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Hasan B. Uzun · Kenneth S. Alexander

Lower bounds for boundary roughness for droplets in Bernoulli percolation

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Abstract. We consider boundary roughness for the "droplet" created when supercritical two-dimensional Bernoulli percolation is conditioned to have an open dual circuit surrounding the origin and enclosing an area at least l^2, for large l. The maximum local roughness is the maximum inward deviation of the droplet boundary from the boundary of its own convex hull; we show that for large l this maximum is at least of order $l^{1/3}(\log l)^{-2/3}$. This complements the upper bound of order $l^{1/3}(\log l)^{2/3}$ proved in [Al3] for the average local roughness. The exponent 1/3 on l here is in keeping with predictions from the physics literature for interfaces in two dimensions.

1. Introduction

We consider Bernoulli bond percolation on the square lattice at supercritical density, conditioned to have a large dual circuit enclosing the origin; we denote the outermost such circuit by Γ_0. (Complete definitions and the basic properties of the model will be given in the next section.) The supercritical, or percolating, regime of Bernoulli percolation is the analog of the low-temperature phase of a spin system, and the region enclosed by the dual circuit is the analog of the droplet that occurs with high probability in the Ising magnet below the critical temperature in a finite box with minus boundary condition, when it is conditioned to have a number of plus spins somewhat larger than is typical [DKS]. In fact, the droplet boundary in the Ising magnet appears as a circuit of open dual bonds in the corresponding Fortuin-Kastelyn random cluster model (briefly, the FK model) of [FK], in view of the construction given in [ES]. One can gain information for the study of the Ising droplet by studying the FK model conditioned on Γ_0 enclosing at least a given area l^2, as is done in [Al3]. The droplet boundary in this FK model thus corresponds to an interface; the heuristics in the case of Bernoulli percolation are the same, but the mathematics is more tractable. We therefore refer to Γ_0 and its interior as a

H.B. Uzun: ALEKS Corporation, 400 N. Tustin Avenue, Suite 300, Santa Ana, CA 92705, USA. e-mail: huzun@aleks.com

K.S. Alexander: Department of Mathematics, DRB 155, University of Southern California, Los Angeles, CA 90089-1113, USA. e-mail: alexandr@math.usc.edu

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droplet. Our main result is a lower bound on the maximum local roughness of the droplet, that is, the maximum inward deviation of the boundary of the droplet from the boundary of its convex hull. Related upper bounds were proved in [Al3].

The study of the shapes of such droplets is related to a classical problem: When a fixed volume of one phase is immersed in another, what is the equilibrium shape of the droplet, or crystal, having minimal surface tension? When the surface tension is known, this is an isoperimetric problem. The solution of the continuum version of the problem is given by Wulff [Wu]: Let $\tau(\mathbf{n})$ be the surface tension of a flat interface orthogonal to the outward normal $\mathbf{n}$. For a fixed crystal volume, the equilibrium shape is given by the convex set

$$W = \{\mathbf{x} \in \mathbf{R}^d \mid \mathbf{x} \cdot \mathbf{n} \leq \tau(\mathbf{n}), \text{ for all } \mathbf{n}\}. \tag{1.1}$$

In the two-dimensional Ising model, say with minus boundary condition and conditioned to have an excess of pluses, a rigorous justification of the Wulff construction has been given for the resulting droplet of plus phase. Minlos and Sinai considered an instance in which the temperature T tends to zero as the volume grows to infinity, and proved that most of the excess plus spins form a single droplet of essentially square shape ([MS1], [MS2]); the Wulff shape W also tends to a square as $T \to 0$. Dobrushin, Kotecky and Shlosman [DKS] then provided a justification of the Wulff construction at very low fixed temperatures. Moreover, they showed that the Hausdorff distance between the droplet boundary γ and the boundary of the Wulff shape W is bounded by a power of the linear scale of the droplet. This Hausdorff distance is related but not equivalent to local roughness; see [Al3]. The very-low-temperature restriction was removed by Ioffe and Schonmann [IS], who proved Dobrushin-Kotecky-Schlosman theorem up to the critical temperature. For Bernoulli percolation the Wulff construction was justified in [ACC], and for the FK model this was done in [Al3]. For these models the surface tension is given by the inverse of the exponential rate of decay of the dual connectivity.

Boundary roughness has been a topic of considerable interest in the physics literature (see e.g. [KS]). The heuristics for the local roughness of Γ_0, described in [Al3], are related to the boundary-roughness heuristics for two-dimensional growth models such as first-passage percolation that are believed to be governed by the "KPZ" theory ([KPZ], [LNP], [NP]), to polymers in two-dimensional random environments [Pi], and, as noted in [Al3], to the heuristics of rigorously proved results on longest increasing subsequences of random permutations [BDJ], which in turn are related to the fluctuations of eigenvalues of random matrices (see [Jo]). In all cases for an object of linear scale l there is known or believed to be roughness of order $l^{1/3}$ and a longitudinal correlation length of order $l^{2/3}$. In the percolation droplet this correlation length should appear as the typical separation between adjacent extreme points of the convex hull of Γ_0.

In [Al3] the *average local roughness*, denoted $\mathrm{ALR}(\Gamma_0)$, for the percolation droplet was defined as the area between the droplet and its convex hull boundary, divided by the Euclidean length of the convex hull boundary. It was proved there that with high probability, for a droplet conditioned to have area at least l^2, $\mathrm{ALR}(\Gamma_0)$ is $O(l^{1/3}(\log l)^{2/3})$. The main feature of interest is the exponent 1/3 matching the

KPZ heuristic; the power of $\log l$ may be considered an artifact of the proof. Here we consider not average but *maximum local roughness*, denoted MLR(Γ_0) and defined as the maximum distance from any point of Γ_0 to the convex hull boundary, and we show that for the Bernoulli percolation droplet, for some $c_0 > 0$, with high probability it is at least $c_0 l^{1/3}(\log l)^{-2/3}$. It was proved in [Al3] that with high probability MLR(Γ_0) is $O(l^{2/3}(\log l)^{1/3})$, but this is a presumably a very crude bound, lacking the right power of l; it is more reasonable to compare the lower bound here on MLR(Γ_0) to the upper bound for ALR(Γ_0), as the two should differ by at most a multiplicative factor that is a power of $\log l$, as we explain next.

One way to obtain more-detailed heuristics for the droplet boundary is to view it as having Gaussian fluctuations about a fixed Wulff shape of area l^2, a point of view justified in part by the results in [DH] and [Hr]. This point of view suggests that if we take a Brownian bridge on [0,1], rescale it by $2\pi l$ horizontally and $l^{1/2}$ vertically, and wrap it around a circle of radius l, joining $(0, 0)$ and $(2\pi l, 0)$, the result should resemble the droplet boundary. In [Uz] it was proved that for this wrapped Brownian bridge the maximum local roughness is with high probability bounded between $c_1 l^{1/3}(\log l)^{2/3}$ and $c_2 l^{1/3}(\log l)^{2/3}$ for some $0 < c_1 < c_2 < \infty$. The exponent 2/3 on $\log l$ here is related to the Lévy modulus of continuity for Brownian motion. The wrapped-Brownian-bridge heuristic suggests that ALR(Γ_0) should be of order $l^{1/3}$, without a power of $\log l$, supporting the idea that ALR(Γ_0) and MLR(Γ_0) differ by only a multiplicative factor that is roughly a power of $\log l$. The circle provides a reasonable heuristic here because Ioffe and Schonmann [IS] showed that for fixed p the curvature of the boundary of the unit-area Wulff shape is bounded away from 0 and ∞.

2. Definitions, preliminaries, statement of main result

A *bond*, denoted $\langle xy \rangle$, is an unordered pair of nearest neighbor sites $x, y \in \mathbb{Z}^2$. The set of all bonds between the nearest neighbor sites of $\mathbb{Z}^2$, will be denoted by $\mathbb{B}_2$. Let $\{\omega(b), b \in \mathbb{B}_2\}$ be an i.i.d. family of Bernoulli random variables with $P(\omega(b) = 1) = p$. Given a realization of ω, a bond $b \in \mathbb{B}_2$ is said to be *open* if $\omega(b) = 1$ and *closed* if $\omega(b) = 0$. Consider the random graph containing the vertex set of $\mathbb{Z}^2$ and the open bonds only; the connected components of this graph are called *open clusters*. For p below the critical probability $p_c = 1/2$ [Ke] all open clusters are finite with probability one and when $p > p_c$, there exists a unique infinite cluster of open bonds with probability one.

For $x \in \mathbb{Z}^2$ let x^* denote $x+(1/2, 1/2)$. The lattice with vertex set $\{x^* : x \in \mathbb{Z}^2\}$ and all nearest neighbor bonds is called the *dual lattice*. Each bond b has a unique dual bond, denoted b^*, which is its perpendicular bisector; b^* is defined to be open precisely when b is closed, so that the dual configuration is Bernoulli percolation at density $1 - p$. A (*dual*) *path* is a sequence $(x_0, \langle x_0 x_1 \rangle, x_1, \cdots, \langle x_{n-1}, x_n \rangle)$ of alternating (dual) sites and bonds. A *(dual) circuit* is a path with $x_n = x_0$ which has all bonds distinct and does not cross itself (in the obvious sense). Note we allow a circuit to touch itself without crossing, i.e. nondistinct sites are not restricted to $x_n = x_0$. For a (dual) circuit γ, the interior $\mathrm{Int}(\gamma)$ is the union of the bounded components of the complement of γ in $\mathbb{R}^2$. An open dual circuit γ is called an *exterior dual* *circuit* in a configuration ω if $\gamma \cup \mathrm{Int}(\gamma)$ is maximal among all open dual circuits in ω. A site x is surrounded by at most one exterior dual circuit; when this circuit exits, it is denoted by Γ_x. $|\cdot|$ denotes the Euclidean norm for vectors, cardinality for finite sets and Lebesgue measure for regions in $\mathbb{R}^2$, depending on the context. For $x, y \in \mathbb{R}^2$, let dist$(\cdot,\cdot)$ and diam$(\cdot)$ denote Euclidean distance and Euclidean diameter, respectively. Let $B_r(x)$, denote the open Euclidean ball of radius r about x. For $A, B \subset \mathbb{R}^2$, define $\mathrm{dist}(A, B) = \inf\{\mathrm{dist}(x, y) : x \in A, y \in B\}$ and $\mathrm{dist}(x, A) = \mathrm{dist}(\{x\}, A)$. We define the *average local roughness* of a circuit γ by

$$\mathrm{ALR}(\gamma) = \frac{|\mathrm{Co}(\gamma) \setminus \mathrm{Int}(\gamma)|}{|\partial \mathrm{Co}(\gamma)|},$$

where Co$(\cdot)$ denotes the convex hull. The *maximum localroughness* is

$$\mathrm{MLR}(\gamma) = \sup\{\mathrm{dist}(x, \partial\mathrm{Co}(\gamma)) \ : \ x \in \gamma\}$$

Throughout the paper, $K_1, K_2, \ldots$ represent constants which depend only on p. Our main result is the following.

Theorem 2.1. *Let $1/2 < p < 1$. There exists $K_1 > 0$ such that, under the measure $P\big(\ \cdot \ \big|\ |\mathrm{Int}(\Gamma_0)| \geq l^2\big)$, with probability approaching 1 as $l \to \infty$ we have*

$$\mathrm{MLR}(\Gamma_0) \geq K_1 l^{1/3} (\log l)^{-2/3} \tag{2.1}$$

The main ingredients of the proof will be coarse graining concepts, the renewal structure of long dual connections in the supercritical regime and exchangeability of the increments between regeneration points, all of which will be discussed below. The basic idea is that $\mathrm{MLR}(\Gamma_0) < K_1 l^{1/3}(\log l)^{-2/3}$ implies that Γ_0 stays in a narrow tube along its own convex hull, which is a highly unlikely event, due to the Gaussian fluctuations of connectivities. More precisely, if w and w' are extreme points of the convex hull Co(Γ_0) separated by a distance of order $l^{2/3}(\log l)^{-1/3}$, then $\mathrm{MLR}(\Gamma_0) < K_1 l^{1/3}(\log l)^{-2/3}$ requires that Γ_0 stay confined within $O(l^{1/3}(\log l)^{-2/3})$ of the straight line from w to w'. Gaussian fluctuations, though, would say that the typical deviation from the straight line is of order $l^{1/3}(\log l)^{-1/6}$, which is the square root of the length of the line. Thus the confinement for the segment between w and w' is analogous to keeping the maximum magnitude of a Brownian bridge below $O((\log l)^{-1/2})$, and such confinement along the entire boundary of Γ_0 is very unlikely. The Brownian bridge analogy is an underlying heuristic but does not enter directly into our proofs.

We use some notation, results and techniques introduced in [Al3]. For a family of bond percolation models including Bernoulli percolation and the FK model, upper bounds have been established in [Al3] for ALR(Γ_0), MLR(Γ_0) and the deviation between $\partial\Gamma_0$ and Wulff shape. We denote the unit Wulff shape (i.e. the set W of (1.1), normalized to have area 1) by $\mathbf{K}_1$. There exists constants K_i such that the following hold with probability approaching to 1, as $l \to \infty$, under the measure $P(\cdot \mid |\mathrm{Int}(\Gamma_0)| \geq l^2)$:

$$\mathrm{ALR}(\Gamma_0) \leq K_2 l^{1/3}(\log l)^{2/3}, \tag{2.2}$$

$$\inf_x \ \mathrm{dist}_H\left(\partial\mathrm{Co}(\Gamma_0), x + \partial(l\mathbf{K}_1)\right) \leq K_3 l^{2/3}(\log l)^{1/3}, \tag{2.3}$$

$$\mathrm{MLR}(\Gamma_0) \leq K_4 l^{2/3}(\log l)^{1/3}, \tag{2.4}$$

where dist_H denotes Hausdorff distance. Together, (2.1) and (2.2) suggest that local roughness is of order $l^{1/3}$, up to a possible logarithmic correction factor, for sufficiently large l.

We will use two standard inequalities for percolation: the Harris-FKG inequality [Ha] and the BK inequality [vdBK]. Let $\mathbb{D} \subset \mathbb{B}_2$ and $\omega, \widetilde{\omega} \in \{0,1\}^{\mathbb{D}}$. We write $\widetilde{\omega} \geq \omega$ if all open bonds in ω are also open in $\widetilde{\omega}$. An event $A \subset \{0,1\}^{\mathbb{D}}$ is *increasing (decreasing)* if its indicator function δ_A is nondecreasing (nonincreasing) according to this partial order.

Harris-FKG inequality. For Bernoulli percolation, if $A_1, A_2, \cdots, A_n$ are all increasing, or all decreasing, events, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq P(A_1)P(A_2)\cdots P(A_n).$$

For sets $S \subset \mathbb{B}_2$, we will denote by ω_S the restriction of ω to S. The event A is said to *occur on* the set S in the configuration ω if $\omega'_S = \omega_S$ implies $\omega' \in A$. Two events A_1 and A_2 *occur disjointly* in ω, denoted by $A_1 \circ A_2$, if there exist disjoint sets S_1, S_2 (depending on ω) such that A_1 occurs on S_1, and A_2 occurs on S_2, in ω. The event that A_1 and A_2 occur disjointly is denoted $A_1 \circ A_2$.

BK inequality. If $A_1, \cdots, A_n$ are all increasing, or all decreasing, events then

$$P(A_1 \circ A_2 \circ \cdots \circ A_n) \leq P(A_1)P(A_2)\cdots P(A_n).$$

Two points $x, y \in (\mathbb{Z}^2)^*$ are connected, an event written $\{\, x \longleftrightarrow y \,\}$, if there exists a path of open dual bonds leading from x to y. The Harris-FKG inequality implies that $-\log P(0 \leftrightarrow x)$ is a subadditive function of x, and therefore the limit

$$\tau(x) = \lim_{n\to\infty} -\frac{1}{n} \log P(0^* \leftrightarrow (nx)^*),$$

exists for $x \in \mathbb{Q}^2$, where the limit is taken through the values of n satisfying $nx \in \mathbb{Z}^2$. This definition extends to $\mathbb{R}^2$ by continuity (see [ACC]). τ is a strictly convex norm on $\mathbb{R}^2$; the strict convexity is shown in [CI]. The τ-norm for unit vectors serves as the surface tension for our context. Let $\mathbb{S}$ denote the unit circle in $\mathbb{R}^2$. It is known ([Al2],[Me]) that for $1/2 < p < 1$,

$$0 < \min_{x\in\mathbb{S}} \tau(x) \leq \max_{x\in\mathbb{S}} \tau(x) < \infty, \tag{2.5}$$

$$\beta_1 |x|^{-\beta_2} \exp(-\tau(x)) \leq P(0^* \leftrightarrow x^*) \leq \exp(-\tau(x)) \tag{2.6}$$

for some constants $\beta_1, \beta_2 > 0$ and

$$\frac{\tau(e)}{\sqrt{2}} \leq \frac{\tau(x)}{|x|} \leq \sqrt{2}\,\tau(e), \tag{2.7}$$

where e is a coordinate vector.

For $x, y \in \mathbb{R}^2$, let $\mathrm{dist}_\tau(\cdot,\cdot)$ and $\mathrm{diam}_\tau(\cdot)$ denote the τ-distance and the τ-diameter, respectively. Some of the properties of connectivities and geometry of Wulff shapes will be given next. Denote the unit τ-unit ball by $\mathbf{U}_1$:

$$\mathbf{U}_1 = \left\{x \in \mathbb{R}^2 \ :\ \tau(x) \leq 1\right\}$$

and the Wulff shape by $\mathbf{W}_1$:

$$\mathbf{W}_1 = \{t \in \mathbb{R}^2 \ : \ (t, z)_2 \leq \tau(z) \text{ for all } z \in \mathbb{S}\},$$

so that $0 \in \mathrm{Int}(\mathbf{W}_1)$ and $\mathbf{K}_1 = \mathbf{W}_1/|\mathbf{W}_1|$. Here $(\cdot, \cdot)_2$ denotes the Euclidean inner product. We also refer to multiples of $\mathbf{W}_1$ as Wulff shapes. For the functional

$$\mathcal{W}(\gamma) = \int_\gamma \tau(v_x)\, dx,$$

$\mathbf{K}_1$ minimizes $\mathcal{W}(\partial V)$ over all regions V with piecewise C^1 boundary, subject to the constraint $|V| = 1$; here v_x is the unit forward tangent vector at x and dx is arc length. (A class larger than the regions with piecewise C^1 boundary can be used here, but is not relevant for our purposes; for specifics see [Ta1], [Ta2].) We define the Wulff constant $\mathcal{W}_1 = \mathcal{W}(\partial\mathbf{K}_1)$. For every $t \in \partial\mathbf{W}_1$ and $x \in \partial\mathbf{U}_1$, we have

$$1 = \max_{y \in \mathbf{U}_1} (t, y)_2 = \max_{s \in \partial\mathbf{W}_1} (s, x)_2.$$

Definition 2.2. *Given $x \in \mathbb{R}^2 \backslash \{0\}$, a point $t \in \partial\mathbf{W}_1$ is* polar *to x if*

$$(t, x)_2 = \tau(x) = \max_{s \in \partial\mathbf{W}_1} (s, x)_2$$

3. Renewal structure of connectivities

For the remainder of the paper we assume we have fixed $1/2 < p < 1$.

This section will follow Section 4 of [CI]. For $x, y \in (\mathbb{Z}^2)^*$ and $t \in \partial\mathbf{W}_1$, we define the line

$$\mathcal{H}_x^t = \{z \in \mathbb{R}^2 \mid (t, z)_2 = (t, x)_2\}$$

and the slab

$$\mathcal{S}_{x,y}^t = \{z \in \mathbb{R}^2 \mid (t, x)_2 \leq (t, z)_2 \leq (t, y)_2\}.$$

When x and y are connected in the restriction of the percolation configuration to the slab $\mathcal{S}_{x,y}^t$ (excluding the bonds that are only partially in $\mathcal{S}_{x,y}^t$), $\mathbf{C}_{x,y}^t$ denotes the set of sites in the corresponding common cluster inside $\mathcal{S}_{x,y}^t$. Let $e = e(t)$ be a unit vector in the direction of one of the axes such that the scalar product of e with t is maximal.

Definition 3.1. *For $x, y \in (\mathbb{Z}^2)^*$ satisfying $(t, x)_2 < (t, y)_2$, let $\left\{ x \overset{\widetilde{h}_t}{\longleftrightarrow} y \right\}$ denote the event that x and y are* $\widetilde{h}_t$-connected, *meaning x and y are connected by an open dual path in $\mathcal{S}_{x,y}^t$. Let $\left\{ x \overset{h_t}{\longleftrightarrow} y \right\}$ denote the event that x and y are* h_t-connected, *meaning x and y are connected inside $\mathcal{S}_{x,y}^t$ and*

$$\mathbf{C}_{x,y}^t \cap \mathcal{S}_{x,x+e}^t = \{x, x+e\} \ \textit{and} \ \mathbf{C}_{x,y}^t \cap \mathcal{S}_{y-e,y}^t = \{y - e, y\}.$$

Let $\left\{ x \overset{f_t}{\longleftrightarrow} y \right\}$ denote the event that x and y are f_t-connected, *meaning $x \overset{h_t}{\longleftrightarrow} y$ and for no $z \in \mathrm{Int}(\mathcal{S}_{x,y}^t)$ do both $x \overset{h_t}{\longleftrightarrow} z$ and $z \overset{h_t}{\longleftrightarrow} y$.*

Definition 3.2. *Given a configuration and given x, y with $x \leftrightarrow y$, we say that $z \in (\mathbb{Z}^2)^*$ is a* regeneration point *if $(t,x)_2 < (t,z)_2 < (t,y)_2$ and $\mathbf{C}^t_{x,y} \cap \mathcal{S}^t_{z-e,z+e} = \{z-e, z, z+e\}$.*

Let $\mathcal{R}^t_{x,y}$ denote the random set of regeneration points of $\mathbf{C}^t_{x,y}$. Next, a probabilistic bound on the size of $\mathcal{R}^t_{x,y}$ will be given. For our purposes, we need a different formulation of Lemma 4.1 of [CI]: we use $\left\{ x \overset{\widetilde{h_t}}{\longleftrightarrow} y \right\}$ instead of $\left\{ x \overset{h_t}{\longleftrightarrow} y \right\}$ to state the lemma, but the proof is same with minor changes.

Lemma 3.3. *For every $\epsilon \in (0, \frac{1}{2})$, there exists $\lambda > 0$, $\delta > 0$ and $\nu > 0$ such that for all $t_0 \in \partial \mathbf{W}_1$, $t \in B_\lambda(t_0)$ and all x satisfying $(t,x)_2 \geq (1-\epsilon)\tau(x)$ we have*

$$P\left(|\mathcal{R}^{t_0}_{0,x}| < \delta|x| \,;\ 0 \overset{\widetilde{h_{t_0}}}{\longleftrightarrow} x\right) \leq \exp\{-(t,x)_2 - \nu|x|\}. \tag{3.1}$$

4. Coarse graining and related preliminaries

We will use the coarse graining setup and results of [Al3]. For $s > 0$, and any contour with a τ-diameter of at least $2s$, the coarse graining algorithm selects a subset $\{w_0, w_1, \cdots, w_{m+1}\}$ of the extreme points of $\mathrm{Co}(\gamma)$, with $w_{m+1} = w_0$, called the *s-hull skeleton* of γ and denoted $\mathrm{HSkel}_s(\gamma)$. The points w_i of $\mathrm{HSkel}_s(\gamma)$ appear in order as one traces γ in the direction of positive orientation. We denote the polygonal path $w_0 \to w_1 \to \cdots \to w_{m+1}$ by $\mathrm{HPath}_s(\gamma)$. The specifics of the algorithm for choosing the s-hull skeleton are not important to us here; we refer the reader to [Al3]. What we need are the following properties, also from [Al3].

Lemma 4.1. *There exist constants $K_5, K_6, K_7, K_8 > 0$ such that for every $s > 0$ and every circuit γ having τ-diameter at least $2s$, the s-hull skeleton $\mathrm{HSkel}_s(\gamma) = \{w_0, w_1, \cdots, w_{m+1}\}$ satisfies*

$$m + 1 < \frac{K_5 \,\mathrm{diam}(\gamma)}{s}, \tag{4.1}$$

$$|\,\mathrm{Int}(\gamma) \setminus \mathrm{Int}(\mathrm{HPath}_s(\gamma))| \leq K_6 s^2, \tag{4.2}$$

$$\sup_{x \in \mathrm{Co}(\gamma)} \mathrm{dist}(x, \mathrm{Int}(\mathrm{HPath}_s(\gamma))) \leq \frac{K_7 s^2}{\mathrm{diam}(\gamma)}, \tag{4.3}$$

$$\mathcal{W}(\partial\, \mathrm{Co}(\gamma)) \leq \mathcal{W}(\mathrm{HPath}_s(\gamma)) + \frac{K_8 s^2}{\mathrm{diam}(\gamma)}. \tag{4.4}$$

For $0 < \theta < 1$ a small constant to be specified later, our choice of s is

$$s = \left(\frac{\theta\sqrt{\pi}}{2K_7}\right)^{1/2} l^{2/3} (\log l)^{-1/3}.$$

Suppose $\mathrm{HSkel}_s(\Gamma_0) = \{w_0, w_1, \ldots, w_{m+1}\}$ with $w_{m+1} = w_0$. We define

$$\mathcal{L} = \left\{ i : |w_{i+1} - w_i| \geq \frac{s\sqrt{\pi}}{16K_5} \right\}$$

For $i \in \mathcal{L}$, we call the side between w_i and w_{i+1} *long*. The next lemma gives a lower bound on the sum of the lengths of long sides when $\mathrm{diam}(\Gamma_0)$ is not abnormally large. From [Al3], for some $K_9, K_{10}, K_{11} > 0$, for $T > 0$,

$$P(\mathrm{diam}_\tau(\Gamma_0) \geq T) \leq K_9 T^4 e^{-T}$$

and

$$P\big(|\operatorname{Int}(\Gamma_0)| \geq l^2\big) \geq K_{10} \exp\big(-\mathcal{W}_1 l - K_{11} l^{1/3} (\log l)^{2/3}\big), \tag{4.5}$$

so that for large l,

$$P\big(\mathrm{diam}_\tau(\Gamma_0) \geq 2\mathcal{W}_1 l \;\big|\; |\operatorname{Int}(\Gamma_0)| \geq l^2\big) \leq e^{-\mathcal{W}_1 l/2}.$$

Also using (2.7), we have

$$\mathrm{diam}(\Gamma_0) \leq \frac{\sqrt{2}}{\tau(e)}\, \mathrm{diam}_\tau(\Gamma_0) \leq \frac{4\sqrt{2}}{\mathcal{W}_1}\, \mathrm{diam}_\tau(\Gamma_0)$$

where in the second inequality we use $\mathcal{W}_1 \leq 4\tau(e)$, which follows from the fact that the unit square encloses the unit area. Therefore

$$P\big(\mathrm{diam}(\Gamma_0) \geq 8\sqrt{2} l \;\big|\; |\operatorname{Int}(\Gamma_0)| \geq l^2\big) \leq e^{-\mathcal{W}_1 l/2}. \tag{4.6}$$

so to prove Theorem 2.1 we need only consider configurations with $\mathrm{diam}(\Gamma_0) < 8\sqrt{2}\, l$. We say that $\{w_0, .., w_{m+1}\}$ is *l-regular* if there exists a configuration in which $|\operatorname{Int}(\Gamma_0)| \geq l^2$, $\mathrm{diam}(\Gamma_0) < 8\sqrt{2}\, l$ and $\mathrm{HSkel}_s(\Gamma_0) = \{w_0, .., w_{m+1}\}$.

Lemma 4.2. *If $\{w_0, w_1, \ldots, w_{m+1}\}$ is l-regular and l is sufficiently large, then*

$$\sum_{i \in \mathcal{L}} |w_{i+1} - w_i| \geq \sqrt{\frac{\pi}{2}}\, l \tag{4.7}$$

Proof. (4.2) implies that for some K_{12}, and Γ_0 as in the definition of l-regular,

$$|\operatorname{Int}(\mathrm{HPath}_s(\Gamma_0))| \geq l^2 - K_{12} l^{4/3} (\log l)^{-2/3} \geq \frac{l^2}{2},$$

where the last inequality is satisfied for sufficiently large l. By the standard isoperimetric inequality, it follows that

$$\sum_{i \in \mathcal{L}} |w_{i+1} - w_i| + \sum_{i \in \mathcal{L}^c} |w_{i+1} - w_i| \geq l\sqrt{2\pi}.$$

Using (4.1), the total number of sides can be bounded above:

$$m + 1 \leq \frac{K_5\, \mathrm{diam}(\Gamma_0)}{s} \leq \frac{8\sqrt{2}\, K_5\, l}{s}.$$

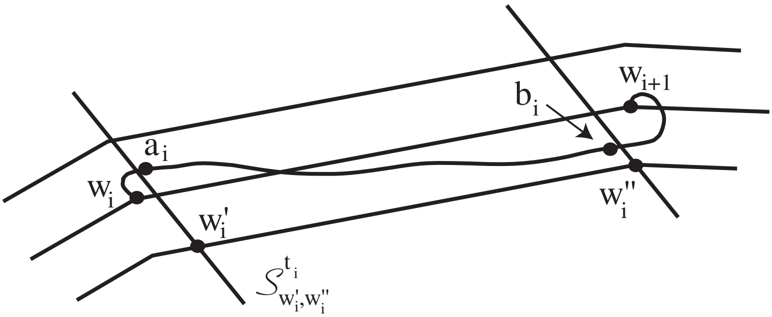

Fig. 1. A section of A_d, and a connection from w_i to w_{i+1} which includes a cylinder connection from a_i to b_i

Therefore

$$\sum_{i \in \mathcal{L}^c} |w_{i+1} - w_i| \leq (m+1) \frac{s\sqrt{\pi}}{16K_5} \leq \sqrt{\frac{\pi}{2}} l,$$

and the lemma follows. □

We next need to specify the vector t_i which will be used to define slabs and regeneration points for the connection from w_i to w_{i+1}. The natural choice is to take t_i polar to $w_{i+1} - w_i$, but in order to avoid some technicalities in upcoming proofs we will choose t_i to be close to the polar value, but having rational slope. Let $V \subset \mathbb{R}^2$ denote the wedge consisting of those vectors x such that the angle from the positive horizontal axis to x is in $[0, \pi/4]$. Due to lattice symmetries we may assume that $w_{i+1} - w_i \in V$. Let $\widetilde{t_i} \in \partial\mathbf{K}_1 \cap V$ be such that $\widetilde{t_i}$ is polar to $w_{i+1} - w_i$. Then the angular difference between $\widetilde{t_i}$ and $w_{i+1} - w_i$ is at most $\pi/4$. The existence of a polar point with such properties is guaranteed by symmetries of $\mathbf{K}_1$. Let us fix $\epsilon \in (0, 1/2)$, and let $\lambda = \lambda(\epsilon)$ as in (3.1). We choose $t_i \in V \cap B_\lambda(\widetilde{t_i}) \cap \partial\mathbf{K}_1$ so that the slope of t_i is r/q, with $q = [1/\lambda] + 1$ and $r \in \mathbf{Z}$. Choosing t_i this way will allow us to use (3.1), with the parameters t_0 and t chosen as t_i and $\widetilde{t_i}$, respectively. Note that $e(t_i) = (1, 0)$, which we denote by e_i.

By (4.3) for our chosen s, the deviation between $\mathrm{Co}(\Gamma_0)$ and $\mathrm{Int}(\mathrm{HPath}_s(\Gamma_0))$ inside it does not exceed $\theta l^{1/3}(\log l)^{-2/3}$. Let l_i be the line through w_i and w_{i+1}. We set $d = 2\theta l^{1/3}(\log l)^{-2/3}$, and define A_d, the annular tube of diameter $2d$ around $\mathrm{HSkel}_s(\Gamma_0)$, as follows. Denote the line parallel to l_i which is d units outside of $\mathrm{HSkel}_s(\Gamma_0)$ by l_i^+ and the line parallel to l_i which is d units in the opposite direction by l_i^-. Let H_{l_i} be the half space bounded by l_i that contains $\mathrm{HSkel}_s(\Gamma_0)$, let $H_{l_i^\pm}$ be the halfspaces bounded by $l_i^\pm$ such that $H_{l_i^-} \subset H_{l_i} \subset H_{l_i^+}$ and let

$$A_d = A_d(w_0, .., w_{m+1}) = \left(\bigcap_{i=1}^{m} H_{l_i^+}\right) \setminus \left(\bigcap_{i=1}^{m} H_{l_i^-}\right)$$

(see Figure 1.) Let T_d^i denote the (infinite) tube with diameter $2d$, bounded by l_i^+ and l_i^-. Let w_i' and w_i'' be the points on l_i^- such that $\mathcal{S}^{t_i}_{w_i', w_i''}$ is the largest slab satisfying

$$\mathcal{S}^{t_i}_{w_i', w_i''} \cap T_d^i \cap A_d = \mathcal{S}^{t_i}_{w_i', w_i''} \cap\ T_d^i.$$

Let B_i be the event that there exist $a_i \in \mathcal{S}^{t_i}_{w_i', w_i'+e_i} \cap T_d^i$ and $b_i \in \mathcal{S}^{t_i}_{w_i''-e_i, w_i''} \cap T_d^i$ such that the event

$$\{w_i \longleftrightarrow a_i\} \circ \{a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T_d^i\} \circ \{ b_i \longleftrightarrow w_{i+1}\}$$

occurs. For configurations in $\big\{w_i \longleftrightarrow w_{i+1} \text{ in } A_d\big\}\backslash B_i$, every open path from w_i to w_{i+1} must go "the long way around A_d"; presuming l is large and $\{w_0, .., w_{m+1}\}$ is l-regular, for some K_{13} this implies that $w_i \leftrightarrow z$ for some $z \in \mathcal{S}^t_{w_i, w_i+e_i}$ with $\text{dist}(z, w_i) \geq K_{13}l$. By ([Al3], Lemma 7.1) we then have for some K_{14}, K_{15},

$$P\big(B_i^c \bigm| w_i \longleftrightarrow w_{i+1}\big) \leq K_{14}e^{-K_{15}l}. \tag{4.8}$$

Lemma 4.3. *There exists constants $K_{14}, K_{15} > 0$ such that for $\{w_0, .., w_{m+1}\}$ l-regular and ϵ, t_i as in the preceeding, we have*

$$\begin{aligned} &P(w_i \leftrightarrow w_{i+1} \textit{ in } A_d \,\big|w_i \leftrightarrow w_{i+1}) \\ &\qquad \leq K_{14} \exp\left(-K_{15}l\right) + \sum_{a_i, b_i} P(a_i \leftrightarrow b_i \textit{ in } T_d^i \bigm| a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i) \end{aligned} \tag{4.9}$$

where the sum is over all $a_i \in \mathcal{S}^{t_i}_{w_i', w_i'+e_i} \cap T_d^i \cap (\mathbb{Z}^2)^*$ and $b_i \in \mathcal{S}^{t_i}_{w_i'', w_i''-e_i} \cap T_d^i \cap (\mathbb{Z}^2)^*$.

Proof. By (4.8) we can bound $P(w_i \leftrightarrow w_{i+1} \text{ in } A_d)$ by

$$\begin{aligned} &K_{14}e^{-K_{15}l} P(w_i \leftrightarrow w_{i+1}) \\ &\quad + \sum_{a_i, b_i} P\big(\{w_i \leftrightarrow a_i\} \circ \{a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T_d^i\} \circ \{b_i \leftrightarrow w_{i+1}\}\big), \end{aligned}$$

where the sum is over all $a_i \in \mathcal{S}^{t_i}_{w_i', w_i'+e_i} \cap T_d^i \cap (\mathbb{Z}^2)^*$ and $b_i \in \mathcal{S}^{t_i}_{w_i'', w_i''-e_i} \cap T_d^i \cap (\mathbb{Z}^2)^*$. We now apply the BK and FKG inequalities:

$$\begin{aligned}
&\sum_{a_i,b_i} P\Big(\{w_i \leftrightarrow a_i\} \circ \{a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i \text{ in } T_d^i\} \circ \{b_i \leftrightarrow w_{i+1}\}\Big) \\
&\le \sum_{a_i,b_i} P(w_i \leftrightarrow a_i)\, P(a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i \text{ in } T_d^i)\, P(b_i \leftrightarrow w_{i+1}), \\
&= \sum_{a_i,b_i} P(w_i \leftrightarrow a_i)\, P(a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i)\, P(a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i \text{ in } T_d^i \mid a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i)\, P(b_i \leftrightarrow w_{i+1}) \\
&\le \sum_{a_i,b_i} P(w_i \leftrightarrow w_{i+1})\, P(a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i \text{ in } T_d^i \mid a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i),
\end{aligned}$$

and (4.9) follows. □

In order to bound the probability of the event $\left\{a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T_d^i\right\}$ using the renewal structure of cylinder connectivities, we need control of the size of $|b_i - a_i|$ to apply (3.1). The parallelogram $\mathcal{S}^{t_i}_{w_i'+e,w_i''-e} \cap T_d^i$ has 2 short sides (the sides not parallel to $w_{i+1} - w_i$), one near w_i and the other near w_{i+1} (see Figure 1). It follows easily from the fact that $w_{i+1} - w_i, t_i$ are in the wedge V that for every a in the short side near w_i we have $|w_i - a| \le 2d\sqrt{2}$, and analogously for w_{i+1}. Therefore

$$|w_i - a_i| \le 2d\sqrt{2} + 1, \qquad |w_{i+1} - b_i| \le 2d\sqrt{2} + 1,$$

and hence

$$\big|(w_{i+1} - w_i) - (b_i - a_i)\big| \le 4d\sqrt{2} + 2. \tag{4.10}$$

Since

$$\tau(w_{i+1} - w_i) = (\widetilde{t_i}, w_{i+1} - w_i)_2, \tag{4.11}$$

provided l is large we have

$$(\widetilde{t_i}, b_i - a_i)_2 \ge (1 - \epsilon)\tau(b_i - a_i) \tag{4.12}$$

for our chosen ϵ.

Lemma 4.4. *Given ϵ, t_i, a_i, b_i as in the preceeding and δ as in* (3.1)*, there exists $\nu' > 0$ such that provided l is sufficiently large,*

$$P\big(|\mathcal{R}^{t_i}_{a_i,b_i}| < \delta|b_i - a_i| \;\big|\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\big) \le \exp(-\nu'|b_i - a_i|). \tag{4.13}$$

Proof. From ([Al3] equation (7.6)), for some $K_{16}, K_{17} > 0$, we have

$$P\big(a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\big) \ge K_{16}|b_i - a_i|^{-K_{17}} \exp\big(-\tau(b_i - a_i)\big). \tag{4.14}$$

By (4.12), Lemma 3.3 applies; with (4.14) this shows that for some $\nu > 0$,

$$P\big(|\mathcal{R}^{t_i}_{a_i,b_i}| < \delta|b_i - a_i| \bigm| a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\big)$$
$$\leq \frac{1}{K_{16}}|b_i - a_i|^{K_{17}} \exp\big(-(\widetilde{t_i}, b_i - a_i)_2 + \tau(b_i - a_i) - \nu|b_i - a_i|\big). \quad (4.15)$$

By (4.10) and (4.11), we have

$$\begin{aligned} &-(\widetilde{t_i}, b_i - a_i)_2 + \tau(b_i - a_i) - \nu|b_i - a_i| \\ &\quad\leq 2\tau(w_{i+1} - w_i - b_i + a_i) - \nu|b_i - a_i| \\ &\quad\leq K_{18}(4d\sqrt{2} + 2) - \nu|b_i - a_i| \end{aligned}$$

for some $K_{18} > 0$. Since d is small compared to $|b_i - a_i|$, using this bound in (4.15), for some constant $\nu' < \nu$ we have (4.13). □

Next, we will define orthogonal increments between adjacent regeneration points. There is no canonical choice of direction relative to which increments are defined; we will use the direction orthogonal to the line joining w_i and w_{i+1}.

Definition 4.5. *For any $x \in \mathcal{S}^{t_i}_{w_i,w_{i+1}}$, define $f : \mathcal{S}^{t_i}_{w_i',w_i''} \to \mathbb{R}$ as follows:*

$$f(x) = \begin{cases} \operatorname{dist}(x, l_i), & \text{if } x \text{ is above the line } l_i, \text{ joining } w_i \text{ and } w_{i+1}, \\ -\operatorname{dist}(x, l_i), & \text{if } x \text{ is on or below the line } l_i. \end{cases}$$

For the following definitions assume $a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i$. The regeneration points between a_i and b_i have a natural ordering according to their distance from $\mathcal{H}^{t_i}_{a_i}$.

Definition 4.6. *For $r' \in \mathcal{S}^{t_i}_{a_i,b_i}$ define $\Delta : \mathcal{S}^{t_i}_{a_i,b_i} \to \mathbb{R}$ as follows:*

$$\Delta(r') = \begin{cases} f(r'), & \text{if } r' \text{ is the first regeneration point}, \\ f(r') - f(\tilde{r}), & \text{if } \tilde{r}, r' \text{ are successive regeneration points}, \\ 0 & \text{if } r' \text{ is not a regeneration point.} \end{cases}$$

Definition 4.7. *For $\mathcal{H}^{t_i}_z \subset \mathcal{S}^{t_i}_{a_i,b_i}$ define*

$$\widetilde{\Delta}(\mathcal{H}^{t_i}_z) = \begin{cases} \Delta(r') & \text{if there is a regeneration point } r' \in \mathcal{H}^{t_i}_z, \\ 0 & \text{otherwise.} \end{cases}$$

We will refer to the values $\Delta(r)$ as *increments*. We need to show that, given $a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i$, there are unlikely to be too many small increments. This will be proved by showing that a positive proportion of increments have magnitude greater than equal to $1/2$, with high probability. This result will be used to bound the variance of sums of increments from below.

For δ as in Lemma 3.3, and a_i, b_i fixed, let $N = \lfloor \delta|b_i - a_i| \rfloor$, and $R = \lfloor N/8 \rfloor$. Let $\mathcal{U}$ be the collection of all $(z_1, \cdots, z_R)$ such that for $j = 1, \cdots, R$, we have

(i) $z_j \in \mathcal{S}^{t_i}_{a_i,b_i}$; z_j is on the line through w_i, parallel to t_i,

(ii) $(t_i, z_1)_2 < (t_i, z_2)_2 < \cdots < (t_i, z_R)_2$,
(iii) $(\mathrm{Int}\, \mathcal{S}^{t_i}_{z_j-4e_i, z_j+4e_i})$ and $(\mathrm{Int}\, \mathcal{S}^{t_i}_{z_k-4e_i, z_k+4e_i})$ are disjoint for $j \neq k$.

By property (i), there is a bijection pairing $\{z_1, z_2, \cdots, z_R\} \in \mathcal{U}$ and the set of lines $\mathcal{H}^{t_i}_{z_j}$ passing through the points $\{z_1, z_2, \cdots, z_R\}$. Suppose $\mathcal{R}^{t_i}_{a_i,b_i} = \{r_1, r_2, \cdots, r_I\}$, with $I \geq N$. Next, we define $\mathcal{Q}^{t_i}_{a_i,b_i} = \{\sigma_1, .., \sigma_R\} \subset \mathcal{R}^{t_i}_{a_i,b_i}$ according to the following algorithm:

(1) $\sigma_1 = r_{k_1}$, where k_1 is the smallest integer satisfying $(t_i, a_i + 4e_i)_2 \leq (t_i, r_{k_1})_2$,
(2) $\sigma_j = r_{k_j}$, where k_j is the smallest integer satisfying $(t_i, \sigma_{j-1} + 8e_i)_2 \leq (t_i, r_{k_j})_2$, for $j = 2, 3, \cdots, R$.

For $j \geq 2$, this algorithm can skip at most 7 regeneration points after σ_{j-1} before it selects σ_j; under the assumption that there are at least N regeneration points, it will successfully choose exactly R regeneration points. ($\mathcal{Q}^{t_i}_{a_i,b_i}$ is undefined when there are fewer than N regeneration points, so $|\mathcal{Q}^{t_i}_{a_i,b_i}| = R$ whenever $\mathcal{Q}^{t_i}_{a_i,b_i}$ is defined.) Notice that, for some $(z_1, z_2, \cdots, z_R) \in \mathcal{U}$, the regeneration point σ_j occurs on $\mathcal{H}^{t_i}_{z_j}$, for $j = 1, 2, \cdots, R$. Also, since the slope of t_i is rational, the line $\mathcal{H}^{t_i}_{\sigma_j}$ contains other lattice points, which are also possible locations for the jth regeneration point, when only $\mathcal{H}^{t_i}_{\sigma_j}$ is specified.

Lemma 4.8. *Given ϵ, t_i, a_i, b_i as in the preceeding, for $\delta > 0$ from* (3.1)*, there exist $\gamma, \varphi > 0$ such that*

$$P\left(\sum_{k=2}^{N} \delta_{\{|\Delta(r_k)| \geq \frac{1}{2}\}} \leq \gamma |b_i - a_i| \,;\; |\mathcal{R}^{t_i}_{a_i,b_i}| > \delta |b_i - a_i| \;\middle|\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\right)$$
$$\leq \exp\big(-\varphi |b_i - a_i|\big). \tag{4.16}$$

Proof. For some $\gamma > 0$ to be specified later, we write

$$\begin{aligned}
&P\left(\sum_{k=2}^{N} \delta_{\{|\Delta(r_k)| \geq \frac{1}{2}\}} \leq \gamma |b_i - a_i| \,;\; |\mathcal{R}^{t_i}_{a_i,b_i}| > N \;\middle|\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\right) \\
&\leq \sum_{(z_1,\cdots,z_R)\in\mathcal{U}} P\left(\mathcal{Q}^{t_i}_{a_i,b_i} \subset \bigcup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j} \,;\; \sum_{k=2}^{N} \delta_{\{|\Delta(r_j)| \geq \frac{1}{2}\}} \leq \gamma |b_i - a_i| \;\middle|\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\right) \\
&\leq \sum_{(z_1,\cdots,z_R)\in\mathcal{U}} P\left(\mathcal{Q}^{t_i}_{a_i,b_i} \subset \bigcup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j} ; \sum_{j=2}^{R} \delta_{\{|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| \geq \frac{1}{2}\}} \leq \gamma |b_i - a_i| \;\middle|\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\right) \\
&\leq \sum_{(z_1,\cdots,z_R)\in\mathcal{U}} P\left(\mathcal{Q}^{t_i}_{a_i,b_i} \subset \bigcup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j} \;\middle|\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\right) \\
&\qquad \times P\left(\sum_{j=2}^{R} \delta_{\{|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| \geq \frac{1}{2}\}} \leq \gamma |b_i - a_i| \;\middle|\; \mathcal{Q}^{t_i}_{a_i,b_i} \subset \bigcup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j} \,;\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\right).
\end{aligned} \tag{4.17}$$

We will bound the second probability in the last sum. In order to do this, we will describe a "renewal shifting" procedure. For $\omega \in \{\mathcal{Q}^{t_i}_{a_i,b_i} \subset \cup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j}\ ;\ a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\}$, satisfying $|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| < \frac{1}{2}$ for some fixed $j \geq 2$, this procedure will produce a configuration $\widetilde{\omega} \in \{\mathcal{Q}^{t_i}_{a_i,b_i} \subset \cup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j}\ ;\ a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\}$, which has at most a bounded number of bonds different from ω, and which satisfies $|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| \geq \frac{1}{2}$. Moreover, this procedure maps at most 2^m configurations to the same $\widetilde{\omega}$, where m is the number of possibly-adjusted bonds. Once this procedure is described, for constants $c_1, c_2, \cdots, c_{j-1}$ we get

$$P\left(|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| < 1/2 \,\middle|\, \mathcal{Q}^{t_i}_{a_i,b_i} \subset \bigcup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j}\ ;\ a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\ ;\ \widetilde{\Delta}(\mathcal{H}^{t_i}_{z_k}) = c_k,\ 1 \leq k < j\right)$$
$$\leq \lambda'\ P\left(|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| \geq 1/2 \,\middle|\, \mathcal{Q}^{t_i}_{a_i,b_i} \subset \bigcup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j}\ ;\ a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\ ;\ \widetilde{\Delta}(\mathcal{H}^{t_i}_{z_k}) = c_k,\ 1 \leq k < j\right), \tag{4.18}$$

where $\lambda' = \lambda'(p) > 0$. This yields

$$P\left(|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| \geq 1/2 \,\middle|\, \mathcal{Q}^{t_i}_{a_i,b_i} \subset \bigcup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j}\ ;\ a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i;\ \widetilde{\Delta}(\mathcal{H}^{t_i}_{z_k}) = c_k,\ \text{for } 1 \leq k < j\right)$$
$$\geq \frac{1}{1+\lambda'}$$

which is sufficient to bound the last probability in (4.17) by $P\big(X < \gamma|b_i - a_i|\big)$, where X is binomially distributed with parameters $R-1$ and $p^* = \frac{1}{1+\lambda'}$. Taking $\gamma < p^*$ and using a bound from [Ho] we have

$$P\big(X < \gamma|b_i - a_i|\big) \leq \exp\left(-\frac{(R-1)(p^*-\gamma)^2}{2}\right) \leq \exp(-\varphi|b_i - a_i|),$$

for some $\varphi > 0$. Using this in the right side of (4.17) and observing that the events $\{\mathcal{Q}^{t_i}_{a_i,b_i} \subset \bigcup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j}\}$ are disjoint for distinct $(z_1, z_2, \cdots, z_R) \in \mathcal{U}$, we obtain (4.16), after summing over all $(z_1, z_2, \cdots, z_R) \in \mathcal{U}$.

The proof will be completed by description of the "renewal shifting" procedure. For a given configuration $\omega \in \{\mathcal{Q}^{t_i}_{a_i,b_i} \subset \bigcup_{j=1}^{R} \mathcal{H}^{t_i}_{z_j}\ ;\ a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\}$ and a fixed $j \leq R$, let us assume $|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| < \frac{1}{2}$, for some j. We will define $\widetilde{\omega}$ by modifying some dual bonds inside $\mathcal{S}^{t_i}_{z_j-4e_i,z_j+4e_i}$. Since t_i has slope $\frac{r}{q}$, there exists infinitely many equally spaced lattice points on the line $\mathcal{H}^{t_i}_{z_j}$. We will use one of the two lattice points on $\mathcal{H}^{t_i}_{z_j}$ closest to the regeneration point σ_j. Call these locations u_j and v_j, with $u_j = \sigma_j + (-r, q)$ and $v_j = \sigma_j + (r, -q)$. The configuration ω has open dual bonds $\langle\sigma_j - e_i, \sigma_j\rangle$ and $\langle\sigma_j, \sigma_j + e_i\rangle$.

There exists a path γ_j^L from $\sigma_j - 2e_i$ to $u_j - e_i$ in $\mathcal{S}^{t_i}_{z_j-3e_i,z_j-e_i}$ having all steps upward or leftward, with $\gamma_j^L \cap \mathcal{H}^{t_i}_{\sigma_j-e_i} = \{u_j - e_i\}$, and similarly a path γ_j^R from $\sigma_j + 2e_i$ to $u_j + e_i$ in $\mathcal{S}^{t_i}_{z_j+e_i,z_j+3e_i}$ having all steps upward or leftward with $\gamma_j^R \cap \mathcal{H}^{t_i}_{\sigma_j+e_i} = \{u_j + e_i\}$. Let A_j be the closed region bounded by γ_j^L, γ_j^R and the horizontal lines through σ_j and u_j. To make our choice of γ_j^L, γ_j^R unique, let us specify that A_j be maximal under the constraints we have imposed on γ_j^L, γ_j^R. Let $\mathcal{D}_j$ be the set of all dual bonds having one endpoint in ∂A_j and the other outside A_j. Note there are at most $12q$ dual bonds contained in A_j, and at most $2r + 2q + 10$ dual bonds in $\mathcal{D}_j$. Let $\widetilde{\omega}$ be such that

(1) all dual bonds in $\partial A_j \backslash \{\langle \sigma_j - e_i, \sigma_j\rangle, \langle \sigma_j, \sigma_j + e_i\rangle\}$ are open;
(2) all other dual bonds contained in A_j are closed;
(3) all dual bonds in $\mathcal{D}_j \cap \mathbf{C}^{t_i}_{a_i,b_i}(\omega)$ are open;
(4) all dual bonds in $\mathcal{D}_j \backslash \mathbf{C}^{t_i}_{a_i,b_i}(\omega)$ are closed;
(5) all other dual bonds retain their state from ω.

In the altered configuration $\widetilde{\omega}$, the regeneration point is still on $\mathcal{H}^{t_i}_{z_j}$ but shifted from σ_j to u_j. After these alterations, if $|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| \geq \frac{1}{2}$, then we are done. It is possible that $|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| < \frac{1}{2}$, for the following reason. Let k be such that $z_j = r_k$. If there are other regeneration points in $\mathcal{S}^{t_i}_{z_j-3e_i,z_j+3e_i}$ in ω, shifting the regeneration point to u_j will destroy these regeneration points; any regeneration points in $\mathcal{S}^{t_i}_{z_j-4e_i,z_j+4e_i} \backslash \mathcal{S}^{t_i}_{z_j-3e_i,z_j+3e_i}$ in ω may or may not be destroyed, depending on the exact geometry of the situation. At any rate, if r_{k-1} is destroyed, the new "preceding regeneration point" for z_j will be outside the slab $\mathcal{S}^{t_i}_{z_j-3e_i,z_j+3e_i}$, equal to r_{k-2} or r_{k-3}, and we may have $|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| < \frac{1}{2}$ in $\widetilde{\omega}$, depending on the location of this new preceding regeneration point relative to l_i. If this is the case we shift the regeneration point from σ_j to v_j instead of u_j. For this we use paths $\tilde{\gamma}_j^L$ from $\sigma_j - 2e_i$ to $v_j - e_i$ and $\tilde{\gamma}_j^R$ from $\sigma_j + 2e_i$ to $v_j + e_i$ in place of γ_j^L and γ_j^R, under an analogous maximality constraint. Let x_j^L (respectively x_j^R) be the site in γ_j^L (respectively γ_j^R) closest to $\mathcal{H}^{t_i}_{\sigma_j-3e_i}$ (respectively $\mathcal{H}^{t_i}_{\sigma_j+3e_i}$). Due to the maximality constraints we have imposed, since all our slabs have boundaries with slope $-q/r$, $x_j^L + (r, -q)$ is the site in $\tilde{\gamma}_j^L$ closest to $\mathcal{H}^{t_i}_{\sigma_j-3e_i}$, and $x_j^R + (r, -q)$ is the site in $\tilde{\gamma}_j^R$ closest to $\mathcal{H}^{t_i}_{\sigma_j+3e_i}$. This means that γ_j^L and $\tilde{\gamma}_j^L$ intersect the same slabs orthogonal to t_i, and similarly for γ_j^R and $\tilde{\gamma}_j^R$. As a consequence, the same regeneration points are destroyed, regardless of whether we shift to u_j or v_j, so $\widetilde{\omega}$ has the same preceding regeneration point either way. It follows that if shifting to u_j results in $|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| < \frac{1}{2}$, then shifting to v_j results in $|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| \geq \frac{1}{2}$, i.e. there is always a shift (the one we choose to create $\widetilde{\omega}$) which results in $|\widetilde{\Delta}(\mathcal{H}^{t_i}_{z_j})| \geq \frac{1}{2}$.

Note that only a bounded number of different configurations may map to the same configuration $\widetilde{\omega}$. In any case, $\widetilde{\omega}$ and ω yield the same value of $\mathcal{Q}^{t_i}_{a_i,b_i}$, and the probabilities of ω and $\widetilde{\omega}$ are within a bounded factor (depending on p), which yields (4.18), completing the proof. □

5. Exchangeability of increments

The core idea in our proof of (2.1) is to make use of the renewal structure of connectivities, for connections between any two consecutive extreme points w_i, w_{i+1} in the s-hull skeleton with $i \in \mathcal{L}$, to see that the increments $\Delta(r_j)$, $2 \le j \le N$, form an exchangeable sequence under certain conditioning, that is, the joint distribution is permutation invariant. The partial sums of this sequence behave like those of an i.i.d. sequence, and from this we can show that with high probability, the path of open dual bonds will not stay in the "narrow tube" from w_i to w_{i+1} with diameter $2d = 4\theta l^{1/3}(\log l)^{-2/3}$. In this section we will prove this exchangeability. Let ϵ, t_i, a_i, b_i be as in the preceeding, δ as in (3.1) and γ as in (4.16). Define the event $E = E(a_i, b_i, t_i, \gamma, \delta)$ by

$$E = \left\{ a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \right\} \cap \left\{ |\mathcal{R}^{t_i}_{a_i,b_i}| \ge \delta |b_i - a_i| \right\} \cap \left\{ \sum_{k=2}^{N} \delta_{\{|\Delta(r_k)| \ge \frac{1}{2}\}} \ge \gamma |b_i - a_i| \right\}.$$

For $v, w \in T^i_d \cap \mathcal{S}^{t_i}_{a_i,b_i} \cap (\mathbb{Z}^2)^*$, define the sets

$$\begin{aligned} V(v, w) \; = \Bigg\{ & \zeta = (\zeta_2, ..., \zeta_N) \in \mathbb{R}^{N-1} : |\zeta_2| \ge |\zeta_3| \\ & \ge \cdots \ge |\zeta_N| \, ; \sum_{k=2}^{N} \zeta_i = f(w) - f(v) \Bigg\}. \end{aligned}$$

For given $\zeta' \in V(v, w)$, let $F = F(v, w, \zeta')$ denote the event that the following all hold:

(i) $a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i$,
(ii) the first and N-th regeneration points are at v and w, respectively,
(iii) for some permutation $\pi : \{2, \cdots, N\} \to \{2, \cdots, N\}$, we have

$$\Delta(r_2) = \zeta'_{\pi(2)}, \Delta(r_3) = \zeta'_{\pi(3)}, \cdots, \Delta(r_N) = \zeta'_{\pi(N)}.$$

Observe that condition (iii) determines the values of the $\Delta(r_k)$'s up to an ordering, and (ii) and (iii) imply $\sum_{k=2}^{N} \Delta(r_k) = f(w) - f(v)$.

Lemma 5.1. *For fixed* $a_i, b_i, t_i, \gamma, \delta, v, w, \zeta', E, F$ *as in the preceding,* $\Delta(r_2), \cdots, \Delta(r_N)$ *are exchangeable under the measure* $P(\cdot \mid E \cap F)$.

Proof. $E \cap F$ determines the location of first and N-th regeneration points, and values of increments in between them, up to an ordering. We will first show how to exchange any two adjacent increments. Consider a configuration $\omega \in E \cap F$, with $\Delta(r_2) = \zeta'_2, \Delta(r_3) = \zeta'_3, \cdots, \Delta(r_N) = \zeta'_N$. For fixed $k \ge 2$, let us consider increments $\Delta(r_k)$ and $\Delta(r_{k+1})$. By definition of regeneration points, the bonds that are only partially in the slab $\mathcal{S}^{t_i}_{r_k,r_{k+1}}$ or have exactly one endpoint in the within-slab cluster containing r_k and r_{k+1} are all vacant. We construct a configuration $\widetilde{\omega}$ such that outside $\mathcal{S}^{t_i}_{r_{k-1},r_{k+1}}$ we have $\widetilde{\omega} = \omega$. We obtain $\widetilde{\omega}$ by interchanging the relative positions of the configurations $\omega_{\mathcal{S}^{t_i}_{r_{k-1},r_k}}$ and $\omega_{\mathcal{S}^{t_i}_{r_k,r_{k+1}}}$ and moving the bonds crossing $\mathcal{H}^{t_i}_{r_k}$ so that they cross $\mathcal{H}^{t_i}_{r_{k-1}+(r_{k+1}-r_k)}$ instead. The latter move is done in such a way that the relative positions of the bonds remain the same, with the old position relative to r_k becoming the new position relative to $r_{k-1}+(r_{k+1}-r_k)$. More precisely, the configuration in $\mathcal{S}^{t_i}_{r_{k-1},r_k}$ is translated by $r_k - r_{k-1}$, the configuration in $\mathcal{S}^{t_i}_{r_k,r_{k+1}}$ is translated by $r_k - r_{k+1}$, and each bond touching or crossing $\mathcal{H}^{t_i}_{r_k}$ is translated by $r_{k-1}+(r_{k+1}-2r_k)$. This moves the k-th regeneration point from r_k to $r_{k-1}+(r_{k+1}-r_k)$, without altering the locations of other regeneration points. The configuration $\widetilde{\omega}$ is in $E \cap F$ and the increments of $\widetilde{\omega}$ satisfy

$$\Delta(r_k) = \zeta'_{j+1},\ \Delta(r_{k+1}) = \zeta'_j,\ \text{and}\ \Delta(r_m) = \zeta'_m,\ \text{for}\ 2 \le m \le N,\ k \ne m, m+1.$$

Moreover, replacing ω with $\widetilde{\omega}$ does not affect probability under the measure $P(\cdot \mid E \cap F)$, due to shift invariance. We can repeat the exchanging of adjacent increments until we achieve the desired permutation of $\{2, 3 \cdots, N\}$, and the lemma follows. $\square$

6. Staying in the narrow tube

In this section, we will show that there is an extra probabilistic cost associated to the event that Γ_0 stays in the narrow tube T^i_d, between a_i and b_i . The proof involves randomization of the order of the increments, using exchangeability.

Lemma 6.1. *Let $i \in \mathcal{L}$ and let ϵ, t_i, a_i, b_i be as in the preceeding. Let δ be as in (3.1) and γ as in (4.16). There exists $\kappa = \kappa(\gamma) > 0$ such that for all $v, w \in T^i_d \cap \mathcal{S}^{t_i}_{a_i,b_i} \cap (\mathbb{Z}^2)^*$ and $\zeta' \in V(v,w) \cap [-2d, 2d]^N$, for $E = E(a_i, b_i, t_i, \gamma, \delta)$, $F = F(v, w, \zeta')$, provided l is large we have*

$$P\left(a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T^i_d \,\middle|\, E \cap F\right) \le 2\exp\left(\frac{-\kappa|w_{i+1}-w_i|}{d^2}\right). \tag{6.1}$$

Proof. Observe that when $a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i$ in T^i_d, every open path from a_i to b_i in T^i_d must pass through all regeneration points. Thus

$$P\big(a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T^i_d \mid\ E \cap F\big) \le P(\{r_1, r_2, \cdots, r_N\} \subset T^i_d \cap \mathcal{S}^{t_i}_{a_i,b_i} \mid E \cap F). \tag{6.2}$$

We can relate the last probability to an event involving increments. If $1 \le k_1 < k_2 \le N$ and $\left|\sum_{j=k_1}^{k_2-1} \Delta(r_{j+1})\right| > 2d$, then the k_1-th or k_2-th regeneration point must lie outside of T^i_d. Therefore,

$$\begin{aligned} &P(\{r_1, r_2, \cdots, r_{N+1}\} \in T^i_d \cap \mathcal{S}^{t_i}_{a_i,b_i} \mid E \cap F) \\ &\quad\le\ P\left(\left|\sum_{j=k_1}^{k_2-1} \Delta(r_{j+1})\right| \le 2d,\ \text{for all}\ 1 \le k_1 < k_2 \le N \,\middle|\, E \cap F\right). \end{aligned} \tag{6.3}$$

Instead of looking at partial sums for all possible values of k_1, k_2, we will consider disjoint blocks of increments with random lengths $X_1, X_2, \cdots, X_B$ satisfying $X_1 + \cdots + X_B < N$, for some $B \in \mathbb{N}$. Let $S_n = \sum_{k=1}^{n} X_k$, for $n = 1, \cdots, B$, and let $S_0 = 0$. Then (6.3) is bounded by

$$P\left(\bigcap_{k=1}^{B} \left\{ \max_{1 \le m \le X_k} \left| \sum_{j=S_{k-1}+1}^{S_{k-1}+m} \Delta(r_{j+1}) \right| \le 2d \right\} \middle| E \cap F \right), \tag{6.4}$$

where we define $X_0 = 0$. If we take X_k, for $1 \le k \le B$, to be deterministic, the increments on these disjoint blocks will not be independent of the increments on other blocks. In order to reduce the dependence between these disjoint blocks, we will take the X_k's to be (non-independent) binomially distributed random variables. Next, we use exchangeability of increments to write (6.4) in an equivalent form. For binomially distributed X_1 with parameters, $N - 1$ and p_0, with p_0 to be specified later,

$$\sum_{j=2}^{X_1+1} \Delta(r_j) \stackrel{d}{=} \sum_{j=2}^{N} \delta_{j1} \zeta'_j$$

where the $\delta_{j1},\ j = 2, .., N$ are i.i.d. Bernoulli random variables with parameter p_0. That is, the sum of first X_1 increments have the same distribution as the sum of increments randomly selected according to the δ_{j1}'s. Continuing this way, for each following random block, we replace the sum of increments corresponding to that block with a sum of increments that are randomly selected from those increments remaining after the earlier steps of the increment–selection process. More precisely, we do the following. Define p_0 and the number of blocks by

$$p_0 = \frac{K_{19} d^2}{|w_{i+1} - w_i|}, \qquad B = \lfloor \frac{1}{2p_0} \rfloor,$$

where $K_{19} = K_{19}(\gamma)$ is sufficiently large constant, to be specified later. Observe that $p_0 = O((\log l)^{-1})$. For all $2 \le j \le N$, define

$$\delta_{jk} = \begin{cases} 0 & \text{with probability } p_k = \frac{1-kp_0}{1-(k-1)p_0} \\ 1 & \text{with probability } 1 - p_k = \frac{p_0}{1-(k-1)p_0}, \end{cases} \tag{6.5}$$

with $\{\delta_{jk},\ j = 2, \cdots N\ ,\ k = 1, \cdots, B\}$ independent random variables. Also define $Y_{jk} = (1 - \delta_{j1})(1 - \delta_{j2}) \cdots (1 - \delta_{jk})$, for $j = 2, .., N$, $k = 1, .., B$. Then we have

$$Y_{jk} = \begin{cases} 0 & \text{with probability } kp_0 \\ 1 & \text{with probability } 1 - kp_0 \end{cases} \tag{6.6}$$

The random variable $Y_{jk} = 1$ says that the j-th increment is not selected for the first k blocks, and $Y_{j(k-1)}\delta_{jk} = 1$ says that the j-th increment is selected for the k-th block. We define the length of the k-th block X_k as

$$X_k = \sum_{j=2}^{N} Y_{j(k-1)} \delta_{jk}, \qquad \text{for } k = 1, 2, \cdots, B.$$

It can be easily seen from (6.5) and (6.6) that the X_k's are binomially distributed (but not independent) with parameters $N-1$ and p_0. By exchangeability, we can rewrite (6.4) as

$$P\Big(\bigcap_{k=1}^{B}\Big\{\max_{2\le m\le N}\Big|\sum_{j=2}^{m} Y_{j(k-1)}\delta_{jk}\zeta_j\Big|\le 2d\Big\}\ \Big|\ E\cap F\Big), \tag{6.7}$$

since $\sum_{k=1}^{B} X_k \le N$, which holds because no j can be chosen for more than one block. We let

$$D_k=\Big\{\omega:\max_{2\le m\le N}\Big|\sum_{j=2}^{m} Y_{j(k-1)}\delta_{jk}\zeta_j\Big|\le 2d\Big\}.$$

We need to control the number of increments $|\zeta_j|\ge 1/2$ which remain after some blocks have been selected. By definition of E, there are at least $\lfloor\gamma|b_i-a_i|\rfloor$ such increments before the first block is selected. Let $g=\sqrt{B\lfloor\gamma|b_i-a_i|\rfloor}$, let $G_1=E\cap F$, and for $k=2,\cdots,B$ define

$$G_k=\Big\{\omega:\Big|\Big(\sum_{j=2}^{\lfloor\gamma|b_i-a_i|\rfloor} Y_{j(k-1)}\Big)-(1-(k-1)p_0)\lfloor\gamma|b_i-a_i|\rfloor\Big|\le g\Big\}.$$

Let $I_k=\{j: Y_{j(k-1)}=1, 1\le j\le N-1\}$, be the random set of remaining increment indices before the k-th block is selected. Then G_{k-1} provides control over $\big|I_k\cap\{1,2,\cdots,\lfloor\gamma|b_i-a_i|\rfloor\}\big|$, the number of remaining increments that are greater than or equal to $1/2$; here we use the monotonicity of the $|\zeta_j|$'s, and the fact that at least $\lfloor\gamma|b_i-a_i|\rfloor$ increments are greater than or equal to $1/2$. By (6.2)–(6.4) we have

$$\begin{aligned}
&P\big(a_i \xleftrightarrow{\widetilde{h_{t_i}}} b_i \text{ in } T_d^i\ \big|\ E\cap F\big)\\
&\quad\le P\Big(\bigcap_{k=1}^{B}(D_k\cap G_k)\ \Big|\ E\cap F\Big)+P\Big(\Big[\bigcap_{k=1}^{B}G_k\Big]^c\ \Big|\ E\cap F\Big)
\end{aligned} \tag{6.8}$$

First we bound the probability in (6.8) of a large deviation for some block for the number of available large increments, using a bound from [Ho]:

$$\begin{aligned}
P\Big(\Big[\bigcap_{k=1}^{B}G_k\Big]^c\ \Big|\ E\cap F\Big)&\le\sum_{k=2}^{B}P(G_k^c\mid E\cap F)\\
&\le 2B\exp\Big(\frac{-2g^2}{\lfloor\gamma|b_i-a_i|\rfloor}\Big)\\
&=2B\exp(-2B)\\
&\le\exp(-B),
\end{aligned} \tag{6.9}$$

where the last inequality holds for l sufficiently large. Next, we consider the probability the probability of staying in the narrow tube in the absence of such a large deviation. This probability from (6.8) can be written

$$P\big(D_1 \mid E \cap F\big) \times \prod_{k=2}^{B} P\Big(D_k \cap G_k \,\Big|\, E \cap F \cap \bigcap_{j=1}^{k-1}(D_j \cap G_j)\Big)$$
$$\leq P\big(D_1 \mid E \cap F\big) \times \prod_{k=2}^{B} P\Big(D_k \,\Big|\, E \cap F \cap \bigcap_{j=1}^{k-1}(D_j \cap G_j)\Big). \tag{6.10}$$

We will conclude by showing

$$P\Big(D_k \,\Big|\, E \cap F \cap \bigcap_{j=1}^{k-1}(D_j \cap G_j)\Big) \leq 2/3, \tag{6.11}$$

for $k \geq 2$. The proof that $P\big(D_1 \mid E \cap F\big) \leq 2/3$ follows by the same technique. Let us fix $k \geq 2$. We define a family of sets of indices:

$$\mathcal{I}_k = \Big\{\Upsilon \subset \{2, \cdots, N-1\} : \Big| \sum_{j=2}^{\lfloor \gamma |b_i - a_i| \rfloor} \delta_{\{j \in \Upsilon\}}$$
$$-(1-(k-1)p_0)\lfloor \gamma |b_i - a_i| \rfloor \Big| \leq g \Big\}.$$

For $\Upsilon \in \mathcal{I}_k$, and $n \leq N-1$, define

$$\Upsilon_n = \Upsilon \cap \{2, 3, \cdots, n\}$$

Observe that if G_{k-1} occurs then $I_k \in \mathcal{I}_k$. It follows that

$$P\Big(D_k \,\Big|\, E \cap F \cap \bigcap_{j=1}^{k-1}(D_j \cap G_j)\Big)$$
$$= \sum_{\Upsilon \in \mathcal{I}_k} P\Big(D_k \cap \{I_k = \Upsilon\} \,\Big|\, E \cap F \cap \bigcap_{j=1}^{k-1}(D_j \cap G_j)\Big). \tag{6.12}$$

Fix $\Upsilon \in \mathcal{I}_k$ and define the event $H_k = [I_k = \Upsilon] \cap E \cap F \cap \bigcap_{j=1}^{k-1}(D_j \cap G_j)$. Define

$$Q(k, \Upsilon_n) = \left[\frac{1}{2}\left(\frac{p_0(1-kp_0)}{(1-(k-1)p_0)^2}\right) \sum_{j \in \Upsilon_n} (\zeta_j')^2\right]^{1/2},$$

so that

$$\mathrm{Var}\Big(\sum_{j \in \Upsilon_n} \delta_{jk} \zeta_j' \,\Big|\, H_k\Big) = 2[Q(k, \Upsilon_n)]^2.$$

For any index set $\Upsilon \in \mathcal{I}_k$, one of three possibilities has to hold:

(1) for all $n = 2, 3, ..., N$

$$\left|\mathbb{E}\left(\sum_{j \in \Upsilon_n} \delta_{jk} \zeta_j' \;\middle|\; H_k \right)\right| = \left| \sum_{j \in \Upsilon_n} \frac{p_0}{1-(k-1)p_0} \zeta_j' \right| \le 2d + Q(k, \Upsilon_n);$$

(2) for some n_0, $2 \le n_0 \le N$

$$\mathbb{E}\left(\sum_{j \in \Upsilon_{n_0}} \delta_{jk} \zeta_j' \;\middle|\; H_k \right) > 2d + Q(k, \Upsilon_{n_0});$$

(3) for some n_0, $2 \le n_0 \le N$

$$-\mathbb{E}\left(\sum_{j \in \Upsilon_{n_0}} \delta_{jk} \zeta_j' \;\middle|\; H_k \right) > 2d + Q(k, \Upsilon_{n_0}).$$

In case (2),

$$\begin{aligned} &P\big(D_k \mid H_k\big) \\ &\quad \le P\left(-2d \le \sum_{j \in \Upsilon_{n_0}} \delta_{jk} \zeta_j' \le 2d \;\middle|\; H_k \right) \\ &\quad \le P\left(\sum_{j \in \Upsilon_{n_0}} \left[\delta_{jk} \zeta_j' - \frac{p_0}{1-(k-1)p_0} \zeta_j' \right] < -Q(k, \Upsilon_{n_0}) \;\middle|\; H_k \right) \end{aligned}$$

By Chebyshev's inequality, the last probability is bounded by

$$\frac{1}{1 + \frac{[Q(k, \Upsilon_{n_0})]^2}{2[Q(k, \Upsilon_{n_0})]^2}} = \frac{2}{3}.$$

In case (3), similarly, $P(D_k \mid H_k) \le 2/3$. Case (1) requires some extra work. Using Kolmogorov's inequality we get

$$\begin{aligned} &P(D_k \mid H_k) \\ &\quad \le P\left(\max_{2 \le m \le N} \left| \sum_{j \in \Upsilon_m} \left(\delta_{jk} \zeta_j' - \frac{p_0}{1-(k-1)p_0} \zeta_j' \right) \right| < 4d + Q(k, \Upsilon_N) \;\middle|\; H_k \right) \\ &\quad \le \frac{\big[6d + Q(k, \Upsilon_N)\big]^2}{2[Q(k, \Upsilon_N)]^2}. \end{aligned}$$

The proof of (6.11) will be concluded by showing

$$\frac{d^2}{2[Q(k, \Upsilon_N)]^2} \le \frac{1}{98},$$

since this implies

$$\frac{\big[6d + Q(k, \Upsilon_N)\big]^2}{2[Q(k, \Upsilon_N)]^2} \le 2/3.$$

Since $\Upsilon \in \mathcal{I}_k$ we have

$$\begin{aligned}\sum_{j\in\Upsilon}(\zeta_j')^2 &\geq \frac{1}{4}\sum_{j\in\Upsilon}\delta_{\left\{|\zeta_j'|\geq 1/2\right\}}\\ &\geq \frac{1}{4}|\Upsilon_{\lfloor\gamma|b_i-a_i|\rfloor}|\\ &\geq \frac{1}{4}\Big((1-(k-1)p_0)\lfloor\gamma|b_i-a_i|\rfloor - g\Big)\\ &\geq \frac{1}{8}\Big(\lfloor\gamma|b_i-a_i|\rfloor - 2g\Big)\\ &\geq \frac{1}{16}\gamma|b_i-a_i|\end{aligned}$$

since $\frac{1}{2} \leq 1 - kp_0 \leq 1$, for sufficiently large l. Therefore,

$$\frac{d^2}{2[Q(k,N)]^2} \leq \frac{16d^2((1-(k-1)p_0)^2)}{p_0(1-kp_0)\gamma|b_i-a_i|} \leq \frac{32d^2}{p_0\gamma|b_i-a_i|} = \frac{32|w_{i+1}-w_i|}{K_{19}\gamma|b_i-a_i|}$$

By (4.10), we can choose $K_{19} = K_{19}(\gamma)$ (from the definition of p_0) sufficiently large so that the last expression is less than 1/98, for large l. Under each case (1)-(3) we have shown $P(D_k \mid H_k) \leq 2/3$, for arbitrary $\Upsilon \in \mathcal{I}_k$. With (6.12) this proves (6.11). Using (6.8)–(6.10) we get

$$\begin{aligned}P\big(a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T_d^i \;\big|\; E\cap F\big) &\leq (2/3)^B + e^{-B}\\ &\leq 2\exp\big(\frac{-\kappa|w_{i+1}-w_i|}{d^2}\big),\end{aligned}$$

for some $\kappa > 0$, which concludes the proof of the lemma. □

Lemma 6.2. *Let ϵ, t_i, a_i, b_i be as in the preceeding, with $i \in \mathcal{L}$. Let δ be as in (3.1), γ as in (4.16) and κ as in (6.1). Provided l is sufficiently large we have*

$$P\left(a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T_d^i \;\Big|\; a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i\right) \leq 3\exp\left(\frac{-\kappa|w_{i+1}-w_i|}{d^2}\right). \quad (6.13)$$

Proof. Let ν' be as in (4.13) and φ as in (4.16). We will consider intersections of the event $\{a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T_d^i\}$ with the events $E = E(a_i, b_i, t_i, \delta, \gamma)$ and E^c, separately. First we have

$$\begin{aligned}&P\left(E^c \mid a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i\right)\\ &\leq P\left(|\mathcal{R}^{t_i}_{a_i,b_i}| < \delta|b_i-a_i| \;\Big|\; a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i\right)\\ &\quad + P\left(\sum_{j=1}^{N}\delta_{\{|\Delta(r_j)|\geq\frac{1}{2}\}} \leq \gamma|b_i-a_i|\,;\ |\mathcal{R}^{t_i}_{a_i,b_i}| > \delta|b_i-a_i| \;\Bigg|\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\right)\\ &\leq \exp(-\nu'|b_i-a_i|) + \exp(-\varphi|b_i-a_i|), \qquad (6.14)\end{aligned}$$

by (4.13) and (4.16). Since $i \in \mathcal{L}$, this bound is small compared to the right side of (6.13). Next, we have

$$P\Big(\{a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T_d^i\} \cap E \;\Big|\; a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i\Big) \le \sum_{v,w \in T_d^i \cap \mathcal{S}^{t_i}_{a_i+3e_i,b_i} \cap (\mathbb{Z}^2)^*} \times \Big[\sum_{\zeta' \in V(v,w)} P\Big(\{a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i \text{ in } T_d^i\} \cap E \cap F(v,w,\zeta') \;\Big|\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\Big)\Big], \tag{6.15}$$

where the first sum is over all possible locations of first and N-th regeneration points, and the second sum is over all possible sets of increments between v and w. If the magnitude of one of these increments is greater than $2d$, this implies at least one regeneration point must be outside the tube T_d^i. Therefore, we can restrict the second sum to $\zeta' \in V(v,w) \cap [-2d, 2d]^N$, and the last sum is bounded by

$$\sum_{v,w \in T_d^i \cap \mathcal{S}^{t_i}_{a_i+3e_i,b_i} \cap (\mathbb{Z}^2)^*} \Big[\sum_{\zeta' \in V \cap [-2d,2d]^N} \times P\Big(\{a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i \text{ in } T_d^i\} \cap E \cap F(v,w,\zeta') \;\Big|\; a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\Big)\Big]. \tag{6.16}$$

For the remainder of the proof, our sums are over $v, w \in T_d^i \cap \mathcal{S}^{t_i}_{a_i+3e_i,b_i} \cap (\mathbb{Z}^2)^*$ and $\zeta' \in V(v,w) \cap [-2d, 2d]^N$. We can write the last expression as

$$\sum_{v,w}\Big[\sum_{\zeta'} P\big(E \cap F(v,w,\zeta') |\, a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\big)\; P\big(a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i \text{ in } T_d^i \;\big|\; E \cap F(v,w,\zeta')\big)\Big]$$
$$\le 2\exp\Big(\frac{-\kappa|w_{i+1}-w_i|}{d^2}\Big)\sum_{v,w}\sum_{\zeta'} P\big(E \cap F(v,w,\zeta') |\, a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i\big),$$

using (6.1). Taking the double sum over the probabilities of disjoint events, in view of (6.15) and (6.16) we get

$$P\Big(\{a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i \text{ in } T_d^i\} \cap E \;\Big|\; a_i \overset{\widetilde{h_{t_i}}}{\longleftrightarrow} b_i\Big) \le 2\exp\Big(\frac{-\kappa|w_{i+1}-w_i|}{d^2}\Big).$$

Combining this with (6.14) completes the proof. □

7. Assembling the segments

In the last section, we proved that on every long facet of the $\partial\mathrm{HSkel}_s(\Gamma_0)$, there is an extra probabilistic cost for staying in the narrow tube. In this section, we will bring the pieces in the preceding sections together to deduce that, there is an extra probabilistic cost of staying in the annular region A_d (with diameter $2d$), throughout the boundary of the $\mathrm{HSkel}_s(\Gamma_0)$. We will show that in light of the inequality (4.3), leaving the annular region A_d implies that $\mathrm{MLR}(\Gamma_0) > \theta l^{1/3}(\log l)^{-2/3}$. Then by bounding the number of possible skeletons, we will prove (2.1).

Lemma 7.1. *There exists $K_{20} = K_{20}(\delta, \gamma)$ such that for sufficiently large l, for all l-regular s-hull skeletons $\{w_0, .., w_{m+1}\}$,*

$$\begin{aligned} &P\big(\{w_0 \leftrightarrow w_1\} \circ \cdots \circ \{w_m \leftrightarrow w_{m+1}\} \text{ in } A_d\big) \\ &\quad \le \exp(-\mathcal{W}_1 l - \frac{K_{20}}{\theta^2} l^{1/3} (\log l)^{4/3}). \end{aligned} \tag{7.1}$$

Proof. Using the BK-inequality, we have

$$P\big(\{w_0 \leftrightarrow w_1\} \circ \cdots \circ \{w_m \leftrightarrow w_{m+1}\} \text{ in } A_d\big) \le \prod_{i=0}^{m} P(w_i \leftrightarrow w_{i+1} \text{ in } A_d). \tag{7.2}$$

This last product can be written as products over long and short sides separately. We will bound the product over long sides further. As before for $i \in \mathcal{L}$, let $a_i \in \mathcal{S}^{t_i}_{w_i', w_i'+e_i} \cap T_d^i$ and $b_i \in \mathcal{S}^{t_i}_{w_i'', w_i''-e_i} \cap T_d^i$; note there are at most $2d$ choices each for a_i and b_i. Using (4.9), (6.13) and l-regularity we have

$$\begin{aligned} &\prod_{i\in\mathcal{L}} P(w_i \leftrightarrow w_{i+1} \text{ in } A_d) \\ &\le \prod_{i\in\mathcal{L}} P(w_i \leftrightarrow w_{i+1}) \Big[K_{14} \exp(-K_{15} l) + 4d^2 \max_{a_i, b_i} P(a_i \leftrightarrow b_i \text{ in } T_d^i \,|\, a_i \overset{\widetilde{h_{t_i}}}{\leftrightarrow} b_i) \Big] \\ &\le \prod_{i\in\mathcal{L}} P(w_i \leftrightarrow w_{i+1}) \left[K_{14} \exp(-K_{15} l) + 12 d^2 \exp\left(\frac{-\kappa |w_{i+1} - w_i|}{d^2} \right) \right] \\ &\le \prod_{i\in\mathcal{L}} P(w_i \leftrightarrow w_{i+1}) \left[13 d^2 \exp\left(\frac{-\kappa |w_{i+1} - w_i|}{d^2} \right) \right]. \end{aligned} \tag{7.3}$$

Now we place a condition on the as-yet-unspecified constant θ; recall that

$$s = \left(\frac{\theta\sqrt{\pi}}{2K_7} \right)^{1/2} l^{2/3} (\log l)^{-1/3}, \qquad d = 2\theta l^{1/3} (\log l)^{-2/3}.$$

For some $\beta = \beta(K_5, K_7, \kappa)$, provided θ is sufficiently small we have

$$13 d^2 \exp\left(\frac{-\kappa |w_{i+1} - w_i|}{2d^2} \right) \le 20\theta^2 l^{2/3} (\log l)^{-4/3} \exp(-\beta\theta^{-3/2} \log l) \le 1,$$

so that

$$13 d^2 \exp\left(\frac{-\kappa |w_{i+1} - w_i|}{d^2} \right) \le \exp\left(\frac{-\kappa |w_{i+1} - w_i|}{2d^2} \right).$$

Therefore, using Lemma 4.2, the right side of (7.3) is bounded by

$$\begin{aligned} &\exp\left(-\frac{\kappa}{2d^2} \cdot \sum_{i\in\mathcal{L}} |w_{i+1} - w_i| \right) \cdot \prod_{i\in\mathcal{L}} P(w_i \leftrightarrow w_{i+1}) \\ &\quad \le \exp\left(-\frac{\kappa\sqrt{\pi}}{8\theta^2\sqrt{2}} l^{1/3} (\log l)^{4/3} \right) \cdot \prod_{i\in\mathcal{L}} P(w_i \leftrightarrow w_{i+1}). \end{aligned}$$

Using (7.3) and (2.6) we then obtain

$$\prod_{i=0}^{m} P(w_i \leftrightarrow w_{i+1} \text{ in } A_d) \le \exp\biggl(-\frac{\kappa\sqrt{\pi}}{8\theta^2\sqrt{2}}\, l^{1/3}(\log l)^{4/3}\biggr)\prod_{i=0}^{m} P(w_i \leftrightarrow w_{i+1})$$
$$\le \exp\biggl(-\frac{\kappa\sqrt{\pi}}{8\theta^2\sqrt{2}}\, l^{1/3}(\log l)^{4/3} - \sum_{i=0}^{m}\tau(w_{i+1}-w_i)\biggr). \tag{7.4}$$

By l-regularity there exists a dual circuit γ_0 with $|\operatorname{Int}(\gamma_0)| \ge l^2$, $\operatorname{diam}(\gamma_0) \le 8\sqrt{2}\, l$ and $\operatorname{HSkel}_s(\gamma_0) = \{w_0, .., w_{m+1}\}$. The first condition implies $\operatorname{diam}(\gamma_0) \ge l$, and by definition of $\mathcal{W}_1$ we have $\mathcal{W}(\partial\operatorname{Co}(\gamma_0) \ge \mathcal{W}_1 l$. Therefore by (4.4), for some K_{21},

$$\sum_{i=0}^{m}\tau(w_{i+1}-w_i) \ge \mathcal{W}_1 l - K_{21}\theta l^{1/3}(\log l)^{-2/3}.$$

The lemma now follows from this together with (7.2) and (7.4). □

Proof of Theorem 2.1. By (4.6),

$$P\Bigl(\ \operatorname{MLR}(\Gamma_0) \le \theta l^{1/3}(\log l)^{-2/3}\ \Big|\ |\operatorname{Int}(\Gamma_0)| \ge l^2\ \Bigr)$$
$$\le P\Bigl(\ \{\operatorname{MLR}(\Gamma_0) \le \theta l^{1/3}(\log l)^{-2/3}\} \cap \{\operatorname{diam}(\Gamma_0) \le 8\sqrt{2}l\}\ \Big|$$
$$|\operatorname{Int}(\Gamma_{\ge l^2}\ \Bigr) + \exp(-\mathcal{W}_1 l/2). \tag{7.5}$$

This means we need only consider l-regular skeletons $\{w_0, \cdots, w_{m+1}\}$ for Γ_0. When $|\operatorname{Int}(\Gamma_0)| \ge l^2$ we have $\operatorname{diam}(\Gamma_0) \ge l$ and therefore

$$\frac{K_7 s^2}{\operatorname{diam}(\Gamma_0)} < d.$$

Presuming $\operatorname{HSkel}_s(\Gamma_0) = \{w_0, \cdots, w_{m+1}\}$, this implies $\partial\operatorname{Co}(\Gamma_0) \subset A_d$. This means that in order to have $\operatorname{MLR}(\Gamma_0) \le \theta l^{1/3}(\log l)^{-2/3}$, Γ_0 must be entirely inside A_d. Thus

$$P\Bigl(\ \{\operatorname{MLR}(\Gamma_0) \le \theta l^{1/3}(\log l)^{-2/3}\} \cap \{\operatorname{diam}(\Gamma_0) \le 8\sqrt{2}l\} \cap \{|\operatorname{Int}(\Gamma_0)| \ge l^2\}\Bigr)$$
$$\le \sum_{\{w_0,\cdots,w_{m+1}\}} P\Biggl(\ \Bigl\{\operatorname{HSkel}_s(\Gamma_0) = \{w_0, \cdots, w_m\}\Bigr\}$$
$$\cap\Biggl\{\ \Bigl\{w_0 \leftrightarrow w_1\Bigr\} \circ \cdots \circ \Bigl\{w_m \leftrightarrow w_{m+1}\Bigr\} \text{ in } A_d(w_0, .., w_{m+1}\ \Bigr\}\Biggr\}\Biggr), \tag{7.6}$$

where the sum is over all l-regular skeletons. By (4.1) the number of l-regular skeletons is at most

$$(K_{22}l^2)^{K_{23}\theta^{-1/2}l^{1/3}(\log l)^{1/3}} \le \exp(K_{24}\theta^{-1/2}l^{1/3}(\log l)^{4/3}),$$

for some K_{22}, K_{23}, K_{24}. This together with (7.1) and (7.6) gives

$$P\Big(\{\mathrm{MLR}(\Gamma_0) \le \theta l^{1/3}(\log l)^{-2/3}\} \cap \{\mathrm{diam}(\Gamma_0) \le 8\sqrt{2}l\} \cap \{|\operatorname{Int}(\Gamma_0)| \ge l^2\}\Big)$$
$$\le \exp\left(-\mathcal{W}_1 l - \left(\frac{K_{20}}{\theta^2} - \frac{K_{24}}{\theta^{1/2}}\right) l^{1/3}(\log l)^{4/3}\right)$$

which with (4.5) and (7.5) yields

$$P\Big(\ \mathrm{MLR}(\Gamma_0) \le \theta l^{1/3}(\log l)^{-2/3} \ \Big|\ |\operatorname{Int}(\Gamma_0)| \ge l^2\ \Big)$$
$$\le \exp\left(-\left(\frac{K_{20}}{\theta^2} - \frac{K_{24}}{\theta^{1/2}}\right) l^{1/3}(\log l)^{4/3} + K_{11} l^{1/3}(\log l)^{2/3}\right)$$
$$+ \ \exp\left(-\frac{\mathcal{W}_1 l}{2}\right).$$

For $\theta > 0$, sufficiently small, the last bound tends to 0, as $l \to \infty$. □

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