-\beta}$, $\sum\limits_{k=1}^{\infty} Q(X_1\le Cg(\rho N^{-k}))=\infty$, where $g(t)=|\log|\log t||^{\beta}$. Moreover, $\overline{\lim\limits_{k\to\infty}}\sum\limits_{i=[\log k]}^{k} Q(X_1\le Cg(\rho N^{-i}))-D\log k=\infty$ for any $D>0$ and $C>(3t_0)^{-\beta}$.*

Proof. Take $C>t_0^{-\beta}$, then $C^{-1/\beta}=t>t_0$. Let $z=(t-t_0)/2$, $c=(|\beta|/ez)^{\beta}$. It is easy to see that for all $x>0$, $xe^{zx^{1/\beta}}\ge c$. Hence, $xe^{tx^{1/\beta}}=xe^{zx^{1/\beta}}e^{(t_0+z)x^{1/\beta}}\ge ce^{(t_0+z)x^{1/\beta}}$. Therefore

$$E_Q[e^{tX_k^{1/\beta}}]=E\left[\sum_{|\sigma|=k} l_\sigma^{\alpha}X_\sigma e^{tX_\sigma^{1/\beta}}\right]=E\left[Xe^{tX^{1/\beta}}\right]\ge cE\left[e^{(t_0+z)X^{1/\beta}}\right]=\infty.$$

Let $h(x)=Q(e^{tX_1^{1/\beta}}\ge x)$, by [14], theorem 1.15, $\int\limits_0^{\infty} h(x)dx=\infty$. Since $h(x)$ is non-increasing, we obtain that for all $0<\rho<1$ and $N>1$, $\sum\limits_{k=1}^{\infty} h(|\log\rho|+k\log N)=\sum\limits_{k=1}^{\infty} Q\left(X_1\le Cg(\rho N^{-k})\right)=\infty$.

Take $C > (3t_0)^{-\beta}$ and let $h(x) = Q\left(e^{C^{-1/\beta}X^{1/\beta}/3} \geq x\sqrt[3]{\log N}\right)$. By the argument above, $\int_0^\infty h(x)dx = \infty$. By lemma 3.2 in [7],

$$\overline{\lim_{k\to\infty}} \int_{\sqrt[3]{[\log k]}}^{\sqrt[3]{k}} h(x)x^2 dx - D\log k = \infty.$$

The result now follows by easy computation. □

Part of the next lemma can be also found in [12] or [22].

Lemma 3. *In case of polynomial decay with parameter β, there exists $K_1 > 0$ such that for all $k \in \mathbb{N}$, $Q(X_k \leq a) = E\left[X\mathbf{1}_{\{X\leq a\}}\right] \geq K_1 a^{\beta+1}$ for all $a \in (0,1)$. In case of exponential decay with parameter β there exists $K_1 > 0$ such that $Q(X \leq a) \geq e^{-K_1 a^{1/\beta}}$ for all $a \in (0,1)$.*

Proof. Let $z \in (0,1)$, then

$$Q(X_k \leq a) = E\left[\sum_{|\sigma|=k} l_\sigma^\alpha X_\sigma \mathbf{1}_{\{X_\sigma \leq a\}}\right] = E\left[X\mathbf{1}_{\{X\leq a\}}\right]$$
$$\geq E\left[X\mathbf{1}_{\{az<X\leq a\}}\right] \geq az(P(0 < X \leq a) - P(0 < X \leq az)).$$

In case of polynomial decay, take $z < (C_1/C_2)^{1/\beta}$. Then $Q(X_k \leq a) \geq az(C_1 a^\beta - C_2 a^\beta z^\beta) = a^{\beta+1}z(C_1 - C_2 z^\beta)$.

In case of exponential decay choose $0 < z < \min\{1/2, (3C_1/2C_2)^\beta\}$. Then $Q(X \leq a) \geq aze^{-C_1 a^{1/\beta}}(1 - e^{a^{1/\beta}(C_1 - C_2 z^{1/\beta})}) \geq K_3 a e^{-C_1 a^{1/\beta}}$ for some $K_3 > 0$ for all $a \in (0,1)$. Thus there exists $K_1 > 0$ such that $Q(X \leq a) \geq e^{-K_1 a^{1/\beta}}$ for all $a \in (0,1)$. □

Theorem 1. *Fix $c > E_Q[|\log T_1|]$ and let $N = e^c$. There are $M_1, M_2 > 0$ such that $M_1 N^{k\alpha} \leq E[\mathrm{card}\{\sigma | N^{-k-1} < l_\sigma \leq N^{-k}\}] \leq M_2 N^{k\alpha}$ for all $k \in \mathbb{N}$. In particular, $\underline{\lim}_{k\to\infty} \sum_{j=1}^\infty Q(N^{-k-1} < l_j \leq N^{-k}) > 0$.*

Proof. Fix $k \in \mathbb{N}$. The upper bound was proved in [3], lemma 2. To make the lower estimate we also begin as in that lemma:

$$N^{-k\alpha} E[\mathrm{card}\{\sigma | N^{-k-1} < l_\sigma \leq N^{-k}\}] = N^{-k\alpha} \sum_{j=1}^\infty \sum_{|\sigma|=j} E[\mathbf{1}_{\{N^{-k-1}<l_\sigma\leq N^{-k}\}}]$$
$$= N^{-k\alpha} \sum_{j=1}^\infty E\left[\sum_{|\sigma|=j} l_\sigma^\alpha X_\sigma l_\sigma^{-\alpha} \mathbf{1}_{\{N^{-k-1}<l_\sigma\leq N^{-k}\}}\right] \geq \sum_{j=1}^\infty E_Q\left[\mathbf{1}_{\{N^{-k-1}<l_j\leq N^{-k}\}}\right]$$
$$= \sum_{j=1}^\infty Q(N^{-k-1} < l_j \leq N^{-k}) = E_Q[\mathrm{card}\{j \in \mathbb{N}: kc \leq |\log l_j| < (k+1)c\}]$$

$$= \sum_{m=1}^{\infty} Q(\mathrm{card}\{j \in \mathbb{N} \colon kc \le |\log l_j| < (k+1)c\} \ge m)$$
$$\ge 1 - Q(\mathrm{card}\{j \in \mathbb{N} \colon kc \le |\log l_j| < (k+1)c\} = 0).$$

Let $\tau = \sup\{j : |\log l_j| < kc\} + 1$. It is easy to see that $E_Q[\tau] < \infty$. Now using Markov's inequality and Wald's identity, we obtain

$$\begin{aligned} &Q(\mathrm{card}\{j \in \mathbb{N} \colon kc \le |\log l_j| < (k+1)c\} = 0) = Q(|\log l_\tau| \ge (k+1)c) \\ &\quad \le Q(|\log l_\tau| - |\log l_{\tau-1}| > c) \le E_Q[|\log l_\tau| - |\log l_{\tau-1}|]/c \\ &\quad = \frac{E_Q[\tau]E_Q[|\log T_1|] - E_Q[\tau-1]E_Q[|\log T_1|]}{c} = E_Q[|\log T_1|]/c < 1. \end{aligned}$$

□

Proposition 1. *The series* $\sum_{k=1}^{\infty} Q\left(l_k^\alpha X_{k+s_0} < C\varphi(l_k\rho)\right)$ *diverges for all* $C > 0$ *in polynomial case, and for all* $C > \rho^{-\alpha} e^{\alpha E_Q[|\log T_1|]} t_0^{-\beta}$ *in exponential case.*

Proof. In exponential case we can find $N > e^{E_Q[|\log T_1|]}$ such that $C > N^\alpha \rho^{-\alpha} t_0^{-\beta}$. Conditioning on the value of l_k, we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} Q(l_k^\alpha X_{k+s_0} < C\varphi(l_k\rho)) \\ &\quad \ge \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} Q(l_k^\alpha X_{k+s_0} < C\varphi(l_k\rho) | N^{-j-1} < l_k \le N^{-j}) Q(N^{-j-1} < l_k \le N^{-j}) \\ &\quad \ge \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} Q(N^{-j\alpha} X_{k+s_0} < C\varphi(N^{-j-1}\rho)) Q(N^{-j-1} < l_k \le N^{-j}) \\ &\quad = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} Q(X_1 < CN^{-\alpha}\rho^\alpha g(N^{-j-1}\rho)) Q(N^{-j-1} < l_k \le N^{-j}) \\ &\quad = \sum_{j=1}^{\infty} Q(X_1 < CN^{-\alpha}\rho^\alpha g(N^{-j-1}\rho)) \sum_{k=1}^{\infty} Q(N^{-j-1} < l_k \le N^{-j}) \\ &\quad \ge M_1 \sum_{j=1}^{\infty} Q(X_1 < CN^{-\alpha}\rho^\alpha g(N^{-j-1}\rho)) = \infty, \end{aligned}$$

where M_1 is taken from theorem 1, and the latter sum diverges in case of polynomial decay by lemmas 1 and 3, and in case of exponential decay by lemma 2 and by the choice of C. □

Lemma 4. *For all* $k \in \mathbb{N}$ *and* $m \ge k + s_0$, X_m *is independent of* $\hat{R}_k$.

Proof. Let $B \subset \mathbb{R}$ be a Borel set.

$$\begin{aligned} Q(\{X_m \in B\} \cap \hat{R}_k) &= E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha X_\sigma \; 1_{\{X_\sigma \in B\}} \mathbf{1}_{\hat{R}_{\sigma|_k}}\Bigg] \\ &= \sum_{|\sigma|=m} E\left[X_\sigma \mathbf{1}_{\{X_\sigma \in B\}}\right] E\left[l_\sigma^\alpha \mathbf{1}_{\hat{R}_{\sigma|_k}}\right] \\ &= E\left[X \mathbf{1}_{\{X \in B\}}\right] E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha X_\sigma \mathbf{1}_{\hat{R}_{\sigma|_k}}\Bigg] \\ &= Q(X_k \in B) Q(\hat{R}_k). \end{aligned}$$

□

Suppose further that the following assumption holds. For random self-similar sets instead of reduction ratios in the assumption we use corresponding similarity maps and their Lipschitz constants.

Assumption 1. *There exist* $p_0, \rho > 0$, $s_0 \geq 0$ *and a collection of events* R_σ *in the* σ*-algebra generated by random vectors* $(T_{\tau*1}, \ldots, T_{\tau*n})$, $\sigma \prec \tau$, $|\tau| < |\sigma| + s_0$, *such that* $R_\sigma \cap \{K_\sigma \neq \emptyset\} \neq \emptyset$, *for every* $w \in R_\sigma \cap \{K_\sigma \neq \emptyset\}$ *there exists* $x \in K_\sigma$ *with* $\operatorname{dist}(x, \partial J_\sigma) \geq \rho l_\sigma$, *and* $\int_{R_\sigma} \sum_{|\tau|=s_0} \prod_{i=1}^{s_0} T_{\sigma*\tau|_i}^\alpha dP = p_0$.

Proposition 2. *Random self-similar sets satisfy assumption 1 under random strong open set condition.*

Proof. Denote by $\hat{S}_\sigma = (S_{\sigma*1}, S_{\sigma*2}, \ldots, S_{\sigma*n})$, $\sigma \in \Delta^*$ a sequence of i.i.d. random vectors of similarity maps, such that for $j = 1, 2, \ldots, n$, $S_{\sigma*j}(J_\sigma) = J_{\sigma*j}$ provided $J_\sigma \neq \emptyset$. Let $\tilde{S}_\sigma : \Omega \to [0, 1]^{n+\cdots+n^{s_0}}$ be a random vector consisting of all vectors $\hat{S}_\tau$ with $\sigma \prec \tau$, $|\tau| < |\sigma| + s_0$ listed in lexicographical order of τ. The random strong open set condition for random self-similar sets says that there exists $B \subset [0, 1]^{n+\cdots+n^{s_0}}$ such that $P(\tilde{S}_\emptyset \in B | K \neq \emptyset) > 0$ and for every $w \in \tilde{S}_\emptyset^{-1}(B) \cap \{K \neq \emptyset\}$ there exists $x \in K(w)$ such that $\operatorname{dist}(x, \partial J) \geq \rho_0$. Denote the Lipschitz constant of a similarity map S by $|S|$, let $R_\sigma = \tilde{S}_\sigma^{-1}(B)$, and $p_0 = \int_{\tilde{S}_\emptyset^{-1}(B)} \sum_{|\tau|=s_0} |S_\tau|^\alpha dP > 0$. Since the random vectors $\tilde{S}_\sigma$ have the same distribution, we have

$$\begin{aligned} E\Bigg[\sum_{|\tau|=s_0} \prod_{j=1}^{s_0} |S_{\sigma*\tau|_i}|^\alpha \mathbf{1}_{R_\sigma}\Bigg] &= \int_{\tilde{S}_\sigma^{-1}(B)} \sum_{|\tau|=s_0} \prod_{j=1}^{s_0} |S_{\sigma*\tau|_i}|^\alpha dP \\ &= \int_{\tilde{S}_\emptyset^{-1}(B)} \sum_{|\tau|=s_0} \prod_{j=1}^{s_0} |S_{\tau|_i}|^\alpha dP = p_0. \end{aligned}$$

□

Let $R_k = \{(\eta, w) \in \Delta^{\mathbb{N}} \times \Omega | w \in R_{\eta|_k}\} = \bigcup_{|\sigma|=k} A(\sigma) \times R_\sigma \subset \hat{R}_k$. Fix ρ so that assumption 1 holds, fix an arbitrary $C > 0$ in case of polynomial decay and $C > (3t_0)^{-\beta} \rho^{-\alpha} e^{\alpha E_Q[|\log T_1|]}$ in case of exponential decay.

Lemma 5. *For every $\sigma \in \Delta^*$, $E\left[X_\sigma \mathbf{1}_{R_\sigma}\right] = p_0$. For every $k \in \mathbb{N}$, $Q(R_k) = p_0$.*

Proof.

$$E\left[X_\sigma \mathbf{1}_{R_\sigma}\right] = E\left[\sum_{|\tau|=s_0} \prod_{i=1}^{s_0} T^\alpha_{\sigma*\tau|_i} X_{\sigma*\tau} \mathbf{1}_{R_\sigma}\right] = \int_{R_\sigma} \sum_{|\tau|=s_0} \prod_{i=1}^{s_0} T^\alpha_{\sigma*\tau|_i} dP = p_0.$$

Finally, $Q(R_k) = E\left[\sum_{|\sigma|=k} l^\alpha_\sigma X_\sigma \mathbf{1}_{R_\sigma}\right] = p_0 E\left[\sum_{|\sigma|=k} l^\alpha_\sigma\right] = p_0.$ □

Lemma 6. *For $j \leq k$, l_j is independent of R_k.*

Proof. Suppose $B \subset \mathbb{R}$, then

$$\begin{aligned} Q(\{l_j \in B\} \cap R_k) &= E\left[\sum_{|\sigma|=k} l^\alpha_\sigma X_\sigma \mathbf{1}_{R_\sigma} \mathbf{1}_{\{l_{\sigma|_j} \in B\}}\right] \\ &= \sum_{|\sigma|=k} E\left[l^\alpha_\sigma X_\sigma \mathbf{1}_{\{l_{\sigma|_j} \in B\}}\right] E\left[X_\sigma \mathbf{1}_{R_\sigma}\right] = p_0 Q(l_j \in B). \end{aligned}$$

□

For $k \in \mathbb{N}$, let $B_k = \{l^\alpha_k X_{k+s_0} \leq C\varphi(l_k \rho)\} \cap R_k$. Since R_k is independent of l_k and X_{k+s_0}, we have $Q(B_k) = p_0 Q(l^\alpha_k X_{k+s_0} \leq C\varphi(l_k\rho))$, and $\sum_{k=1}^{\infty} Q(B_k) = \infty$ by proposition 1. We would like to prove that B_k occurs i.o. Q-almost surely. To do that in polynomial case, we use Borel-Cantelli lemma generalised by Ortega and Wschebor in [16]. The exponential case requires another version of Borel-Cantelli lemma by Talagrand ([20]). For the latest results and a thorough review of what has been done in this direction the reader is referred to an article of Petrov ([19]).

Extended Borel-Cantelli lemmas. *1. Let $\{B_k\}_{k=1}^{\infty}$ be a sequence of events in a probability space such that $\sum_{k=1}^{\infty} Q(B_k) = \infty$, and*

$$\varliminf_{k\to\infty} \frac{\sum_{1\leq i<j\leq k} \left(Q(B_i B_j) - Q(B_i)Q(B_j)\right)}{\left(\sum_{i=1}^k Q(B_i)\right)^2} \leq 0, \tag{1}$$

then $Q\left(\varlimsup B_k\right) = 1$.

2. Let $\{B_k\}_{k=1}^{\infty}$ be a sequence of events in a probability space. If there exist positive constants M, ε, and positive integers k_0, J such that for $k_0 \leq j < J$,

$$\sum_{i=j+1}^{J} Q(B_j \cap B_i) \leq Q(B_j)\left(M + (1+\varepsilon)\sum_{i=j+1}^{J} Q(B_i)\right) \tag{2}$$

and

$$\sum_{i=k_0}^{J} Q(B_i) \geq (1+2M)/\varepsilon, \tag{3}$$

then

$$Q\left(\bigcup_{i=k_0}^{J} B_i\right) \geq 1/(1+2\varepsilon).$$

Let δ be a positive number. For $k < m$ and $v \in \mathbb{N}$, let $N_{k,m}^v = \{(\eta, w) \in \Delta^{\mathbb{N}} \times \Omega | \text{card}\{\tau \in \Delta^m \setminus \{\eta|_m\} \colon \eta|_k \prec \tau, l_\tau \geq \delta^{m-k} l_{\eta|_k} \text{ and } X_\tau > 0\} = v\}$. For $\sigma \in \Delta^m$, let $N_{k,\sigma}^v = \{w | \text{card}\{\tau \in \Delta^m \setminus \{\sigma\} \colon \sigma|_k \prec \tau, l_\tau \geq \delta^{m-k} l_{\sigma|_k} \text{ and } X_\tau > 0\} = v\}$.

Lemma 7. $Q(N_{k,m}^v) = Q(N_{0,m-k}^v)$.

Proof.

$$\begin{aligned} Q(N_{k,m}^v) &= E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha X_\sigma \mathbf{1}_{N_{k,\sigma}^v}\Bigg] = E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha \mathbf{1}_{N_{k,\sigma}^v}\Bigg] \\ &= \sum_{|\sigma|=k} E\left[l_\sigma^\alpha\right] E\Bigg[\sum_{|\tau|=m-k} \prod_{i=1}^{m-k} T_{\sigma*\tau|_i}^\alpha \mathbf{1}_{N_{k,\sigma*\tau}^v}\Bigg] \\ &= \sum_{|\sigma|=k} E\left[l_\sigma^\alpha\right] E\Bigg[\sum_{|\tau|=m-k} l_\tau^\alpha \mathbf{1}_{N_{0,\tau}^v}\Bigg] \\ &= \sum_{|\sigma|=k} E\left[l_\sigma^\alpha\right] Q(N_{0,m-k}^v) = Q(N_{0,m-k}^v). \end{aligned}$$

□

Now consider a function $v \colon \mathbb{N} \to \mathbb{N}$ and denote $N_{k,m} = \{(\eta, w) \in \Delta^{\mathbb{N}} \times \Omega | \text{card}\{\tau \in \Delta^m \setminus \{\eta|_m\} \colon \eta|_k \prec \tau, l_\tau \geq \delta^{m-k} l_{\eta|_k} \text{ and } X_\tau > 0\} \leq v(m-k)\}$, and for $\sigma \in \Delta^m$, let $N_{k,\sigma} = \{w | \text{card}\{\tau \in \Delta^m \setminus \{\sigma\} \colon \sigma|_k \prec \tau, l_\tau \geq \delta^{m-k} l_{\sigma|_k} \text{ and } X_\tau > 0\} \leq v(m-k)\}$.

Lemma 8. *If there exists* $\delta > 0$ *such that* $P(T_i \geq \delta | T_i \neq 0) = 1$ *for all* i *and* $v(m) \leq \mu_0^m$ *for some* $\mu_0 < \mu$, *then there exists* $M' > 0$ *such that* $\sum_{m=1}^{\infty} Q(N_{0,m}) < M'$.

Proof.

$$\begin{aligned} Q(N_{0,m}) &= E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha X_\sigma \mathbf{1}_{N_{0,\sigma}}\Bigg] \\ &= E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha \mathbf{1}_{\{X_\sigma>0\}} \mathbf{1}_{\{l_\sigma \geq \delta^m\}} \sum_{\substack{A\subset\Delta^m\setminus\{\sigma\} \\ \text{card}(A)\leq v(m)}} \prod_{\tau\in A} \mathbf{1}_{\{l_\tau\geq\delta^m, X_\tau>0\}} \prod_{\tau\in\Delta^m\setminus A} \mathbf{1}_{\{l_\tau<\delta^m \text{ or } X_\tau=0\}}\Bigg] \\ &= E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha \sum_{\substack{\sigma\in A\subset\Delta^m \\ \text{card}(A)\leq v(m)+1}} \prod_{\tau\in A} \mathbf{1}_{\{l_\tau\geq\delta^m\}} \mathbf{1}_{\{X_\tau>0\}} \prod_{\tau\in\Delta^m\setminus A} \mathbf{1}_{\{l_\tau=0 \text{ or } X_\tau=0\}}\Bigg] \end{aligned}$$

$$\leq \sum_{j=1}^{v(m)+1} \sum_{\substack{A\subset\Delta^m \\ \mathrm{card}(A)=j}} jP\,(K\neq\emptyset, \forall\tau\in A,\ l_\tau\geq\delta^m,\ \text{and}\ \forall\tau\in\Delta^m\setminus A,\ l_\tau=0 \text{ or } X_\tau=0)$$
$$\leq (v(m)+1)P(1\leq \mathrm{card}\{\tau\in\Delta^m|l_\tau>0\}\leq v(m)+1 \text{ and } K\neq\emptyset)$$
$$+(v(m)+1)P(\mathrm{card}\{\tau\in\Delta^m|X_\tau=0\}\geq n^m-v(m)-1)$$
$$\leq (v(m)+1)P(K\neq\emptyset)P(1\leq \mathrm{card}\{\tau\in\Delta^m|l_\tau>0\}\leq v(m)+1|K\neq\emptyset)$$
$$+(v(m)+1)\frac{n^mP(X=0)P(X>0)}{(n^mP(X>0)-v(m)-1)^2}$$
$$\leq P(K\neq\emptyset)\frac{(v(m)+1)E\left[(\mathrm{card}\{\tau\in\Delta^m|l_\tau>0\}-\mu^m)^2\right]}{(\mu^m-v(m)-1)^2}$$
$$+(v(m)+1)\frac{n^mP(X=0)P(X>0)}{(n^mP(X>0)-v(m)-1)^2}\asymp\frac{v(m)+1}{\mu^m}+\frac{v(m)+1}{n^m}$$

as we know from [1]. The result follows. □

Later we will choose $\mu_0<\mu$ and put $v(m)=\mu_0^m$. To start the proof that conditions of extended Borel-Cantelli lemmas hold in our case, we represent $Q(B_k\cap B_m)=Q(B_k\cap B_m\cap N_{k+s_0,m})+Q(B_k\cap B_m\cap\overline{N_{k+s_0,m}})$. Suppose that the second assumption holds. Note that it implies $Q(l_k=\delta^k)=1$.

Assumption 2. *There exists* $\delta>0$ *such that* $P(T_i=\delta|T_i\neq 0)=1$ *for all* i.

Lemma 9. $Q(B_k\cap B_m\cap N_{k+s_0,m})\leq Q(B_m)Q(N_{k+s_0,m})$.

Proof. Fix $b>0$, then

$$Q(\{X_{m+s_0}\leq b\}\cap R_m\cap N_{k+s_0,m})$$
$$=E\Bigg[\sum_{|\sigma|=m+s_0} l_\sigma^\alpha X_\sigma \mathbf{1}_{\{X_\sigma\leq b\}}\mathbf{1}_{R_{\sigma|_m}}\mathbf{1}_{\overline{N_{k+s_0,\sigma|_m}}}\Bigg]$$
$$=Q(X_{m+s_0}\leq b)\sum_{|\sigma|=m}E\left[l_\sigma^\alpha\mathbf{1}_{\overline{N_{k+s_0,\sigma}}}\right]E\Bigg[\sum_{|\tau|=s_0}\mathbf{1}_{R_\sigma}\prod_{i=1}^{s_0}T^\alpha_{\sigma*\tau|_i}\Bigg]$$
$$=Q(X_{m+s_0}\leq b)p_0E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha X_\sigma\mathbf{1}_{\overline{N_{k+s_0,\sigma}}}\Bigg]$$
$$=Q(\{X_{m+s_0}\leq b\}\cap R_m)Q(N_{k+s_0,m}).$$
□

Remark. If the function g is monotone, lemma 9 can be proven in the form $Q(B_k\cap B_m\cap N_{k+s_0,m})\leq Q(B_k)Q(N_{k+s_0,m})$ without assumption 2, and that is enough for the purposes of the extended Borel-Cantelli lemma.

For natural numbers $k+s_0<m$, let $Y_{k,m}=X_{k+s_0}-X_m\prod_{j=k+s_0+1}^{m}T_j^\alpha$. For a code σ of length at least m, let $Y_{k,\sigma}=X_{\sigma|_{k+s_0}}-X_{\sigma|_m}\prod_{j=k+s_0+1}^{m}T^\alpha_{\sigma|_j}$. Then it is easy to see that $Y_{k,\sigma}$ and X_τ with $\sigma|_m\prec\tau\prec\sigma$ are P-independent, which results in Q-independence of $Y_{k,m}$ and X_m.

Lemma 10. $Q(B_k \cap B_m \cap \overline{N_{k+s_0,m}}) = p_0 Q(B_m) Q(\{Y_{k,m} \le C\rho^\alpha g(l_k\rho)\} \cap \overline{N_{k+s_0,m}})$.

Proof. Fix $a, b > 0$, then

$$\begin{aligned}
&Q(\{Y_{k,m} \le a\} \cap R_k \cap \{X_{m+s_0} \le b\} \cap R_m \cap \overline{N_{k+s_0,m}}) \\
&\quad = E\Bigg[\sum_{|\sigma|=m+s_0} l_\sigma^\alpha X_\sigma \mathbf{1}_{\{Y_{k,\sigma} \le a\}} \mathbf{1}_{\{X_\sigma \le b\}} \mathbf{1}_{R_{\sigma|_k}} \mathbf{1}_{R_{\sigma|_m}} \mathbf{1}_{\overline{N_{k+s_0,\sigma|_m}}}\Bigg] \\
&\quad = Q(\{X_{m+s_0} \le b\} \cap R_m) E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha X_\sigma \mathbf{1}_{\{Y_{k,\sigma} \le a\}} \mathbf{1}_{R_{\sigma|_k}} \mathbf{1}_{\overline{N_{k+s_0,\sigma}}}\Bigg] \\
&\quad = Q(\{X_{m+s_0} \le b\} \cap R_m) \\
&\qquad \times \sum_{|\sigma|=k+s_0} E\left[l_\sigma^\alpha \mathbf{1}_{R_{\sigma|_k}}\right] E\Bigg[\sum_{|\tau|=m-k-s_0} X_{\sigma*\tau} \mathbf{1}_{\{Y_{k,\sigma*\tau} \le a\}} \mathbf{1}_{\overline{N_{k+s_0,\sigma*\tau}}} \prod_{i=1}^{m-k-s_0} T^\alpha_{\sigma*\tau|_i}\Bigg] \\
&\quad = p_0 Q(\{X_{m+s_0} \le b\} \cap R_m) E\Bigg[\sum_{|\sigma|=m} l_\sigma^\alpha X_\sigma \mathbf{1}_{\{Y_{k,\sigma} \le a\}} \mathbf{1}_{\overline{N_{k+s_0,\sigma}}}\Bigg] \\
&\quad = p_0 Q(\{X_{m+s_0} \le b\} \cap R_m) Q(\{Y_{k,m} \le a\} \cap \overline{N_{k+s_0,m}}). \qquad \square
\end{aligned}$$

Lemma 11. *In case of polynomial decay for every $\varepsilon \in (0, 1)$ there exists $M_\varepsilon \in (0, 1)$ so that $Q(X \le a(1 + M_\varepsilon)) \le (1 + \varepsilon)Q(X \le a)$ for all $a \in (0, 1/2)$. In case of exponential decay there exists $W > 0$ such that for every $\varepsilon \in (0, 1)$, if we let $M_{\varepsilon,a} = \varepsilon W^{-1} e^{-K_1 a^{1/\beta}}$, then $Q(X \le a(1 + M_{\varepsilon,a})) \le (1 + \varepsilon)Q(X \le a)$ for all $a \in (0, 1/2)$.*

Proof. By lemma 3, $Q(X \le a) = \int_0^a t P(X \in dt)$. From [1] we know that P-density of X, $w_P(t)$, is continuous on $(0, 1]$.

In polynomial case by theorem 1 in [4], $C_1 t^{\beta-1} \le w_P(t) \le C_2 t^{\beta-1}$ for all $t \in (0, 1)$. Suppose that $M \in (0, 1)$, then there exists $y_a \in [a, a(1 + M)]$ such that $Q(X \le a(1 + M)) - Q(X \le a) = aM y_a w_P(y_a) \le MC_2(2a)^{\beta+1}$. Thus by lemma 3 it is enough to choose $M_\varepsilon < \min\{\varepsilon 2^{-\beta-1} K_1/C_2, 1\}$.

In exponential case we know from [1] that $w_P(t)$ is uniformly continuous. Thus the Q-density of X, $w_Q(t)$, is bounded on $[0, 1]$. Denoting $W = \sup\{w_Q(t) | t \in [0, 1]\} < \infty$, we can estimate

$$Q(X \le a(1 + M_{a,\varepsilon})) - Q(X \le a) \le a M_{\varepsilon,a} W < \varepsilon e^{-K_1 a^{1/\beta}} \le \varepsilon Q(X \le a).$$

$\square$

Proposition 3. *In case of polynomial decay, for an appropriate choice of μ_0 for any $\varepsilon \in (0, 1)$ there exists a finite set $A_\varepsilon \subset \mathbb{N}$ such that the inequality $Q(\{Y_{k,m} \le a\} \cap \overline{N_{k+s_0,m}}) \le (1 + \varepsilon)Q(X_k \le a)$ holds for all $a \in (0, 1/2)$ and all $m > k$, where $m - k \notin A_\varepsilon$. In case of exponential decay for any $\varepsilon \in (0, 1)$, $Q(Y_{k,m} \le a) \le (1 + \varepsilon)Q(X_k \le a)$ for all $m - k > \max\{s_0, \frac{|\log a| + |\log(M_{\varepsilon/3,a}\varepsilon/3)| + |\log E[X^2]|}{\alpha |\log \delta|}\}$.*

Proof. We will consider only $m > k + s_0$, thus $1, \ldots, s_0 \in A_\varepsilon$. Take an $\varepsilon > 0$. In case of polynomial decay we choose $M = M_{\varepsilon/3}$ by lemma 11. In case of exponential decay we let $M = M_{\varepsilon/3,a}$ using the same lemma. Suppose that $m - k \geq \frac{|\log a| + |\log(M\varepsilon/3)| + |\log E[X^2]|}{\alpha|\log\delta|}$. Then

$$\begin{aligned}
&Q(\{Y_{k,m} \leq a\} \cap \overline{N_{k+s_0,m}}) \leq Q(Y_{k,m} \leq a)\\
&\quad = Q(Y_{k,m} \leq a, \delta^{\alpha(m-k)} X_m \leq aM) + Q(Y_{k,m} \leq a, \delta^{\alpha(m-k)} X_m > aM)\\
&\quad \leq Q(X_k \leq a(1+M)) + Q(Y_{k,m} \leq a) Q(\delta^{\alpha(m-k)} X_m > aM)\\
&\quad \leq (1+\varepsilon/3) Q(X_k \leq a) + Q(Y_{k,m} \leq a) E_Q[X] \delta^{\alpha(m-k)}/aM\\
&\quad \leq (1+\varepsilon/3) Q(X_k \leq a) + Q(Y_{k,m} \leq a)\varepsilon/3.
\end{aligned}$$

Thus $Q(Y_{k,m} \leq a) \leq Q(X_k \leq a)(1+\varepsilon/3)/(1-\varepsilon/3) < (1+\varepsilon) Q(X_k \leq a)$.

To perform the estimate when $m - k < \frac{|\log a|}{\alpha|\log\delta|}$ in polynomial case, assume that $m - k$ is large enough so that $v(m-k) = \mu_0^{m-k} > \max\{4, 4/\beta\}$. Let $\{X_i'\}_{i\in\mathbb{N}}$ be a sequence of i.i.d.r.v. distributed as X. Let $j = m - k - s_0$ and note that

$$\begin{aligned}
&Q(\{Y_{k,m} \leq a\} \cap \overline{N_{k+s_0,m}})\\
&\quad = E\Bigg[\sum_{|\sigma|=j} l_\sigma^\alpha X_\sigma \mathbf{1}_{\{Y_{0,\sigma} \leq a\}} \mathbf{1}_{\overline{N_{0,\sigma}}}\Bigg]\\
&\quad \leq E\Bigg[\sum_{|\sigma|=j} l_\sigma^\alpha \mathbf{1}_{\overline{N_{0,\sigma}}} \mathbf{1}_{\left\{\delta^{(m-k)\alpha} \sum_{i=1}^{v(m-k)} X_i' \leq a\right\}} \prod_{i=1}^{v(m-k)} \mathbf{1}_{\{X_i' > 0\}}\Bigg]\\
&\quad \leq P\left(0 < \sum_{i=1}^{v(m-k)} X_i' \leq a/\delta^{\alpha(m-k)}\right)\\
&\quad \leq P(0 < X \leq a/\delta^{\alpha(m-k)})^{v(m-k)}.
\end{aligned} \tag{4}$$

Now take $r > 4\beta + 4$, and assume that μ_0 has been chosen to satisfy the inequality $\mu^{\frac{4\beta+4}{r}} < \mu_0 < \mu$. Suppose that $m - k \leq \frac{|\log a|}{2\alpha|\log\delta|} - \frac{|\log C_2 - \log K_1|}{4\alpha|\log\delta|}$. From inequality (4) we see that

$$\begin{aligned}
&Q(\{Y_{k,m} \leq a\} \cap \overline{N_{k+s_0,m}}) \leq P(0 < X \leq a/\delta^{\alpha(m-k)})^{v(m-k)}\\
&\quad \leq C_2 a^{\beta v(m-k)} \delta^{-\alpha\beta(m-k)v(m-k)} \leq K_1 a^{\beta+1} \leq Q(X \leq a).
\end{aligned}$$

Now denote by Φ the distribution function of standard normal random variable and by ς the P-variance of X. Again by inequality (4) we obtain

$$\begin{aligned}
&P(\{Y_{k,m} \leq a\} \cap \overline{N_{k+s_0,m}})\\
&\quad \leq P\left(0 < \sum_{i=1}^{v(m-k)} X_i' \leq a/\delta^{\alpha(m-k)}\right) \leq P\left(\sum_{i=1}^{v(m-k)} X_i' \leq 1\right)\\
&\quad = P\left(\frac{\sum_{i=1}^{v(m-k)} X_i' - v(m-k)}{\varsigma\sqrt{v(m-k)}} \leq \frac{1 - v(m-k)}{\varsigma\sqrt{v(m-k)}}\right).
\end{aligned}$$

Since X has finite moment of order r, we can apply a theorem about non-uniform estimate of speed of convergence in central limit theorem (see, e.g. [18]). Using that $v(m-k) > 2$, we continue

$$\leq \Phi\left(-\mu_0^{(m-k)/2}/2\varsigma\right) + C_3\mu_0^{-(m-k)r/2} \leq C_4\mu_0^{-(m-k)r/2} \tag{5}$$

for some constants $C_3, C_4 > 0$ that depend only on distribution of X and r and all $m-k$ large enough. Since $\log\mu_0 > \frac{4\beta+4}{r}\log\mu$ by the choice of μ_0 and $\log\mu = \alpha|\log\delta|$, we see that the last term in inequality (5) does not exceed $K_1 a^{\beta+1}$ and thus $Q(X \leq a)$, if $m-k \geq \frac{2(\beta+1)|\log a|}{r\log\mu_0} + \frac{2|\log C_4 - \log K_1|}{r\log\mu_0} \geq \frac{|\log a|}{2\alpha|\log\delta|} + \frac{2|\log C_4 - \log K_1|}{r\log\mu_0}$. The result follows. □

Remark. Boundedness of the Q-density of X in a neighbourhood of 0 is sufficient for the proofs of lemma 11 and proposition 3.

Lemma 12. *Under assumption 2* $Q(\overline{\lim}\, B_k) = 1$.

Proof. In case of polynomial decay without loss of generality we may assume that $\lim_{0+} g = 0$. Thus there exists $k_0 \in \mathbb{N}$ such that in any case $C\rho^\alpha g(l_k\rho) < 1/2$ for all $k \geq k_0$. Take $\varepsilon > 0$.

Suppose that we are in case of polynomial decay. Since $\sum_{i=1}^{\infty} Q(B_i) = \infty$, it suffices to prove inequality (1) when summation starts with k_0. Denote by s_1 the cardinality of set $A_{\varepsilon/2}$ from proposition 3. By lemmas 9, 7 and 8,

$$\begin{aligned}\Sigma_{1,k} &= \sum_{\substack{k_0 \leq i < j \leq k \\ i+s_0 < j}} Q(B_i \cap B_j \cap N_{i+s_0,j}) \\ &\leq \sum_{j=k_0+s_0+1}^{k} Q(B_j) \sum_{i=k_0}^{j-s_0-1} Q(N_{i+s_0,j}) \leq M' \sum_{j=k_0}^{k} Q(B_j).\end{aligned}$$

By proposition 3 and lemma 10,

$$\begin{aligned}\Sigma_{2,k} &= \sum_{\substack{k_0 \leq i < j \leq k \\ j-i \notin A_{\varepsilon/2}}} \left(Q(B_i \cap B_j \cap \overline{N_{i+s_0,j}}) - Q(B_i)Q(B_j)\right) \\ &\leq \frac{\varepsilon}{2} \sum_{k_0 \leq i < j \leq k} Q(B_i)Q(B_j) \leq \frac{\varepsilon}{2}\left(\sum_{j=k_0}^{k} Q(B_j)\right)^2.\end{aligned}$$

Obviously, $\Sigma_{3,k} = \sum_{\substack{k_0 \le i < j \le k \\ j-i \in A_{\varepsilon/2}}} Q(B_i \cap B_j) \le s_1 \sum_{j=k_0}^{k} Q(B_j)$. Thus

$$\frac{\sum_{k_0 \le i < j \le k} \big(Q(B_i B_j) - Q(B_i)Q(B_j)\big)}{\left(\sum_{i=k_0}^{k} Q(B_i)\right)^2} \le \frac{\Sigma_{1,k} + \Sigma_{2,k} + \Sigma_{3,k}}{\left(\sum_{i=k_0}^{k} Q(B_i)\right)^2}$$

$$\le \varepsilon/2 + (s_1 + M')\left(\sum_{i=k_0}^{k} Q(B_i)\right)^{-1} < \varepsilon$$

for all sufficiently large k by proposition 1, and thus condition (1) of the first extended Borel-Cantelli lemma is satisfied.

Now suppose that we are in case of exponential decay, $k_0 > |\log \rho|$. By proposition 3 and lemma 10, there exist constants $D_1 = D_1(\varepsilon, \rho, C) > s_0$ and $D_2 = D_2(\varepsilon, \rho, C)$ such that for all $j, i \ge k_0$, $j - i > D_1 + D_2 \log i$ implies that $Q(B_i \cap B_j \cap \overline{N_{i+s_0,j}}) \le (1+\varepsilon)Q(B_i)Q(B_j)$. By lemma 2 and observation after lemma 6, for any $k_1 > e^{k_0}$ there exists $k > e^{k_1}$ such that $\sum_{i=[\log k]}^{k} Q(B_i) - \frac{2D_2}{\varepsilon}\log k > (2 + 2D_1 + 2M')/\varepsilon$. By remark after lemma 9, for every $[\log k] \le j < k$

$$\sum_{i=j+1}^{k} Q(B_j \cap B_i) \le Q(B_j)\left(D_1 + M' + D_2 \log k + (1+\varepsilon)\sum_{i=j+1}^{k} Q(B_i)\right),$$

and we see that conditions (2) and (3) of the second extended Borel-Cantelli lemma are satisfied, thus $Q\left(\bigcup_{i=k_1}^{\infty} B_i\right) \ge Q\left(\bigcup_{i=[\log k]}^{k} B_i\right) \ge 1/(1+2\varepsilon)$. The result follows. □

Lemma 13. *Assume that* $Q(\overline{\lim}\, B_k) = 1$, *then there exists a sequence of natural numbers* $\{\tilde{J}(k)\}_{k=1}^{\infty}$, $\tilde{J}(k) \ge k + s_0$, *such that if we define for* $k \in \mathbb{N}$ *and* $w \in \Omega$

$$A_k(w) = \{\sigma \in \Delta^{\tilde{J}(k)} | \textit{for all } j = k + s_0, \ldots, \tilde{J}(k)$$
$$X_{\sigma|_j} > \rho^{\alpha} C g(\rho l_{\sigma|_{j-s_0}}) \textit{ or } w \notin R_{\sigma|_{j-s_0}} \textit{ or } K_{\sigma|_{j-s_0}} = \emptyset\},$$

then there exists a sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ *such that*

$$\lim_{i\to\infty} \sum_{\sigma \in A_{k_i}} l_\sigma^{\alpha} X_\sigma = 0 \textit{ a.s.}$$

Proof. For $j \in \mathbb{N}$, let $\tilde{R}_j = \{(\eta, w) \in \Delta^{\mathbb{N}} \times \Omega | K_{\eta|_j}(w) \ne \emptyset\}$. We see that

$$Q(\tilde{R}_j) = E\left[\sum_{|\sigma|=j} l_\sigma^{\alpha} X_\sigma \mathbf{1}_{\{K_\sigma \ne \emptyset\}}\right]$$
$$= E\left[\sum_{|\sigma|=j} l_\sigma^{\alpha} X_\sigma \mathbf{1}_{\{l_\sigma > 0\}} \mathbf{1}_{\{X_\sigma > 0\}}\right] = E\left[\sum_{|\sigma|=j} l_\sigma^{\alpha} X_\sigma\right] = 1.$$

It is enough to show that $\varliminf_{k\to\infty} E\Big[\sum_{\sigma\in A_k} l_\sigma^\alpha X_\sigma\Big] = 0$. Indeed,

$$E\Big[\sum_{\sigma\in A_k} l_\sigma^\alpha X_\sigma\Big] = E\Big[\sum_{|\sigma|=\tilde{J}(k)} l_\sigma^\alpha X_\sigma \mathbf{1}_{\{\sigma\in A_k\}}\Big]$$

$$\leq E\Big[\sum_{|\sigma|=\tilde{J}(k)} l_\sigma^\alpha X_\sigma \prod_{j=k+s_0}^{\tilde{J}(k)} \mathbf{1}_{\{X_{\sigma|_j}(w)>\rho^\alpha Cg(\rho l_{\sigma|_{j-s_0}}) \text{ or } w\notin R_{\sigma|_{j-s_0}} \text{ or } K_{\sigma|_{j-s_0}}=\emptyset\}}\Big]$$

$$= E_Q\left[\prod_{j=k+s_0}^{\tilde{J}(k)} \mathbf{1}_{\{X_j>\rho^\alpha Cg(\rho l_{\sigma|_{j-s_0}}) \text{ or } (\eta,w)\notin R_{j-s_0} \text{ or } (\eta,w)\notin \tilde{R}_{j-s_0}\}}\right]$$

$$= Q\Big(\bigcap_{j=k}^{\tilde{J}(k)-s_0} \overline{B_j\cap\tilde{R}_j}\Big) = 1 - Q\Big(\bigcup_{j=k}^{\tilde{J}(k)-s_0} B_j\cap\tilde{R}_j\Big) = 1 - Q\Big(\bigcup_{j=k}^{\tilde{J}(k)-s_0} B_j\Big).$$

Therefore it suffices to choose $\tilde{J}(k)$ so that $Q\left(\bigcup_{j=k}^{\tilde{J}(k)-s_0} B_j\right) \geq 1 - 1/k$. □

Theorem 2. *Suppose $Q(\overline{\lim}\, B_k) = 1$ and that the construction satisfies assumption 1, then*

1. *If $C_1a^\beta \leq P(0 < X \leq a) \leq C_2a^\beta$ for all $a \in (0,1)$ and $\varphi(t) = t^\alpha g(t)$ is a function, then $\int_{0^+} \frac{g^{\beta+1}(s)}{s}ds = +\infty$ implies $P(\mathcal{P}^\varphi(K(w)) = +\infty | K(w) \neq \emptyset) = 1$.*
2. *If $C_1a^{1/\beta} \leq -\log P(0 < X \leq a) \leq C_2a^{1/\beta}$ for all $a \in (0,1)$, then for $\varphi(t) = t^\alpha g(t) = t^\alpha |\log|\log t||^\beta$, $P(\mathcal{P}^\varphi(K(w)) > 0 | K(w) \neq \emptyset) = 1$.*

Proof. Suppose that the conclusion of the theorem is false for the φ-packing premeasure, which we denote by $\mathcal{P}_0^\varphi$. Then in case 1 we can find $M > 0$ such that $p = P(\mathcal{P}_0^\varphi(K(w)) < M | K(w) \neq \emptyset) > 0$, and in the second case we let $p = P(\mathcal{P}_0^\varphi(K(w)) = 0 | K(w) \neq \emptyset) > 0$. Since $\operatorname{ess\,inf}_{\{K\neq\emptyset\}} X = 0$, there exists $\varepsilon > 0$ such that $P(X > \varepsilon | K \neq \emptyset) > 1 - p/4$. In case 1 choose $C < \varepsilon/(2Mn^{s_0})$, and in case 2 choose $C > (3t_0)^{-\beta} e^{\alpha E_Q[|\log T_1|]}\rho^{-\alpha}$. Fix w such that the following three conditions hold:

$$\forall \sigma \in \Delta^{\mathbb{N}} \lim_{k\to\infty} l_{\sigma|_k}(w) = 0,$$

$$\text{for every maximal antichain } \Gamma \subset \Delta^*,\ \varepsilon < X(w) = \sum_{\sigma\in\Gamma} l_\sigma^\alpha(w) X_\sigma(w) < \infty,$$

$$\text{and } \lim_{i\to\infty} \sum_{\sigma\in A_{k_i}} l_\sigma(w)^\alpha X_\sigma(w) = 0,$$

where k_i and A_{k_i} are from lemma 13.

Take an arbitrary $\gamma > 0$. Then there exists $k_0 \in \mathbb{N}$ such that for all $\sigma \in \Delta^{\mathbb{N}}$ and all $j \geq k_0$ $l_{\sigma|_j}(w) < \gamma$. Next we choose $i \in \mathbb{N}$ such that $k_i \geq k_0$, $\sum_{\sigma \in A_{k_i}} l_\sigma^\alpha(w) X_\sigma(w) < \varepsilon/2$. Let

$$A = \{\sigma \in \Delta^{\mathbb{N}} | \text{ for all } j = k_i + s_0, \ldots, \tilde{J}(k_i)$$
$$X_{\sigma|_j} > \rho^\alpha C g(l_{\sigma|_{j-s_0}} \rho) \text{ or } w \notin R_{\sigma|_{j-s_0}} \text{ or } K_{\sigma|_{j-s_0}} = \emptyset\}. \tag{6}$$

For $\sigma \in \Delta^{\mathbb{N}} \setminus A$, let $k(\sigma) = \min\{j | k_i + s_0 \leq j \leq \tilde{J}(k_i)$ and condition in line (6) fails$\}$. Set $\Gamma_1 = \{\sigma|_{k(\sigma)} : \sigma \in \Delta^{\mathbb{N}} \setminus A\}$, $\Gamma_2 = \{\sigma|_{\tilde{J}(k_i)} : \sigma \in A\}$. Then $\Gamma_1 \cup \Gamma_2$ is a maximal antichain. For $\sigma \in \Gamma_1$, there exists $x_\sigma \in K_{\sigma||\sigma|-s_0}$ such that $\operatorname{dist}(x_\sigma, \partial J_{\sigma||\sigma|-s_0}) \geq l_{\sigma||\sigma|-s_0}\rho$. Thus we can produce a packing of K by $B(x_\sigma, l_{\sigma||\sigma|-s_0}\rho)$, $\sigma \in \Gamma_1$. Since for each $\sigma_0 = \sigma||\sigma|-s_0$, there can be no more than n^{s_0} elements in Γ_1 extending code σ_0, we obtain with probability greater than $1 - p/4$ for every $\gamma > 0$:

$$\begin{aligned}\mathcal{P}^{\varphi}_{0,\gamma}(K(w)) &\geq n^{-s_0} \sum_{\sigma \in \Gamma_1} \varphi(l_{\sigma||\sigma|-s_0}\rho) = n^{-s_0}\rho^\alpha \sum_{\sigma \in \Gamma_1} l^\alpha_{\sigma||\sigma|-s_0} g(l_{\sigma||\sigma|-s_0}\rho) \\ &\geq n^{-s_0} C^{-1} \sum_{\sigma \in \Gamma_1} l_\sigma^\alpha X_\sigma \geq n^{-s_0} C^{-1} \left(\sum_{\sigma \in \Gamma} l_\sigma^\alpha X_\sigma - \sum_{\sigma \in \Gamma_2} l_\sigma^\alpha X_\sigma \right) \\ &= n^{-s_0} C^{-1} \left(X - \sum_{\sigma \in \Gamma_2} l_\sigma^\alpha X_\sigma \right) > \begin{cases} M \text{ in case 1} \\ n^{-s_0} C^{-1} \varepsilon/2 > 0 \text{ in case 2} \end{cases}.\end{aligned}$$

This is a contradiction. Now using Baire's category theorem we can spread the result obtained for φ-packing premeasure onto φ-packing measure. □

Remark. The proofs of theorem 2 and theorem 6 in [3] go through for random recursive constructions in any complete separable metric space. Proposition 1 and theorem 2 suggest that the result remains valid without assumption 2.

3. Examples

Example 1. Mandelbrot percolation.

Suppose the square is divided into n^2 equal subsquares and each survives with probability p. Inside each square that survives the procedure repeats. The fractal dimension in this case is $\alpha = 2 + (\log p / \log n)$. The exact Hausdorff gauge function is $t^\alpha (|\log|\log t||)^{1-(\alpha/2)}$ as determined in [7], example 6.2. By theorem 1 from the article of Dubuc ([4]), we are in case of polynomial decay with parameter β, where β is the solution of equation $p_1 \mu^\beta = 1$, $p_1 = P(\exists!\, i : T_i \neq 0) = n^2 p (1-p)^{n^2-1}$ and $\mu = n^2 p$ is the expected number of offspring. In this case $\beta = -1 - \frac{\log(1-p)^{n^2-1}}{\log n^2 p}$. Without loss of generality $n \geq 3$. The random strong open set condition is satisfied with $s_0 = 1$ because with positive probability all offspring touching the boundary of the parent "die out." According to [3], example 1, for the gauge function $\varphi(t) = t^\alpha g(t)$ such that $\int_{0+} \frac{g(s)^{\beta+1}}{s} ds < \infty$, $\mathcal{P}^\varphi(K) = 0$ a.s. By theorem 2 and lemma 12, $\int_{0^+} \frac{g(s)^{\beta+1}}{s} ds = \infty$ implies $P(\mathcal{P}^{\varphi}(K) = \infty | K \neq \emptyset) = 1$, i.e. the exact packing dimension does not exist.

Example 2. Modified Mandelbrot percolation.

Fix $n \in \mathbb{N}$ (without loss of generality $n \geq 3$) and a probability measure ν on the power set of $\{1, ..., n^2\}$. Let $J_1, ..., J_{n^2}$ be a labelling of the partition of $[0, 1] \times [0, 1]$ into congruent subsquares. If the square J_σ has been constructed, then choose $A \subset \{1, ..., n^2\}$ according to ν and let $J_{\sigma * i}, i \in A$ be the subsquares of J_σ obtained by scaling J_i into J_σ via the natural map.

This construction clearly satisfies the random strong open set condition with $s_0 = 1$, if we can get with positive probability an offspring that does not touch the boundary of the parent. If all offspring touch the boundary but there is positive probability of them touching different sides of the square, then the random strong open set condition is satisfied with $s_0 = 2$. Finally, if all offspring touch only one side of the square almost surely, then the limit set can be realised as a random self-similar set on the line with $J = [0, 1]$ that satisfies our condition.

If $\underline{\mu}$, the essential infimum of the number of offspring, is at least 2, then according to Biggins and Bingham([2], proposition 7) we are in the case of exponential decay with parameter $\beta = 1 - \log \mu / \log \underline{\mu}$, and according to example 4 in [3], for the gauge function $\varphi(t) = t^\alpha |\log|\log t||^\beta$, we have $\mathcal{P}^\varphi(K) < \infty$ a.s. By theorem 2 and lemma 12, $P(\mathcal{P}^\varphi(K) > 0 | K \neq \emptyset) = 1$ and thus φ is the exact packing dimension.

If $\underline{\mu} = 1$, the picture is the same as in example 1, i.e. there is no exact packing dimension.

Example 3. Self-avoiding stochastic process on the Sierpinski gasket.

This process was introduced in [10], and its almost sure exact Hausdorff dimension was found in [9]. Here we give an alternative definition, which allows to apply already known theorems about random recursive constructions.

Let J be an equilateral triangle of diameter 1 with one vertex O at the origin and another vertex B at a point with coordinates (1,0). By A we denote the third vertex of this triangle. J_1, J_2, J_3 are those three equilateral triangles of diameter 1/2 out of 4 partitioning J that have as one of their vertices O, A or B correspondingly. Then the process is iterated, and we obtain a (non-random) self-similar set which is called *the Sierpinski Gasket*, $G = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in \{1,2,3\}^n} J_\sigma$.

Fix $1 > p > 0$. Let $f_n(x) : [0, 1] \to \bigcup_{\sigma \in \{1,2,3\}^n} J_\sigma$ be a collection of random maps such that for all $\sigma \in \{1, 2, 3\}^n$, $J_\sigma \cap f_n([0, 1])$ coincides with a side of triangle J_σ or is empty in the following way:

(i) For $n = 0$, $f_0(0) = O$, $f_0(1) = A$, and the map f_0 is linear.

(ii) Suppose that the random function f_n has been defined. For a fixed $\sigma \in \{1, 2, 3\}^n$, let $[a_\sigma, b_\sigma] = f_n^{-1}(J_\sigma \cap f_n([0, 1]))$. Let $m \in \{1, 2, 3\}$ be such that $J_{\sigma * m} \cap J_\sigma \cap f_n([0, 1]) = \emptyset$, $k \in \{1, 2, 3\}$ such that $f(a_\sigma) \in J_{\sigma * k}$ and l such that $f(b_\sigma) \in J_{\sigma * l}$. Define f_{n+1} so that $f_{n+1}(a_\sigma) = f_n(a_\sigma)$, $f_{n+1}(b_\sigma) = f_n(b_\sigma)$.

With probability p, we let $f_{n+1}((a_\sigma + b_\sigma)/2) = J_{\sigma*k} \cap J_{\sigma*l}$, and with probability $1-p$, $f_{n+1}((a_\sigma + b_\sigma)/3) = J_{\sigma*k} \cap J_{\sigma*m}$ and $f_{n+1}(2(a_\sigma + b_\sigma)/3) = J_{\sigma*m} \cap J_{\sigma*l}$. Then the map f_{n+1} is extended by linearity. Inside all J_σ's, the process of refining of f_n to f_{n+1} is independent.

Finally we define a random map $f : [0, 1] \to G$ by setting $f(x) = \lim_{n\to\infty} f_n(x)$. It is easy to see that the map f is well defined, continuous, one-to-one and $f([0, 1])$ is a random arc that coincides with the limit set of the random recursive construction obtained by redefining the triangles J_σ so that for each σ, if only two triangles out of $J_{\sigma*1}$, $J_{\sigma*2}$, $J_{\sigma*3}$ intersect $f_{|\sigma|+1}([0, 1])$ along an edge, these two triangles are denoted by $J_{\sigma*1}$ and $J_{\sigma*2}$, and $J_{\sigma*3} = \emptyset$. For each σ in this random recursive construction, the random vector of reduction ratios is $(1/2, 1/2, 0)$ with probability p, and $(1/2, 1/2, 1/2)$ with probability $1-p$. Note that this is not a random self-similar set.

By theorem 1 in [3] the Hausdorff, packing and Minkowski dimensions of $f([0, 1])$, $\alpha = \log_2(3-p)$ almost surely. Assumption 1 is satisfied with $s_0 = 2$, $R_\sigma = \{T_{\sigma*3} \neq 0, T_{\sigma*33} = 0\}$, assumption 2 is satisfied with $\delta = 1/2$. By [2], proposition 7, we are in case of exponential decay with parameter $\beta = 1 - \log_2(3-p)$. Thus by lemma 12, theorem 2 and theorem 6 from [3], for the gauge function $\varphi(t) = t^\alpha |\log|\log t||^\beta$, $P(0 < \mathcal{P}^\phi(f[0, 1]) < \infty) = 1$.

Example 4. Boundary of a Galton-Watson tree.

Let N_σ, $\sigma \in \Delta^*$, be a sequence of i.i.d.r.v., $N_\sigma \in \mathbb{N} \cup \{0\}$, $E[N_\emptyset \log N_\emptyset] < \infty$. The Galton-Watson tree T corresponding to this sequence is a subset of Δ^* such that $\emptyset \in T$ and $\sigma \in T \iff \sigma * i \in T$ for all $1 \le i \le N_\sigma$. The boundary, ∂T, of the random tree is the set of all infinite paths through the tree. The tree metric on ∂T is defined by setting for $\sigma, \tau \in \partial T$, $d_T(\sigma, \tau) = c^{|\sigma\wedge\tau|}$ when $\sigma \neq \tau$ and $d_T(\sigma, \tau) = 0$ if $\sigma = \tau$, where $c \in (0, 1)$ and $\sigma \wedge \tau$ denotes the largest common subsequence of σ and τ. We require that $E[X^2] < \infty$.

Assumption 1 is satisfied with $s_0 = 0$ and $R_\sigma = \Omega$. Note that because of $s_0 = 0$ the proof of theorem 2 still holds in case of polynomial decay with parameter β if the number of offspring is unbounded and X has finite moment of order $r > 4\beta + 4$. The proof of theorem 6 in [3] holds for unbounded number of offspring under assumption 2.

If the probability of $N_\sigma = 1$ is positive, we are in the case of polynomial decay, and there exists no exact packing dimension. In case the number of offspring has geometric distribution, $P(N_\emptyset = k) = p(1-p)^k$ for $k \in \mathbb{N}$, by a result of Hawkes in [8] $P(0 < X \le x) = 1 - e^{-x}$ for $x \ge 0$. The rate of polynomial decay $\beta = 1$, and we obtain the result of Xiao from [22] because X has moments of all orders.

If $N_\sigma \ge 2$ almost surely, then we are in the case of exponential decay with parameter $\beta = 1 - \log\mu/\log\underline{\mu}$. Thus, by theorem 2 and lemma 12 the exact packing dimension function is given by $\varphi(t) = t^\alpha|\log|\log t||^\beta$. This proves the conjecture of Liu in [13] who has studied the exact packing dimension of ∂T in case of exponential decay and made a mistake in the proof of the lower bound as it was pointed out in [3], theorem 7.

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