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Pinning class of the Wiener measure by a functional: related martingales and invariance properties

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Abstract. For a given functional Y on the path space, we define the pinning class of the Wiener measure as the class of probabilities which admit the same conditioning given Y as the Wiener measure. Using stochastic analysis and the theory of initial enlargement of filtration, we study the transformations (not necessarily adapted) which preserve this class. We prove, in this non Markov setting, a stochastic Newton equation and a stochastic Noether theorem. We conclude the paper with some non canonical representations of Brownian motion, closely related to our study.

Contents

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1. Introduction

Given a constant time horizon $T \in (0, +\infty]$, we denote by I the interval $[0, T]$ (resp. $[0, T[$) if $T < +\infty$ (resp. $T = +\infty$). We work on the Wiener space of continuous paths

$$\mathbb{W} = \left(\mathcal{C}_T, (\mathcal{F}_t)_{t\in I}, (x_t)_{t\in I}, \mathbb{P}\right)$$

(1) $\mathcal{C}_T$ is the space of continuous functions $f : I \to \mathbb{R}$, such that $f(0) = 0$
(2) $(x_t)_{t\in I}$ is the coordinate process defined by $x_t(f) = f(t)$
(3) $(\mathcal{F}_t)_{t\in I}$ is the natural filtration of $(x_t)_{t\in I}$
(4) $\mathbb{P}$ is the Wiener measure.

On $\mathbb{W}$ we consider an $\mathcal{F}_T$-measurable functional Y and we study the set of probabilities $\mathbb{Q}$ on $\mathbb{W}$ such that for any $A \in \mathcal{F}_T$, $\mathbb{Q}(A \mid Y) = \mathbb{P}(A \mid Y)$. We call this set the pinning class of Wiener measure under Y and denote it by $\mathcal{R}_Y(\mathbb{P})$.

This class has been the object of study of previous papers with various approaches:

(1) In [3] it was proved that $\mathcal{R}_Y(\mathbb{P})$ is the set of laws of weak solutions of stochastic differential equations (called Conditioned Stochastic Differential Equations)
(2) When $T < +\infty$ and $Y = x_T$, our pinning class coincides with the set of Markov processes starting from 0 and belonging to the so called reciprocal class of the Wiener measure (see [13] and references therein). Actually the general reciprocal class is defined in the same way as the pinning class except that the conditioning is made by the two functionals x_0 and x_T. Let us recall that any element of the reciprocal class enjoys the Markov field property with respect to time (or is a reciprocal process). If its initial value is deterministic, then it is Markovian. This is the case here since all elements in $\mathcal{R}_Y(\mathbb{P})$ start at 0.
(3) The use of $\mathcal{R}_Y(\mathbb{P})$ in mathematical finance was developed in [3] and [5], in the topic of asymmetric information between different insiders.
(4) In the case where $Y = x_T$ the invariance of a reciprocal class under some one parameter families of transformations was investigated in [18]. This study was based on symmetries for linear second order parabolic p.d.e. Another point of view based on symmetries for an action functional related to $\mathcal{R}_Y(\mathbb{P})$ provided a stochastic Noether theorem in [20]. At this point let us mention that symmetries of Markov processes were also considered with motivation from filtering theory (cf. [6]) and in the framework of potential theory (cf. [11]). These studies were concerned with global symmetries whereas here we also study local ones (these two notions are defined in the next paragraph).

The framework of the present paper differs from the above results since for a general functional Y the elements of $\mathcal{R}_Y(\mathbb{P})$ are not Markovian. Our tools belong to Stochastic Analysis and the theory of initial enlargement of a filtration. With these tools we first associate to $\mathcal{R}_Y(\mathbb{P})$ two families of martingales. The first one is a consequence of a stochastic Newton equation, the second one of a stochastic Noether Theorem. These two results provide a stochastic Mechanics interpretation of $\mathcal{R}_Y(\mathbb{P})$ (see [17] for Newton equation in a reciprocal class). When it is possible, we state our results both in the canonical filtration and in the enlarged one. We also characterize the transformations on the path space which leave $\mathcal{R}_Y(\mathbb{P})$ infinitesimally invariant. We consider local (resp. global) transformations which preserve a given element of $\mathcal{R}_Y(\mathbb{P})$ (resp. globally the whole class $\mathcal{R}_Y(\mathbb{P})$) which we call symmetries. We allow anticipating transformations since in our non Markov setting, non trivial adapted symmetries may fail to exist. Since we are not in a Markov setting any longer, we do not rely on the symmetries of an action functional neither of a partial differential equation. All through the paper we explicit our results on three examples: $Y = y_T$, the terminal value of a diffusion process $(y_t)_{0\leq t\leq T}$, $Y = \int_0^T f(s)\, dx_s$, the Wiener integral of f, and $Y = \int_0^{+\infty} e^{2x_s - 2\mu s} ds$ $(\mu > 2)$, the exponential functional of the geometric Brownian motion which appears for instance in finance (Asian options), in optics and in the theory of diffusions in a random environment. We conclude the paper by showing that our results allow to recover some non canonical representations: singular linear Volterra transforms (cf. [1]) and Pitman's exponential theorem (cf. [15], [4]).

The present paper is organized as follows. Section 2 is devoted to some preliminary results and the description of our basic assumptions. In Section 3 we present our generalization of the stochastic Newton equation. In Section 4 we investigate the local symmetries of the pinning class in order to prove a Noether Theorem in our setting. Section 5 concerns global symmetries; in particular we study the existence of non adapted symmetries with the techniques of anticipative calculus. This section ends with non canonical representations. Section 6 ends the paper by recalling some results of the Markovian setting which are generalized in this paper.

2. Preliminaries and assumptions

We will denote $\mathbb{H}$ the Cameron-Martin space associated with $\mathbb{W}$. We recall (see [16] pp. 26) that the Banach space $\mathbb{D}^{1,n}$ is the closure of the class of smooth cylindric random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,n} = \left(\mathbb{P}\left(|F|^n\right) + \mathbb{P}\left(\|\mathbf{D}F\|^n_{L^2}\right)\right)^{\frac{1}{n}},$$

where $\mathbf{D}$ is the Malliavin's differential. For a vector field $u : \mathbb{W} \to \mathbb{H}$, we will sometimes denote by $(\delta u)_t$ the Skorohod integral process $\int_0^t \frac{du_s}{ds} dx_s$ (which coincides with Itô classical integral as soon as u is adapted) and by $D_u F$, with $F \in \mathbb{D}^{1,2}$, the directional derivative $\int_0^t \frac{du_s}{ds} \mathbf{D}_s F ds$.

On $\mathbb{W}$ we consider a real valued functional of the trajectories $Y \in \mathbb{D}^{1,2}$ which is measurable with respect to $\mathcal{F}_T$. The law of Y under $\mathbb{P}$ shall be denoted $\mathbb{P}_Y$. Moreover, all through the paper the functional Y is assumed to satisfy the following conditions:

(A1) *The law $\mathbb{P}_Y$ of Y under $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure and the density, which will be denoted by p can be chosen strictly positive in the interior of the support of $\mathbb{P}_Y$ and continuously differentiable. Moreover, there exists a version of the regular conditional probabilities given Y such that the map $y \to \mathbb{P}(\cdot \mid Y = y)$, is continuous in the weak topology of convergence of probability measures.*

(A2) *There exists an $\mathcal{F}-$adapted and jointly measurable process*

$$\eta_t^y \ , 0 \le t < T, \, y \in \mathbb{R}$$

satisfying for any random variable Z bounded and $\mathcal{F}_t-$measurable, $t < T$, and $\mathbb{P}_Y - a.e.\ y \in \mathbb{R}$,

$$\mathbb{E}\,(Z \mid Y = y) = \mathbb{E}\left(\eta_t^y Z\right).$$

(A3) *For $\mathbb{P}_Y - a.e.\ y \in \mathbb{R}$ and $0 \le t < T$, $\eta_t^y \in \mathbb{D}^{1,2}$, and there exists a measurable version of the two parameter process $\left(\mathbf{D}_s \eta_t^y\right)_{0 \le s,t < T}$ verifying for $\tau < T$,*

$$\mathbb{P}\left(\int\int_{[0,\tau]^2} \left(\mathbf{D}_s \eta_t^y\right)^2 ds\ dt\right) < +\infty.$$

Remark 1. Under these assumptions, the topological support of $\mathbb{P}_Y$ is a closed interval (see [16], 1.2.12) and if, moreover, $Y \in \mathbb{D}^{1,n}$ with $n > 2$, then p is strictly positive in the interior of this interval (see [16], 2.1.2).

We now turn to the definition of the main object of our study.

Definition 2. *We call pinning class of the Wiener measure over the functional Y the closure in the weak topology (i.e. the topology of convergence in law) of the set of probabilities*

$$\mathcal{R}_Y\,(\mathbb{P}) = \{\mathbb{Q},\ \mathbb{Q} \sim \mathbb{P}, \forall\ A \in \mathcal{F}_T, \mathbb{Q}\,(A \mid Y) = \mathbb{P}\,(A \mid Y)\}.$$

Remark 3. (1) $\mathcal{R}_Y\,(\mathbb{P})$ is a convex set whose extremal points are the probabilities

$$\mathbb{P}\,(\cdot \mid Y = y)\,,\ \ y \in Supp\,(\mathbb{P}_Y)\,.$$

(2) In order to justify a careful study of $\mathcal{R}_Y\,(\mathbb{P})$, let us recall that the elements of $\mathcal{R}_Y\,(\mathbb{P})$ are optimal for the following set of variational problems which arise naturally, for instance in mathematical finance (see [3] and [5]). Let ν be a probability measure on $\mathbb{R}$ which is absolutely continuous with respect to $\mathbb{P}_Y$. Consider now a convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} \left|\varphi\left(\frac{d\nu}{d\mathbb{P}_Y}\right)\right| d\mathbb{P}_Y < +\infty,$$

and denote by $\mathcal{E}^{\nu,\varphi}$ the set of probability measures on $\mathcal{F}_T$ which are absolutely continuous with respect to $\mathbb{P}$ and such that:
(a)

$$\mathbb{P}\left(\left|\varphi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right|\right) < +\infty$$

(b) The law of Y under $\mathbb{Q}$ is ν.

We have then,

$$\inf_{\mathbb{Q}\in\mathcal{E}^{\nu,\varphi}} \mathbb{P}\left(\varphi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right) = \mathbb{P}\left(\varphi\left(\frac{d\mathbb{P}^{\nu}}{d\mathbb{P}}\right)\right)$$

where $\mathbb{P}^{\nu}$ is the element of $\mathcal{R}_Y(\mathbb{P})$ defined by

$$\mathbb{P}^{\nu} = \int_{\mathbb{R}} \mathbb{P}\left(\cdot \mid Y = y\right) \nu\left(dy\right).$$

Let us now recall some elements from the theory of initial enlargement of the Itô filtration $\mathcal{F}$ by the functional Y. In what follows, $\mathcal{P}(\mathcal{F})$ denotes the predictable σ-field associated with the filtration $\mathcal{F}$. We shall also denote, slightly abusively, $\mathcal{F} \vee \sigma(Y)$ the filtration $\mathcal{F}$ initially enlarged with Y, i.e. the $\mathbb{P}-$completion of $\bigcap_{\varepsilon>0}\left(\mathcal{F}_{t+\varepsilon} \vee \sigma(Y)\right), t < T$.

Proposition 4. *(See [3]) There exists a* $\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R})$ *measurable process*

$$\begin{array}{llll} [0,T[\times & \Omega \times & \mathbb{R} & \to \mathbb{R} \\ (t, & \omega, & y) & \to \alpha_t^y(\omega) \end{array}$$

such that:

(1) For $\mathbb{P}_Y - a.e.\ y \in \mathbb{R}$ *and for* $0 \le t < T$,

$$\mathbb{P}\left(\int_0^t \left(\alpha^y\right)_s^2 ds < +\infty\right) = 1$$

(2) For $\mathbb{P}_Y - a.e.\ y \in \mathbb{R}$ *and for* $0 \le t < T$,

$$\mathbf{D}_t \eta_t^y = \alpha_t^y \eta_t^y$$

(3) For $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$, *the process*

$$x_t - \int_0^t \alpha_s^Y ds,\ t < T$$

is a $\mathbb{Q}$ *Brownian motion in the enlarged filtration* $\mathcal{F} \vee \sigma(Y)$.

Assumption (A1). *implies that we can furthermore chose versions for* α *and* η *which are continuous with respect the variable* y*. Of course, we shall always use such versions. Now remember that, according to the following proposition in [3],* $\mathcal{R}_Y(\mathbb{P})$ *can be seen as a set of laws of weak solutions of stochastic differential equations (called Conditioned Stochastic Differential Equations).*

Proposition 5. *(See [3]) If* $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$, *then the process*

$$x_t - \int_0^t \frac{\int_{\mathbb{R}} \alpha_s^y \eta_s^y\ \mathbb{Q}\left(Y \in dy\right)}{\int_{\mathbb{R}} \eta_s^y\ \mathbb{Q}\left(Y \in dy\right)} ds,\ t < T$$

is a $\mathbb{Q}$ *standard Brownian motion.*

To conclude this preliminary section, we give some examples of functionals Y which satisfy the assumptions **(A1), (A2),** and **(A3)** and for which the processes $\left(\alpha_t^y\right)_{0\leq t<T;\ y\in\mathbb{R}}$ can be explicitly computed.

Example 6. (See [3] and [9]) Assume that there exist two functions

$$b : \mathbb{R} \to \mathbb{R}$$

and

$$\sigma : \mathbb{R} \to \mathbb{R}_+^*$$

which are infinitely continuously differentiable with bounded derivatives, such that the solution $(y_t)_{0\leq t\leq T}$ of the stochastic differential equation

$$y_t = \int_0^t b\left(y_s\right) ds + \int_0^t \sigma\left(y_s\right) dx_s \tag{2.1}$$

satisfies

$$y_T = Y,$$

then for $0 \leq t < T,\ y \in Supp\ \mathbb{P}_Y$

$$\eta_t^y = \frac{p_{T-t}\left(y_t, y\right)}{p_T\left(0, y\right)}$$

$$\alpha_t^y = \sigma\left(y_t\right) \frac{\partial}{\partial x} \ln p_{T-t}\left(y_t, y\right)$$

where $p_t\left(x, y\right)$ is the density with respect of the Lebesgue measure of the semi-group associated with the diffusion (2.1) .

Example 7. (See [3]) Let $f \in L^2\left([0, T]\right)$ then for the functional

$$Y = \int_0^T f\left(s\right) dx_s$$

for $0 \leq t < T,\ y \in \mathbb{R}$

$$\eta_t^y = \sqrt{\frac{\int_0^T f\left(s\right)^2 ds}{\int_t^T f\left(s\right)^2 ds}} \exp\left[\frac{y^2}{2\int_0^T f\left(s\right)^2 ds} - \frac{\left(y - \int_0^t f\left(s\right) dx_s\right)^2}{2\int_t^T f\left(s\right)^2 ds}\right],$$

and

$$\alpha_t^y = \frac{y - \int_0^t f\left(s\right) dx_s}{\int_t^T f\left(s\right)^2 ds} f\left(t\right).$$

Example 8. (See [3] and [4]) For the exponential Wiener functional $Y = \int_0^{+\infty} e^{2x_s - 2\mu s} ds$ ($\mu > 2$), for $t \geq 0$, $y > 0$

$$\eta_t^y = \left(e^{x_t - \mu t}\right)^{2\mu} \left(\frac{y}{y - \int_0^t e^{2x_s - 2\mu s} ds}\right)^{1+\mu} e^{\frac{1}{2y} - \frac{e^{2x_t - 2\mu t}}{2\left(y - \int_0^t e^{2x_s - 2\mu s} ds\right)}} 1_{\int_0^t e^{2x_s - 2\mu s} ds < y},$$

and

$$\alpha_t^y = 2\mu - \frac{e^{2x_t - 2\mu t}}{y - \int_0^t e^{2x_s - 2\mu s} ds}.$$

3. Newton martingales

In this paragraph, our motivation is to give a natural generalization of the following very simple discussion which starts from a proposition first due to P. Lévy.

Proposition 9. *Under the Wiener measure $\mathbb{P}$, the process*

$$M_t := \frac{x_T - x_t}{T - t}, \ t < T$$

is a martingale in the enlarged filtration $\mathcal{F} \vee \sigma(x_T)$.

Let us consider $Y = x_T$. An immediate corollary of the preceding proposition is that $(M_t)_{0 \leq t < T}$ is also a martingale in $\mathcal{F} \vee \sigma(x_T)$ under each probability $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$. Indeed, for $s < t < T$,

$$\mathbb{Q}(M_t \mid \mathcal{F}_s \vee \sigma(x_T)) = \mathbb{P}(M_t \mid \mathcal{F}_s \vee \sigma(x_T)) = M_s.$$

Hence if $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$, the process

$$N_t := \mathbb{Q}(M_t \mid \mathcal{F}_t), \ t < T$$

is a martingale in the filtration $\mathcal{F}$ under the probability $\mathbb{Q}$. Now, it is known from [10] or [20] that if

$$\mathbb{Q}\left(\int_0^T N_t^2 dt\right) < +\infty$$

(a sufficient condition for that is that the relative entropy of the law of x_T under $\mathbb{Q}$ with respect to $\mathbb{P}_Y$ is finite) then the following limits exist in L^2

$$\left(\mathcal{D}_{\mathbb{Q}} x\right)_t := \lim_{h \to 0^+} \mathbb{Q}\left(\frac{x_{t+h} - x_t}{h} \mid \mathcal{F}_t\right), \ t < T$$

$$\left(\mathcal{D}_{\mathbb{Q}}^2 x\right)_t := \lim_{h \to 0^+} \mathbb{Q}\left(\frac{\left(\mathcal{D}_{\mathbb{Q}} x\right)_{t+h} - \left(\mathcal{D}_{\mathbb{Q}} x\right)_t}{h} \mid \mathcal{F}_t\right), \ t < T$$

and, furthermore, from a classical filtering formula that

$$\left(\mathcal{D}_{\mathbb{Q}}x\right)_t = N_t,\ t < T$$

Hence, as N is a martingale in $\mathcal{F}$ under $\mathbb{Q}$, we deduce

$$\mathcal{D}^2_{\mathbb{Q}}x = 0$$

which is the stochastic analog of the Newton equation

$$\frac{d^2x}{dt^2} = 0$$

which governs the dynamics of a mobile point in a free potential field in classical mechanics.

3.1. Newton martingales in the Itô filtration

In order to construct some martingales related to $\mathcal{R}_Y\ (\mathbb{P})$, we make in this section the following additional assumption:

Assumption (B). *The process* $(\mathbf{D}_t Y)_{0\leq t\leq T}$ *has a continuous version* $\mathbf{D}Y$ *which satisfies*

1. $\mathbf{D}Y \neq 0$
2. $\frac{\mathbf{D}Y}{\mathbf{D}_0 Y}$ *is* $\mathcal{F}-$ *adapted.*

The following proposition shows that this assumption is not as restrictive as it might seem.

Proposition 10. *Assume that there exist two functions*

$$b : \mathbb{R} \to \mathbb{R}$$

and

$$\sigma : \mathbb{R} \to \mathbb{R}^*_+$$

which are infinitely continuously differentiable with bounded derivatives, such that the solution $(y_t)_{0\leq t\leq T}$ *of the stochastic differential equation*

$$y_t = \int_0^t b\,(y_s)\,ds + \int_0^t \sigma\,(y_s)\,dx_s$$

satisfies

$$y_T = Y,$$

then Assumption B is satisfied.

Proof. It is well-known (see [16], 2.2.) that, under the assumptions of the proposition, $y_T \in \mathbb{D}^{1,2}$

$$\mathbf{D}_t y_T = \sigma(y_t) \exp\left[\int_t^T \sigma'(y_s)\,dx_s + \int_t^T \left(b' - \frac{1}{2}(\sigma')^2\right)(y_s)\,ds\right]$$

so that Assumption B is immediately satisfied since σ never vanishes. □

Remark 11. Of course, if $f \in L^2([0,T])$ is continuous and such that for any $t \in [0,T]$, $f(t) \neq 0$, then the Assumption B is satisfied for the functional

$$Y = \int_0^T f(s)\,dx_s$$

Theorem 12. *If* $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ *then the process* $\left(\frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y} \frac{\int \alpha_t^y \eta_t^y\, \nu(dy)}{\int \eta_t^y\, \nu(dy)}\right)_{0 \le t < T}$ *is a* $\mathbb{Q}$ *martingale (not uniformly integrable in general).*

Proof. We prove first our theorem for a dense subset (in the weak topology) of $\mathcal{R}_Y(\mathbb{P})$.

Let $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ defined by

$$d\mathbb{Q} = \xi(Y)\,d\mathbb{P}$$

where ξ is a strictly positive continuously differentiable function with bounded derivative such that

$$\int \xi(y)\,p(y)\,dy = 1$$

Let us denote by Z_t the density process of $\mathbb{Q}$ with respect to $\mathbb{P}$ defined by $Z_t := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$. We now compare two different expressions of Z_t. By assumption $Z_t = \mathbb{P}(\xi(Y) \mid \mathcal{F}_t)$. Since $\xi(Y) \in \mathbb{D}^{1,2}$ the Clark-Ocone formula applies and the following identity holds:

$$\xi(Y) = 1 + \int_0^T \mathbb{P}(D_s \xi(Y) \mid \mathcal{F}_s)\,dx_s$$

Moreover from Proposition 5 we deduce that

$$\langle Z, x\rangle_t = \int_0^t Z_s \frac{\int_{\mathbb{R}} \alpha_s^y \eta_s^y\, \mathbb{Q}(Y \in dy)}{\int_{\mathbb{R}} \eta_s^y\, \mathbb{Q}(Y \in dy)} ds.$$

Therefore $\mathbb{P}$-a.s. the following identity holds:

$$\frac{\mathbb{P}\left(\xi'(Y)\,\mathbf{D}_t Y \mid \mathcal{F}_t\right)}{Z_t} = \frac{\int_{\mathbb{R}} \alpha_t^y \eta_t^y\, \mathbb{Q}(Y \in dy)}{\int_{\mathbb{R}} \eta_t^y\, \mathbb{Q}(Y \in dy)}.$$

The left hand side of this identity as a process belongs to $L^1(\mathbb{Q})$; the right hand side can be rewritten using the filtering formula. These two remarks result in:

$$\left(\mathcal{D}_{\mathbb{Q}}x\right)_t = \mathbb{Q}\left(\frac{\xi'(Y)}{\xi(Y)}\mathbf{D}_t Y \mid \mathcal{F}_t\right).$$

Hence,

$$\frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\left(\mathcal{D}^{\mathbb{Q}}x\right)_t = \mathbb{Q}\left(\frac{\xi'(Y)}{\xi(Y)}\mathbf{D}_0 Y \mid \mathcal{F}_t\right)$$

and our result follows.

Let us now prove our statement for an arbitrary $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$. From the previous result and assumption **(A2)**, we have for $0 \le s \le t < T$ and $A \in \mathcal{F}_s$

$$\mathbb{P}\left(\int \eta_t^y \nu(dy)\left[\frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\frac{\int \alpha_t^y \eta_t^y\ \nu(dy)}{\int \eta_t^y\ \nu(dy)} - \frac{\mathbf{D}_0 Y}{\mathbf{D}_s Y}\frac{\int \alpha_s^y \eta_s^y\ \nu(dy)}{\int \eta_s^y\ \nu(dy)}\right]1_A\right) = 0$$

with

$$\nu(dy) = \xi(y)\, p(y)\, dy.$$

Hence,

$$\int \mathbb{P}\left(\left[\eta_t^y \frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\alpha_t^y - \eta_s^y \frac{\mathbf{D}_0 Y}{\mathbf{D}_s Y}\alpha_s^y\right]1_A\right)\nu(dy) = 0.$$

By taking a sequence ξ_n such that

$$\xi_n(y)\, p(y)\, dy \to_{n\to+\infty}^{weakly} \delta_y$$

we deduce

$$\mathbb{P}\left(\left[\eta_t^y \frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\alpha_t^y - \eta_s^y \frac{\mathbf{D}_0 Y}{\mathbf{D}_s Y}\alpha_s^y\right]1_A\right) = 0.$$

This implies that for $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$

$$\int \mathbb{P}\left(\left[\eta_t^y \frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\alpha_t^y - \eta_s^y \frac{\mathbf{D}_0 Y}{\mathbf{D}_s Y}\alpha_s^y\right]1_A\right)\mathbb{Q}(Y \in dy) = 0.$$

Hence,

$$\mathbb{Q}\left(\left[\frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\frac{\int \alpha_t^y \eta_t^y\ \mathbb{Q}(Y \in dy)}{\int \eta_t^y\ \mathbb{Q}(Y \in dy)} - \frac{\mathbf{D}_0 Y}{\mathbf{D}_s Y}\frac{\int \alpha_s^y \eta_s^y\ \mathbb{Q}(Y \in dy)}{\int \eta_s^y\ \mathbb{Q}(Y \in dy)}\right]1_A\right) = 0$$

which gives the expected result. □

Now, we would like to rewrite the previous theorem as a stochastic Newton equation.

Proposition 13. *Let* $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ *. Assume that* $\mathbb{Q}$ *is absolutely continuous with respect to* $\mathbb{P}$ *and that the law* $\mathbb{Q}_Y$ *of* Y *under* $\mathbb{Q}$ *has a relative entropy with respect to* $\mathbb{P}_Y$ *which is finite, i.e.*

$$\int \ln\left(\frac{d\mathbb{Q}_Y}{d\mathbb{P}_Y}\right) d\mathbb{Q}_Y < +\infty$$

then the following limits exist in L^2

$$\left(\mathcal{D}_{\mathbb{Q}} x\right)_t := \lim_{h\to 0} \mathbb{Q}\left(\frac{x_{t+h}-x_t}{h} \mid \mathcal{F}_t\right),\ t<T$$

$$\left(\mathcal{D}_{\mathbb{Q}}\left(\frac{\mathbf{D}_0 Y}{\mathbf{D}Y}\mathcal{D}_{\mathbb{Q}} x\right)\right)_t := \lim_{h\to 0} \mathbb{Q}\left(\frac{\frac{\mathbf{D}_0 Y}{\mathbf{D}_{t+h} Y}\left(\mathcal{D}_{\mathbb{Q}} x\right)_{t+h} - \frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\left(\mathcal{D}_{\mathbb{Q}} x\right)_t}{h} \mid \mathcal{F}_t\right),\ t<T,$$

and we have

$$\mathcal{D}_{\mathbb{Q}}\left(\frac{\mathbf{D}_0 Y}{\mathbf{D}Y}\mathcal{D}_{\mathbb{Q}} x\right) = 0. \tag{3.1}$$

Proof. It is easily seen that

$$\int \ln\left(\frac{d\mathbb{Q}_Y}{d\mathbb{P}_Y}\right) d\mathbb{Q}_Y < +\infty$$

implies

$$\int \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) d\mathbb{Q} < +\infty.$$

Thus, as in Theorem 2.4. of [20],

$$\mathbb{Q}\left(\left(\frac{\int_{\mathbb{R}} \alpha_t^y \eta_t^y\ \mathbb{Q}(Y\in dy)}{\int_{\mathbb{R}} \eta_t^y\ \mathbb{Q}(Y\in dy)}\right)^2\right) < +\infty.$$

Hence from [10], the following limits exist in L^2

$$\left(\mathcal{D}_{\mathbb{Q}} x\right)_t := \lim_{h\to 0} \mathbb{Q}\left(\frac{x_{t+h}-x_t}{h} \mid \mathcal{F}_t\right),\ t<T$$

and is precisely equal to

$$\frac{\int_{\mathbb{R}} \alpha_t^y \eta_t^y\ \mathbb{Q}(Y\in dy)}{\int_{\mathbb{R}} \eta_t^y\ \mathbb{Q}(Y\in dy)}$$

which immediately gives the second part of our proposition from the previous theorem. □

Remark 14. (1) It may be of interest to remark that, in all generality, according to the “méthode des Laplaciens approchés” (see [7]) the bounded variation part A of a continuous semimartingale $X = M + A$ in a filtration $\mathcal{G}$ under a probability measure $\mathbb{Q}$ is:

$$A_t = \lim_{h\to 0}\int_0^t \frac{\mathbb{Q}\left(X_{s+h} - X_s \mid \mathcal{G}_s\right)}{h} ds$$

as soon as the right hand side exists in the weak topology $\sigma\left(L^1, L^\infty\right)$.

(2) From a mechanical point of view, the term $\frac{\mathbf{D}_0 Y}{\mathbf{D} Y}$ in the Newton equation can be interpreted as a stochastic ”friction” term. Indeed, the equation of the motion of a free mobile point which is submitted to frictions is

$$\frac{d^2 x}{dt^2} + k(t)\frac{dx}{dt} = 0,$$

or equivalently

$$\frac{d}{dt}\left(e^{\int_0^t k(s)ds}\frac{dx}{dt}\right) = 0.$$

3.2. Newton martingales in the enlarged filtration

The following theorem shows that we can also write a Newton martingale in the enlarged filtration, this martingale being the generalization of P. Lévy’s martingale considered at the beginning of the section.

Theorem 15. *Assume that Assumption B is satisfied, then for each* $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ *the process* $\left(\frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\alpha_t^Y\right)_{0\le t<T}$ *is a martingale adapted to the enlarged filtration* $\mathcal{F} \vee \sigma(Y)$.

Proof. Let $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ and

$$M_t := \frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\alpha_t^Y, \; t < T$$

For $s < t < T$, $A \in \mathcal{F}_s$ and $\Lambda \in \mathcal{B}(\mathbb{R})$ we have

$$\mathbb{Q}\left((M_t - M_s)\, 1_{A\cap(Y\in\Lambda)}\right) = \int_\Lambda \mathbb{P}^y\left((M_t - M_s)\, 1_A\right)\mathbb{Q}(Y \in dy)$$

where $\mathbb{P}^y \in \mathcal{R}_Y(\mathbb{P})$ is defined by

$$\mathbb{P}^y = \mathbb{P}(\cdot \mid Y = y).$$

Under $\mathbb{P}^y$ we have almost surely

$$M_t = \frac{\mathbf{D}_0 Y}{\mathbf{D}_t Y}\alpha_t^y, \; t < T.$$

Hence, according to proposition 12, $(M_t)_{t<T}$ is a $\mathbb{P}^y-$martingale. This implies

$$\int_\Lambda \mathbb{P}^y\left((M_t - M_s)\, 1_A\right) \mathbb{Q}\,(Y \in dy) = 0$$

and finally

$$\mathbb{Q}\left((M_t - M_s)\, 1_{A\cap(Y\in\Lambda)}\right) = 0.$$

Therefore $(M_t)_{t<T}$ is a $\mathbb{Q}-$martingale. □

We are now going to specify the above results in two of the examples quoted in section 2.

Example 16. Assume that there exist two functions

$$b : \mathbb{R} \to \mathbb{R}$$

and

$$\sigma : \mathbb{R} \to \mathbb{R}_+^*$$

which are infinitely continuously differentiable with bounded derivatives, such that the solution $(y_t)_{0\le t\le T}$ of the stochastic differential equation

$$y_t = \int_0^t b\,(y_s)\, ds + \int_0^t \sigma\,(y_s)\, dx_s$$

satisfies

$$y_T = Y.$$

Since,

$$\mathbf{D}_t\, y_T = \sigma\,(y_t) \exp\left[\int_t^T \sigma'\,(y_s)\, dx_s + \int_t^T \left(b' - \frac{1}{2}\left(\sigma'\right)^2\right)(y_s)\, ds\right]$$

we deduce that the process

$$\exp\left[\int_0^t \sigma'\,(y_s)\, dx_s + \int_0^t \left(b' - \frac{1}{2}\left(\sigma'\right)^2\right)(y_s)\, ds\right] \frac{\partial}{\partial x} \ln p_{T-t}\,(y_t, Y)$$

$t < T$, is a martingale in the enlarged filtration $\mathcal{F} \vee \sigma\,(Y)$.

Example 17. Let $f \in \mathcal{C}_T$ such that for any $t \in [0, T]$, $f\,(t) \neq 0$ then the process

$$\frac{\int_t^T f\,(s)\, dx_s}{\int_t^T f\,(s)^2\, ds}, t < T$$

is a martingale in the enlarged filtration $\mathcal{F} \vee \sigma\left(\int_0^T f\,(s)\, dx_s\right)$.

4. Local symmetries of the pinning class and Noether martingales

Let us first recall that we only consider random variables Y which satisfy assumptions **(A1)** to **(A3)** given in section 2.

4.1. Symmetry of an element of the pinning class

We first turn to the definition of symmetry with respect to an element of the pinning class. A vector field $u : \mathbb{W} \to \mathbb{H}$ is $\mathbb{H}$-continuously differentiable if and only if, for all $\omega \in \mathbb{W}$, the mapping $h :\to u(\omega + h)$ is continuously differentiable in $\mathbb{H}$ (cf. [16] p 192).

Definition 18. *Let $u : \mathbb{W} \to \mathbb{H}$ an $\mathbb{H}$-continuously differentiable and adapted vector field satisfying the following Novikov's condition, for $\varepsilon > 0$ small enough*

$$\mathbb{P}\left(\exp\left(\frac{\varepsilon^2}{2}\int_0^T \left(\frac{du_s}{ds}\right)^2 ds\right)\right) < +\infty. \tag{4.1}$$

Let $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$. Then u will be called an order-one symmetry of $\mathbb{Q}$ with respect to Y if for any $A \in \mathcal{F}_T$ such that $\mathbb{P}(A \mid Y) = 0$

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} \left(T^{\varepsilon}\right)^{-1} \mathbb{Q}(A) = 0$$

where T^{ε} denotes the change of variable on the path space

$$T^{\varepsilon}(x)_t = x_t + \varepsilon \int_0^t \frac{du_s}{ds} ds,$$

and $(T^{\varepsilon})^{-1}\mathbb{Q}$ is the direct image law of $\mathbb{Q}$ by $(T^{\varepsilon})^{-1}$.

The set of the order-one symmetries of $\mathbb{Q}$ with respect to Y shall be denoted $Sym_Y(\mathbb{Q})$.

4.2. Noether martingales and general form of the symmetries

In the Markov setting symmetry groups have been used to build martingales (cf. [20]); such martingales were called Noether martingales since they are the stochastic analog of the constants of motion associated to the symmetry group of an action functional by Noether theorem in classical Mechanics. In the present section we state our generalization of the Noether stochastic theorem obtained in [20]. We will give a short survey of the Markovian case in Section 6.

Theorem 19. *Let us consider $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ and $u : \mathbb{W} \to \mathbb{H}$ an $\mathbb{H}$-continuously differentiable adapted vector field satisfying (4.1) such that $u_t = \int_0^t \frac{du_s}{ds} ds$. Let us denote by $(\delta u)_t$ the stochastic integral $\int_0^t \frac{du_s}{ds} dx_s$. Then $u \in Sym_Y(\mathbb{Q})$ if and only if there exists a signed ($\sigma-$finite) measure μ such that*

$$\int_{\mathbb{R}} |\mu|(dy) < +\infty, \quad \int_{\mathbb{R}} \mu(dy) = 0$$

and

$$D_u Z_t - (\delta u)_t \, Z_t = \int_{\mathbb{R}} \eta_t^y \mu(dy) \ , t < T$$

where Z is the density process of $\mathbb{Q}$ with respect to $\mathbb{P}$. In particular,

$$N_t := D_u Z_t - (\delta u)_t \, Z_t, \ t < T$$

is a $\mathbb{P}$-martingale (non uniformly integrable in general).

Remark 20. The process $D_u Z_t - (\delta u)_t \, Z_t$ is a martingale for any continuously differentiable adapted vector field (see the proof below). What characterizes the local symmetries is the representation of this martingale using η. Further properties of this martingale are given in Corollary 21 and Proposition 22 below.

Proof. Let $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ and u an $\mathbb{H}$-continuously differentiable adapted vector field. We denote by ν the law of Y under $\mathbb{Q}$. Let us denote by S^ε the inverse of

$$T_t^\varepsilon = x_t + \varepsilon \int_0^t \frac{du_s}{ds} ds \ , t \leq T.$$

From (4.1) and Girsanov's theorem, we have at the first order in ε, for all $t < T$, and all bounded and $\mathcal{F}_t-$measurable functional F,

$$\begin{aligned}
\mathbb{Q}(F \circ S^\varepsilon) &= \mathbb{P}\left[\textstyle\int_{\mathbb{R}} \eta_t^y \nu(dy) \, F \circ S^\varepsilon\right] \\
&= \mathbb{P}\left[\left(1 - \varepsilon \textstyle\int_0^t \frac{du_s}{ds} dx_s\right) \textstyle\int_{\mathbb{R}} (\eta^y \circ T^\varepsilon)_t \, \nu(dy) \, F\right] \\
&= \mathbb{P}\left[\left(1 - \varepsilon \textstyle\int_0^t \frac{du_s}{ds} dx_s\right) \left(\textstyle\int_{\mathbb{R}} \left(\eta_t^y + \varepsilon \textstyle\int_0^t \frac{du_s}{ds} \mathbf{D}_s \eta_t^y ds\right) \nu(dy)\right) F\right] \\
&= \mathbb{P}\left[\textstyle\int_{\mathbb{R}} \eta_t^y \nu(dy) \, F\right] + \varepsilon \, \mathbb{P}[N_t F]
\end{aligned}$$

where

$$N_t = \int_0^t \frac{du_s}{ds} \left(\int_{\mathbb{R}} \mathbf{D}_s \eta_t^y \ \nu(dy)\right) ds - \left(\int_{\mathbb{R}} \eta_t^y \ \nu(dy)\right) \int_0^t \frac{du_s}{ds} \, dx_s.$$

Let us now show that N is a $\mathbb{P}$ martingale. Indeed, by the above identity, if G is a $\mathcal{F}_s-$measurable, $s < t$, bounded functional then

$$\mathbb{P}\left[\int_{\mathbb{R}} \eta_t^y \nu(dy) \, G\right] + \varepsilon \, \mathbb{P}[N_t G] = \mathbb{P}\left[\int_{\mathbb{R}} \eta_s^y \nu(dy) \, G\right] + \varepsilon \, \mathbb{P}[N_s G]$$

which implies, because $\left(\int_{\mathbb{R}} \eta_s^y \nu(dy)\right)_{0 \leq s < T}$ is a martingale

$$\mathbb{P}[N_t G] = \mathbb{P}[N_s G]$$

and so N is a martingale.

Assume now $u \in Sym_Y(\mathbb{Q})$. In this case

$$\mathbb{P}[N_t F] = \mathbb{P}\left[N_t \varrho(Y)\right]$$

with

$$\varrho(y) = \mathbb{P}\left[\eta_t^y F\right].$$

Indeed by writing $F = (F - \mathbb{P}[F \mid Y]) + \mathbb{P}[F \mid Y]$ and using the fact that

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} \left(T^\varepsilon\right)^{-1} \mathbb{Q}\left(\left(F - \mathbb{P}[F \mid Y]\right)\right) = 0$$

which holds since $u \in Sym_Y(\mathbb{Q})$, one obtains

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} \left(T^\varepsilon\right)^{-1} \mathbb{Q}(F) = \left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} \left(T^\varepsilon\right)^{-1} \mathbb{Q}\left(\mathbb{P}[F \mid Y]\right)$$

which is equivalent to the identity $\mathbb{P}[N_t F] = \mathbb{P}[N_t \mathbb{P}[F \mid Y]]$ thanks to the preceding computation. Moreover from assumption **(A2)**,

$$\mathbb{P}[F \mid Y = y] = \mathbb{P}\left[\eta_t^y F\right]$$

and trivially $\mathbb{P}[N_t F] = \mathbb{P}[\mathbb{P}[N_t \mid Y]\,\mathbb{P}[F \mid Y]]$; it follows that

$$\mathbb{P}[N_t F] = \mathbb{P}\left[\left(\int_{\mathbb{R}} \eta_t^y \mathbb{P}(N_t \mid Y = y)\, p(y)\, dy\right) F\right].$$

Thus, we get

$$N_t = \int_{\mathbb{R}} \eta_t^y \mathbb{P}(N_t \mid Y = y)\, p(y)\, dy$$

because the previous relationship holds for all F. Let us now set

$$\mu_t(dy) = \mathbb{P}(N_t \mid Y = y)\, p(y)\, dy, \; t < T.$$

Since N is a martingale, for $s < t < T$

$$\int_{\mathbb{R}} \eta_s^y \mu_t(dy) = \int_{\mathbb{R}} \eta_s^y \mu_s(dy)$$

which implies

$$\mu_t = \mu_s.$$

In order to conclude, it is enough to note that

$$\int_{\mathbb{R}} |\mu|(dy) \le \mathbb{P}(|N_t|) < +\infty, \; \int_{\mathbb{R}} \mu(dy) = \mathbb{P}(N_t) = 0.$$

Now, on the other hand if there exists a signed measure μ such that

$$\int_{\mathbb{R}} |\mu|(dy) < +\infty, \; \int_{\mathbb{R}} \mu(dy) = 0$$

and

$$\int_0^t \frac{du_s}{ds}\left(\int_{\mathbb{R}} \mathbf{D}_s\eta_t^y\ \nu(dy)\right) ds - \left(\int_{\mathbb{R}} \eta_t^y\ \nu(dy)\right)\int_0^t \frac{du_s}{ds}\ dx_s$$
$$= \int_{\mathbb{R}} \eta_t^y \mu(dy),\ t < T$$

from the previous computations, for all bounded functional F and $\mathcal{F}_t-$measurable, $t < T$

$$\begin{aligned}\left(\tfrac{d}{d\varepsilon}\right)_{\varepsilon=0} \mathbb{Q}(F \circ S^\varepsilon) &= \textstyle\int_{\mathbb{R}} \mathbb{P}\left(\eta_t^y F\right)\mu(dy)\\ &= \textstyle\int_{\mathbb{R}} \mathbb{P}(F \mid Y = y)\,\mu(dy).\end{aligned}$$

So that the result is also true for $t = T$. This concludes the proof of our theorem. □

We also deduce from the proof of the previous theorem.

Corollary 21. *Let us consider* $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ *and* $u : \mathbb{W} \to \mathbb{H}$ *a continuously differentiable adapted vector field satisfying (4.1). Let us denote by Z the density process of* $\mathbb{Q}$ *with respect to* $\mathbb{P}$*. Then* $u \in Sym_Y(\mathbb{Q})$ *if and only if*

$$N_t = D_u Z_t - (\delta u)_t\, Z_t,\ t < T$$

is a martingale such that for any bounded and $\mathcal{F}_T-$*measurable* F *which satisfies* $\mathbb{P}(F \mid Y) = 0$

$$\lim_{t\to T^-} \mathbb{P}(N_t F) = 0.$$

As it was noticed in the previous theorem, in general, the martingale N is not uniformly integrable and hence not closable, nevertheless:

Proposition 22. *Let* $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ *and* $u : \mathbb{W} \to \mathbb{H}$ *an* $\mathbb{H}$*- continuously differentiable vector field satisfying (4.1). Then, with the notations of Theorem 19, the following limit exists* $\mathbb{P}-$*a.s. and*

$$\lim_{t\to T^-} N_t = \frac{d\mu_a}{d\mathbb{P}_Y}(Y)$$

where

$$\mu = \mu_a + \mu_s$$

is the Lebesgue decomposition of μ *with respect to* $\mathbb{P}_Y$ *(*μ_a *denotes the absolutely continuous part and* μ_s *the singular part of the decomposition).*

Proof. We can, because of the Hahn-Jordan decomposition, assume that μ is positive. We shall denote by Λ the $\sigma-$finite measure defined by

$$\Lambda = \int_{\mathbb{R}} \mathbb{P}(\cdot \mid Y = y)\,\mu(dy).$$

The proof proceeds now in two steps.

First step.
For $r \in \mathbb{R}^+$, let

$$B_r = \{\omega, \ \lim \inf_{0 \le t < T} N_t \le r\}$$

and

$$B^r = \{\omega, \ \lim \sup_{0 \le t < T} N_t \ge r\}.$$

Let now $(r_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence which converges to r. We have

$$B_r = \lim_{\tau \to T} \nearrow \bigcap_{n>0} \bigcup_{s<\tau} \{ \sup_{s \le u \le \tau} N_u \le r_n\}.$$

Since N is continuous, the indexes above can be taken in $\mathbb{Q}$.

We have,

$$\Lambda \left(\bigcup_{s<\tau} \{ \sup_{s \le u \le \tau} N_u \le r_n\} \right) \le r_n \mathbb{P} \left(\bigcup_{s<\tau} \{ \sup_{s \le u \le \tau} N_u \le r_n\} \right)$$

hence for $A \in \mathcal{F}_T$

$$\Lambda \left(A \cap B_r\right) \le r \ \mathbb{P} \left(A \cap B_r\right).$$

In the same way, it is shown that

$$\Lambda \left(A \cap B^r\right) \ge r \ \mathbb{P} \left(A \cap B^r\right).$$

Second step.
The martingale convergence theorem implies that when $t \to T^-$, N_t converges $\mathbb{P}-a.s.$ to an integrable variable N_T. Let us now introduce

$$C = \{\omega, \lim_{t \to T^-} N_t(\omega) = N_T(\omega)\}.$$

On $\Omega \backslash C$, which has $\mathbb{P}-$measure zero, we set

$$N_T := +\infty.$$

Let us now consider the following sequence

$$\varphi_n = \sum_k \frac{k}{2^n} \, 1_{A_{n,k}},$$

with

$$A_{n,k} = \{\omega, \ \frac{k}{2^n} \le N_T \le \frac{k+1}{2^n}\}.$$

It is easy to verify the following pointwise convergence

$$\varphi_n \to N_T$$

but, we also have

$$\varphi_n \leq N_T$$

which implies by the dominated convergence theorem

$$\mathbb{E}\left(N_T 1_A\right) = \lim_{n\to+\infty} \mathbb{E}\left(\varphi_n 1_A\right).$$

Now,

$$\mathbb{E}\left(\varphi_n 1_A\right) = \sum_k \frac{k}{2^n}\mathbb{P}\left(A \cap A_{n,k}\right)$$

and

$$A_{n,k} = B_{\frac{k+1}{2^n}} \cap B^{\frac{k}{2^n}} \cap C.$$

Hence, from the first step

$$\frac{2^n}{k+1}\Lambda\left(A \cap A_{n,k}\right) \leq \mathbb{P}\left(A \cap A_{n,k}\right) \leq \frac{2^n}{k}\mathbb{P}\left(A \cap A_{n,k}\right),$$

and hence

$$\sum_k \Lambda\left(A \cap A_{n,k}\right) = \Lambda\left(A \cap C\right)$$

$$\sum_k \frac{k}{k+1}\Lambda\left(A \cap A_{n,k}\right) \leq \mathbb{E}\left(\varphi_n 1_A\right) \leq \Lambda\left(A \cap C\right).$$

Now, since

$$\sum_k \frac{k}{k+1}\Lambda\left(A \cap A_{n,k}\right) \to_{n\to+\infty} \Lambda\left(A \cap C\right),$$

we deduce

$$\lim_{n\to+\infty} \mathbb{E}\left(\varphi_n 1_A\right) = \Lambda\left(A \cap C\right)$$

and finally

$$\mathbb{E}\left(N_T 1_A\right) = \Lambda\left(A \cap C\right)$$

which yields the expected result. □

5. Global symmetries

Up to now we have been studying the symmetries of a fixed element of $\mathcal{R}_Y(\mathbb{P})$. We will now look for global symmetries. Precisely, we are going to study

$$\bigcap_{\mathbb{Q}\in\mathcal{R}_Y(\mathbb{P})} Sym_Y(\mathbb{Q}).$$

As it will be seen, this intersection can be trivial (i.e. reduced to $u = 0$), but if non-adapted symmetries are allowed, this is not the case. That is why we use the tools of anticipative stochastic calculus (see [16] and [21]) to investigate global symmetries.

On α, we shall make the following additional assumptions **(A4)**:

(1) For $0 \le t < T$ and $Z \in \mathbb{D}^{1,2}$, $\alpha_t^Z \in \mathbb{D}^{1,2}$
(2) For $\mathbb{P}_Y - a.e.\ y \in \mathbb{R}$ there exists a measurable version of the two parameter process $\left(\mathbf{D}_s\alpha_t^y\right)_{0\le s,t<T}$ verifying for $\tau < T$

$$\mathbb{P}\left(\int\int_{[0,\tau]^2}\left(\mathbf{D}_s\alpha_t^y\right)^2 ds\, dt\right) < +\infty$$

(3) For $\mathbb{P}_Y - a.e.\ y \in \mathbb{R}$, the transformation

$$x_t - \int_0^t \alpha_s^y ds, t < T$$

is bijective.
(4) $dt \otimes \mathbb{P}-a.s.$, the function $y \to \alpha_t^y$ is infinitely differentiable on the support of $\mathbb{P}_Y$ (which is assumed to be an interval, cf. Section Preliminaries and assumptions).

Remark 23. According to Proposition 4, the transformation

$$x_t - \int_0^t \alpha_s^{Y(x)} ds, t < T$$

can not be bijective. For instance, in the case $Y = x_T$, then we have $\alpha_t^y = \frac{y-x_t}{T-t}$ and

$$x_t - \int_0^t \frac{x_T - x_s}{T-s} ds = x_t + \mu t - \int_0^t \frac{(x_T + \mu T) - (x_s + \mu s)}{T-s} ds$$

for $\mu \neq 0$. Nevertheless, if we freeze the variable Y, then the transformation becomes bijective because, for a process z,

$$x_t + \int_0^t \frac{y - x_s}{T-s} ds = z_t$$

easily implies

$$x_t = \frac{t}{T} y + (T-t)\int_0^t \frac{dz_s}{T-s}.$$

Moreover, notice that

$$\det_2 \left(I_{\mathbb{H}} - \mathbf{D}_* \alpha_*^Y \right) = 0$$

$\det_2$ being the Carleman-Fredholm determinant, whereas for $\mathbb{P}_Y - a.e.\ y \in \mathbb{R}$

$$\det_2 \left(I_{\mathbb{H}} - \mathbf{D}_* \alpha_*^y \right) = 1.$$

Let us recall how the Carleman-Fredholm determinant is defined. Let K be a linear operator from $\mathbb{H}$ to $\mathbb{H}$ with discrete spectrum and let λ_i be the sequence of eigenvalues of K repeated according to their multiplicity. The Carleman-Fredholm determinant of K is defined as

$$\det_2 \left(I_{\mathbb{H}} + K \right) = \prod_{i=1}^{\infty} (1 + \lambda_i)\, e^{-\lambda_i},$$

and this product is known to converge for Hilbert-Schmidt operators.

Notice furthermore that, as easily checked, Assumption **(A4)** is satisfied for the examples of functionals treated up to now.

5.1. Definition and characterization of the symmetries

Before we give the definition of a global symmetry of the pinning class, let us recall the well-known Ramer-Kusuoka theorem (see [14] pp. 191, [16] pp. 202, or [21]).

Theorem 24. *(Ramer-Kusuoka) Let $(K_s)_{0 \le s \le T}$ a process taking values on $\mathbb{H}$ (adapted or not) which is $\mathbb{H}-$continuously differentiable (we recall that $\mathbb{H}$ is the Cameron-Martin space). Let us consider the change of variable*

$$T^{\lambda} (x)_t = x_t + \lambda \int_0^t K_s ds\ ,\ t \le T$$

with $\lambda \in [0, 1]$.

Assume that

(1) *T^{λ} is bijective*
(2) *The operator $I_{\mathbb{H}} + \lambda D_* K_*$ is invertible*

Then,

$$\mathbb{P}(F) = \mathbb{P}\left(D^{\lambda}\ F \circ T^{\lambda} \right)$$

where

$$D^{\lambda} := \det_2 \left(I + \lambda D_* K_* \right) \exp \left[-\lambda \int_0^T K_s dx_s - \frac{\lambda^2}{2} \int_0^T |K_s|^2 ds \right]$$

$\det_2$ *being the Carleman-Fredholm determinant and $\int_0^T K_s dx_s$ the Skorohod integral of K.*

Definition 25. *Let $u : \mathbb{W} \to \mathbb{H}$ an $\mathbb{H}$-continuously differentiable vector field, u will be called an order-one symmetry of $\mathcal{R}_Y(\mathbb{P})$ if the change of variable*

$$T^{\varepsilon}(x)_t = x_t + \varepsilon \int_0^t \frac{du_s}{ds} ds$$

satisfies for $\varepsilon > 0$ small enough the assumptions of the Ramer-Kusoka theorem and if for all $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ and $A \in \mathcal{F}_T$ which satisfies $\mathbb{P}(A \mid Y) = 0$

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} \left(T^{\varepsilon}\right)^{-1} \mathbb{Q}(A) = 0,$$

$(T^{\varepsilon})^{-1}\mathbb{Q}$ denoting the direct image law of $\mathbb{Q}$ by $(T^{\varepsilon})^{-1}$.

Theorem 26. *$u : \mathbb{W} \to \mathbb{H}$ is an order-one symmetry if and only if*

$$D_u Y := \int_0^T \frac{du_s}{ds} \mathbf{D}_s Y \, ds$$

and

$$\delta u := \int_0^T \frac{du_s}{ds} dx_s$$

are deterministic functions of Y.

Proof. Let $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ defined by

$$d\mathbb{Q} = \xi(Y)\, d\mathbb{P}$$

where ξ is a strictly positive continuously differentiable function with bounded derivative such that

$$\int \xi(y)\, p(y)\, dy = 1.$$

Let us denote by S^{ε} the inverse of

$$T^{\varepsilon}(x)_t = x_t + \varepsilon \int_0^t \frac{du_s}{ds} ds \,,\, t \le T$$

Here $K_s = \frac{du_s}{ds}$ and the Carleman-Fredholm determinant satisfies for ε close to 0,

$$\det_2 (I + \varepsilon D_* K_*) = 1 + \rho(\varepsilon)$$

where $\rho(\varepsilon)$ is negligible w.r.t. ε. This can be deduced from the following identity

$$\det_2 (I + \lambda D_* K_*) = 1 + \sum_{n=2}^{+\infty} \lambda^n \frac{\gamma^n}{n!}$$

valid for all kernel $K \in L^2([0,T]\times[0,T])$ with $\gamma_n = \int_{[0,T]^n} \det K'(t_i,t_j)dt_1\dots dt_n$ where $K'(t_i,t_j) = K(t_i,t_j)$ if $i \neq j$ and $K(t_i,t_i) = 0$ (cf. [16] pp. 239 formula (A.11)). Therefore at the first order in ε, we have for all bounded functional F of the Wiener space

$$\begin{aligned}\mathbb{Q}[F\circ S^\varepsilon] &= \mathbb{P}[\xi(Y)(F\circ S^\varepsilon)]\\ &= \mathbb{P}[\xi(Y\circ T^\varepsilon)F(1-\varepsilon\delta u)]\\ &= \mathbb{P}\left[\left(\xi(Y)+\varepsilon\xi'(Y)D_uY\right)(1-\varepsilon\delta u)F\right]\\ &= \mathbb{P}\left[\xi(Y)F+\varepsilon\left[\xi'(Y)D_uY-\xi(Y)\delta u\right]F\right].\end{aligned}$$

Hence,

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} S^\varepsilon\mathbb{Q}(F) = \mathbb{P}\left[\left(\xi'(Y)D_uY-\xi(Y)\delta u\right)F\right].$$

Thus if u is an order-one symmetry,

$$\mathbb{P}\left[\left(\xi'(Y)D_uY-\xi(Y)\delta u\right)F\right] = \mathbb{P}\left[\left(\xi'(Y)D_uY-\xi(Y)\delta u\right)\mathbb{P}(F\mid Y)\right].$$

Since this relationship must hold for all F, it implies

$$\mathbb{P}\left[\xi'(Y)D_uY-\xi(Y)\delta u\mid Y\right] = \xi'(Y)D_uY-\xi(Y)\delta u.$$

By taking $\xi = 1$ we see that δu is a deterministic function of Y. It follows that D_uY is also a deterministic function of Y.

Conversely, let us now assume that D_uY and δu are deterministic functions of Y. Let F a bounded functional such that $\mathbb{P}(F\mid Y)=0$.

Let us again consider $\mathbb{Q}\in\mathcal{R}_Y(\mathbb{P})$ defined by

$$d\mathbb{Q} = \xi(Y)\,d\mathbb{P}$$

We have, from the previous computations

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} S^\varepsilon\mathbb{Q}(F) = 0.$$

But,

$$S^\varepsilon\mathbb{Q}(F) = \int S^\varepsilon\mathbb{P}^y(F)\;\xi(y)\,p(y)\,dy$$

where $\mathbb{P}^y\in\mathcal{R}_Y(\mathbb{P})$ is the disintegrated probability defined by

$$\mathbb{P}^y = \mathbb{P}(\cdot\mid Y=y).$$

Hence,

$$\int_{\mathbb{R}}\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} S^\varepsilon\mathbb{P}^y(F)\;\xi(y)\,p(y)\,dy = 0.$$

By taking a sequence ξ_n such that

$$\xi_n(y)\,p(y)\,dy \rightarrow_{n\to+\infty}^{weakly} \delta_y$$

we deduce,

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} S^{\varepsilon}\mathbb{P}^{y}(F) = 0.$$

Hence, if $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ then

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} \int S^{\varepsilon}\mathbb{P}^{y}(F)\ \mathbb{Q}(Y \in dy) = 0,$$

which gives

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} S^{\varepsilon}\mathbb{Q}(F) = 0$$

□

Remark 27. This proposition shows that in the search of the order-one symmetries, the enlarged filtration $\mathcal{F} \vee \sigma(Y)$ plays a central role. From a dual point of view, this role is strengthened by the fact that all the L^2 functionals F which satisfy $\mathbb{P}(F \mid Y) = 0$ can be represented (See [2]) as a stochastic integral with respect to the Brownian motion of the enlarged filtration, i.e. the process

$$x_t - \int_0^t \alpha_s^Y\, ds.$$

This remark will be enlightened by the structure theorem below.

Corollary 28. *Let* $u : \mathbb{W} \to \mathbb{H}$ *an order-one symmetry, then there exists* $\phi \in L^2(p)$ *which satisfies*

$$\delta u = \phi(Y)$$

and

$$D_u Y = \frac{-1}{p(Y)} \int_{-\infty}^{Y} p(y)\,\phi(y)\,dy.$$

Proof. From the previous proposition , there exists ϕ which satisfies

$$\delta u = \phi(Y).$$

First, we check that $\phi \in L^2(p)$. Indeed,

$$\int \phi(y)^2\, p(y)\, dy = \mathbb{P}\left(\phi(Y)^2\right) = \mathbb{P}\left((\delta u)^2\right) < +\infty.$$

Now, from the integration by parts formula on the Wiener space, for any function f which is infinitely continuously differentiable and which has a compact support included in the support of $\mathbb{P}_Y$, we have

$$\mathbb{P}(D_u f(Y)) = \mathbb{P}(f(Y)\,\delta u).$$

Hence,

$$\mathbb{P}\left(f'(Y)\,D_u Y\right) = \mathbb{P}\left(f(Y)\,\delta u\right).$$

By denoting φ the function such that

$$D_u Y = \varphi(Y)$$

we have

$$\int f'(y)\,\varphi(y)\,p(y)\,dy = \int f(y)\,\phi(y)\,p(y)\,dy$$

and the conclusion follows readily. □

Example 29. Let $f \in L^2([0,T])$. For the functional

$$Y = \int_0^T f(s)\,dx_s$$

the (deterministic) vector field

$$u_t = \int_0^t f(s)\,ds,\, t \leq T$$

is a one-order symmetry. Indeed,

$$\delta u = Y$$

and

$$D_u Y = \int_0^T f(s)^2\,ds.$$

From this one-order symmetry, we deduce the following one-parameter family of global symmetries

$$T^\lambda(x)_t = x_t + \lambda \int_0^t f(s)\,ds\;,\, t \leq T,\; \lambda \in \mathbb{R}.$$

Example 30. For the exponential Wiener functional $Y = \int_0^{+\infty} e^{2x_s - 2\mu s} ds$ $(\mu > 2)$, the vector field

$$u_t = \int_0^t e^{2x_s - 2\mu s} ds,\; t \geq 0$$

is a one-order symmetry. Indeed,

$$\delta u = \int_0^{+\infty} e^{2x_s - 2\mu s} dx_s$$

But from Itô formula,

$$\int_0^{+\infty} e^{2x_s - 2\mu s} dx_s = (\mu - 1)\int_0^{+\infty} e^{2x_s - 2\mu s} ds - \frac{1}{2}.$$

Hence,

$$\delta u = (\mu - 1)\, Y - \frac{1}{2}.$$

We also have

$$\begin{aligned} D_u Y &= 2 \int_0^{+\infty} e^{2x_s - 2\mu s} \int_s^{+\infty} e^{2x_u - 2\mu u} du ds \\ &= 2Y^2. \end{aligned}$$

5.2. Structure theorem

In this paragraph, we describe the structure of a general order-one symmetry. For $\mathbb{P}_Y - a.e.$ y, let us denote by Γ^y the inverse transformation of

$$\gamma_t^y = x_t - \int_0^t \alpha_s^y ds, \ t < T$$

and we shall assume that for $dt \otimes \mathbb{P}{-}a.s.$ the function $y \to \Gamma_t^y$ is differentiable (see Assumption **(A4)**).

Theorem 31. *For $\phi \in L^2(p)$, there exists an order-one symmetry u which satisfies*

$$\delta u = \phi(Y).$$

Moreover, all the order-one symmetries u which satisfy

$$\delta u = \phi(Y)$$

can be written as

$$u_t = \frac{-1}{p(Y)} \int_{-\infty}^{Y} p(y)\, \phi(y)\, dy \left(\frac{d\Gamma^y}{dy} \right)_{y=Y} \left(\gamma^Y \right)_t + u_t^0 \,,\, t < T$$

where u^0 satisfies $\delta u^0 = 0$.

Before we prove this theorem, let us state a previous lemma interesting for itself:

Lemma 32. *Let $u : \mathbb{W} \to \mathbb{H}$ a vector field adapted to the enlarged filtration $\mathcal{F} \vee \sigma(Y)$, then u is a one-order symmetry if and only if*

$$\frac{du_t}{dt} = \int_0^T \frac{du_s}{ds} \left(\mathbf{D}_s \alpha_t^y \right)_{y=Y} ds + (D_u Y) \left(\frac{d\alpha_t^y}{dy} \right)_{y=Y} ,\ t < T. \tag{5.1}$$

Proof of the Lemma. Let $u : \mathbb{W} \to \mathbb{H}$ be a vector field adapted to the enlarged filtration $\mathcal{F} \vee \sigma(Y)$. Let now $(F_s)_{0 \le s \le T}$ be a process adapted to the enlarged filtration $\mathcal{F} \vee \sigma(Y)$ and such that $\mathbb{P}\left(\int_0^T F_s^2 ds \right) < +\infty$. Consider now $\mathbb{Q} \in \mathcal{R}_Y(\mathbb{P})$ defined by

$$d\mathbb{Q} = \xi(Y)\, d\mathbb{P}$$

where ξ is a strictly positive function such that

$$\int \xi(y)\, p(y)\, dy = 1$$

and denote by S^{ε} the inverse of

$$T^{\varepsilon}(x)_t = x_t + \varepsilon \int_0^t \frac{du_s}{ds}(x)\, ds\, , t \leq T.$$

We have, for $\varepsilon > 0$, small enough

$$\left(T^{\varepsilon}\right)^{-1} \mathbb{Q}\left(\int_0^T F_s d\gamma_s^Y\right) = \mathbb{P}\left(\xi(Y) \int_0^T F\left(S^{\varepsilon}\right)_u d\gamma^Y \left(S^{\varepsilon}\right)_u\right).$$

But, in the first order in ε we have

$$\left(\gamma^Y\right)\left(S^{\varepsilon}\right) = \left(\gamma^Y\right)\left(Id - \varepsilon \int_0^{\cdot} \frac{du_s}{ds} ds\right)$$

and so,

$$\begin{aligned}\left(\gamma^Y\right)\left(S^{\varepsilon}\right) &= Id - \varepsilon \int_0^{\cdot} \frac{du_s}{ds} ds \\ &+ \varepsilon \int_0^{\cdot} \left[\int_0^T \frac{du_s}{ds}\left(\mathbf{D}_s \alpha_v^y\right)_{y=Y} ds + \left(D_u Y\right)\left(\frac{d\alpha_v^y}{dy}\right)_{y=Y}\right] dv.\end{aligned}$$

If we denote

$$A := \int_0^{\cdot} \frac{du_s}{ds} ds - \int_0^{\cdot} \left[\int_0^T \frac{du_s}{ds}\left(\mathbf{D}_s \alpha_v^y\right)_{y=Y} ds + \left(D_u Y\right)\left(\frac{d\alpha_v^y}{dy}\right)_{y=Y}\right] dv$$

we have then,

$$\left(T^{\varepsilon}\right)^{-1} \mathbb{Q}\left(\int_0^T F_s d\gamma_s^Y\right) = \mathbb{P}\left(\xi(Y)\left(\int_0^T F\left(S^{\varepsilon}\right)_u d\gamma_u^Y + \varepsilon \int_0^T F\left(S^{\varepsilon}\right)_u dA_u\right)\right).$$

But, since

$$\mathbb{P}\left(\int_0^T F\left(S^{\varepsilon}\right)_u d\gamma_u^Y \mid Y\right) = 0$$

we get in the first order in ε

$$\left(T^{\varepsilon}\right)^{-1} \mathbb{Q}\left(\int_0^T F_s d\gamma_s^Y\right) = \varepsilon \mathbb{P}\left(\xi(Y) \int_0^T F_u dA_u\right)$$

and so, if u is an order-one symmetry

$$\mathbb{P}\left(\xi\left(Y\right)\int_0^T F_u dA_u\right)=0.$$

Since this relationship must hold for all F and all ξ, we get $A=0$ i.e.

$$\frac{du_t}{dt}=\int_0^T \frac{du_s}{ds}\left(\mathbf{D}_s\alpha_t^y\right)_{y=Y}\,ds+\left(D_uY\right)\left(\frac{d\alpha_t^y}{dy}\right)_{y=Y},\ t<T.$$

On the other hand, since every bounded random variable H which is $\mathcal{F}_T-$adapted and which satisfies

$$\mathbb{P}\left(H\mid Y\right)=0$$

can be expressed as $\int_0^T F_s d\gamma_s$ for some $(F_s)_{0\leq s\leq T}$ we easily deduce that the nullity of A is also a sufficient condition to ensure that u is an order-one symmetry. □

With this characterization of the order-one symmetries which are adapted to the enlarged filtration $\mathcal{F}\vee\sigma\left(Y\right)$, we are now able to give the proof of our theorem.

Proof of the Theorem 31. Let us consider the vector field

$$u_t=\frac{-1}{p\left(Y\right)}\int_{-\infty}^Y p\left(y\right)\phi\left(y\right)dy\left(\frac{d\Gamma^y}{dy}\right)_{y=Y}\left(\gamma^Y\right)_t,\ t\leq T.$$

As it is adapted to the enlarged filtration $\mathcal{F}\vee\sigma\left(Y\right)$, in order to show that u is an order-one symmetry, it suffices to check that

$$\frac{du_t}{dt}=\int_0^T \frac{du_s}{ds}\left(\mathbf{D}_s\alpha_t^y\right)_{y=Y}\,ds+\left(D_uY\right)\left(\frac{d\alpha_t^y}{dy}\right)_{y=Y},\ t<T.$$

As easily shown, we have

$$\frac{dP^y}{dy}\left(\gamma^y\right)_t=\int_0^t\int_0^T \frac{d\Gamma^y}{ds}\left(\gamma^y\right)_s\mathbf{D}_s\alpha_v^y ds dv+\int_0^t \frac{d\alpha_s^y}{dy}ds,\ t<T.$$

Let us now show that

$$D_uY=\varphi\left(Y\right)$$

where

$$\varphi\left(Y\right)=\frac{-1}{p\left(Y\right)}\int_{-\infty}^Y p\left(y\right)\phi\left(y\right)dy.$$

We have, for all $y\in Supp\ \mathbb{P}_Y$,

$$Y\left(\Gamma^y\right)=y$$

hence, by differentiating with respect to y

$$\int_0^T \frac{d}{dt}\left(\frac{d\Gamma^y}{dy}\left(\gamma^y\right)\right)\mathbf{D}_t Y\, dt = 1$$

which gives

$$D_u Y = \varphi\left(Y\right).$$

We can now conclude that u is an order-one symmetry. By the integration by parts formula, this symmetry satisfies

$$\delta u = \phi\left(Y\right)$$

Let now $\widetilde{u}$ be an order-one symmetry which satisfies

$$\delta \widetilde{u} = \phi\left(Y\right).$$

The vector field $u - \widetilde{u}$ is an order-one symmetry such that $\delta\left(u - \widetilde{u}\right) = 0$. □

From the structure theorem, we deduce the following immediate corollaries:

Corollary 33. *Let us denote by $\mathcal{SR}_Y\left(\mathbb{P}\right)$ the vector space of the order-one symmetries and $\mathcal{S}^*\mathcal{R}_Y\left(\mathbb{P}\right)$ the subspace of the order-one symmetries which are adapted to the filtration $\mathcal{F} \vee \sigma\left(Y\right)$, then*

$$\mathcal{SR}_Y\left(\mathbb{P}\right) \simeq \mathcal{S}^*\mathcal{R}_Y\left(\mathbb{P}\right) \oplus \mathcal{R}^0\left(\mathbb{P}\right)$$

where $\mathcal{R}^0\left(\mathbb{P}\right) = \{u :\in \mathcal{SR}_Y\left(\mathbb{P}\right), \delta u = 0\}$.

Corollary 34. *There exist adapted global symmetries (i.e. $\bigcap_{\mathbb{Q}\in\mathcal{R}_Y(\mathbb{P})} Sym_Y\left(\mathbb{Q}\right)$ is not trivial) if and only if there exist a Borel function f and an $\mathcal{F}-$adapted process A such that*

$$\frac{d\Gamma^y}{dy}\left(\gamma^y\right)_t = f\left(y\right) A_t.$$

Remark 35. (1) Moreover, if there exist adapted order-one symmetries, then they form a one-dimensional vector space because for $\mathbb{P}_Y - a.e.\ y \in \mathbb{R}$

$$\det_2\left(I_{\mathbb{H}} - \mathbf{D}_*\alpha_*^y\right) = 1.$$

(2) The corollary above, which states that in the case where global symmetries exist a certain flow is stationnary, suggests that there is an ergodic counterpart to our study. More precisely, it is tempting to study $\sigma-$finite measures which are infinitesimally invariant by $\mathcal{S}^*\mathcal{R}_Y\left(\mathbb{P}\right)$ (see examples below).

Example 36. Let $f \in L^2([0,T])$ and

$$Y = \int_0^T f(s)\,dx_s.$$

In this case, as easily seen, we have

$$\Gamma_t^y = \frac{\int_0^t f(s)\,ds}{\int_0^T f^2(s)\,ds}\, y + \Gamma_t^0.$$

Hence, all the order-one symmetries can be written as

$$u_t = \varphi\left(\int_0^T f(s)\,dx_s\right)\int_0^t f(s)\,ds + u_t^0$$

And it is easily seen that the $\sigma-$finite measure

$$\int_{\mathbb{R}} \mathbb{P}\left(\cdot \mid \int_0^T f(s)\,dx_s = y\right) dy$$

is infinitesimally invariant by the adapted symmetries

$$u_t = k\int_0^t f(s)\,ds.$$

Example 37. For the exponential Wiener functional $Y = \int_0^{+\infty} e^{2x_s - 2\mu s} ds$ ($\mu > 2$) we have (see [4] and [8])

$$\Gamma_t^y = 2\mu t + x_t + \ln y - \ln\left(y + \int_0^t e^{2x_s + 2\mu s} ds\right).$$

Hence,

$$\left(\frac{d\Gamma^y}{dy}\right)_{y=Y}\left(\gamma^Y\right)_t = \frac{\int_0^t e^{2\gamma_s^Y + 2\mu s} ds}{\int_0^{+\infty} e^{2x_s - 2\mu s} ds\left(\int_0^{+\infty} e^{2x_s - 2\mu s} ds + \int_0^t e^{2\gamma_s^Y + 2\mu s} ds\right)}.$$

But, from Dufresne's identity (see [4] and [8])

$$\frac{1}{\int_0^t e^{2\gamma_s^Y + 2\mu s} ds} + \frac{1}{\int_0^{+\infty} e^{2x_s - 2\mu s} ds} = \frac{1}{\int_0^t e^{2x_s - 2\mu s} ds}.$$

This implies

$$\left(\frac{d\Gamma^y}{dy}\right)_{y=Y}\left(\gamma^Y\right)_t = \frac{\int_0^t e^{2x_s - 2\mu s} ds}{\left(\int_0^{+\infty} e^{2x_s - 2\mu s} ds\right)^2}.$$

Hence all the order-one symmetries can be written as

$$u_t = \varphi\left(\int_0^{+\infty} e^{2x_s - 2\mu s} ds\right) \int_0^t e^{2x_s - 2\mu s} ds + u_t^0.$$

Furthermore, it is easily seen that the σ−finite measure

$$\int_{\mathbb{R}_+^*} \mathbb{P}\left(\cdot \mid \int_0^{+\infty} e^{2x_s - 2\mu s} ds = y\right) \frac{dy}{y^2}$$

is infinitesimally invariant by the adapted symmetries

$$u_t = k \int_0^t e^{2x_s - 2\mu s} ds.$$

5.3. Applications: Non-canonical representations of Brownian motion

We now show that the previous study of the order-one global symmetries of the pinning class enables us to recover some well-known non-canonical representations of Brownian motion. This point of view is new in the topic of non-canonical representations of the Brownian motion.

Proposition 38. (1) *(See [1]) Let $(B_t)_{0\le t\le T}$ a standard Brownian motion and $f \in L^2([0, T])$. The process*

$$\left(B_t - \int_0^t \frac{\int_0^u f(v)\, dB_v}{\int_0^u f(v)^2\, dv} f(u)\; du\right)_{0\le t<T}$$

is well-defined and its natural filtration is strictly included in the natural filtration of B, moreover it is independent of $\int_0^T f(s)\, dB_s$.

(2) *(See [4] and [15]) Let $(B_t)_{0\le t\le T}$ a standard Brownian motion and $\mu > 2$* [1]*.* *The process*

$$\left(\frac{\int_0^t e^{2B_s - 2\mu s} ds}{e^{B_t - \mu t}}\right)_{t\ge 0}$$

has a natural filtration which is strictly included in those of B, moreover it is independent of $\int_0^{+\infty} e^{2B_s - 2\mu s} ds$.

Proof. (1) It is enough to show that the transformation

$$\mathcal{G}(x)_t = x_t - \int_0^t \frac{\int_0^u f(v)\, dx_v}{\int_0^u f(v)^2\, dv} f(u)\; du$$

[1] This result is also true for $0 < \mu \le 2$.

is invariant under the action of a one-parameter family of one-order symmetries of $\mathcal{R}_{\int_0^T f(s)dB_s}(\mathbb{P})$. If

$$u_t = k \int_0^t f(s)\, ds$$

with $k \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{G}(x+u)_t &= x_t + k\int_0^t f(s)\,ds - \int_0^t \frac{\int_0^u f(v)dx_v}{\int_0^u f(v)^2 dv} f(u)\ du - k\int_0^t f(s)\,ds \\ &= \mathcal{G}(x)_t\,.\end{aligned}$$

(2) It is enough to show that the transformation

$$\mathcal{H}(x)_t = \frac{\int_0^t e^{2x_s - 2\mu s} ds}{e^{x_t - \mu t}}$$

is invariant in the first order in ε under the action of the following one-order symmetry of $\mathcal{R}_{\int_0^t e^{2x_s-2\mu s}ds}(\mathbb{P})$

$$u_t = \int_0^t e^{2x_s - 2\mu s} ds.$$

We have, at the first order in ε,

$$\begin{aligned}\mathcal{H}(x+\varepsilon u)_t &= \exp\left(-x_t + \mu t - \varepsilon \int_0^t e^{2x_s-2\mu s} ds\right) \\ &\quad \times \int_0^t \exp\left(2x_s - 2\mu s + \varepsilon \int_0^s e^{2x_u - 2\mu u} du\right) ds \\ &= \mathcal{H}(x)_t + \varepsilon \exp(-x_t + \mu t) \\ &\quad \times \left[-\left(\int_0^t e^{2x_s-2\mu s} ds\right)^2 + 2\int_0^t e^{2x_s - 2\mu_s} \int_0^s e^{2x_u - 2\mu u} du ds\right] \\ &= \mathcal{H}(x)_t\end{aligned}$$

which gives the expected result. □

6. About the Markovian Noether theorem

To conclude, we show how the results of the present paper generalize the Markovian case $Y = x_T$. We give some details for the heat equation with a potential. For a general second order parabolic p.d.e. with time dependent coefficients we refer the reader to [19].

Let us recall that in classical Mechanics it is well known that the Newton equation $\frac{d^2x}{dt^2} = 0$ holds as a consequence of the fact that the Lagrangian $L(t, x_t, \frac{dx}{dt}) = \frac{1}{2}\left(\frac{dx}{dt}\right)^2$ does not depend on the position x_t which implies that it is invariant under space translations. Such property has been generalized in the Markovian case for diffusions in [20] where the invariance of a stochastic Lagrangian under transformations provides martingales (the stochastic counterparts of the classical constants of motion; this is why this result has been called stochastic Noether Theorem). The framework of [20] corresponds in our case to $Y = x_T$. Let us first recall the definition of symmetries for an action functional.

Definition 39. *Let us denote by $\mathcal{P}$ the set of probabilities $\mathbb{Q}$ on $\mathbb{W}$ such that the coordinate process is a semi-martingale under $\mathbb{Q}$:*

$$dx_t = \mathcal{D}_{\mathbb{Q}} x_t \, dt + dW_t$$

with W a Brownian motion. Given a scalar potential V the action functional J is defined on $\mathcal{P}$ by

$$J(\mathbb{Q}) = \mathbb{Q}\left(\int_0^T \left(\frac{1}{2}\left(\mathcal{D}_{\mathbb{Q}} x_t\right)^2 + V(t, x_t)\right) dt\right)$$

whenever this quantity is finite. The one-parameter family of deterministic space-time infinitesimal transformations

$$(t, x) \to \left(\bar{t}, \bar{x}\right) = (t + \varepsilon\theta(t, x), x + \varepsilon U(t, x))$$

is a symmetry of J if and only if there exists a function ϕ such that for any $T > 0$, for any $\mathbb{Q} \in \mathcal{P}$ it holds

$$\left(\frac{d}{d\varepsilon}\right)_{\varepsilon=0} J\left(\overline{\mathbb{Q}}\right) = \mathbb{Q}\left(\phi(T, x_T) - \phi(0, x_0)\right)$$

where

$$J\left(\overline{\mathbb{Q}}\right) = \mathbb{Q}\left(\int_0^T \left(\frac{1}{2}\left(\mathcal{D}_{\mathbb{Q}} x_t + \varepsilon\left(\mathcal{D}_{\mathbb{Q}} U - \mathcal{D}_{\mathbb{Q}} x_t \mathcal{D}_{\mathbb{Q}} \theta\right)\right)^2 + V\left(\bar{t}, \bar{x}_t\right)\right) \mathcal{D}_{\mathbb{Q}} \bar{t} \, dt\right)$$

with $\mathcal{D}_{\mathbb{Q}} U = \mathcal{D}_{\mathbb{Q}}(U(\cdot, x.)), \mathcal{D}_{\mathbb{Q}} \theta = \mathcal{D}_{\mathbb{Q}}(\theta(\cdot, x.))$ obtained by Itô formula and

$$\left(\bar{t}, \bar{x}\right) = (t + \varepsilon\theta(t, x_t), x_t + \varepsilon U(t, x_t)).$$

Proposition 40. *The symmetries of the action J coincide with the symmetries of the p.d.e.*

$$\frac{\partial}{\partial t} + \frac{1}{2}\Delta - V = 0.$$

In the case $V = 0$, the above symmetries are also symmetries of space-time harmonics of the Brownian motion x. Now, we know that the set of the laws of the Doob's $h-$transforms is dense (for the topology of weak convergence of probability measures) in $\mathcal{R}_{x_T}(\mathbb{P})$, hence these symmetries are global symmetries of the pinning class in the sense of Section 5. That is why we consider the results of the Section 5 as a generalization of the Markovian setting.

Remark 41. It is important to note that in the preceding sections we have considered only symmetries which act on the space variable and not on time (i.e. $\bar{t} = t$).

Let us now recall that, when V belongs to the Kato class, given any pair (h,k) of strictly positive functions satisfying $\frac{\partial h}{\partial t}+\frac{1}{2}\Delta h - Vh = 0$ as well as $\frac{\partial k}{\partial t}-\frac{1}{2}\Delta k + Vk = 0$ and $\int_{\mathbb{R}} k(0,x)h(0,x) = 1$, the Schrödinger process is the Markov diffusion solution of the stochastic differential equation $dx_t = dW_t + \frac{\partial}{\partial x}\ln h\,(t,x_t)$ such that the law of x_0 has density $k(0,x)h(0,x)$ w.r.t. Lebesgue measure (cf. [12], [22]).

In this Markovian setting, Noether stochastic theorem is the following:

Theorem 42. *(see [20]) Let (θ,U) and ϕ define a symmetry of J in the sense of Definition 39. Let us denote by $\mathbb{S}$ the law of the Schrödinger process associated to the pair (h,k) as above. Then the process*

$$\theta\,(t,x_t)\,\frac{\partial}{\partial t}\ln h\,(t,x_t) + U\,(t,x_t)\,\frac{\partial}{\partial x}\ln h\,(t,x_t) - \phi\,(t,x_t)$$

is an $\mathbb{S}$-martingale.

Let us notice that when the potential V vanishes, h is a solution of the heat equation $\frac{\partial h}{\partial t}+\frac{1}{2}\Delta h = 0$; $(h\,(t,x_t))$ is a $\mathbb{P}$ martingale and if M is a $\mathbb{S}$ martingale, then $(h(t,x_t)M_t)$ is a $\mathbb{P}$ martingale. This provides another way of stating Noether theorem with respect to $\mathbb{P}$:

Corollary 43. *With the previous notations*

$$\theta\,(t,x_t)\,\frac{\partial}{\partial t}h\,(t,x_t) + U\,(t,x_t)\,\frac{\partial}{\partial x}h\,(t,x_t) - \phi\,(t,x_t)\,h\,(t,x_t)$$

is a $\mathbb{P}$ martingale.

To prove this theorem, the system of *determining equations* played an important role. We recall it below for the heat equation in $\mathbb{R}^d$.

Proposition 44. *The following system (system of determining equations) characterizes the symmetries of J when $V=0$ (or equivalently of $\frac{\partial h}{\partial t}+\frac{1}{2}\Delta h = 0$):*

$$\begin{array}{lll}
\partial_i\theta & = 0 & 1 \le i \le d\\
\partial_i U_i & = \frac{1}{2}\partial_t\theta & 1 \le i \le d\\
\partial_i U_j + \partial_j U_i & = 0 & 1 \le i < j \le d\\
\partial_t\phi + \frac{1}{2}\Delta\phi & = 0 & \\
\partial_i\phi & = \partial_t U_i & 1 \le i \le d.
\end{array}$$

We notice that in dimension 1, when $\theta = 0$, this system reduces to

$$\begin{array}{ll}
\partial_x U & = 0\\
\partial_x\phi & = \partial_t U\\
\partial_t\phi + \frac{1}{2}\partial_x^2\phi & = 0
\end{array}$$

which is equivalent to

$$\begin{array}{ll}
U & = U\,(t)\\
\phi & = \phi\,(x)\\
\frac{\partial U}{\partial t} & = \partial_x\phi
\end{array}$$

In this simple case of heat equation without drift this system can be solved explicitly. The solutions are the pairs $(U(t), \phi(x)) = (at + b, ax + c)$ with a, b, c real constants. In the more general framework of our present paper the counterpart of the above system of determining equations lies in theorem 26. We explicit below the computations for $d = 1$ for simplicity. Indeed let $U(t)$ be a deterministic vector field. From Theorem 26, U is a global symmetry of $\mathcal{R}_{x_T}(\mathbb{P})$ if and only if there exists a function ϕ such that $\delta(U) = \int_0^T U_s' dx_s = \phi(x_T)$ (the second condition holds true since $D_U x_T = U(T)$). Assuming ϕ smooth, this is equivalent to require that the two processes (U_s') ans (ϕ_s') are equal to the same constant. Thus $(U(t), \phi(x)) = (at + b, ax + c)$; we recover the system of determining equations when $d = 1$. A similar argument holds for $d > 1$. For any $\mathbb{Q}$ in $\mathcal{R}_{x_T}(\mathbb{P})$ such that for any $t < T$ and for (U, ϕ) solution of the system of determining equations, the Noether martingale of Theorem 41 coincides with that of Theorem 19.

In the general case, which has been treated here the equation which determines the local symmetries is the following

$$D_u Z_t - (\delta u)_t \, Z_t = \int_{\mathbb{R}} \eta_t^y \mu(dy) \ , t < T$$

where Z, μ and η are known and u is unknown (cf. Theorem 19). If Z does not vanish, this equation can be written under the form

$$\int_0^t K(\omega, s, t) \, Y_s ds + \int_0^t Y_s dx_s = \Lambda_t, \ t < T$$

with straightforward notations ($Y = \frac{du_s}{ds}$ is unknown, K and Λ are known) which has the form of a stochastic linear integral equation.

7. Open question

It would be really interesting to know exactly the variables Y for which the set of global symmetries is non trivial. As far as we know, up to now, the only examples which are known are (up to elementary transformations) those studied in this paper i.e. the case where Y is a Wiener integral or an exponential functional.

References

[1] Alili, L.: Canonical decomposition of certain generalized Brownian bridges. Electronic Commun. Probab. **7**, 27–35 (2002)

[2] Amendinger, J.: Martingale representation theorems for initially enlarged filtrations. Stochastic Processes and their Applications **89**, 101–116 (2000)

[3] Baudoin, F.: Conditioned stochastic differential equations: theory, examples and application to finance. Stochastic Processes and their Applications **100**, 109–145 (2002)

[4] Baudoin, F.: Further exponential generalization of Pitman's 2M-X theorem. Electronic Commun. Probab. **7**, (2002)

[5] Baudoin, F., Nguyen Ngoc, L.: The financial value of a weak information flow, May, 2002. Preprint

[6] Cohen de Lara, M.: Reduction of the Zakai equation by invariance group technics. Stochastic Processes and their Applications **73**, 119–130 (1998)
[7] Dellacherie, C.: Capacités et processus stochastiques. Springer, 1972
[8] Donati-Martin, C., Matsumoto, H., Yor, M.: Some absolute continuity relationships for certain anticipative transformations of geometric Brownian motions. Publ. RIMS Kyoto **37**, 295–326 (2001)
[9] Fitzsimmons, P., Pitman, J.W., Yor, M.: Markov bridges, construction, palm interpretation and splicing. Seminar on Stochastic Processes Birkhauser, 101–134 (1993)
[10] Föllmer, H.: Time-reversal on Wiener space. Proc. Bielefeld, LNM **1158**, Berlin: Springer, 1984, pp. 119–120
[11] Glover, J.: Symmetry groups and translation invariant representation of Markov processes. Annals of Probab. **19**, 562–586 (1991)
[12] Jamison, B.: The Markov processes of Schrödinger. Z. Wahrscheinlichkeitstheorie verw. Gebiete **32**, 323-331 (1975)
[13] Krener, A.J.: Reciprocal diffusions in flat space. Probab. Theory Relat. Fields **107**, 243–281 (1997)
[14] Malliavin, P.: Stochastic Analysis. Grundlehren der mathematischen Wissenschaften **313**, Springer, 1997
[15] Matsumoto, H., Yor, M.: A version of Pitman's $2M - X$ theorem for geometric Brownian motions. C.R. Acad. Sci. Paris, t. I **328**, 1067–1074 (1999)
[16] Nualart, D.: The Malliavin calculus and related topics. Berlin-Heidelberg-New-York: Springer, 1995
[17] Thieullen, M.: Second order stochastic differential equations and non Gaussian reciprocal diffusions. Probab. Theory Relat. Fields **97**, 231–257 (1993)
[18] Thieullen, M.: Reciprocal diffusions and symmetries. Stochastics and Stochastic Reports **65**, 41–77 (1998)
[19] Thieullen, M.: Reciprocal diffusions and symmetries of parabolic p.d.e.: the non flat case. Potential Analysis **16**, 1–28 (2002)
[20] Thieullen, M., Zambrini, J.C.: Symmetries in the stochastic calculus of variations. Probab. Theory Relat. Fields **107**, 401–427 (1997)
[21] Üstünel, A.S., Zakai, M.: The change of variables formula on Wiener space. Séminaire de Probabilités XXXI, LNM **1655**, Azéma, J., Emery, M., Yor, M., Eds, 1997
[22] Zambrini, J.-C.: Variational processes and stochastic versions of mechanics. J. Math. Phys **27**, 2307–2330 (1986)