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Krzysztof Bogdan · Krzysztof Burdzy · Zhen-Qing Chen

Censored stable processes

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Abstract. We present several constructions of a "censored stable process" in an open set $D \subset \mathbf{R}^n$, i.e., a symmetric stable process which is not allowed to jump outside D.

We address the question of whether the process will approach the boundary of D in a finite time – we give sharp conditions for such approach in terms of the stability index α and the "thickness" of the boundary. As a corollary, new results are obtained concerning Besov spaces on non-smooth domains, including the critical exponent case.

We also study the decay rate of the corresponding harmonic functions which vanish on a part of the boundary. We derive a boundary Harnack principle in $C^{1,1}$ open sets.

Contents

1. Introduction

We will introduce "censored" stable processes and present some of their basic properties. A censored stable process in an open set $D \subset \mathbf{R}^n$ is obtained from the symmetric stable process by suppressing its jumps from D to the complement of D, i.e., by restricting its Lévy measure to D. In other words, a censored stable process Y in a domain D is a stable process "forced" to stay inside D. We have not used the word "conditioned" because this usually indicates conditioning in the sense of Doob's h-transform; it will be shown in this paper that in fact these two processes are different. To study censored stable processes, we introduce yet another process

K. Bogdan: Institute of Mathematics, Wrocław University of Technology, 50-370 Wrocław, Poland. e-mail: bogdan@im.pwr.wroc.pl

K. Burdzy, Z.-Q. Chen: Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA. e-mail: {burdzy,zchen}@math.washington.edu

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Y^* on $\overline{D}$, which we call reflected stable process on D. In a sense, Y^* is the maximal extension of Y. The relation between Y^* and Y is similar to that between the reflected Brownian motion in D and the killed Brownian motion in D. We will show that the censored stable process may be obtained by piecing together of killed stable processes but it can be also represented as the reflected stable process killed upon hitting of the boundary of D.

We believe that censored and reflected stable processes deserve to be studied because their classical counterparts, killed and reflected Brownian motions, are important models in both pure mathematics (RBM is related to the Laplacian with Neumann boundary conditions) and in applied probability (RBM arises naturally in queueing theory). This paper is mainly focussed on censored stable processes; see [17] for some results on reflected α-stable processes, including heat kernel estimates and a parabolic Harnack inequality.

Brownian motion is at the extreme end of the class of symmetric α-stable processes, corresponding to $\alpha = 2$. The potential theory for the Brownian motion and that for the (discontinuous) symmetric stable processes share many similarities but also exhibit numerous important differences. One of our goals is to compare potential theories for censored stable processes and killed Brownian motion (see question (Q2) below). We will not limit ourselves to comparisons, though, and we will address some problems which are specific to censored processes (see question (Q1)).

We will first show that the censored stable process can be constructed in three different but equivalent ways and each construction is useful under different circumstances. We will also take the first few steps in the analysis of the new model. The paper contains a large number of various estimates and other results on censored processes but the two main questions we address in the paper are

(Q1) Does the censored process approach the boundary of the set to which it is constrained in a finite time?
(Q2) Do harmonic functions corresponding to the censored process satisfy a boundary Harnack principle analogous to the boundary Harnack principle for the classical harmonic functions, i.e., those corresponding to Brownian motion?

We will now give a semi-formal presentation of our main results—see the main body of the paper for the fully rigorous version.

An almost complete answer to question (Q1) is given in Corollary 2.6 and Theorems 2.7 and 2.9 in Section 2. The following theorem is a special case of those much more general results.

Theorem 1.1. *Suppose that $D \in \mathbf{R}^n$ is a bounded Lipschitz open set, i.e., D lies above the graph of a Lipschitz function in a neighborhood of every boundary point.*

(1) If $\alpha \le 1$ then the censored symmetric α-stable process Y in D is conservative and will never approach ∂D;
(2) If $\alpha > 1$ then the process Y has a finite lifetime ζ and will approach ∂D at ζ; that is, $\lim_{t\uparrow\zeta} Y_t$ exists and takes a value in ∂D.

Our main results in Section 2 yield new information on Besov spaces. Corollary 2.6 gives necessary and sufficient conditions for an open n-set $D \subset \mathbf{R}^n$ so that the

Besov or Sobolev spaces of fractional orders $W^{s,2}(D)$ and $W_0^{s,2}(D)$ are the same, where $0 < s < 1$. The explicit results in terms of Hausdorff dimension of ∂D given in Corollary 2.8 not only recover but also extend some results established recently in Caetano [12] and in Farkas and Jacob [27]. In particular, our result covers the critical case which is left open in both [12] and [27]. Our approach is quite different from those in [12] and [27] and, we believe, is more elementary. We would like to point out that [12] and [27] contain a number of other interesting results besides the ones we referred to.

The proofs of Corollary 2.6, Theorems 2.7 and 2.9 are based on some powerful results from the theory of Dirichlet spaces and Sobolev (or Besov) spaces of fractional order, so they might not be accessible to some readers. Sections 3–7 provide a gentler approach to the process Y. The alternative technique developed in these sections is applied to a certain class of relatively smooth domains being of independent interest in the boundary theory of stable processes.

In Section 3 we derive basic properties of harmonic functions of Y such as the Harnack inequality. To this end we use a characterization of Y as a Feynman-Kac transform of the symmetric stable process X killed off D.

In Section 4 we consider the problem of hitting the boundary when D is a half-line. The methods used in this section are very different and less technical than the multidimensional methods of Sections 5 and 6 and so this section may serve as an elementary introduction to the problem of estimating the hitting probabilities for the process Y.

In Sections 5 and 6 we study in detail powers of the distance function $x \to \mathrm{dist}(x, D^c)$. We show that some of these functions are super- or subharmonic for the process Y at the boundary of $C^{1,\beta-1}$ domains (see Section 5 for definitions). Here $\beta \in (1, 2]$. We focus on the case $\beta = 2$ in Section 6.

An open set D in $\mathbf{R}^n$ is said to be $C^{1,1}$ if there is a localization radius $r_0 > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there is a $C^{1,1}$-function $\phi = \phi_Q : \mathbf{R}^{n-1} \to \mathbf{R}$ satisfying $\phi(0) = 0$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \le \Lambda|x - z|$, and an orthonormal coordinate system $y = (y_1, \cdots, y_{n-1}, y_n) = (\tilde{y}, y_n)$ such that $B(Q, r_0) \cap D = B(Q, r_0) \cap \{y : y_n > \phi(\tilde{y})\}$. The pair (r_0, Λ) is called the characteristics of the $C^{1,1}$-open set D. In dimension $n = 1$, a $C^{1,1}$ open set is the union of (at most countably many) disjoint open intervals I_j with lengths $|I_j|$ and distances d_{ij} between any two distinct intervals I_i and I_j bounded below by a positive constant. For definiteness, we may put $r_0 = \min(\inf |I_j|, \inf d_{ij})/2$ and $\Lambda = 0$ in this case. The main result concerning $C^{1,1}$ open sets is the following boundary Harnack principle, which gives a sharp estimate for the rate of decay at the boundary of nonnegative harmonic functions of the censored symmetric α-stable process Y when $\alpha \in (1, 2)$. This is our partial answer to question (Q2) stated above.

Theorem 1.2. *Let D be a $C^{1,1}$ open set in $\mathbf{R}^n$ with characteristics $r_0 \le 1$ and Λ, and let $\rho(x) = \mathrm{dist}(x, D^c)$. Let Y be the censored stable process in D with index of stability $\alpha \in (1, 2)$. Let $Q \in \partial D$ and $r \in (0, r_0)$.*

Assume that $u \ge 0$ is a function on D which is not identically equal to 0, vanishes continuously on $\partial D \cap B(Q, r)$ and is harmonic on $D \cap B(Q, r)$ for Y. Then *there is a constant* $C = C(n, \alpha, \Lambda) > 1$ *such that*

$$\frac{u(x)}{u(y)} \le C \, \frac{\rho(x)^{\alpha-1}}{\rho(y)^{\alpha-1}} \qquad \textit{for } x, y \in D \cap B(Q, r/2). \tag{1.1}$$

The notation $C = C(n, \alpha, \Lambda)$ means that the positive real constant C depends only on n, α, Λ. This convention will be in force throughout the paper.

Theorem 1.2 played a key role in Chen and Kim [16] in obtaining sharp two-sided Green function estimates for censored α-stable processes in bounded $C^{1,1}$-open sets when $\alpha > 1$. It can also be used to answer question (Q1) when D is a $C^{1,1}$ open set and $\alpha \in (1, 2)$—this is shown in Subsection 4.1 in the special case when D is a half-line.

The present paper is a contribution to the boundary potential theory for discontinuous non-Lévy Markov processes on open subsets $D \subset \mathbf{R}^n$. For some recent results on the boundary potential theory for discontinuous Lévy processes on open subsets $D \subset \mathbf{R}^n$, see [7]–[13], [18]–[22], [33], [40] and [45]. We also refer the reader to [4], [42] and to the references therein for an account of the potential theory of Lévy processes on the whole of $\mathbf{R}^n$.

In this paper, we use ":=" to indicate a definition. For functions f and g, the notation "$f \approx g$" means that there exist constants $c_2 > c_1 > 0$ such that $c_1 g \le f \le c_2 g$. For two real numbers a and b, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

2. Boundary behavior in non-smooth open sets

In the first part of this section we will define a censored stable process. We will start by reviewing some standard definitions and results for the classical symmetric stable processes.

Let $X = \{X_t\}$ denote the symmetric α-stable process in $\mathbf{R}^n$ with $\alpha \in (0, 2)$ and $n \ge 1$, that is, let X_t be a Lévy process whose transition density $p(t, y - x)$ relative to the Lebesgue measure is given by the following Fourier transform,

$$\int_{\mathbf{R}^n} e^{ix\cdot\xi} p(t, x) dx = e^{-t|\xi|^\alpha}.$$

It follows that X has a scaling property. Namely, if $\{X_t, t \ge 0\}$ has the distribution $\mathbf{P}_x$ then the distribution of $\{cX_{t/c^\alpha}, t \ge 0\}$ is $\mathbf{P}_{cx}$.

It is well known (cf. (I.2.20) of [5] and Example 1.4.1 of [29]) that the Dirichlet form $(\mathcal{C}, \mathcal{F}^{\mathbf{R}^n})$ associated with X is given by

$$\mathcal{C}(u, v) = \frac{1}{2}\mathcal{A}(n, -\alpha) \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}} \, dx dy, \tag{2.1}$$

$$\mathcal{F}^{\mathbf{R}^n} = \left\{ u \in L^2(\mathbf{R}^n) : \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} \, dx dy < \infty \right\}, \tag{2.2}$$

where

$$\mathcal{A}(n, -\alpha) = \frac{|\alpha| \, 2^{\alpha-1} \Gamma(\frac{\alpha+n}{2})}{\pi^{n/2} \Gamma(1 - \frac{\alpha}{2})}.$$

It is also well known that space $C_c^\infty(\mathbf{R}^n)$ of smooth functions in $\mathbf{R}^n$ with compact support is dense in $\mathcal{F}^{\mathbf{R}^n}$ with respect to the inner product $\mathcal{C}_1 := \mathcal{C} + (\cdot,\cdot)_{L^2(\mathbf{R}^n,dx)}$. Every function u in $\mathcal{F}^{\mathbf{R}^n}$ has a quasi-continuous version and it is this version that will be used hereafter for $u \in \mathcal{F}^{\mathbf{R}^n}$.

Given an open set $D \subset \mathbf{R}^n$, define $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Let $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and set $X_t^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where ∂ is a coffin state added to $\mathbf{R}^n$. The process X^D, i.e., the process X killed upon leaving open set D is called the symmetric α-stable process in D. Note that X^D is irreducible even when D is disconnected. The Dirichlet form of X^D on $L^2(D, dx)$ is $(\mathcal{C}, \mathcal{F}^D)$, where

$$\mathcal{F}^D = \{f \in \mathcal{F}^{\mathbf{R}^n} : f = 0 \text{ q.e. on } D^c\}.$$

Here q.e. is the abbreviation for quasi-everywhere (cf. [29]). For $u, v \in \mathcal{F}^D$, by (2.1),

$$\begin{aligned}\mathcal{C}(u,v) = {} & \frac{1}{2}\mathcal{A}(n,-\alpha)\int_D\int_D \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy \\ & + \int_D u(x)v(x)\kappa_D(x)dx,\end{aligned}$$

where

$$\kappa_D(x) = \mathcal{A}(n,-\alpha)\int_{D^c} \frac{1}{|x-y|^{n+\alpha}}dy \tag{2.3}$$

is the density of the killing measure of X^D. We will use $C_c(D)$ ($C_c^\infty(D)$) to denote the space of continuous (smooth) functions in D with compact support. It is well known that $\mathcal{F}^D$ is the $\mathcal{C}_1$-closure of $C_c^\infty(D)$, where $\mathcal{C}_1 = \mathcal{C} + (\cdot,\cdot)_{L^2(D)}$.

Note that $\lim_{t\uparrow\tau_D} X_t$ exists and typically belongs to D. We would like to extend X^D beyond its lifetime τ_D. To this end, define a bilinear form $\mathcal{E}$ on $C_c^\infty(D)$:

$$\mathcal{E}(u,v) = \frac{1}{2}\mathcal{A}(n,-\alpha)\int_D\int_D \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy, \quad u,v \in C_c^\infty(D). \tag{2.4}$$

By Fatou's lemma, $(C_c^\infty(D), \mathcal{E})$ is closable in $L^2(D, dx)$; that is, whenever $\{u_k\}_{k\geq 1} \subset C_c^\infty(D)$ is an $\mathcal{E}$-Cauchy sequence such that $u_k \to 0$ in $L^2(D, dx)$, then $\mathcal{E}(u_k, u_k) \to 0$. Let

$\mathcal{F}$ be the closure of $C_c^\infty(D)$ under the Hilbert inner product $\mathcal{E}_1 = \mathcal{E} + (\cdot,\cdot)_{L^2(D)}$.

It is well known that for every $\varepsilon > 0$, there is a $\phi_\varepsilon \in C_c^\infty(\mathbf{R})$ taking values in $(-\varepsilon, 1+\varepsilon)$ such that $\phi_\varepsilon(t) = t$ on $[0,1]$, and $0 \leq \phi_\varepsilon(t) - \phi_\varepsilon(s) \leq t - s$ whenever $t > s$. Clearly for each $u \in C_c^\infty(D)$, $\phi_\varepsilon \circ u \in C_c^\infty(D)$ and $\mathcal{E}(\phi_\varepsilon \circ u, \phi_\varepsilon \circ u) \leq \mathcal{E}(u,u)$. Thus $(\mathcal{F}, \mathcal{E})$ is Markovian and hence a regular Dirichlet form on $L^2(D, dx)$ (cf. Theorem 3.1.1 of [29]). Therefore there is an associated symmetric Hunt process $(Y, \mathbf{P}_x)$ taking values in D and with lifetime ζ (cf. Theorem 7.2.1 of [29]). As $(\mathcal{F}, \mathcal{E})$ has no killing measure,

$$\mathbf{P}_x(Y_{\zeta-} \in D) = 0 \qquad \text{for } \mathcal{E}\text{-q.e. } x \in D. \tag{2.5}$$

Here $\mathcal{E}$-q.e. is the abbreviation for quasi-everywhere with respect to the Dirichlet form $(\mathcal{F}, \mathcal{E})$. See Section 2 for details.

We will now show that the above construction of Y is equivalent to an alternative construction called the "Ikeda-Nagasawa-Watanabe piecing together procedure." Note that $\kappa_D(x)dx$ is a Radon measure on D. It is the Revuz measure for the following positive continuous additive functional of Y,

$$A_t = \int_0^t \kappa_D(Y_s)ds, \quad t \geq 0.$$

Here we used the convention that $\kappa_D(\partial) = 0$. The decreasing multiplicative functional e^{-A_t} uniquely determines a probability measure $\widehat{P}_x$ on Ω for $\mathcal{E}$-q.e. $x \in D$, which satisfies the following condition for any bounded Borel measurable function f on D,

$$\widehat{\mathbf{E}}_x[f(Y_t)] = \mathbf{E}_x[e^{-A_t} f(Y_t)],$$

and which makes Y_t a right Markov process (we used the fact that $Y_t = \partial$ for $t \geq \zeta$ and the convention that $f(\partial) = 0$). Let (Y^κ, ζ^κ) denote the process with distributions $\widehat{P}_x$. By Theorems 6.1.1 and 6.1.2 of [29], Y^κ is a symmetric strong Markov process with associated Dirichlet form $(\mathcal{F}^\kappa, \mathcal{E}^\kappa)$ that is regular on $L^2(D, dx)$. Here $\mathcal{F}^\kappa = \mathcal{F} \cap L^2(D, \kappa_D(x)dx)$ and

$$\mathcal{E}^\kappa(u, v) = \mathcal{E}(u, v) + \int_D u(x)v(x)\kappa_D(x)dx, \quad u, v \in \mathcal{F}^\kappa.$$

Thus $\mathcal{E}^\kappa = \mathcal{C}$ on $\mathcal{F}^\kappa \cap \mathcal{F}^D$. Note that $C_c(D) \cap \mathcal{F} = C_c(D) \cap \mathcal{F}^D = C_c(D) \cap \mathcal{F}^\kappa$. Since $(\mathcal{F}^\kappa, \mathcal{E}^\kappa)$ is regular on $L^2(D, dx)$, we conclude that $\mathcal{F}^\kappa$ is the $\mathcal{C}_1$-completion of $C_c(D) \cap \mathcal{F}^D$. Therefore $(\mathcal{F}^\kappa, \mathcal{E}^\kappa) = (\mathcal{F}^D, \mathcal{C})$ and so the process Y^κ has the same distribution as the symmetric stable process X^D in D. We can construct Y and Y^κ on the same probability space with Y^κ being the process Y killed at a random time $\zeta^\kappa \leq \zeta$ (see section III.3 of [5]) that has the property

$$\mathbf{P}_x(\zeta^\kappa > 0) = 1 \quad \text{for all } x \in D \quad \text{and} \quad t + \zeta^\kappa \circ \theta_t = \zeta^\kappa \quad \text{for all } t < \zeta^\kappa, \tag{2.6}$$

where θ_t is the time shift operator. Define $\tau_1 = \zeta^\kappa$ and $\tau_{j+1} = \tau_j + \zeta^\kappa \circ \tau_j$ for $j \geq 1$, with the convention that if $\tau_j = \zeta$ then $\tau_m = \zeta$ for all $m > j$. Clearly $\eta := \lim_{j\to\infty} \tau_j$ has property (2.6) with ζ^κ being replaced by η and, moreover, we see that a.s. $\eta \leq \zeta$. We claim that $\eta = \zeta$ a.s. To see this, we define a subprocess Z of Y by $Z_t(\omega) = Y_t(\omega)$ for $t < \eta(\omega)$ and $Z_t(\omega) = \partial$ if $t \geq \eta(\omega)$. By Corollary III.3.16 of [5], Z is a Hunt process, and so by its quasi-left continuity,

$$\begin{aligned}\mathbf{P}(\eta < \zeta) &= \mathbf{P}\left(Z_{\eta-} \in D\right) = \mathbf{P}\left(\lim_{j\to\infty} Z_{\tau_j} \in D,\ \tau_j < \tau_{j+1} \text{ for all } j \geq 1\right)\\ &\leq \mathbf{P}\left(Z_\eta \in D\right) = 0.\end{aligned}$$

This proves that $\eta = \zeta$ a.s. It follows that the process Y can also be obtained by extending X^D beyond its lifetime τ_D by the Ikeda-Nagasawa-Watanabe piecing together procedure described as follows. Let $Y_t(\omega) = X_t^D(\omega)$ for $t < \tau_D(\omega)$. If

$X^D_{\tau_D-}(\omega) \notin D$, set $Y_t(\omega) = \partial$ for $t \geq \tau_D(\omega)$. If $X^D_{\tau_D-}(\omega) \in D$, let $Y_{\tau_D}(\omega) = X^D_{\tau_D-}(\omega)$ and glue an independent copy of X^D starting from $X^D_{\tau_D-}(\omega)$ to $Y_{\tau_D}(\omega)$. Iterating this procedure countably many times, we obtain a process on D which is a version of the strong Markov process Y; the procedure works for every starting point in D. Thus constructed process Y may be called a "resurrected" process, see, e.g., [29]. Hence, our "censored" stable process Y is an example of a "resurrected" Markov process.

Our final construction of Y is based on the Feynman-Kac transform. Note that for any bounded function f on D, by **62** of Sharpe [43],

$$\mathbf{E}_x[e^{\int_0^t \kappa_D(Y^\kappa_s)ds} f(Y^\kappa_t)] = \mathbf{E}_x\left[e^{-A_t}e^{A_t} f(Y_t)\right] = \mathbf{E}_x[f(Y_t)]. \tag{2.7}$$

Hence Y can also be obtained from X^D by "creation" at the rate κ_D through the Feynman-Kac transform $e^{\int_0^t \kappa_D(X^D_s)ds}$. We summarize the three constructions of Y presented above as a theorem.

Theorem 2.1. *The following processes have the same distribution.*

(1) The symmetric Hunt process Y associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D, dx)$;
(2) The strong Markov process Y obtained from the symmetric α-stable process X^D in D through the Ikeda–Nagasawa–Watanabe piecing together procedure;
(3) The process Y obtained from X^D through the Feynman-Kac transform $e^{\int_0^t \kappa_D(X^D_s)ds}$.

We will now investigate the problem of whether $\lim_{t\uparrow\zeta} Y_t \in \partial D$ on $\{\zeta < \infty\}$. In other words, we will seek an answer to question (Q1) posed in the Introduction.

Let $(\mathcal{E}^{\text{ref}}, \mathcal{F}^{\text{ref}}_a)$ be the Dirichlet space on $L^2(D, dx)$ defined by

$$\mathcal{F}^{\text{ref}}_a = \left\{u \in L^2(D) : \int_D\int_D \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dxdy < \infty\right\},$$
$$\mathcal{E}^{\text{ref}}(u, v) = \frac{1}{2}\mathcal{A}(n, -\alpha)\int_D\int_D \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dxdy, \quad u, v \in \mathcal{F}^{\text{ref}}_a.$$

We will show below that $(\mathcal{E}^{\text{ref}}, \mathcal{F}^{\text{ref}}_a)$ is the active reflected Dirichlet form for $(\mathcal{E}, \mathcal{F})$, introduced by Silverstein in [44]. To this end, we first relate spaces $\mathcal{F}$ and $\mathcal{F}^{\text{ref}}_a$ to Sobolev (or Besov) spaces of fractional order $W_0^{\alpha/2,2}(D)$ and $W^{\alpha/2,2}(D)$. To simplify notation, let $s = \alpha/2$. Recall that $W^{s,2}(\mathbf{R}^n) = \mathcal{F}^{\mathbf{R}^n}$ with the Sobolev norm $\|u\|_{s,2} = \sqrt{\mathcal{C}_1(u,u)}$, and

$$W^{s,2}(D) = \left\{u \in L^2(D, dx) : \ u = v \text{ a.e. on } D \text{ for some } v \in W^{s,2}(\mathbf{R}^n)\right\},$$
$$\|u\|_{s,2;D} = \inf\left\{\|v\|_{s,2} : \ v \in W^{s,2}(\mathbf{R}^n) \text{ with } v = u \text{ a.e. on } D\right\}.$$

It is known that $(W^{s,2}(D), \ \|\cdot\|_{s,2;D})$ is a Hilbert space. Let $(W_0^{s,2}(D), \ \|\cdot\|_{s,2;D})$ be the smallest closed subspace of $W^{s,2}(D)$ containing $C_c^\infty(D)$.

We will state our main results (Corollary 2.6, Theorems 2.7 and 2.9 below) in the greatest possible generality given our technical tools. In order to do so, we will recall the definition of a d-set. Readers who are not interested in very rough open sets may want to limit their attention to Lipschitz domains, each of which is an example of a d-set.

For $0 < d \le n$, we will use $\mathcal{H}^d$ to denote the d-dimensional Hausdorff measure in $\mathbf{R}^n$.

Definition 2.1. *A Borel set $\Gamma \subset \mathbf{R}^n$ is called a d-set for some $0 < d \le n$ if there exist positive constants c_1 and c_2 such that for all $x \in \Gamma$ and $r \in (0,\ 1]$,*

$$c_1 r^d \le \mathcal{H}^d(\Gamma \cap B(x,r)) \le c_2 r^d.$$

The notion of a d-set arises both in the theory of function spaces and in fractal geometry. It is well known that (see Proposition 1 in Chapter VIII of [38]) that if Γ is a d-set, then its Euclidean closure $\overline{\Gamma}$ is also a d-set and $\overline{\Gamma} \setminus \Gamma$ has zero $\mathcal{H}^d$-measure. If an open set D is an n-set, then by Theorem 1 on page 103 of [38],

$$W^{s,2}(D) = \mathcal{F}_a^{\mathrm{ref}} \quad \text{and the Sobolev norm } \|\cdot\|_{s,2;D} \text{ is equivalent to } \sqrt{\mathcal{E}_1^{\mathrm{ref}}}. \tag{2.8}$$

Consequently,

$$W_0^{s,2}(D) = \mathcal{F}. \tag{2.9}$$

Theorem 2.2. *For any open set $D \subset \mathbf{R}^n$, the Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$ on $L^2(D,dx)$ defined above is the active reflected Dirichlet form of $(\mathcal{E},\mathcal{F})$ in the sense of [14] and [44]; that is,*

$$\mathcal{F}_a^{\mathrm{ref}} = \left\{ u \in L^2(D) : u_k = ((-k)\vee u)\wedge k \in \mathcal{F}_{\mathrm{loc}} \text{ and } \sup_{k\ge 1} \mathcal{E}^{\mathrm{ref}}(u_k,u_k) < \infty \right\}, \tag{2.10}$$

$$\mathcal{E}^{\mathrm{ref}}(u,u) = \lim_{k\to\infty} \mathcal{E}^{\mathrm{ref}}(u_k,u_k). \tag{2.11}$$

Here $f \in \mathcal{F}_{\mathrm{loc}}$ means that for any relatively compact open subset D_0 of D, there is some $f_0 \in \mathcal{F}$ such that $f = f_0$ a.e. on D_0.

Proof. For (2.10), it suffices to show that $\mathcal{F}_a^{\mathrm{ref}} \subset \mathcal{F}_{\mathrm{loc}}$. Without loss of generality, assume that $u \in \mathcal{F}_a^{\mathrm{ref}}$ is bounded. For any relatively compact open subset D_0 of D, there is a $\phi \in C_c^\infty(D)$ and a relatively compact smooth open subset U_0 of D such that $\overline{D}_0 \subset U_0$, $\phi = 1$ on D_0 and $\mathrm{supp}[\phi] \subset U_0$. As U_0 is smooth, by applying (2.8) to $u|_{U_0}$ with U_0 in place of D, we see that there is some $v \in W^{s,2}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ such that $v = u$ a.e. on U_0. Since $C_c^\infty(\mathbf{R}^n)$ is $\|\cdot\|_{s,2}$-dense in $W^{s,2}(\mathbf{R}^n)$, there is a sequence $\{v_k\}_{k\ge 1} \subset C_c^\infty(\mathbf{R}^n)$ that is $\|\cdot\|_{s,2}$-convergent to v with $\sup_{k\ge 1} \|v_k\|_\infty \le 1 + \|v\|_\infty$. This implies that

$$\sup_{k\ge 1} \mathcal{E}_1(\phi v_k, \phi v_k) < \infty.$$

Hence by the Banach-Saks theorem there is a subsequence $\{k_m\}$ such that the Cesàro means $\left\{\frac{1}{j}\sum_{m=1}^{j} \phi v_{k_m}\right\}_{j\ge 1}$ of $\{\phi v_{k_m}\}_{m\ge 1} \subset C_c^\infty(D)$ are $\mathcal{E}_1$-convergent to some

$f \in \mathcal{F}$. Clearly $f = \phi v = u$ a.e. on D_0. This proves $u \in \mathcal{F}_{\rm loc}$ and therefore (2.10). Property (2.10) follows from the Lebesgue dominated convergence theorem. □

Lemma 2.3. *Let $\mathcal{F}_b := \mathcal{F} \cap L^\infty(D, dx)$ and $\mathcal{F}^{\rm ref}_{a,b} := \mathcal{F}^{\rm ref}_a \cap L^\infty(D, dx)$. Then $\mathcal{F}_b$ is an ideal of $\mathcal{F}^{\rm ref}_{a,b}$; that is, $uv \in \mathcal{F}_b$ whenever $u \in \mathcal{F}_b$ and $v \in \mathcal{F}^{\rm ref}_{a,b}$.*

Proof. Let $u \in \mathcal{F}_b$ and $v \in \mathcal{F}^{\rm ref}_{a,b}$. Then one can find a sequence $\{u_k\}_{k\ge 1} \subset C_c^\infty(D) \cap \mathcal{F}$ such that u_k's converge to u in $\mathcal{E}_1$-norm and $\sup_{k\ge 1} \|u_k\|_\infty \le 1 + \|u\|_\infty$. It follows from Theorem 2.2 that $u_k v \subset \mathcal{F}$ with $\mathcal{E}_1$-norm bounded uniformly in k. Hence there is a subsequence $\{k_m\}$ such that the Cesàro means of $\{u_{k_m} v\} \subset \mathcal{F}$ are $\mathcal{E}_1$-convergent to some $f \in \mathcal{F}$. Clearly $f = uv$ a.e. on D. □

Let us present a probabilistic interpretation for the actively reflected Dirichlet space $(\mathcal{E}^{\rm ref}, \mathcal{F}^{\rm ref}_a)$ (see also Remark 2.1 below). By Theorem 20.1 of Silverstein [44], Theorem 2.2 and Lemma 2.3 above, there is a compactification D^* of D and a symmetric Hunt process Y^* on D^* with associated Dirichlet form $(\mathcal{F}^{\rm ref}_a, \mathcal{E}^{\rm ref})$ such that the process Y^* killed upon leaving D has the same distribution as Y. If D is an open subset of $\mathbf{R}^n$ having finite Lebesgue measure, then Y^* is recurrent as $1 \in \mathcal{F}^{\rm ref}_a$ and $\mathcal{E}^{\rm ref}(1, 1) = 0$. If $1 \in \mathcal{F}$, then $\mathcal{E}(1, 1) = 0$. Thus by Theorem 1.6.3 of [29], Y is recurrent and therefore it is conservative. It follows then that $Y = Y^*$ and $(\mathcal{F}, \mathcal{E}) = (\mathcal{F}^{\rm ref}_a, \mathcal{E}^{\rm ref})$. If $1 \notin \mathcal{F}$ and D has finite Lebesgue measure, then $\mathcal{F}$ is strictly contained in $\mathcal{F}^{\rm ref}_a$ as $1 \in \mathcal{F}^{\rm ref}_a$. In such a case $D^* \setminus D$ is non-polar and so it will be hit by Y^* infinitely many times with probability 1 (cf. Theorem 4.6.6 of [29]). This implies that Y is transient having finite lifetime ζ. We summarize these remarks as a theorem.

Theorem 2.4. *Suppose that D is an open set in $\mathbf{R}^n$ with finite Lebesgue measure. Then the following conditions are equivalent*

(1) $\mathbf{P}_x(\zeta < \infty) > 0$ *for some (and hence for all) $x \in D$;*
(2) $\mathbf{P}_x(\zeta < \infty) = 1$ *for some (and hence for all) $x \in D$;*
(3) $1 \notin \mathcal{F}$;
(4) $\mathcal{F} \neq \mathcal{F}^{\rm ref}_a$.

Remark 2.1. (1) When $D \subset \mathbf{R}^n$ is an open n-set and $0 < s < 1$, by Theorem 1 of Chapter V in [38], $W^{s,2}(D)$ is the restriction (or trace) of $W^{s,2}(\mathbf{R}^n)$ on D. More precisely, there is a restriction operator $R : W^{s,2}(\mathbf{R}^n) \to W^{s,2}(D)$ such that there is a constant $c_1 > 0$ so that

$$Rf = f \text{ a.e. on } D \quad \text{and} \quad \|Rf\|_{s,2;D} \le c_1 \|f\|_{s,2} \quad \text{for any } f \in W^{s,2}(\mathbf{R}^n), \tag{2.12}$$

and there is an extension operator $S : W^{s,2}(D) \to W^{s,2}(\mathbf{R}^n)$ such that there is a constant $c_2 > 0$ so that

$$Su = u \text{ a.e. on } D, \quad Su \in W^{s,2}(\mathbf{R}^n) \quad \text{and} \quad \|Su\|_{s,2} \le c_2 \|u\|_{s,2;D} \quad \text{for any } u \in W^{s,2}(D). \tag{2.13}$$

As $C_c^\infty(\mathbf{R}^n)$, the space of smooth functions in $\mathbf{R}^n$ with compact support, is $\|\cdot\|_{s,2}$-dense in $W^{s,2}(\mathbf{R}^n)$, $C_c(\overline{D}) \cap \mathcal{F}^{\rm ref}_a$ is dense both in $(\mathcal{F}^{\rm ref}_a, \mathcal{E}^{\rm ref}_1)$ and in $(C_c(\overline{D}), \|\cdot\|_\infty)$. Hence $(\mathcal{E}^{\text{ref}}, \mathcal{F}_a^{\text{ref}})$ is a regular Dirichlet form on $\overline{D}$ and its associated Hunt process Y^* lives on $\overline{D}$. A similar construction transforms the usual Brownian motion into a "reflected" Brownian motion so it is natural to call process Y^* a *reflected* α-stable process in D. One can take $D^* = \overline{D}$ and then Y can be identified with the process Y^* killed upon leaving D. As X^D is irreducible, we have by Theorem 2.1 that Y is irreducible and so is Y^*. When $W_0^{s,2}(D) \neq W^{s,2}(D)$, Y is a proper subprocess of Y^* and ∂D is not a polar set for Y^*. This implies that almost surely on $\{\zeta < \zeta^{Y^*}\}$, $Y_{\zeta-}$ exists in the Euclidean topology, where ζ^{Y^*} is the lifetime of Y^*. In particular, if $W_0^{s,2}(D) \neq W^{s,2}(D)$, we have

$$\mathbf{P}_x(Y_{\zeta-} \in \partial D,\ \zeta < \infty) > 0 \qquad \text{for all } x \in D. \tag{2.14}$$

If D has finite Lebesgue measure, then Y^* is recurrent. In this case, if $1 \notin W^{s,2}(D)$, then almost surely $Y_{\zeta-}$ exists in the Euclidean topology and, moreover, by (2.5),

$$\mathbf{P}_x(Y_{\zeta-} \in \partial D,\ \zeta < \infty) = 1 \qquad \text{for all } x \in D. \tag{2.15}$$

(2) The reflected α-stable process Y^* mentioned in (1) above in general differs from the following two processes, which can also be candidates for the title of "reflected α-stable process". The first one, denoted as U_t, is the $\alpha/2$-subordination of reflecting Brownian motion in $\overline{D}$. The second one, denoted as V_t, is obtained from the symmetric α-stable process in $\mathbf{R}^n$ by solving the corresponding Skorohod equation in D, whenever the latter is uniquely solvable. Consider the case $n = 1$ and $D = [0, \infty)$. It is shown in Lemma 3.1 of Burdzy, Chen and Sylvester [11] that the deterministic Skorohod equation in D is uniquely solvable and so process V_t can be defined. It is easy to see that both U_t and V_t have the same law as $|X_t|$, where X_t is the one-dimensional symmetric α-stable process. As the symmetric Dirichlet form corresponding to $|X_t|$ is of the form

$$\mathcal{C}(u,u) = c \int_{D\times D} (u(x) - u(y))^2 \left(|x-y|^{-1-\alpha} + |x+y|^{-1-\alpha} \right) dx\, dy,$$

we see that both U_t and V_t differ from Y^*.

We will show next that when $D \subset \mathbf{R}^n$ is an open n-set, Y is the censored α-stable process in D and Y^* is the reflected α-stable process on $\overline{D}$, then Y^* will not visit those sets in $\overline{D}$ which are not visited by the symmetric α-stable process X in $\mathbf{R}^n$, and vice versa. In particular, $Y_{\zeta-}$ will not visit subsets of the boundary ∂D which are avoided by X. To this end, we recall the following terminology.

A set A is called polar for the process Y^* if there is a nearly Borel measurable set $B \supset A$ such that $\mathbf{P}_x(\sigma_B^{Y^*} < \infty) = 0$ for every $x \in \overline{D}$, where $\sigma_B^{Y^*} := \inf\{t > 0 : Y^* \in B\}$. Polar sets for Y and X are defined similarly, with $\overline{D}$ replaced by D and $\mathbf{R}^n$, respectively. Recall that $(\mathcal{E}, W_0^{s,2}(D))$, $(\mathcal{E}^{\text{ref}}, W^{s,2}(D))$ and $(\mathcal{C}, W^{s,2}(\mathbf{R}^n))$ are the Dirichlet spaces for Y, Y^* and X respectively. A set A is called $\mathcal{E}^{\text{ref}}$-polar if there is a nearly Borel measurable set $B \supset A$ such that $\int_D \mathbf{P}_x(\sigma_B^{Y^*} < \infty)\, dx = 0$. $\mathcal{E}$-polar sets and $\mathcal{C}$-polar sets are defined in a similar way. As transition probabilities of Y (cf. Theorem 2.1) and X have densities with respect to the Lebesgue measures in D and in $\mathbf{R}^n$ respectively, Theorem 4.1.2 of [29] shows that a set A is $\mathcal{E}$-polar (respectively, $\mathcal{C}$-polar) if and only if it is polar for the process Y (respectively, X). A statement is said to be true $\mathcal{E}^{\text{ref}}$-q.e. (respectively, $\mathcal{C}$-q.e.) if it holds everywhere except on a $\mathcal{E}^{\text{ref}}$-polar set (respectively, $\mathcal{C}$-polar set). It is well known (cf. [29]) that functions in $W^{s,2}(D)$ (respectively, $W^{s,2}(\mathbf{R}^n)$) have $\mathcal{E}^{\text{ref}}$-quasi-continuous versions (respectively, $\mathcal{C}$-quasi-continuous versions) and in the sequel they are always represented by their quasi-continuous versions, which are unique up to $\mathcal{E}^{\text{ref}}$-polar sets (respectively, $\mathcal{C}$-polar sets).

Theorem 2.5. *Suppose that $D \subset \mathbf{R}^n$ is an open n-set, and that Y and Y^* are censored α-stable process in D and reflected α-stable process on $\overline{D}$ respectively.*

(1) A set $A \subset \overline{D}$ is $\mathcal{E}^{\text{ref}}$-polar if and only if it is polar for the process X.

(2) A set $A \subset D$ is polar for process Y if and only if it is polar for the process X. If a set $A \subset \partial D$ is polar for the process X, then

$$\mathbf{P}_x(Y_{\zeta-} \in A) = 0 \quad \textit{for every } x \in D. \tag{2.16}$$

Proof. Let $s = \alpha/2$. Let Cap_X and Cap_{Y^*} denote the 1-capacity for processes X and Y^* respectively. Capacity Cap_X is also called Riesz capacity of order $n-\alpha$ or Bessel capacity of order $(s, 2)$ in [1]. It is well known from the theory of Dirichlet forms (cf. Theorem 4.2.1 [29]) that a set A is $\mathcal{E}^{\text{ref}}$-polar ($\mathcal{C}$-polar) if and only if $\text{Cap}_{Y^*}(A) = 0$ (respectively, $\text{Cap}_X(A) = 0$).

(1) For any relatively open subset U in $\overline{D}$,

$$\begin{aligned}\text{Cap}_{Y^*}(U) &:= \inf\{\mathcal{E}_1^{\text{ref}}(u,u) : u \in W^{s,2}(D) \text{ with } u \geq 1 \text{ a.e. on } U\} \\ &\leq \inf\{\mathcal{C}_1(u,u) : u \in W^{s,2}(\mathbf{R}^n) \text{ with } u \geq 1 \ \mathcal{C}\text{-q.e. on } U\} \\ &= \text{Cap}_X(U)\end{aligned}$$

(cf. Theorem 2.1.5 of [29]). This implies by the definition of the capacity (cf. page 64 of [29]) that for any set $A \subset \overline{D}$,

$$\begin{aligned}\text{Cap}_{Y^*}(A) &= \inf\left\{\text{Cap}_{Y^*}(U) : U \text{ is a relatively open set in } \overline{D} \text{ containing } A\right\} \\ &\leq \inf\left\{\text{Cap}_X(U) : U \text{ is a relatively open set in } \overline{D} \text{ containing } A\right\} \\ &\leq \text{Cap}_X(A).\end{aligned}$$

We will now obtain a lower bound for $\text{Cap}_{Y^*}(A)$ in terms of $\text{Cap}_X(A)$. Let $S : W^{s,2}(D) \to W^{s,2}(\mathbf{R}^n)$ be the extension operator specified in (2.13). Fukushima and Uemura observed in [30] that S maps a continuous function in $W^{s,2}(D)$ with compact support in $\overline{D}$ into a continuous function in $W^{s,2}(\mathbf{R}^n)$ with compact support in $\mathbf{R}^n$. Note that $C_c^\infty(\overline{D})$, the space of smooth functions with compact support in $\overline{D}$, is the special standard core of $(\mathcal{E}^{\text{ref}}, W^{s,2}(D))$ in the sense of [29] (on page 6). For a compact subset $K \subset \overline{D}$, by Lemma 2.2.7 of [29] and (2.13),

$$\begin{aligned}\text{Cap}_{Y^*}(K) &= \inf\{\mathcal{E}_1^{\text{ref}}(f,f) : f \in C_c^\infty(\overline{D}) \text{ with } f \geq 1 \ \text{ on } K\} \\ &\geq c_2^{-2} \inf\{\mathcal{C}_1(Sf, Sf) : f \in C_c^\infty(\overline{D}) \text{ with } f \geq 1 \ \text{ on } K\} \\ &\geq c_2^{-2} \inf\{\mathcal{C}_1(u,u) : u \in W^{s,2}(\mathbf{R}^n) \text{ with } u \geq 1 \ \mathcal{C}\text{-q.e. on } K\} \\ &\geq c_2^{-2}\text{Cap}_X(K).\end{aligned}$$

Hence for any Borel subset $A \subset \overline{D}$,

$$\begin{aligned}\mathrm{Cap}_{Y^*}(A) &= \sup\left\{\mathrm{Cap}_{Y^*}(K) :\ K \subset A,\ K \text{ is compact}\right\} \\ &\geq c_2^{-2} \sup\left\{\mathrm{Cap}_X(K) :\ K \subset A,\ K \text{ is compact}\right\} \\ &= c_2^{-2}\mathrm{Cap}_X(A)\end{aligned}$$

(cf. (2.1.6) of [29]). Thus we have proved that for any set $A \subset \overline{D}$,

$$c_2^{-2}\mathrm{Cap}_X(A) \leq \mathrm{Cap}_{Y^*}(A) \leq \mathrm{Cap}_X(A). \tag{2.17}$$

Therefore a subset A in $\overline{D}$ is $\mathcal{E}^{\mathrm{ref}}$-polar if and only if it is polar for the process X.

(2) Note that Y is the subprocess of Y^* killed upon leaving the open set D. So by Theorem 4.4.3 in [29], a subset A in D is $\mathcal{E}$-polar if and only if it is $\mathcal{E}^{\mathrm{ref}}$-polar. Hence $A \subset D$ is polar for the process Y if and only if it is polar for X. Next suppose that $A \subset \partial D$ is polar for the process X and therefore it is $\mathcal{E}^{\mathrm{ref}}$-polar. By Theorem A.2.3 of [29],

$$\mathbf{P}_x\left(\text{there is some } t > 0 \text{ such that } Y_t^*(\omega) \in A \text{ or } Y_{t-}^*(\omega) \in A\right) = 0 \quad \text{for a.e. } x \in D.$$

In particular,

$$\mathbf{P}_x(Y_{\zeta-} \in A) = 0 \qquad \text{for a.e. } x \in D.$$

Note that it follows from Theorem 2.1 that the process Y has a transition density function $p(t, x, y)$ with respect to Lebesgue measure in D. So for each $x \in D$, using the Markov property of Y,

$$\begin{aligned}\mathbf{P}_x\left(Y_{\zeta-} \in A\right) &= \lim_{t\downarrow 0}\mathbf{P}_x(Y_{\zeta-} \in A,\ \zeta > t) \\ &= \lim_{t\downarrow 0}\int_D \mathbf{P}_y(Y_{\zeta-} \in A)\, p(t, x, y)dy = 0.\end{aligned}$$

This proves the theorem □

Remark 2.2. (1) Recall that for any increasing function h on $[0, \infty)$ with $h(0) = 0$, one can define a Hausdorff measure $\mathcal{H}_h$ with respect to the gauge h in the following way (see, e.g., p.132 of [1]). For $E \subset \mathbf{R}^n$,

$$\mathcal{H}_h(E) = \liminf_{\varepsilon\downarrow 0}\left\{\sum_{k=1}^{\infty} h(r_k) :\ E \subset \bigcup_{k=1}^{\infty} B(x_k, r_k) \text{ for some } x_k \in \mathbf{R}^n \text{ with } \sup_{1\leq k\leq\infty} r_k \leq \varepsilon\right\}.$$

When $h(r) = r^\beta$ for some $\beta > 0$, the Hausdorff measure $\mathcal{H}_h$ is denoted $\mathcal{H}^\beta$. In the case of $\alpha \leq n$, there is an intimate relationship between the Hausdorff measure $\mathcal{H}_h$ and the Riesz capacity Cap_X of order $n - \alpha$ (it is called Newtonian capacity if $n \geq 3$ and $\alpha = 2$, and is called logarithmic capacity if $\alpha = n$), see Theorems 2.2.7, 5.1.9 and 5.1.13 in [1]. Namely, $\mathcal{H}_h(A) < \infty$ implies that $\mathrm{Cap}_X(A) = 0$ if we take $h(r) = r^{n-\alpha}$ in the case $n > \alpha$ and

$h(r) = (\max\{\log(2/r), 0\})^{-1}$ when $n = \alpha$. On the other hand if $\mathrm{Cap}_X(A) = 0$ then $\mathcal{H}_h(A) = 0$ for every h such that

$$h \text{ is increasing on } [0, \infty) \text{ with } h(0) = 0 \quad \text{and} \int_0^1 \frac{h(r)}{r^{n+1-\alpha}}\, dr < \infty. \tag{2.18}$$

In particular, $\mathrm{Cap}_X(A) = 0$ implies $\mathcal{H}^{\lambda}(A) = 0$ for any $\lambda > n - \alpha$.

(2) The converse to the last statement in Theorem 2.5(2) is not true. Take, for example, D to be the unit ball in $\mathbf{R}^2$ centered at x_0, and $\alpha \in (1, 2)$. Theorem 2.9 below asserts that $\mathbf{P}_x(Y_{\zeta-} \in \partial D, \zeta < \infty) = 1$ for all $x \in D$. By the rotation invariance of Y, it is easy to see that the distribution of $Y_{\zeta-}$ under $\mathbf{P}_{x_0}$ is the normalized surface measure on ∂D. It follows from the Harnack inequality (Theorem 3.2 below) that the distribution of $Y_{\zeta-}$ under $\mathbf{P}_x$ is absolutely continuous with respect to the surface measure on ∂D for every $x \in D$. Let A be a Cantor set embedded into the circle ∂D. It is well known that A has Hausdorff dimension $\log 2/\log 3$ so $\mathbf{P}_x(Y_{\zeta-} \in A) = 0$ for every $x \in D$. However when $\alpha > 2 - (\log 2/\log 3)$, the set A will be visited by the symmetric α-stable process X.

(3) A result similar to Theorem 2.5 can be established for the reflecting Brownian motion in "extension domains" $D \subset \mathbf{R}^n$ and Brownian motion in $\mathbf{R}^n$ by an almost identical proof. Here an extension domain means a domain D on which there is a bounded linear operator $S : W^{1,2}(D) \to W^{1,2}(\mathbf{R}^n)$ with $S(W^{1,2}(D) \cap C_c^{\infty}(\overline{D})) \subset W^{1,2}(\mathbf{R}^n) \cap C_c(\mathbf{R}^n)$. Examples of extension domains are Lipschitz domains and (ε, δ)-domains (see Lemma 3.5 of [37] for a proof). The class of (ε, δ)-domains was introduced by Peter Jones [37] in 1981; this class includes Lipschitz domains and non-tangentially accessible domains. The boundary of an (ε, δ)-domain can be non-rectifiable and highly irregular but always has zero Lebesgue measure.

The following corollary follows immediately from Theorem 2.5 and the relationship between Y and Y^* outlined in Remark 2.1.

Corollary 2.6. *Let $n \geq 1$, $0 < \alpha < 2$, D be an open n-set in $\mathbf{R}^n$, and Y and Y^* be the censored symmetric α-stable process in D and the reflected α-stable process in $\overline{D}$ respectively. Denote by ζ the lifetime of Y. Then the following statements are equivalent.*

(1) $Y \neq Y^*$;
(2) $W_0^{\alpha/2,2}(D) \subsetneq W^{\alpha/2,2}(D)$;
(3) ∂D *is not polar for the symmetric α-stable process in* $\mathbf{R}^n$;
(4) $\mathbf{P}_x(\lim_{t\uparrow\zeta} Y_t \in \partial D, \zeta < \infty) > 0$ *for every* $x \in D$;
(5) $\mathbf{P}_x(\lim_{t\uparrow\zeta} Y_t \in \partial D, \zeta < \infty) > 0$ *for some* $x \in D$.

Proof. Note that Y is the process Y^* killed upon leaving the open set D, and that $W_0^{\alpha/2,2}(D)$ and $W^{\alpha/2,2}(D)$ are the domains of the Dirichlet forms for Y and Y^* respectively. The equivalence of (1)–(4) of the corollary follows immediately from Theorem 2.5 and (2.14). Clearly (4) implies (5). Suppose now (5) holds. As Y has strictly positive transition density function $p(t, x, y)$ with respect to the Lebesgue measure in D and

$$0 < \mathbf{P}_x(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty) = \lim_{t\to 0}\int_D \mathbf{P}_y(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty)p(t, x, y)dy,$$

we have for any $w \in D$,

$$\mathbf{P}_w(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty) = \lim_{t\to 0}\int_D \mathbf{P}_y(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty)p(t, w, y)dy > 0.$$

Thus (4) holds and therefore Corollary 2.6 is established. □

The following result provides an explicit and essentially complete answer to Question 1.1 in terms of Hausdorff dimension and measure.

Theorem 2.7. *Suppose that $n \geq 1$, $\alpha \in (0, 2)$ and $D \subsetneq \mathbf{R}^n$ is an open n-set. Let Y and Y^* be the censored symmetric α-stable process in D and the reflected α-stable process in $\overline{D}$, respectively.*

(1) Suppose that $\alpha \leq n$ and that $\mathcal{H}_h(\partial D \cap K_m) < \infty$ for an increasing sequence of Borel sets K_m such that $\cup_{m=1}^{\infty} K_m \supset \partial D$, where $h(r) = r^{n-\alpha}$ if $\alpha < n$ and $h(r) = \max\{\log \frac{2}{r}, 0\}$ when $\alpha = n = 1$. Then $Y = Y^*$ and so Y does not approach ∂D at any finite time.
(2) Suppose that $\alpha \leq n$ and that $\mathcal{H}_h(\partial D) > 0$ for some h satisfying (2.18). Then Y is a proper subprocess of Y^*, Y is transient with lifetime ζ so that

$$\mathbf{P}_x(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty) > 0 \quad \textit{for all } x \in D.$$

In particular, this statement holds if $\mathcal{H}^d(\partial D) > 0$ for some $d > n - \alpha$.
(3) When $\alpha \geq n = 1$, Y^* is recurrent. If $\alpha > n = 1$, then Y is transient with finite lifetime ζ and

$$\mathbf{P}_x(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty) = 1 \quad \textit{for all } x \in D.$$

(4) When $\alpha < n$ and D is unbounded with compact boundary, then $\mathbf{P}_x(\zeta = \infty) > 0$ for every $x \in D$.

Proof. (1) If the condition of (1) is satisfied, then by Theorem 2.5 and Remark 2.2(1), each $\partial D \cap K_m$ and therefore ∂D is $\mathcal{E}^{\mathrm{ref}}$-polar for Y^*. So $Y = Y^*$.

(2) Suppose now the condition of (2) is satisfied, then by Theorem 2.5 and Remark 2.2(1), ∂D has positive capacity and therefore will be visited by Y^* with positive probability for every starting point in D (cf. Theorem 4.6.6 of [29]), which yields the conclusion of (2).

(3) It is well know (see, e.g., page 83 of [5] and page 34 of [4]) that the one-dimensional α-stable process is recurrent if and only if $\alpha \geq 1$ (it is pointwise recurrent if $\alpha > 1$ and neighborhood recurrent if $\alpha = 1$). This implies by Theorem 1.6.3 in [29] that when $\alpha \geq 1$, there is a sequence $\{u_n\} \subset \mathcal{F}^{\mathbf{R}}$ such that $\lim_{n\to\infty} u_n = 1$ a.e. on $\mathbf{R}$ and $\lim_{n\to\infty} \mathcal{C}(u_n, u_n) = 0$. Let $f_n = u_n|_D$. Then

$f_n \in W^{\alpha/2,2}(D)$, $\lim_{n\to\infty} f_n = 1$ a.e. on D and $\lim_{n\to\infty} \mathcal{E}(f_n, f_n) = 0$. So by Theorem 1.6.3 in [29], Y^* is recurrent when $\alpha \geq 1$. On the other hand, a point is polar for the one-dimensional symmetric α-stable process X on $\mathbf{R}$ if and only if $\alpha \leq 1$. Thus when $\alpha > 1$, as ∂D is not $\mathcal{E}^{\text{ref}}$-polar for Y^*, it is hit by Y^* with probability one for almost every starting point. As Y has a density function with respect to the Lebesgue measure in D, part (3) of the theorem is established.

(4) When $\alpha < n$, the symmetric α-stable process X in $\mathbf{R}^n$ is transient. Let $R > 0$ be such that $\partial D \subset B(0, R)$. Then for $x \in \mathbf{R}^n$ with $|x| > R$, there is a positive $\mathbf{P}_x$-probability that the event $\{\inf\{t > 0 : X_t \in B(0, R)\} = \infty\}$ occurs. This says that there is a positive probability that X wanders to infinity without entering D^c and so by Theorem 2.1, $\mathbf{P}_x(\zeta = \infty) > 0$ for every $x \in D$. □

Corollary 2.8. *Suppose that $n \geq 1$, $s \in (0, 1)$ and $D \subsetneq \mathbf{R}^n$ is an open n-set.*

(1) If $2s \leq n$ and $\mathcal{H}_h(\partial D \cap K_m) < \infty$ for an increasing sequence of Borel sets K_m such that $\cup_{m=1}^{\infty} K_m \supset \partial D$, where $h(r) = r^{n-2s}$ if $2s < n$ and $h(r) = (\max\{\log(2/r), 0\})^{-1}$ when $2s = n = 1$, then $W_0^{s,2}(D) = W^{s,2}(D)$.

(2) If either $2s > n = 1$ or $2s \leq n$ with $\mathcal{H}_h(\partial D) > 0$ for some h satisfying (2.18) with $2s$ in place of α there, then $W_0^{s,2}(D) \subsetneq W^{s,2}(D)$. In particular, $W_0^{s,2}(D) \subsetneq W^{s,2}(D)$ if $\mathcal{H}^d(\partial D) > 0$ for some $d > n - 2s$, with $\mathcal{H}^0$ being interpreted as counting measure on $\mathbf{R}$ in the case of $2s > n = 1$.

Proof. This follows from Theorem 2.7 with $\alpha = 2s$ and the fact that $W_0^{s,2}(D)$ and $W^{s,2}(D)$ are the domains of the Dirichlet forms for Y and Y^* respectively. □

Remark 2.3. The above Corollary not only recovers but also extends the corresponding results in Caetano [12] (Proposition 2.2, Corollary 2.7 and Proposition 3.7) and in Farkas and Jacob [27] (Theorems 3.3 and 3.9). In these two papers, some additional conditions are imposed, for example in Caetano [12], it is required that D is a bounded open set with ∂D being a d-set for some $d \in [n-1, n)$. In Farkas and Jacob [27], D is a bounded (ε, δ)-domain with ∂D being a d-set for some $d \in [n-1, n)$. Our proof is quite different from those in [12] and [27] and more elementary. Furthermore, when D is an open n-set and ∂D "locally" has finite d-dimensional Hausdorff measure for some $d \in [n-1, n)$, our result asserts that in the critical case of $2s = n - d$, $W^{s,2}(D) = W_0^{s,2}(D)$. This critical case is covered in neither [12] nor [27]. Our result also extends substantially Theorem 4.3.2.1(a) in [46] where $W^{1/2,1}(D) = W_0^{1/2,1}(D)$ is proved for bounded C^∞-smooth domains.

Theorem 2.9. *Suppose that $n \geq 1$, $\alpha \in (0, 2)$ and $D \subset \mathbf{R}^n$ is an open n-set having finite Lebesgue measure.*

(1) Suppose that $\alpha \leq n$ and $\mathcal{H}_h(\partial D \cap K_m) < \infty$ for an increasing sequence of Borel sets K_m such that $\cup_{m=1}^{\infty} K_m \supset \partial D$, where $h(r) = r^{n-\alpha}$ if $\alpha < n$ and $h(r) = (\max\{\log(2/r), 0\})^{-1}$ when $n = \alpha = 1$. Then the censored symmetric α-stable process Y in D is recurrent and therefore conservative. It does not approach ∂D at any finite time.

(2) If either $\alpha > n = 1$ or $\alpha \leq n$ with $\mathcal{H}_h(\partial D) > 0$ for some h satisfying (2.18), then the censored symmetric α-stable process Y in D is transient with finite

lifetime ζ. *Moreover,* $P_x(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty) = 1$ *for all* $x \in D$. *In particular, the above statements hold if* $\mathcal{H}^d(\partial D) > 0$ *for some* $d > n - \alpha$ *with* $\mathcal{H}^0$ *being interpreted as the counting measure in* $\mathbf{R}$ *in the case of* $\alpha > n = 1$.

Proof. Let $s = \alpha/2$. As $1 \in W^{s,2}(D)$, the reflected α-stable process Y^* on $\overline{D}$ is recurrent. Claims (1) and (2) follow immediately from Theorem 2.7. □

Proof of Theorem 1.1. When $n = 1$, a bounded Lipschitz open set D in $\mathbf{R}$ is a finite union of bounded intervals with no common endpoints, which is a 1-open set. It is well known (see, e.g. page 83 of [5]) that a point is polar for one-dimensional symmetric α-stable process if and only if $\alpha \le 1$. So the conclusion of the theorem follows from Corollary 2.6. Now for $n \ge 2$, note that a bounded Lipschitz open set $D \subset \mathbf{R}^n$ is an n-set and its boundary ∂D has positive and finite $(n-1)$-dimensional Hausdorff measure. So the conclusion of this theorem follows immediately from Theorem 2.9. □

The localization condition in Theorem 2.7, Corollary 2.8 and Theorem 2.9 is needed so that we can apply those results to open sets such as in Example 2.1 below.

Example 2.1. Let $n = 2$ and D be the unit square $[0,\ 1] \times [0,\ 1]$ with slits $\{1/k\} \times [0,\ 1/2]$, $k \ge 2$, removed. Then clearly D is an open 2-set with $\mathcal{H}^1(\partial D) = \infty$ but $0 < \mathcal{H}^1(\partial D \cap K_m) < \infty$ for each $m \ge 2$, where K_m is the union of $[1/m,\ 1] \times [0, 1]$ and $\{0\} \times [0,\ 1]$. So Theorem 2.7, Corollary 2.8 and Theorem 2.9 apply. In particular, we have $W^{s,2}(D) = W_0^{s,2}(D)$ if and only if $s \le 1/2$; the censored α-stable process Y in D is recurrent and therefore conservative if $\alpha \le 1$, and Y is transient with $\mathbf{P}_x(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty) = 1$ for every $x \in D$ if $\alpha > 1$. Note that ∂D is *not* a d-set.

Example 2.2. Let $n = 1$ and D be the unit interval $[0,\ 1]$ with the Cantor set removed. It is well known that $0 < \mathcal{H}^d(\partial D) < \infty$, where $d = \log 2/\log 3$. Clearly D is an open 1-set. So by Theorem 2.9, the censored α-stable process Y in D is recurrent and therefore conservative if $\alpha \le 1 - \frac{\log 2}{\log 3}$, and Y is transient with $\mathbf{P}_x(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty) = 1$ for every $x \in D$ if $\alpha > 1 - \frac{\log 2}{\log 3}$. Furthermore, $W^{s,2}(D) = W_0^{s,2}(D)$ if and only if $s \le \frac{1}{2}\left(1 - \frac{\log 2}{\log 3}\right)$.

In view of Theorem 2.9(1) it is natural to ask whether a recurrent censored symmetric α-stable process in D has the same distribution as the symmetric α-stable process in D conditioned not to leave D. We will show below that the answer is no.

Let D be a bounded open set in $\mathbf{R}^n$ and $\alpha \in (0,\ 2)$. P_t^D has a symmetric density function $p_D(t, x, y)$, which is bounded by $ct^{-n/\alpha}$. Since D is bounded,

$$\int_D \int_D p_D(t, x, y)^2 dxdy = \int_D p_D(2t, x, x)dx < \infty,$$

that is, P_t^D is a Hilbert-Schmidt operator. So P_t^D is a self-adjoint compact operator in $L^2(D, dx)$ (see Problem 5.1.4 of [28]) and hence it has a discrete spectrum (see Problems 6.7.4 and 6.7.5 in [28]). This implies that the infinitesimal generator L^D

of X^D has a discrete spectrum. It is clear that the first eigenvalue λ_1 of L^D has to be strictly negative. The symmetric stable process conditioned not to leave the open set D is a process Z obtained from X^D via a parabolic Doob's h-transform, i.e., Z is given by

$$\mathbf{E}_x[f(Z_t)] = e^{-\lambda_1 t}\varphi(x)^{-1}\mathbf{E}_x[(\varphi f)(X_t^D)].$$

(See Gong, Qian and Zhao [32] for the diffusion case.) The following theorem characterizes the process Z.

Theorem 2.10. *The process Z is $\varphi^2\,dx$-symmetric and recurrent in D. Let $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ be the Dirichlet space of Z on $L^2(D, \varphi^2\,dx)$. Then*

$$\widetilde{\mathcal{E}}(u, v) = \frac{1}{2}\mathcal{A}(n, -\alpha) \times \int_D\int_D \frac{(u(x)-u(y))(v(x)-v(y))\varphi(x)\varphi(y)}{|x-y|^{n+\alpha}}\,dxdy, \qquad u, v \in \widetilde{\mathcal{F}}, \tag{2.19}$$

where $\widetilde{\mathcal{F}}$ is the closure of $\mathcal{F}^D$ under the norm $\sqrt{\widetilde{\mathcal{E}}_1}$, and $\widetilde{\mathcal{E}}_1(u, u) = \widetilde{\mathcal{E}}(u, u) + \int_D u(x)^2\varphi(x)^2\,dx$.

Proof. Note that $\varphi \in L^2(D)$ and $P_t^D\varphi = e^{\lambda_1 t}\varphi$. So φ is a bounded function in $\mathcal{F}^D$, the Dirichlet space of X^D. As a special case of a result of Fukushima and Takeda (Theorems 6.3.1 and 6.3.2 in [29]), we have that process Z is $\varphi^2\,dx$-symmetric and recurrent, $\widetilde{\mathcal{F}} \supset \mathcal{F}^D$ and (2.19) holds for $u, v \in \mathcal{F}^D$. It follows from a general result recently proved in [15] that $\widetilde{\mathcal{F}}$ is the closure of $\mathcal{F}^D$ with respect to the norm $\sqrt{\widetilde{\mathcal{E}}_1}$. □

We claim that the eigenfunction φ can not be constant. Were φ a constant function, then we would have

$$\mathcal{C}(\varphi, v) = \int_D v(x)\varphi(x)\kappa_D(x)dx \quad \text{for any } v \in \mathcal{F}^D.$$

This would imply that $L^D\varphi = -\kappa_D\varphi$ and thus $\kappa_D(x)$ would have to be constant, i.e., $\kappa_D(x) \equiv -\lambda_1$, which is impossible in view of (2.3). By comparing the Dirichlet spaces $(\mathcal{E}, \mathcal{F})$ and $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ (see (2.4) and (2.19)), we see that Y and Z are different processes for any $\alpha \in (0, 2)$. Furthermore, Y^* differs from Z for any $\alpha \in (0, 2)$ as well.

Remark 2.4. All the results in this section hold for a large class of "pure jump" processes whose jumping measure is comparable to that of the symmetric α-stable process X on $\mathbf{R}^n$. More precisely, the results in this section are valid when X is replaced by a symmetric process $\widetilde{X}$ on $\mathbf{R}^n$ whose Dirichlet form is given by

$$\widetilde{\mathcal{C}}(u, v) = \frac{1}{2}\mathcal{A}(n, -\alpha)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))\,k(x, y)}{|x-y|^{n+\alpha}}\,dxdy,$$

$$\mathcal{F}^{\mathbf{R}^n} = \left\{u \in L^2(\mathbf{R}^n) : \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dxdy < \infty\right\},$$

where $k(x, y)$ is a symmetric Borel measurable function on $\mathbf{R}^n \times \mathbf{R}^n$ satisfying $c^{-1} \le k(x, y) \le c$ for some $1 < c < \infty$ and all x and y. (In fact it is enough for this to hold on $D \times D$.) Our claim can be justified as follows. The Dirichlet form $(\widetilde{\mathcal{C}}, \mathcal{F}^{\mathbf{R}^n})$ is comparable to $(\mathcal{C}, \mathcal{F}^{\mathbf{R}^n})$ in (2.1)–(2.2), so the capacity induced by $\widetilde{X}$ is comparable to that induced by X. On the other hand, $\widetilde{X}$ has continuous transition density functions by Theorem 1 of Komatsu [39]. Since the capacities are comparable, the polar sets for the censored process and reflected process obtained from $\widetilde{X}^D$, the subprocess of $\widetilde{X}$ killed upon leaving an open n-set in $\mathbf{R}^n$, are the same as those of the censored α-stable process Y and reflected α-stable process Y^* on D respectively.

3. Harmonic functions of censored stable processes

We collect a few potential theoretic results for reference in later sections of the paper. Let us recall some notation and definitions. Let $0 < \alpha < 2$ and let D be an open set in $\mathbf{R}^n$. By $\kappa(x) := \kappa_D^\alpha(x) = \mathcal{A}(n, -\alpha) \int_{D^c} |y - x|^{-n-\alpha}\, dy$, we denote the density function of the killing measure for D, and we write Y_t for the censored stable process on D. Clearly, κ is a C^∞ function on D. Let $e_\kappa(t) = \exp[\int_0^t \kappa(X_s^D) ds]$. In particular, $e_\kappa(\tau_B) = \exp[\int_0^{\tau_B} \kappa(X_s)\, ds]$, where τ_B is the exit time from a set B.

Definition 3.1. *Let U be an open subset of D. A Borel measurable function u on $\mathbf{R}^n$ is harmonic on U with respect to the censored process Y_t if*

$$u(x) = \mathbf{E}_x u(Y_{\tau_B}), \quad x \in B, \tag{3.1}$$

for every bounded open set B such that $\overline{B} \subset U$. It is called regular harmonic in U if (3.1) holds with U in place of B. We say that u is superharmonic (subharmonic) on U for Y_t if (3.1) is satisfied with the equality sign replaced by $\ge$ ($\le$, respectively).

Here we use the convention that $Y_\infty = \delta$. We will always make a tacit assumption about all functions that they take value 0 at the cemetery point δ and that the expectation in (3.1) is absolutely convergent, and so finite. As Y does not visit D^c, the value of u on D^c is irrelevant in the above definition and thus one may assume that $u = 0$ on D^c.

Functions "harmonic in U" for the unconstrained symmetric α-stable process X on $\mathbf{R}^n$ are defined in a way analogous to (3.1), see, e.g., [10], [21].

Let $D \subset \mathbf{R}^n$ be a domain. Then using the Lévy system (see [43]) of X it is easy to see that (see [35]) the distribution of the pair $(X_{\tau_D -}, X_{\tau_D})$ restricted to event $\{X_{\tau_D -} \ne X_{\tau_D}, \tau_D < \infty\}$ under $\mathbf{P}_x$, $x \in D$, is concentrated on $D \times D^c$ with the density function $g^x(v, y)$ given by the following explicit formula:

$$g^x(v, y) = \frac{\mathcal{A}(n, -\alpha)}{|y - v|^{n+\alpha}} G_D(x, v), \quad (v, y) \in D \times D^c, \tag{3.2}$$

where $G_D(x, v)$ is the Green function for the process X_t^D. Note that if D is a domain satisfying $\mathbf{P}_x\{X_{\tau_D} \in \partial D; \tau_D < \infty\} = 0$, then $\mathbf{P}_x(X_{\tau_D -} \ne X_{\tau_D}, \tau_D < \infty) = \mathbf{P}_x(\tau_D < \infty)$. This condition is satisfied, for example, when D has the exterior cone property (see Lemma 6 and Lemma 17 of [7]). Assuming that $\mathbf{P}_x(X_{\tau_D -} \ne$

X_{τ_D}, $\tau_D < \infty) = \mathbf{P}_x(\tau_D < \infty)$ and integrating (3.2) in v over D we obtain the density function

$$g^x(y) = \int_D \frac{\mathcal{A}(n, -\alpha)\, G_D(x, v)}{|y - v|^{n+\alpha}}\, dv\,, \quad y \in D^c, \tag{3.3}$$

of the harmonic measure $\mathbf{P}_x\{X_{\tau_D} \in dy;\ \tau_D < \infty\}$ with respect to the Lebesgue measure.

When $r > 0$, B_r is the ball $B(0, r) \subset \mathbf{R}^n$ and $|x| < r$, the $\mathbf{P}_x$-distribution of $X_{\tau_{B_r}}$ has a density function $P_r(x, \cdot)$ (the *Poisson kernel*), explicitly given by a formula of Riesz (cf. [6]):

$$P_r(x, y) = C_{n,\alpha} \left[\frac{r^2 - |x|^2}{|y|^2 - r^2}\right]^{\alpha/2} |x - y|^{-n} \quad \text{provided } |y| > r\,, \tag{3.4}$$

with $C_{n,\alpha} = \Gamma(n/2)\pi^{-n/2-1}\sin(\pi\alpha/2)$, and equal to 0 otherwise. The Green function of the unit ball $B = B(0, 1)$ is given in [6]:

$$G(x, y) = \mathcal{B}_{n,\alpha}\, |x - y|^{\alpha-n} \int_0^{w(x,y)} \frac{s^{\alpha/2-1}}{(s+1)^{n/2}}\, ds, \quad x, y \in B, \tag{3.5}$$

where

$$w(x, y) = (1 - |x|^2)(1 - |y|^2)/|x - y|^2,$$

and $\mathcal{B}_{n,\alpha} = \Gamma(n/2)/(2^\alpha \pi^{n/2}[\Gamma(\alpha/2)]^2)$. Setting (3.3) equal to (3.4) for the unit ball, multiplying both sides by $|y|^{n+\alpha}$, letting $|y| \to \infty$ and using the scaling property of X one easily recovers (cf. [9]) the following formula which originally appeared in [31],

$$\begin{aligned} \mathbf{E}_x \tau_{B(x_0,r)} &= \int_{B(x_0,r)} G_{B(x_0,r)}(x, v)\, dv \\ &= \frac{C_{n,\alpha}}{\mathcal{A}(n, -\alpha)} \left(r^2 - |x - x_0|^2\right)^{\alpha/2}, \quad x \in B(x_0, r)\,. \end{aligned} \tag{3.6}$$

For $\alpha < n$ the symmetric α-stable process X_t is transient and its potential kernel is (see [5], [41]),

$$G^{(\alpha)}(y - x) = \int_0^\infty p(t; x, y)\, dt = \frac{\mathcal{A}(n, \alpha)}{|y - x|^{n-\alpha}}, \qquad x, y \in \mathbf{R}^n. \tag{3.7}$$

Suppose that u is harmonic in $U \subset D$ for censored α-stable process Y. By Theorem 2.1(3) and (2.7), for every bounded open set B with $\overline{B} \subset U$,

$$\mathbf{E}_x u(Y_{\tau_B}) = \mathbf{E}_x[\tau_B < \tau_D\,;\ u(X_{\tau_B})e_\kappa(\tau_B)] = \mathbf{E}_x[\mathbf{1}_D(X_{\tau_B})u(X_{\tau_B})e_\kappa(\tau_B)].$$

If $u = 0$ on D^c then by Theorem 2.1(3), (3.1) is equivalent to the equality

$$u(x) = \mathbf{E}_x[u(X_{\tau_B})e_\kappa(\tau_B)]\,, \quad x \in B\,. \tag{3.8}$$

Such u is κ-harmonic on U for the symmetric stable process X_t, meaning that u is a harmonic function in U of the Feynman-Kac semigroup of X^D obtained through the multiplicative functional e_κ. Consequently, u is continuous in U (cf. [9]). In the discussion below we use the general setting of [9] and [10] to study such functions.

For a function u satisfying the integrability condition

$$\int_{\mathbf{R}^n} \frac{|u(y)|}{(1+|y|)^{n+\alpha}}\, dy < \infty, \tag{3.9}$$

we define, as usual,

$$\Delta_\varepsilon^{\alpha/2} u(x) = \mathcal{A}(n,-\alpha) \int_{|y-x|>\varepsilon} \frac{u(y)-u(x)}{|y-x|^{n+\alpha}}\, dy\,, \tag{3.10}$$

and

$$\Delta^{\alpha/2} u(x) = \mathcal{A}(n,-\alpha) P.V. \int_{\mathbf{R}^n} \frac{u(y)-u(x)}{|y-x|^{n+\alpha}}\, dy := \lim_{\varepsilon\to 0^+} \Delta_\varepsilon^{\alpha/2} u(x)\,, \tag{3.11}$$

whenever the limit exists. Here "$P.V.$" stands for the "principal value." For instance, the limit exists and is finite if u is of class C^2 in a neighborhood of x and satisfies condition (3.9); in such a case,

$$\Delta^{\alpha/2} u(x) = \mathcal{A}(n,-\alpha) \int_{\mathbf{R}^n} \frac{u(y)-u(x)-\nabla u(x)\cdot(y-x)\mathbf{1}_{\{|y-x|<\varepsilon\}}}{|y-x|^{n+\alpha}}\, dy$$

for any $\varepsilon > 0$. Harmonic functions of the symmetric α-stable process X may be characterized as those annihilating $\Delta^{\alpha/2}$, see [10].

Let

$$\begin{aligned} A_D^\alpha \phi(x) &= \mathcal{A}(n,-\alpha) P.V. \int_D \frac{\phi(y)-\phi(x)}{|y-x|^{n+\alpha}} dy \\ &:= \mathcal{A}(n,-\alpha) \lim_{\varepsilon\to 0^+} \int_{\{y\in D\,:\,|y-x|>\varepsilon\}} \frac{\phi(y)-\phi(x)}{|y-x|^{n+\alpha}} dy\,. \end{aligned} \tag{3.12}$$

It is elementary to see that for $\phi \in C_c^\infty(D)$,

$$\mathcal{E}(\phi,\phi) = -(A_D^\alpha \phi, \phi)\,.$$

We can express the relation between Y and the symmetric stable process X in terms of the generators A_D^α and $\Delta^{\alpha/2}$. Namely, for $x \in D$ and a sufficiently regular function ϕ which *vanishes* on D^c we have

$$\begin{aligned} A_D^\alpha \phi(x) &= \mathcal{A}(n,-\alpha) P.V. \int_{\mathbf{R}^n} \frac{\phi(y)-\phi(x)}{|y-x|^{n+\alpha}} dy + \mathcal{A}(n,-\alpha) \int_{D^c} \frac{\phi(x)}{|y-x|^{n+\alpha}}\, dy \\ &= \Delta^{\alpha/2}\phi(x) + \kappa_D(x)\phi(x)\,; \end{aligned} \tag{3.13}$$

so in particular,

$$A_D^\alpha = \Delta^{\alpha/2} + \kappa_D \quad \text{on } C_c^2(D). \tag{3.14}$$

It can be shown using Theorem 2.1 that the L^2-generator of Y is the smallest closed extension of $(A^\alpha_D,\ C^2_c(D))$, which is a self-adjoint operator in $L^2(D, dx)$.

Let $\mathbf{E}^v_x$ denote the expectation for α-stable process X conditioned by the Green function $G_D(\cdot, v)$ if $v \in D$, or the Martin kernel $K_D(\cdot, v)$ if $v \in \partial D$ (see [9] or [22] for a discussion of conditional expectations and processes). By (3.2) and routine arguments we obtain for $\Phi \geq 0$ measurable with respect to $\mathcal{F}_{\tau_D -}$ and any Borel $f \geq 0$, the following useful formula

$$\mathbf{E}_x[f(X_{\tau_D})\,\Phi;\ X_{\tau_D -} \neq X_{\tau_D}] = \mathbf{E}_x\left[f(X_{\tau_D})\,\mathbf{E}_x^{X_{\tau_D -}}[\Phi];\ X_{\tau_D -} \neq X_{\tau_D}\right], \quad x \in D\,. \tag{3.15}$$

The formulas presented so far in this section help perform explicit calculations for harmonic functions of X and some of them can be extended to the censored process Y using the relationship of Y to the killed α-stable process X^D. As an illustration, in Theorem 3.2 below we will prove the Harnack inequality for nonnegative harmonic functions of Y.

By $\rho(x) = \mathrm{dist}(x, D^c)$ we denote the Euclidean distance between x and D^c.

Lemma 3.1. *There is $r_1 = r_1(n, \alpha) \in (0,\ 1)$, independent of domain D, such that for every ball $B = B(x, r_1\rho(x)) \subset D$,*

$$\int_B \frac{G_B(v, y)G_B(y, w)}{G_B(v, w)}\kappa_D(y)\,dy \leq 1/2, \qquad v, w \in B\,, \tag{3.16}$$

where G_B denotes the Green function of B for the symmetric stable process X.

Proof. Let

$$S = \sup_{v,w\in B}\int_B \frac{G_B(v, y)G_B(y, w)}{G_B(v, w)}\kappa_D(y)\,dy\,,$$

and let G be the Green function of the unit ball $B(0, 1) \subset \mathbf{R}^n$. By translation invariance and scaling of X we have that for all $a, b, c \in \mathbf{R}^n$ and $s > 0$

$$G_{B(a,s)}(a + sb, a + sc) = s^{\alpha-n}G(b, c)\,.$$

By the change of variable $v = x + r_1\rho(x)v$, $y = x + r_1\rho(x)y$, $w = x + r_1\rho(x)w$, we obtain

$$S = r_1^\alpha \sup_{v,w\in B(0,1)}\int_{B(0,1)} \frac{G(v, y)G(y, w)}{G(v, w)}\kappa_{\frac{D-x}{\rho(x)}}(r_1 y)\,dy\,. \tag{3.17}$$

We will assume without loss of generality that $x = 0 \in D$ and $\rho(x) = 1$. For $y \in B(0, 1) \subset D$ and $0 < r_1 < 1$ we have

$$\kappa_D(r_1 y) \leq \mathcal{A}(n, -\alpha)\int_{B(r_1 y, 1-r_1)^c} \frac{dz}{|r_1 y - z|^{n+\alpha}} = \mathcal{A}(n, -\alpha)\frac{\omega_n}{\alpha}(1 - r_1)^{-\alpha}\,,$$

which is bounded for $r_1 < 1/2$. Here ω_n is the surface measure of the unit sphere in $\mathbf{R}^n$. By Proposition 3.3 and (3.17) in [9] or by Theorem 1.6 in [18],

$$\int_{B(0,1)} \frac{G(v, y)G(y, w)}{G(v, w)}\,dy \leq \text{const.}, \quad v, w \in B(0, 1)\,.$$

Thus, if we choose r_1 small enough in (3.17) then (3.16) is satisfied. □

If (3.16) holds then by Khasminskii's lemma (see Lemma 3.7 of [24]),

$$1 \le \mathbf{E}_x^v e_\kappa(\tau_B) \le 2\,, \qquad x, v \in B\,. \tag{3.18}$$

Here $\mathbf{E}_x^v$ refers to conditioning with respect to the Green function of B; the quantity $\mathbf{E}_x^v e_\kappa(\tau_B)$ is the so called conditional gauge function. The lower bound in (3.18) is trivial because $\kappa = \kappa_D^\alpha > 0$.

The following result is a scale-invariant version of the Harnack inequality for nonnegative harmonic functions of the censored process Y, see also Theorem 4.1 in [9].

Theorem 3.2. *Let $D \subset \mathbf{R}^n$ and let Y be the censored process on D. Let $x_1, x_2 \in D$, $r > 0$ with $B(x_1, r) \cup B(x_2, r) \subset D$ and $k \in \{1, 2, \ldots\}$, such that $|x_1 - x_2| < 2^k r$. If $u \ge 0$ is harmonic for Y on $B(x_1, r) \cup B(x_2, r)$ then there exists a constant J depending only on n and α, such that*

$$J^{-1} 2^{-k(n+\alpha)} u(x_2) \le u(x_1) \le J 2^{k(n+\alpha)} u(x_2)\,. \tag{3.19}$$

Proof. We can assume as usual that $u = 0$ on D^c. We first consider the case when $|x_1 - x_2| < 2^k r$ but $|x_1 - x_2| \ge r$. Let $B_1 = B(x_1, r_1 r)$, $B_2 = B(x_2, r_1 r)$, where r_1 is the constant in Lemma 3.1. By (3.8) and (3.15),

$$u(y) = \mathbf{E}_y[e_\kappa(\tau_{B_1}) u(X_{\tau_{B_1}})] = \mathbf{E}_y[u(X_{\tau_{B_1}}) \mathbf{E}_y^{X_{\tau_{B_1}-}} e_\kappa(\tau_{B_1})], \quad y \in B_1\,. \tag{3.20}$$

(3.20) and (3.18) yield

$$\mathbf{E}_y u(X_{\tau_{B_1}}) \le u(y) \le 2\mathbf{E}_y u(X_{\tau_{B_1}}), \quad y \in \mathbf{R}^n. \tag{3.21}$$

Let $w(y) = \mathbf{E}_y u(X_{\tau_{B_1}})$, $y \in \mathbf{R}^n$. Note that w is harmonic in B_1 for X, hence by (3.4) and (3.21),

$$w(y) \ge 2^{n-2} 3^{-n+1} w(x_1) \ge 2^{n-3} 3^{-n+1} u(x_1)\,, \quad \text{if } |y - x_1| < r_1 r/2\,. \tag{3.22}$$

Straightforward calculations using (3.4) yield

$$\begin{aligned} u(x_2) &\ge \mathbf{E}_{x_2}[w(X_{\tau_{B_2}});\ X_{\tau_{B_2}} \in B(x_1, r_1 r/2)] \\ &\ge 2^{n-3} 3^{-n+1} u(x_1) \mathbf{P}_{x_2}\{X_{\tau_{B_2}} \in B(x_1, r_1 r/2)\} \\ &\ge c(n,\alpha) u(x_1) (r_1 r)^n \frac{(r_1 r)^\alpha}{(|x_1 - x_2| + r_1 r/2)^\alpha} (|x_1 - x_2| + r_1 r/2)^{-n} \\ &\ge c(n,\alpha)\, 2^{-k(n+\alpha)}\, u(x_1)\,. \end{aligned}$$

This proves the upper bound in (3.19) for the case of $|x_1 - x_2| \ge r$, while the lower bound follows by symmetry.

If $|x_1 - x_2| < r$, we take $r' = |x_1 - x_2|$ and $k' = 1 \le k$; then (3.19) follows from the first part of the proof. □

It is noteworthy that in Theorem 3.2, the open set $B(x_1, r) \cup B(x_2, r)$, where u is harmonic, may be disconnected and the Harnack inequality still holds. This is due to jumps of Y.

For future reference we note that if $x \in B$ and B is an open subset with $\overline{B} \subset D$, then by (3.2), (3.15) and Theorem 2.1(3), the joint $\mathbf{P}_x$-distribution of $(Y_{\tau_B -}, Y_{\tau_B})\mathbf{1}_{\{Y_{\tau_B -} \neq Y_{\tau_B}\}}$ on $B \times (D \setminus B)$ has the density function

$$h^x(v, y) = \frac{\mathcal{A}(n, -\alpha)}{|v-y|^{n+\alpha}} G_B(x, v)\, \mathbf{E}_x^v e_\kappa(\tau_B), \quad (v, y) \in B \times (D \setminus B). \quad (3.23)$$

Here G_B is the Green function of X in B and $\mathbf{E}_x^v$ is expectation under the law of X conditioned by $G_B(\cdot, v)$. Comparing (3.23) with the version of (3.2) for Y, we see that

$$G_B^Y(x, v) := G_B(x, v)\, \mathbf{E}_x^v e_\kappa(\tau_B)$$

is the Green function of Y in B if $\overline{B} \subset D$. As we mentioned earlier, if such B satisfies the exterior cone condition then $\mathbf{P}_x(X_{\tau_B -} = X_{\tau_B}) = 0$ and so by the characterization of Y via Feynman-Kac transform from X^D, we have $\mathbf{P}_x(Y_{\tau_B -} = Y_{\tau_B}) = 0$. In passing we note, by integrating both sides of (3.23), that

$$\begin{aligned} &\text{if} \quad \inf_{v \in B} \int_{D \setminus B} \frac{\mathcal{A}(n, -\alpha)}{|v-y|^{n+\alpha}} dy \geq c > 0, \\ &\text{then} \quad \mathbf{E}_x \tau_B = \int_D G_B^Y(x, v)\, dv \leq 1/c < \infty \end{aligned} \quad (3.24)$$

for every $x \in B$ (and hence for every $x \in D$). This in particular implies that $\tau_B < \infty$, $\mathbf{P}_x$-a.s. for Y.

From Definition 3.1, (3.18) and (3.23), we see that if u is a harmonic, superharmonic or subharmonic function of Y in $U \subset D$, then u must satisfy the following integrability condition

$$\int_D \frac{|u(y)|}{(1+|y|)^{n+\alpha}}\, dy < \infty. \quad (3.25)$$

This is because the expectation in (3.1) is assumed to converge absolutely and by (3.23) the $\mathbf{P}_x$-distribution of Y_{τ_B} on the interior of $D \setminus B$ has the density function

$$K_B(x, y) = \mathcal{A}(n, -\alpha) \int_B \frac{G_B(x, v)\, \mathbf{E}_x^v e_\kappa(\tau_B)}{|v-y|^{n+\alpha}}\, dv \geq const. \cdot (1+|y|)^{-n-\alpha}.$$

On the other hand, it is easy to see from (3.12) and (3.11) that if $A_D^\alpha u$ ($\Delta^{\alpha/2} u$) exists and is less than positive infinity at two points, then u satisfies condition (3.25) ((3.9), respectively).

The following result will be used on several occasions.

Lemma 3.3. *Assume that v is continuous on $U \subset D$ and $A_D^\alpha v(x) \leq 0$ on U. Then v is superharmonic on U for the censored process Y_t on D.*

Proof. Without loss of generality, we may and do assume that $v = 0$ on D^c. As $A^\alpha_D v < \infty$ exists in U, v satisfies the integrability condition (3.25). Let B be an open bounded set such that $\overline{B} \subset U$. We may and do assume that B is regular for the process X, or equivalently for the process Y. We let $w(x) = \mathbf{E}_x v(Y_{\tau_B}) = \mathbf{E}_x[e_\kappa(\tau_B)v(X_{\tau_B})]$, $x \in \mathbf{R}^n$, which is well defined in view of continuity of v, (3.23) and (3.25). The function w is continuous and κ-harmonic for X_t or harmonic for Y_t in B. Using pathwise integration, one has

$$w(x) = \mathbf{E}_x v(X_{\tau_B}) + G_B(\kappa w)(x)\,, \quad x \in \mathbf{R}^n\,. \tag{3.26}$$

Note that w is smooth in B. Indeed, for any ball $B_1 \subset B$ we have

$$w(x) = \mathbf{E}_x v(X_{\tau_{B_1}}) + G_{B_1}(\kappa w)(x), \quad x \in B_1.$$

The function w is smooth in B_1—the smoothness of w follows from the smoothness of κ and smoothing properties of the Green operator and so it can be viewed as a consequence of the explicit formulas (3.4) and (3.5). By Theorem 5.5 and Lemma 3.8 in [10] we have

$$A^\alpha_D w(x) = (\Delta^{\alpha/2} + \kappa)w(x) = 0 \tag{3.27}$$

pointwise in B_1 (and hence in B). To complete the proof of the lemma we only need to verify that $r = v - w \geq 0$ on B. We note that r is continuous on $\mathbf{R}^n$, vanishes on B^c and

$$(\Delta^{\alpha/2} + \kappa)r(x) = A^\alpha_D v(x) \leq 0 \text{ on } \quad B.$$

Suppose that r has a negative minimum at some point $x_0 \in B$. Then

$$(\Delta^{\alpha/2} + \kappa)r(x_0) = \mathcal{A}(n, -\alpha)P.V. \int_D \frac{r(y) - r(x_0)}{|y - x_0|^{n+\alpha}}\, dy > 0$$

because r vanishes on B^c. This is a contradiction. □

Conversely, we note that if, for example, a function v is harmonic on $U \subset D$ for the censored process Y in the sense of Definition 3.1 then (3.26) is satisfied with $w = v$ and so by (3.27) $A^\alpha_D v(x) = 0$ for $x \in U$.

For a C^2 function ϕ on $\mathbf{R}^n$ we write $\|\phi\|_{C^2} = \sum_{|j|\leq 2} \|D^j\phi\|_\infty$; here j ranges over multi-indices.

Lemma 3.4. *Let $\alpha \in (1, 2)$ and D be an open set in $\mathbf{R}^n$. Let ϕ be a C^2 function on $\mathbf{R}^n$. There is $C_1 = C_1(n, \alpha)$ such that*

$$|A^\alpha_D \phi(x)| \leq C_1 \|\phi\|_{C^2}\, [1 + \rho(x)^{1-\alpha}]\,, \quad x \in D\,. \tag{3.28}$$

Proof. By Taylor's expansion with the remainder of order 2,

$$|\Delta^{\alpha/2}\phi(x)| = \left| \mathcal{A}(n, -\alpha)P.V. \int_{\mathbf{R}^n} \frac{\phi(y) - \phi(x)}{|y - x|^{n+\alpha}} dy \right| \leq c_1 \|\phi\|_{C^2}\,, \quad x \in \mathbf{R}^n\,,$$

where $c_1 = c_1(n, \alpha)$. Thus

$$
\begin{aligned}
|A_D^\alpha \phi(x)| &= \left| \mathcal{A}(n, -\alpha) P.V. \int_D \frac{\phi(y) - \phi(x)}{|y - x|^{n+\alpha}} dy \right| \\
&\leq c_1 \|\phi\|_{C^2} + \mathcal{A}(n, -\alpha) \int_{D^c} \frac{|\phi(y) - \phi(x)|}{|y - x|^{n+\alpha}} dy \\
&\leq c_1 \|\phi\|_{C^2} + \|\phi\|_{C^2} \cdot \mathcal{A}(n, -\alpha) \int_{D^c} \frac{|y - x| \wedge 2}{|y - x|^{n+\alpha}} dy \\
&\leq c_1 \|\phi\|_{C^2} + \|\phi\|_{C^2} \cdot \mathcal{A}(n, -\alpha) \int_{|z| > \rho(x)} \frac{|z| \wedge 2}{|z|^{n+\alpha}} dz \\
&\leq C_1 \|\phi\|_{C^2} [1 + \rho(x)^{1-\alpha}].
\end{aligned}
$$

□

(3.28) has a logarithmic analogue for $\alpha = 1$, but we will not need such an estimate for $\alpha \leq 1$.

4. One-dimensional censored stable processes

This section is entirely devoted to the analysis of one-dimensional censored stable processes on the half-line. More specifically, we will address the question of whether such processes hit the boundary of the domain, i.e., the origin. The results are a special case of much more general results presented in Theorems 1.1 and 2.7. In this section two alternative elementary proofs of Theorem 1.1 are given in the one-dimensional setting. We hope that simple techniques presented below will help the reader develop an intuitive picture of the path behavior of censored processes.

In the remainder of the section we assume that $n = 1$ and we denote $D = (0, \infty)$. Let Y be the censored α-stable process in D, defined in Section 2, with $\alpha \in (0, 2)$.

4.1. Kelvin transform

Let

$$
w_p(x) = \begin{cases} x^p & \text{if } x > 0\,, \\ 0 & \text{if } x \leq 0\,, \end{cases} \tag{4.1}
$$

and note that $w_0 = \mathbf{1}_{(0,\infty)}$, the indicator function of $(0, \infty)$. We also let

$$
w_0^*(x) = \begin{cases} \log x & \text{if } x > 0\,, \\ 0 & \text{if } x \leq 0\,. \end{cases}
$$

Lemma 4.1. *If $\alpha \neq 1$ then the functions w_0 and $w_{\alpha-1}$ are harmonic in D with respect to Y. If $\alpha = 1$ then w_0 and w_0^* are harmonic in D with respect to Y.*

Proof. The argument for w_0 is straightforward. If $\alpha \in (0,2)$ and $x \in (0,\infty)$ then

$$A^\alpha_D w_0(x) = \mathcal{A}(1,-\alpha) P.V. \int_0^\infty \frac{w_0(x)-w_0(y)}{|x-y|^{1+\alpha}} dy = 0\,.$$

By Lemma 3.3, w_0 is harmonic on D for Y.

The Kelvin transform of D is $TD := \{x/|x|^2;\, x \in D\}$, so in the present context $TD = D$. The Kelvin transform of $w_0(x)$ is the function Tw_0 given by

$$Tw_0(x) = |x|^{\alpha-1} w_0(x/x^2) = w_{\alpha-1}(x)\,, \quad x \neq 0\,.$$

We have

$$\kappa(x) = \kappa^\alpha_D(x) = \mathcal{A}(1,-\alpha) \int_{-\infty}^0 \frac{dy}{|x-y|^{1+\alpha}} \tag{4.2}$$

$$= \frac{\mathcal{A}(1,-\alpha)}{\alpha} x^{-\alpha}\,. \tag{4.3}$$

The function w_0 is κ-harmonic for the symmetric stable process X on $\mathbf{R}^n$ so Tw_0 is q-harmonic for X on TD, where $q(x) = |x|^{-2\alpha}\kappa(x/|x|^2)$, by Theorem 8.4 of [9]. Here, by (4.3), we have $q(x) = \kappa(x)$, $x \in D$. It follows that $w_{\alpha-1}$ is κ-harmonic for X^D or harmonic for Y in D.

To see that w_0^* is harmonic in the case of $\alpha = 1$, note that

$$\frac{w_{\alpha-1}(x) - w_0(x)}{\alpha - 1} \to w_0^*(x)\,, \text{ as } \alpha \to 1\,, \quad x \in \mathbf{R}^1\,.$$

By a limiting argument, one has

$$(\Delta^{1/2} + \kappa) w_0^*(x) = 0\,, \quad x \in D\,,$$

where $\kappa = \kappa^1_D$. The proof is complete. □

Let ζ be the lifetime of the censored process Y on $D = (0,\infty)$.

Proposition 4.2. *The following statements hold* $\mathbf{P}_x$*-a.s. for every* $x \in (0,\infty)$.

(1) If $\alpha \in (0,1)$ *then* $\zeta = \infty$ *and* $\lim_{t\to\infty} Y_t = \infty$.
(2) If $\alpha \in (1,2)$ *then* $0 < \zeta < \infty$ *and* $\lim_{t\uparrow\zeta} Y_t = 0$.
(3) If $\alpha = 1$ *then* $\zeta = \infty$ *and the limit of* Y_t *as* $t \to \infty$ *does not exist.*

Proof. First we will show that if for some x, $\mathbf{P}_x(\zeta < \infty) > 0$ then $\mathbf{P}_x(\zeta < \infty) = 1$. We will prove this claim for $x = 1$ only—this does not cause any loss of generality, for the following reason. The scaling property of stable processes easily implies that if $\{Y_t, t \ge 0\}$ has the distribution $\mathbf{P}_x$ then the distribution of $\{cY_{t/c^\alpha}, t \ge 0\}$ is $\mathbf{P}_{cx}$. Hence, $\mathbf{P}_x(\zeta < \infty) = 1$ holds for every x or it does not hold for any $x \in (0,\infty)$.

Assume that $\mathbf{P}_1(\zeta < \infty) > 0$ and find $t_0 < \infty$ and $p > 0$ such that $\mathbf{P}_1(\zeta < t_0) > p$. Consider the following sequence of stopping times, $T_0 = \zeta \wedge t_0$, $T_{k+1} = \zeta \wedge (T_k + t_0(Y_{T_k})^\alpha)$, $k \ge 0$. Since $(Y_\zeta)^\alpha$ is undefined, we declare that $T_{k+1} = \zeta$

if $T_k = \zeta$. By the scaling property of Y_t and the strong Markov property applied at T_k,

$$\mathbf{P}_1(\zeta > T_{k+1} \mid \zeta > T_k) \le 1 - p.$$

This implies that $\mathbf{P}_1(\zeta > T_k) \le (1-p)^{k+1}$. We conclude that $\mathbf{P}_1(\zeta < \infty) = 1$ by letting $k \to \infty$.

We will now consider a censored stable process with $\alpha \in (0, 1)$. Let $I_a = [1/2, a]$ for $a > 1$. Note that $\tau_{I_a} < \infty$ a.s. for Y by (3.24). Since $w_{\alpha-1}$ is harmonic,

$$\begin{aligned} 1 &= w_{\alpha-1}(1) = \mathbf{E}_1 w_{\alpha-1}(Y_{\tau_{I_a}}) \\ &= \mathbf{E}_1[Y_{\tau_{I_a}} < 1/2\,;\; w_{\alpha-1}(Y_{\tau_{I_a}})] + \mathbf{E}_1[Y_{\tau_{I_a}} > a\,;\; w_{\alpha-1}(Y_{\tau_{I_a}})] \\ &\ge (1/2)^{\alpha-1}\mathbf{P}_1(Y_{\tau_{I_a}} < 1/2). \end{aligned} \tag{4.4}$$

Let T_A denote the hitting time of a set A. By (4.4), for all $a > 1$,

$$\mathbf{P}_1(T_{(a,\infty)} < T_{(0,1/2)}) = 1 - \mathbf{P}_1(Y_{\tau_{I_a}} < 1/2) \ge 1 - 2^{\alpha-1}. \tag{4.5}$$

Since $t \to Y_t$ is right continuous with left limits for $t < \zeta$, Y_t is bounded on every closed subinterval of $[0, \zeta)$. By this observation and letting $a \to \infty$ in (4.5) we have that $\mathbf{P}_1(T_{(0,1/2)} = \infty) > 0$. This does not immediately imply that $\mathbf{P}_1(\zeta = \infty) > 0$ because, in principle, the process could die in a finite time without hitting $(0, 1/2)$. We will argue that this cannot happen. Note that for $t_0 > 0$, there is $p = p(t_0, \alpha) > 0$ such that the unconstrained symmetric α-stable process X starting from $x \ge 1/2$ stays on the positive half line for at least t_0 units of time with probability p or higher. Thus by the Ikeda-Nagasawa-Watanabe construction of Y and the Borel-Cantelli lemma, we have $\mathbf{P}_1(\zeta = \infty) > 0$, and so, in view of the opening remarks of this proof, we see that $\mathbf{P}_1(\zeta = \infty) = 1$. We will now show that $\lim_{t\to\infty} Y_t = \infty$. Since $w_{\alpha-1}$ is a non-negative harmonic function for Y, $Y_t^{\alpha-1}$ is a non-negative supermartingale and therefore $\lim_{t\to\infty} Y_t^{\alpha-1}$ exists a.s. and

$$\mathbf{E}_x\left[\lim_{t\to\infty} Y_t^{\alpha-1}\right] \le x^{\alpha-1}. \tag{4.6}$$

Since Y_t is a Hunt process, with probability one, the limit $\lim_{t\to\infty} Y_t$ is not a number in $(0, \infty)$. In view of the existence of $\lim_{t\to\infty} Y_t^{\alpha-1}$ and the assumption that $\alpha \in (0, 1)$, (4.6) rules out the possibility that the limit is zero with positive probability, and so $\lim_{t\to\infty} Y_t = \infty$ a.s.

Next we assume that $\alpha \in (1, 2)$ and proceed along similar lines. Let $J_a = [a, 2]$ for $a \in (0, 1)$. It follows from (3.24) that $\tau_{J_a} < \infty$ a.s. We have by the harmonicity of $w_{\alpha-1}$,

$$\begin{aligned} 1 &= w_{\alpha-1}(1) = \mathbf{E}_1 w_{\alpha-1}(Y_{\tau_{J_a}}) \\ &= \mathbf{E}_1[Y_{\tau_{J_a}} < a\,;\; w_{\alpha-1}(Y_{\tau_{J_a}})] + \mathbf{E}_1[Y_{\tau_{J_a}} > 2\,;\; w_{\alpha-1}(Y_{\tau_{J_a}})] \\ &\ge 2^{\alpha-1}\mathbf{P}_1(Y_{\tau_{J_a}} > 2). \end{aligned}$$

This implies that, for every $a \in (0, 1)$,

$$\mathbf{P}_1(T_{(0,a)} < T_{(2,\infty)}) = 1 - \mathbf{P}_1(Y_{\tau_{J_a}} > 2) \ge 1 - (1/2)^{\alpha-1}. \tag{4.7}$$

The unconstrained symmetric α-stable process jumps from any point of $(0, 2]$ to $(2, \infty)$ at a rate bounded below, so the same holds for the process Y, in view of Theorem 2.1(3). This together with (3.24) implies that $\zeta \wedge T_{(2,\infty)} < \infty$, a.s. Letting $a \to 0$ in (4.7), we obtain $\mathbf{P}_1(T_{(2,\infty)} = \infty) > 0$ so $\mathbf{P}_1(\zeta < \infty) > 0$. It follows that $\mathbf{P}_1(\zeta < \infty) = 1$, by the claim at the beginning of the proof. Since $k \mapsto w_{\alpha-1}(Y_{\tau_{[1/k,\,k]}})$ is a positive martingale under $\mathbf{P}_1$, by the martingale convergence theorem, $\lim_{k\to\infty} w_{\alpha-1}(Y_{\tau_{[1/k,\,k]}})$ exists $\mathbf{P}_1$-a.s., and

$$\mathbf{E}_x\left[\lim_{k\to\infty}(Y_{\tau_{[1/k,\,k]}})^{\alpha-1}\right] \le 1.$$

The finiteness of this expectation and $\alpha - 1 > 0$ imply that $Y_{\zeta-} = 0$ $\mathbf{P}_1$-a.s., as $\lim_{k\to\infty} Y_{\tau_{[1/k,\,k]}} = Y_{\zeta-}$. Hence $Y_{\zeta-} = 0$ $\mathbf{P}_x$-a.s., for every $x > 0$.

Finally, we will analyze the case $\alpha = 1$. We start with some preliminary remarks on the exit distribution of Y_t from an interval. Consider an interval $I = I_{[a,b]} = [a, b]$ for some $0 < a < 1 < b < \infty$, and $Y_0 \in (a, b)$. Conditioning on $\{Y_{\tau_I} > b, Y_{\tau_I-} = y\}$, the distribution of Y_{τ_I} has a density $f_y(x) = c(x-y)^{-2}$, for some constant c and $x > b$ (to see this, let the set B shrink to a point in (3.23)). For every $y \in I$ and $x > 2b$,

$$\frac{f_y(x)}{f_y(2b)} \le \frac{f_a(x)}{f_a(2b)} = \frac{(x-a)^{-2}}{(2b-a)^{-2}} \le (x/2b)^{-2}.$$

This shows that the conditional density $g(x)$ of Y_{τ_I} given $\{Y_{\tau_I} > b, Y_0 = z\}$ satisfies $g(x)/g(2b) \le (x/2b)^{-2}$ for $x > 2b$. Hence,

$$\mathbf{E}_z[Y_{\tau_I} > b\ ;\ w_0^*(Y_{\tau_I})] \le \log 2b + c\int_{2b}^{\infty} \log x (x/b)^{-2} dx \le c_1,$$

where $c_1 < \infty$ does not depend on a.

Similarly, the conditional density of Y_{τ_I} given $\{Y_{\tau_I} < a, Y_{\tau_I-} = y\}$ is $f_y(x) = c(y-x)^{-2}$. For every $y \in I$ and $x < a/2$,

$$\frac{f_y(x)}{f_y(a/2)} \le \frac{f_b(x)}{f_b(a/2)} = \frac{(b-x)^{-2}}{(b-a/2)^{-2}} \le 1.$$

Thus the conditional density $g(x)$ of Y_{τ_I} given $\{Y_{\tau_I} < a, Y_0 = z\}$ must satisfy $g(x) \le g(a/2)$ for $x < a/2$. We conclude that,

$$\mathbf{E}_z[Y_{\tau_I} < a\ ;\ w_0^*(Y_{\tau_I})] \ge \log a/2 + c\int_0^{a/2} \log x dx \ge -c_2,$$

where $c_2 < \infty$ does not depend on b.

Recall that the function w_0^* is harmonic for the censored stable process with $\alpha = 1$. Fix some $y \in (0, \infty)$ and consider $0 < a < 1 < b < \infty$ such that $a < y < b$. Then $\tau_I < \infty$ a.s. by (3.24) and we have

$$\begin{aligned}
\log y = w_0^*(y) &= \mathbf{E}_y w_0^*(Y_{\tau_I}) \\
&= \mathbf{E}_y[Y_{\tau_I} < a\ ;\ w_0^*(Y_{\tau_I})] + \mathbf{E}_y[Y_{\tau_I} > b\ ;\ w_0^*(Y_{\tau_I})] \\
&\ge -c_2 + \log b \cdot \mathbf{P}_y(Y_{\tau_I} > b) \\
&= -c_2 + \log b \cdot \mathbf{P}_y(T_{(b,\infty)} < T_{(0,a)}).
\end{aligned}$$

This implies that for any fixed a, $\mathbf{P}_y(T_{(b,\infty)} < T_{(0,a)}) \le (c_2 + \log y)/\log b \to 0$ as $b \to \infty$. Since the unconstrained symmetric α-stable process cannot remain in a compact subset of $(0,\infty)$ forever, by Theorem 2.1(3) and (3.24) neither does Y. The last two observations imply that $\mathbf{P}_y(T_{(0,a)} < \infty) = 1$, for every $a < y$.

Similarly,

$$\begin{aligned}\log y = w_0^*(y) &= \mathbf{E}_y w_0^*(Y_{\tau_I})\\ &\le c_1 + \log a \cdot \mathbf{P}_y(Y_{\tau_I} < a)\\ &= c_1 + \log a \cdot \mathbf{P}_y(T_{(0,a)} < T_{(b,\infty)}).\end{aligned}$$

It follows that for any fixed b, $\lim_{a\to 0} \mathbf{P}_y(T_{(0,a)} < T_{(b,\infty)}) = 0$. This in turn implies that $\mathbf{P}_y(T_{(b,\infty)} < \infty) = 1$ for every $b > y$. So Y_t oscillates between 0 and ∞ as $t \to \zeta$. This does not immediately imply that $\zeta = \infty$ because the process could oscillate between 0 and ∞ on a finite interval.

We will argue that for every bounded interval $I \subset [0, \infty)$, Y_t is bounded by a (random) constant on $I \cap [0, \zeta)$. This will imply that $\zeta = \infty$ a.s. Choose arbitrarily large $p < 1$ and $t_0 < \infty$ and find x_0 so large that the (uncensored) process X starting from any point $x \in [x_0, \infty)$ will stay in $(0, x + x_0)$ for all $t \in [0, t_0]$ with probability p or higher. It follows from the Ikeda-Nagasawa-Watanabe construction of the censored process that Y has the same property. By the strong Markov property applied to Y at the stopping time $T_{(x_0,\infty)}$, the process Y will not leave the interval $(0, Y_{T_{(x_0,\infty)}} + x_0)$ for all $t \in [T_{(x_0,\infty)}, T_{(x_0,\infty)} + t_0]$, with probability p or higher. By letting $p \to 1$ and $t_0 \to \infty$, we see that Y must be bounded on $I \cap [0, \zeta)$ a.s. for every closed interval $I \subset [0, \infty)$. □

4.2. *Logarithmic transform*

We will sketch another proof of Proposition 4.2, based on a logarithmic transform.

Second Proof of Proposition 4.2. By (3.14), the generator of the process Y_t can be written as

$$A_Y\phi(x) = \mathcal{A}(1,-\alpha) \lim_{\varepsilon\to 0+} \int_{(0,x-\varepsilon)\cup(x+\varepsilon,\infty)} \frac{\phi(y)-\phi(x)}{|y-x|^{1+\alpha}} dy \tag{4.8}$$

$$= \mathcal{A}(1,-\alpha) P.V. \int_0^\infty [\phi(y)-\phi(x)]\mu_x^Y(dy), \quad x > 0, \tag{4.9}$$

where $\mu_x^Y(dy) = |y-x|^{-1-\alpha} dy$. Recall that the "principal value" (P.V.) is defined to be the limit of integrals as on the right hand side of (4.8); in other words, the equality (4.9) is a tautology.

We will show that the generator of $Z_t = \log Y_t$ is given by the formula

$$A_Z\phi(x) = \mathcal{A}(1,-\alpha) P.V. \int_{-\infty}^\infty [\phi(y)-\phi(x)]\mu_x^Z(dy), \ x \in \mathbf{R}, \tag{4.10}$$

where $\mu_x^Z([a,b]) = \mu_{e^x}^Y([e^a, e^b])$, and, consequently,

$$\mu_x^Z(dy) = |e^y - e^x|^{-1-\alpha} e^y dy = e^{-\alpha x} e^{y-x} |1 - e^{y-x}|^{-1-\alpha} dy.$$

For $\phi \in C_c^\infty(\mathbf{R})$, the expression in (4.10) may be written as

$$\mathcal{A}(1,-\alpha) \lim_{\varepsilon\to 0+} \int_{(-\infty,x-\varepsilon)\cup(x+\varepsilon,\infty)} [\phi(y)-\phi(x)]\mu_x^Z(dy), \tag{4.11}$$

while (4.8) yields

$$\begin{aligned} A_Z\phi(x) &= \lim_{t\downarrow 0} \frac{\mathbf{E}_x[\phi(Z_t)]-\phi(x)}{t} \\ &= \lim_{t\downarrow 0} \frac{\mathbf{E}_{e^x}[(\phi\circ\log)(Y_t)-(\phi\circ\log)(e^x)}{t} \\ &= A_Y(\phi\circ\log)(e^x) \\ &= \mathcal{A}(1,-\alpha) \lim_{\varepsilon\to 0+} \int_{(-\infty,\log(e^x-\varepsilon))\cup(\log(e^x+\varepsilon),\infty)} [\phi(y)-\phi(x)]\mu_x^Z(dy) \\ &= \mathcal{A}(1,-\alpha) \lim_{\varepsilon\to 0+} \int_{(-\infty,\log(e^x-\varepsilon e^x))\cup(\log(e^x+\varepsilon e^x),\infty)} [\phi(y)-\phi(x)]\mu_x^Z(dy). \end{aligned} \tag{4.12}$$

In view of (4.12), to prove (4.10) it will suffice to show that

$$\begin{aligned} W = \Bigg| &\int_{(-\infty,x-\varepsilon)\cup(x+\varepsilon,\infty)} [\phi(y)-\phi(x)]\mu_x^Z(dy) \\ &- \int_{(-\infty,\log(e^x-\varepsilon e^x))\cup(\log(e^x+\varepsilon e^x),\infty)} [\phi(y)-\phi(x)]\mu_x^Z(dy)\Bigg| \end{aligned}$$

converges to 0 as $\varepsilon \to 0$. Note that for $\varepsilon \in (0,1)$ we have

$$\log(e^x-\varepsilon e^x) = x+\log(1-\varepsilon) < x-\varepsilon < x+\log(1+\varepsilon) = \log(e^x+\varepsilon e^x) < x+\varepsilon\,.$$

Thus,

$$\begin{aligned} W &\le \int_{(x+\log(1-\varepsilon),x-\varepsilon)} |\phi(y)-\phi(x)|\mu_x^Z(dy) \\ &\quad + \int_{(x+\log(1+\varepsilon),x+\varepsilon)} |\phi(y)-\phi(x)|\mu_x^Z(dy) \\ &= O(\varepsilon^2)\cdot O(\varepsilon)\cdot O(\varepsilon^{-1-\alpha}) = O(\varepsilon^{2-\alpha}). \end{aligned}$$

This converges to 0 as $\varepsilon \to 0$ because $\alpha < 2$ and so (4.10) is proved.

We define a clock $C_t = \inf\{r > 0 : \int_0^r \exp(-\alpha Z_s)ds > t\}$ and time-change Z_t to obtain a new process $V_t = Z_{C_t}$. From (4.10) and Theorem 1.3 in Chapter 6 of [26] we see that the generator of V_t is equal to

$$A_V\phi(x) = \mathcal{A}(1,-\alpha)P.V.\int_{-\infty}^{\infty} [\phi(y)-\phi(x)]\mu_x^V(dy),$$

with

$$\mu_x^V(dy) = e^{y-x}|1-e^{y-x}|^{-1-\alpha}dy.$$

In other words, V_t is a Lévy process with no drift nor Gaussian component and has the Lévy measure $\Pi(dy) = e^y|1-e^y|^{-1-\alpha}dy$. Let $\Pi_1(dy) = \min\left\{e^y|1-e^y|^{-1-\alpha}, e^{-y}|1-e^{-y}|^{-1-\alpha}\right\}dy$ and $\Pi_2(dy) = \Pi(dy) - \Pi_1(dy)$. The process V_t may be represented as $V_t = V_t^1 + V_t^2$, where V_t^1 and V_t^2 are independent Lévy processes with Lévy measures Π_1 and Π_2. It follows from the definition that Π_1 is symmetric with respect to 0. Since the tails of the density of Π_1 go to zero exponentially fast, the process V_t^1 has increments with a finite variance and so it obeys the Central Limit Theorem. For $\alpha \in (0,1)$, Π_2 charges only the positive half-line so V_t^2 goes to infinity at a rate not smaller than linear and so $V_t \to \infty$ as $t \to \infty$. Similarly, for $\alpha \in (1,2)$, Π_2 is carried by $(-\infty, 0)$ and the same argument shows that V_t converges to $-\infty$. Finally, when $\alpha = 1$, $\Pi_2(dy)$ is the null measure, so $V_t = V_t^1$. In this case, V_t oscillates between ∞ and $-\infty$.

We will translate these results into the language of Y_t. Since $Y_t = \exp(Z_t)$ and Z_t is a time-change of V_t, the long-time behavior of V_t-trajectories determines the behavior of the paths of Y_t. When $\alpha \in (0,1)$, Y_t converges ∞. A simple argument in the last paragraph of the first proof of Proposition 4.2 has shown that Y_t is bounded on $[0, k \wedge \zeta)$ for every $k \ge 1$. Hence, $\zeta = \infty$ a.s. If $\alpha \in (1,2)$, Y_t converges to 0 at its lifetime. Since positive jumps of Y_t of size greater than 1 occur with intensity bounded below by a positive constant, the process cannot converge to 0 during an infinite time interval, i.e., we must have $\zeta < \infty$ and $\lim_{t\to\zeta-} Y_t = 0$, a.s. Now assume that $\alpha = 1$. In this case Y_t oscillates between 0 and ∞ infinitely many times on every interval $(\zeta - 1/n, \zeta)$, $n \ge 1$, if $\zeta < \infty$, or on every interval (n, ζ), $n \ge 1$, if $\zeta = \infty$. The first alternative is ruled out, as it was shown in the last paragraph of the first proof of Proposition 4.2 that Y_t is bounded on $[0, k \wedge \zeta)$ for every $k \ge 1$. □

Recall that the one-dimensional symmetric α-stable process X_t hits 0 if and only if $\alpha \in (1,2)$; censoring of jumps of X_t has no impact on this property.

5. Analysis in special $C^{1,\beta-1}$ domains

We will continue our investigation of the probability that the censored process approaches the boundary ∂D of a domain D in a finite time. We will shift our attention from one-dimensional processes discussed in the previous section to multidimensional processes in relatively smooth domains. We will derive a number of explicit estimates on our way to Theorem 5.10 which gives an upper bound for the probability of the event that the censored process starting from a point near ∂D visits a large set inside D before being killed at ∂D.

In this section, unless otherwise stated, we will only consider processes in subsets of $\mathbf{R}^n$ with $n \ge 2$. Our arguments will rely on the following explicit calculations for half-spaces.

For a point $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ we write $x = (\widetilde{x}, x_n)$, where $\widetilde{x} = (x_1, \dots, x_{n-1})$. Vectors in $\mathbf{R}^{n-1}$ will be denoted $\widetilde{x}$, $\widetilde{y}$, etc. As usual, $\widetilde{x} \cdot \widetilde{y}$ will stand for the scalar product of $\widetilde{x}$ and $\widetilde{y}$ and $|\widetilde{x}|$ will be the Euclidean norm of $\widetilde{x}$. Let

$\mathbf{R}^n_+ = \{(\widetilde{x}, x_n) \in \mathbf{R}^n : \ x_n > 0\}$. Consider $\alpha \in (0, 2)$ and $p \in (-1, \alpha)$. We define

$$w_p(x) = \mathrm{dist}(x, (\mathbf{R}^n_+)^c)^p = \begin{cases} x_n^p & \text{if } x_n > 0\,, \\ 0 & \text{if } x_n \le 0\,. \end{cases} \tag{5.1}$$

First we will derive a formula for $I(x) := A^\alpha_{\mathbf{R}^n_+} w_p(x) = (\Delta^{\alpha/2} + \kappa_{\mathbf{R}^n_+}) w_p(x)$, where $x \in \mathbf{R}^n_+$. More explicitly we have

$$I(x) = \mathcal{A}(n, -\alpha) \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}^n_+ \setminus B(x,\varepsilon)} \frac{y_n^p - x_n^p}{|y - x|^{n+\alpha}}\, dy\,.$$

The limit exists by the remarks following (3.11). Let e_n denote $(0, \dots, 0, 1) \in \mathbf{R}^n$. By a change of variable $z = (y - (\widetilde{x}, 0))/x_n$, we have

$$I(x) = x_n^{p-\alpha} \mathcal{A}(n, -\alpha) W,$$

where

$$\begin{aligned} W &= P.V.\int_{\mathbf{R}^n_+} \frac{z_n^p - 1}{|z - e_n|^{n+\alpha}}\, dz = \int_{\mathbf{R}^n_+} \frac{z_n^p - 1 - p(z_n - 1)\mathbf{1}_{B(e_n,1)}(z)}{|z - e_n|^{n+\alpha}}\, dz \\ &= \int_{\mathbf{R}^{n-1}} \int_0^\infty \frac{t^p - 1 - p(t-1)\mathbf{1}_{\{(t-1)^2 + |v|^2 < 1\}}}{[|v|^2 + (t-1)^2|]^{(n+\alpha)/2}}\, dt\, dv\,. \end{aligned}$$

A change of variable $v = |t - 1| u$ turns this into

$$\int_{\mathbf{R}^{n-1}} (|u|^2 + 1)^{-(n+\alpha)/2} \int_0^\infty \left(\frac{t^p - 1}{|t-1|^{1+\alpha}} - p \frac{t-1}{|t-1|^{1+\alpha}} \mathbf{1}_{\{(t-1)^2(|u|^2+1) < 1\}} \right) dt\, du\,.$$

The last term in the second integral is anti-symmetric in t with respect to $t = 1$. Thus

$$W = \int_{\mathbf{R}^{n-1}} (|u|^2 + 1)^{-(n+\alpha)/2}\, du \cdot P.V.\int_0^\infty \frac{t^p - 1}{|t-1|^{1+\alpha}}\, dt\,.$$

Using polar coordinates we obtain

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} (|u|^2 + 1)^{-(n+\alpha)/2}\, du &= \omega_{n-1} \int_0^\infty r^{n-2} (r^2 + 1)^{-(n+\alpha)/2}\, dr \\ &= \frac{\omega_{n-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{n-1}{2} \right), \end{aligned}$$

where $\mathcal{B}$ is the beta function and ω_{n-1} denotes the $(n-2)$-dimensional Lebesgue measure of the unit sphere in $\mathbf{R}^{n-1}$.

For $\varepsilon \in (0, 1)$, by a change of variable we have

$$\begin{aligned} &\int_{(0,\infty)\setminus(1-\varepsilon, 1+\varepsilon)} \frac{t^p - 1}{|t-1|^{1+\alpha}}\, dt \\ &\quad = \int_0^{1-\varepsilon} \frac{t^p - 1}{(1-t)^{1+\alpha}}\, dt + \int_0^{1/(1+\varepsilon)} \frac{t^{-p} - 1}{(1/t - 1)^{1+\alpha}} t^{-2}\, dt \\ &\quad = \int_0^{1-\varepsilon} \frac{t^p - 1}{(1-t)^{1+\alpha}} (1 - t^{\alpha - p - 1})\, dt + R\,, \end{aligned}$$

where $R = \int_{1-\varepsilon}^{1/(1+\varepsilon)} (1-t^p)(1-t)^{-1-\alpha} t^{\alpha-p-1}\,dt$. Since $1-\varepsilon < 1/(1+\varepsilon) < 1-\varepsilon+\varepsilon^2 < 1$,

$$
\begin{aligned}
|R| &\le \varepsilon^2 \frac{|1-(1-\varepsilon)^p|}{(1-(1-\varepsilon+\varepsilon^2))^{1+\alpha}} [(1-\varepsilon)^{\alpha-p-1} \vee (1-\varepsilon+\varepsilon^2)^{\alpha-p-1}] \\
&\le \text{const} \cdot \varepsilon^{2-\alpha} \to 0 \qquad \text{as} \qquad \varepsilon \to 0^+ .
\end{aligned}
$$

We conclude that

$$
P.V. \int_0^\infty \frac{t^p - 1}{|t-1|^{1+\alpha}}\,dt = \int_0^1 \frac{(t^p-1)(1-t^{\alpha-p-1})}{(1-t)^{1+\alpha}}\,dt,
$$

the latter integral being absolutely convergent. We denote

$$
\gamma(\alpha, p) = \int_0^1 \frac{(t^p-1)(1-t^{\alpha-p-1})}{(1-t)^{1+\alpha}}\,dt, \qquad \alpha \in (0,2),\ p \in (-1,\alpha)\,, \tag{5.2}
$$

and we observe that

$$
\begin{aligned}
\gamma(\alpha,p) \le 0 \qquad &\text{if and only if} \qquad p(\alpha-p-1) \ge 0\,, \\
\gamma(\alpha,p) \ge 0 \qquad &\text{if and only if} \qquad p(\alpha-p-1) \le 0\,.
\end{aligned} \tag{5.3}
$$

We summarize our calculations as follows,

$$
A^\alpha_{\mathbf{R}^n_+} w_p(x) = x_n^{p-\alpha} \mathcal{A}(n,-\alpha) \frac{\omega_{n-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{n-1}{2}\right) \gamma(\alpha,p)\,,\ \ x \in \mathbf{R}^n_+\,; n \ge 2. \tag{5.4}
$$

In dimension $n = 1$ we simply have (see (3.12), (3.14) and (4.1) for the notation),

$$
A^\alpha_{\mathbf{R}_+} w_p(x) = x^{p-\alpha} \mathcal{A}(n,-\alpha) \gamma(\alpha,p)\,, \qquad x > 0\,. \tag{5.5}
$$

For later reference we also note that by a similar but simpler calculation than that giving (5.4) we have

$$
\begin{aligned}
\kappa_{\mathbf{R}^n_+}(x) &= \mathcal{A}(n,-\alpha) \int_{(\mathbf{R}^n_+)^c} \frac{dy}{|y-x|^{n+\alpha}} = x_n^{-\alpha} \mathcal{A}(n,-\alpha) \int_{(\mathbf{R}^n_+)^c} \frac{dz}{|z-e_n|^{n+\alpha}} \\
&= x_n^{-\alpha} \mathcal{A}(n,-\alpha) \int_0^\infty \int_{\mathbf{R}^{n-1}} [|v|^2 + (t+1)^2]^{-(n+\alpha)/2}\,dv\,dt \\
&= x_n^{-\alpha} \frac{\mathcal{A}(n,-\alpha)}{\alpha} \frac{\omega_{n-1}}{2} \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{n-1}{2}\right) .
\end{aligned} \tag{5.6}
$$

In view of (5.3), the right hand side of (5.4) equals zero for $p = \alpha - 1$. We obtain the following result from Lemma 3.3.

Lemma 5.1. *The function*

$$
w_{\alpha-1}(x) = \begin{cases} x_n^{\alpha-1} & \text{if } x_n > 0\,, \\ 0 & \text{if } x_n \le 0\,, \end{cases}
$$

is harmonic for the censored process Y on $\mathbf{R}^n_+$.

For completeness we define

$$w_0^*(x) = \begin{cases} \log x_n & \text{if } x_n > 0\,, \\ 0 & \text{if } x_n \le 0\,. \end{cases} \tag{5.7}$$

Note that $w_0^*(x) = \lim_{p\to 0^+}(w_p(x) - 1)/p$ for $x \in \mathbf{R}^n_+$. By an easy limiting procedure we have the following result for $\alpha = 1$.

Corollary 5.2. *We have*

$$(\Delta^{1/2} + \kappa^1_{\mathbf{R}^n_+})w_0^*(x) = 0\,, \quad x \in \mathbf{R}^n_+\,. \tag{5.8}$$

The result can also be obtained by a direct calculation.

Using the asymptotics of the harmonic functions as in the first proof of Proposition 4.2, we obtain the following result.

Corollary 5.3. *Let ζ be the lifetime of the censored process Y in $\mathbf{R}^n_+$. The following statements hold $\mathbf{P}_x$-a.s. for every $x \in \mathbf{R}^n_+$. If $\alpha \in (0,1)$ then $\zeta = \infty$ and $\lim_{t\to\infty}(Y_t)_n = \infty$. If $\alpha \in (1,2)$ then $0 < \zeta < \infty$ and $\lim_{t\uparrow\zeta} Y_t$ exists with $\lim_{t\uparrow\zeta}(Y_t)_n = 0$. If $\alpha = 1$ then $\zeta = \infty$ and the limit of $(Y_t)_n$ as $t \to \infty$ does not exist.*

Lemma 5.1, Corollary 5.2, and Corollary 5.3 can also be derived directly from Lemma 4.1 and Proposition 4.2, respectively, as follows. The n-th coordinate $X_t^{(n)}$ of the symmetric α-stable process X_t in $\mathbf{R}^n$ is a symmetric α-stable process in $\mathbf{R}$. It is easy to check that the censored version of $X_t^{(n)}$ in $(0,\infty)$ has the same distribution as the n-th coordinate of Y_t in $\mathbf{R}^n_+$—this follows, for example, from the Ikeda-Nagasawa-Watanabe piecing together procedure.

Corollary 5.3 is a special case of Theorems 2.7 and 2.9.

In the remainder of the section, we will introduce a class of superharmonic functions for the censored process Y on *special $C^{1,\beta-1}$ domains* in $\mathbf{R}^n$, which we define below. In the remainder of the paper we focus on the case $1 < \alpha < 2$, to study the hitting probability of ∂D by Y.

Let $\Gamma : \mathbf{R}^{n-1} \to \mathbf{R}$. For $\beta \in (1,2]$, we say that Γ is a $C^{1,\beta-1}$ function if it is differentiable and

$$\|\Gamma\|_{1,\beta-1} = \sup_{\tilde{x}\neq\tilde{y}} \frac{|\nabla\Gamma(\tilde{x}) - \nabla\Gamma(\tilde{y})|}{|\tilde{x} - \tilde{y}|^{\beta-1}} < \infty\,, \tag{5.9}$$

where $\nabla\Gamma = (\partial\Gamma/\partial x_i)_{i=1}^{n-1}$. Note that $\|\Gamma\|_{1,\beta-1}$ is a seminorm which neither controls the value of Γ nor, more importantly, the value of $\nabla\Gamma$ at, say, $\tilde{0} \in \mathbf{R}^{n-1}$.

However, if (5.9) holds then by the mean-value theorem,

$$|\Gamma(\tilde{x}) - \{\Gamma(\tilde{0}) + \nabla\Gamma(\tilde{0})\cdot\tilde{x}\}| \le \|\Gamma\|_{1,\beta-1}|\tilde{x}|^\beta, \qquad \tilde{x} \in \mathbf{R}^{n-1}\,. \tag{5.10}$$

It is elementary to verify that $\Gamma(\tilde{x}) = |\tilde{x}|^\beta + b\cdot\tilde{x}$ is a $C^{1,\beta-1}$ function on $\mathbf{R}^{n-1}$ and $\|\Gamma\|_{1,\beta-1} \le 16$ for every $b \in \mathbf{R}^{n-1}$.

We shall consider a suitable family of "tangential" regions in $\mathbf{R}^n$. Let $C \ge 1$. We put

$$\mathcal{P} = \left\{x = (\tilde{x}, x_n) \in \mathbf{R}^n : \; C|\tilde{x}|^\beta < x_n < C^{-1}\right\}. \tag{5.11}$$

Using the vector $e_n = (0, \dots, 0, 1) \in \mathbf{R}^n$, $\mathcal{P}$ may be equivalently defined by the inequalities

$$C\left(|x|^2 - (x \cdot e_n)^2\right)^{\beta/2} < x \cdot e_n < C^{-1}.$$

For an arbitrary vector $b \in \mathbf{R}^n$ of unit length, we define a rotated version of $\mathcal{P}$,

$$\mathcal{P}_b = \left\{x \in \mathbf{R}^n : \; C\left(|x|^2 - (x \cdot b)^2\right)^{\beta/2} < x \cdot b < C^{-1}\right\}. \tag{5.12}$$

Every set $\mathcal{P}_b + x$ with $x \in \mathbf{R}^n$ and $\mathcal{P}_b$ as in (5.12) will be called a region of β-tangential approach of size C^{-1}.

A domain D will be called a *special* $C^{1,\beta-1}$ *domain* if for some $C^{1,\beta-1}$ function $\Gamma : \mathbf{R}^{n-1} \to \mathbf{R}$, the domain can be represented as $D = D_\Gamma = \{x = (\widetilde{x}, x_n) \in \mathbf{R}^n : x_n > \Gamma(\widetilde{x})\}$. For every such domain, at every boundary point $Q \in \partial D$, $Q = (\widetilde{Q}, \Gamma(\widetilde{Q}))$, there is a unique inward unit normal vector at Q, namely

$$b = \frac{(-\nabla\Gamma(\widetilde{Q}), 1)}{\sqrt{|\nabla\Gamma(\widetilde{Q})|^2 + 1}}. \tag{5.13}$$

The following elementary result will help us handle sets $\mathcal{P}_b$.

Lemma 5.4. *Let* $\beta \in (1, 2]$ *and suppose that* $0 < v \le u$*,* $c \ge 1$*, and*

$$c(u^2 - v^2)^{\beta/2} < v < c^{-1}. \tag{5.14}$$

Then

$$v > \frac{c^{\beta-1}}{2} u^\beta. \tag{5.15}$$

Proof. From (5.14) we obtain $u^2 - v^2 < (c^{-2})^{2/\beta} = c^{-4/\beta}$, hence $u^2 < c^{-4/\beta} + c^{-2} \le 2c^{-2}$. Suppose that (5.15) does not hold. Then

$$\begin{aligned} u^2 - v^2 &\ge u^2 - (c^{\beta-1}/2)^2 u^{2\beta} = u^2 - u^2 (c^{\beta-1}/2)^2 u^{2(\beta-1)} \\ &> u^2 - u^2 (c^{\beta-1}/2)^2 (2c^{-2})^{\beta-1} \\ &= u^2 (1 - 2^{\beta-3}) > u^2/2. \end{aligned}$$

Substituting this into (5.14), we have that $c(u^2/2)^{\beta/2} < v$. So $v > cu^\beta/2 \ge c^{\beta-1}u^\beta/2$, which proves (5.15). □

Lemma 5.5. *Let* $n \in \{2, 3, \dots\}$ *and* $\beta \in (1, 2]$*. Let* $\Gamma : \mathbf{R}^{n-1} \to \mathbf{R}$ *be a* $C^{1,\beta-1}$ *function and let* $D = D_\Gamma$*. There is* $C = C(\beta, \|\Gamma\|_{1,\beta-1}) \ge 1$ *such that for every* $Q \in \partial D$*, the region* $\mathcal{P}_b + Q$ *of* β*-tangential approach of size* C^{-1} *satisfies* $\mathcal{P}_b + Q \subset D$*, where b is the unit inward vector* (5.13)*. Also,* $\mathcal{P}_{-b} + Q \subset D^c$*.*

Proof. Fix any C satisfying the inequality $C \geq (2\|\Gamma\|_{1,\beta-1})^{\frac{1}{\beta-1}} \vee 1$. We will assume without loss of generality that $Q = 0 \in \partial D$. In view of (5.10), we only need to verify that for every $y = (\widetilde{y}, y_n) \in \mathbf{R}^n$ satisfying

$$C(|y|^2 - (y \cdot b)^2)^{\beta/2} < y \cdot b < C^{-1} \tag{5.16}$$

it holds that

$$y_n > \nabla\Gamma(\widetilde{0}) \cdot \widetilde{y} + \|\Gamma\|_{1,\beta-1}|\widetilde{y}|^\beta .$$

For any y which satisfies (5.16),

$$y_n - \nabla\Gamma(\widetilde{0}) \cdot \widetilde{y} = y \cdot b\sqrt{|\nabla\Gamma(\widetilde{0})|^2 + 1} \geq y \cdot b .$$

We apply Lemma 5.4 with $u = |y|$, $v = y \cdot b$ and $c = C$ to obtain

$$\begin{aligned} y_n - \nabla\Gamma(\widetilde{0}) \cdot \widetilde{y} &> \left[(2\|\Gamma\|_{1,\beta-1})^{\frac{1}{\beta-1}} \vee 1\right]^{\beta-1} |y|^\beta/2 \\ &= [(2\|\Gamma\|_{1,\beta-1}) \vee 1]|y|^\beta/2 \geq \|\Gamma\|_{1,\beta-1}|\widetilde{y}|^\beta . \end{aligned}$$

The last statement of the lemma can be proved similarly. □

In the remainder of the section we assume that $1 < \alpha < \beta \leq 2$ and we fix a $C^{1,\beta-1}$ function $\Gamma : \mathbf{R}^{n-1} \to \mathbf{R}$. We denote $D = D_\Gamma$ and as usual we put $\rho(x) = \mathrm{dist}(x, D^c)$.

Lemma 5.6. *There is $C_2 = C_2(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1})$ such that*

$$\begin{aligned} &\left|\kappa_D(x) - \rho(x)^{-\alpha}\frac{\mathcal{A}(n, -\alpha)}{\alpha}\frac{\omega_{n-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{n-1}{2}\right)\right| \\ &\quad \leq C_2\rho(x)^{-\alpha}[1 \wedge \rho(x)^{\beta-1}], \qquad x \in D . \end{aligned} \tag{5.17}$$

Proof. Let $C = C(\beta, \|\Gamma\|_{1,\beta-1}) \geq 1$ be the constant of Lemma 5.5. Assume that $x \in D$ and first consider the case when $\rho(x) \leq 1/(2C)$. Let $Q \in \partial D$ be such that $|x - Q| = \rho(x)$. Let b be the unit inward vector for D at Q, and let $\Pi_+ = \{x \in \mathbf{R}^n : (x - Q) \cdot b > 0\}$. It follows from (5.6) that

$$\kappa_{\Pi_+}(x) = \rho(x)^{-\alpha}\frac{\mathcal{A}(n, -\alpha)}{\alpha}\frac{\omega_{n-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{n-1}{2}\right) . \tag{5.18}$$

We will assume without loss of generality that $Q = 0$ and consider the regions of β-tangential approach $\mathcal{P}_b \subset D$ and $\mathcal{P}_{-b} \subset D^c$, tangent to ∂D at $Q = 0$, as in Lemma 5.5. Clearly, $x \in \mathcal{P}_b$ and

$$\kappa_{(\mathcal{P}_{-b})^c}(x) \leq \kappa_D(x) \leq \kappa_{\mathcal{P}_b}(x)$$

and

$$\kappa_{(\mathcal{P}_{-b})^c}(x) \leq \kappa_{\Pi_+}(x) \leq \kappa_{\mathcal{P}_b}(x) .$$

Hence we only need to estimate $R = \kappa_{\mathcal{P}_b}(x) - \kappa_{(\mathcal{P}_{-b})^c}(x)$. Without loss of generality we will assume that $b = e_n$, consequently, $\mathcal{P}_b = \mathcal{P}$, where $\mathcal{P}$ is given by (5.11); $\mathcal{P}_{-b} = -\mathcal{P}$ and $x = (\widetilde{0}, x_n)$ with $\rho(x) = x_n$. We have

$$\begin{aligned} R &= \mathcal{A}(n,-\alpha) \int_{(\mathcal{P})^c \cap (-\mathcal{P})^c} \frac{dy}{|y-x|^{n+\alpha}} \\ &= \mathcal{A}(n,-\alpha) \left[\int_{T^c} + \int_{T \setminus (\mathcal{P} \cup (-\mathcal{P}))} \right] = \mathcal{A}(n,-\alpha)\,[I_1 + I_2]\ , \end{aligned}$$

where $T = \{y \in \mathbf{R}^n :\ |\widetilde{y}| < C^{-2/\beta},\ |y_n| < C^{-1}\} \supset \mathcal{P} \cup (-\mathcal{P})$. Using polar coordinates we obtain

$$I_1 \le \int_{B(x, C^{-2/\beta}/2)^c} \frac{dy}{|y-x|^{n+\alpha}} = \frac{\omega_n}{\alpha} 2^\alpha C^{2\alpha/\beta} < \infty.$$

Let $T_1 = \{y \in T \setminus (\mathcal{P} \cup (-\mathcal{P})) :\ |\widetilde{y}| \le C^{-2/\beta} x_n/2\}$ and $T_2 = T \setminus (\mathcal{P} \cup (-\mathcal{P}) \cup T_1)$. Recall that $\rho(x) \le 1/(2C)$, so $x_n < 1$ and note that for $y \in T_1$,

$$x_n - C|\widetilde{y}|^\beta \ge x_n - x_n^\beta/(C2^\beta) \ge x_n - x_n/(C2^\beta) > x_n/2\,.$$

Hence, using polar coordinates, we obtain

$$\begin{aligned} &\int_{T_1} \left[|\widetilde{y}|^2 + (x_n - y_n)^2 \right]^{-(n+\alpha)/2} dy \\ &\quad \le \int_{\{u \in \mathbf{R}^{n-1}:\ |u| < C^{-2/\beta} x_n/2\}} \int_{-C|u|^\beta}^{C|u|^\beta} 2^{n+\alpha} x_n^{-n-\alpha}\, dt\, du \\ &\quad \le \frac{4C}{n+\beta-1} \omega_{n-1} x_n^{-\alpha+\beta-1}\,. \end{aligned} \tag{5.19}$$

On the other hand

$$\begin{aligned} &\int_{T_2} \left[|\widetilde{y}|^2 + (x_n - y_n)^2 \right]^{-(n+\alpha)/2} dy \\ &\quad \le \int_{\{u \in \mathbf{R}^{n-1}:\ C^{-2/\beta} x_n/2 < |u| \le C^{-2/\beta}\}} \int_{-C|u|^\beta}^{C|u|^\beta} |u|^{-n-\alpha}\, dt\, du \\ &\quad = 2C\omega_{n-1} \int_{C^{-2/\beta} x_n/2}^{C^{-2/\beta}} r^{\beta-\alpha-2}\, dr \\ &\quad \le 2^{2+\alpha-\beta} \frac{\omega_{n-1}}{\alpha+1-\beta} C^{(2\alpha+2-\beta)/\beta} x_n^{-\alpha+\beta-1}. \end{aligned} \tag{5.20}$$

As a consequence, $I_2 \le \text{const} \cdot x_n^{-\alpha+\beta-1}$, and so (5.17) holds provided $\rho(x) \le 1/(2C)$.

Next we assume that $\rho(x) > 1/(2C)$. Then

$$\kappa_D(x) \le \mathcal{A}(n,-\alpha) \int_{B(x,\rho(x))^c} \frac{dy}{|y-x|^{n+\alpha}} = \mathcal{A}(n,-\alpha) \frac{\omega_n}{\alpha} \rho(x)^{-\alpha}$$

and

$$\kappa_{\Pi_+}(x) \le \mathcal{A}(n, -\alpha)\frac{\omega_n}{\alpha}\rho(x)^{-\alpha}.$$

It follows that

$$|\kappa_D(x) - \kappa_{\Pi_+}(x)| \le \mathcal{A}(n, -\alpha)\frac{\omega_n}{\alpha}\rho(x)^{-\alpha}.$$

The proof is complete. □

The next theorem is the main technical result of this section. It gives explicit examples of superharmonic functions with respect to $A_D^\alpha = \Delta^{\alpha/2} + \kappa_D$ near the boundary of a special $C^{1,\beta-1}$ domain. Recall that $\rho(x) = \mathrm{dist}(x, D^c)$.

Theorem 5.7. *Let* $\alpha \in (1, 2)$, $p \in (0, \alpha - 1)$ *and*

$$v(x) = \rho(x)^p, \qquad x \in \mathbf{R}^n.$$

There is $C_3 = C_3(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1}, p) \le 1$ *such that for* x *satisfying* $0 < \rho(x) \le C_3$,

$$\left(\Delta^{\alpha/2} + \kappa_D(x)\right) v(x) \le \rho(x)^{p-\alpha}\mathcal{A}(n, -\alpha)\frac{\omega_{n-1}}{2}\mathcal{B}\left(\frac{\alpha+1}{2}, \frac{n-1}{2}\right) \times|\gamma(\alpha, p)|\left[-1 + \left(\frac{\rho(x)}{C_3}\right)^{\beta-1}\right]. \tag{5.21}$$

In particular, $(\Delta^{\alpha/2} + \kappa_D(x))v(x) < 0$ *provided* $0 < \rho(x) < C_3$.

Proof. We fix $\alpha, \beta, p, \Gamma, D$ and v as above. Let $x \in D$ be such that $\rho(x) \le 1/(2C)$ with $C = C(\beta, \|\Gamma\|_{1,\beta-1})$ of Lemma 5.5. We choose $Q \in \partial D$ such that $|x - Q| = \rho(x)$. As usual, b is the unit inward normal vector at $Q \in \partial D$. We consider the regions of β-tangential approach $\mathcal{P}' = \mathcal{P}_b + Q$ and $\mathcal{P}'' = \mathcal{P}_{-b} + Q$ of size C^{-1}. We also denote $\Pi_+ = \{x \in \mathbf{R}^n : (x - Q)\cdot b > 0\}$ and we define

$$w(y) = \mathrm{dist}(y, (\Pi_+)^c)^p = \begin{cases} ((y-Q)\cdot b)^p & \text{if } y \in \Pi_+, \\ 0 & \text{if } y \notin \Pi_+, \end{cases}$$

so that $w(x) = |x - Q|^p = \rho(x)^p = v(x)$. We have, with $\Delta^{\alpha/2}v(x)$ interpreted as in (3.11),

$$\begin{aligned}\Delta^{\alpha/2}v(x) + \kappa_D(x)v(x) &= (\Delta^{\alpha/2} + \kappa_D)(v - w)(x) \\ &\quad + (\Delta^{\alpha/2} + \kappa_{\Pi_+})w(x) + (\kappa_D(x) - \kappa_{\Pi_+}(x))w(x) \\ &= J_1 + J_2 + J_3.\end{aligned} \tag{5.22}$$

By (5.17) and (5.18),

$$|J_3| \le C_2\rho(x)^{p-\alpha+\beta-1} \tag{5.23}$$

with C_2 of Lemma 5.6. By (5.4), (5.6) and translation and rotation invariance of $\Delta^{\alpha/2}$,

$$J_2 = \rho(x)^p\kappa_{\Pi_+}(x)\alpha\gamma(\alpha, p), \tag{5.24}$$

where $\gamma(\alpha, p)$ is given by (5.2).

We will now find an upper bound for J_1. Since $v(x) = w(x)$ we have

$$\begin{aligned} J_1 &= \mathcal{A}(n,-\alpha)P.V. \int_{\mathbf{R}^n} \frac{v(y)-w(y)}{|y-x|^{n+\alpha}} dy \\ &= \mathcal{A}(n,-\alpha)\left[P.V. \int_{\mathcal{P}'} + \int_{\mathcal{P}''} + \int_{\mathbf{R}^n\setminus(\mathcal{P}'\cup\mathcal{P}'')}\right] \\ &= \mathcal{A}(n,-\alpha)\left[K_1+K_2+K_3\right] . \end{aligned} \tag{5.25}$$

Since $v(y) = 0 = w(y)$ for $y \in \mathcal{P}'' \subset D^c \cap (\Pi_+)^c$,

$$K_2 = 0 . \tag{5.26}$$

To estimate K_1 from above let $0 < \varepsilon \le \mathrm{dist}(x, (\mathcal{P}')^c)$ and let

$$K_1^\varepsilon = \int_{\mathcal{P}'\setminus B(x,\varepsilon)} \frac{v(y)-w(y)}{|y-x|^{n+\alpha}}\, dy \le \int_{\mathcal{P}'\setminus B(x,\varepsilon)} \frac{\mathrm{dist}(y,\mathcal{P}'')^p - \mathrm{dist}(y,(\Pi_+)^c)^p}{|y-x|^{n+\alpha}}\, dy . \tag{5.27}$$

We will assume without loss of generality that $Q = 0$, $b = e_n$ and, consequently, $x = (0,\dots,0,\rho(x))$ and $\Pi_+ = \mathbf{R}^n_+$. We have $\mathcal{P}' = \mathcal{P}$, $\mathcal{P}'' = -\mathcal{P}$, where $\mathcal{P}$ is given by (5.11). By (5.11) we obtain

$$K_1^\varepsilon \le \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{(y_n + C|\widetilde{y}|^\beta)^p - y_n^p}{|y-x|^{n+\alpha}}\, dy .$$

For $y \in \mathcal{P}$ we have $C|\widetilde{y}|^\beta < y_n$ and by Taylor's expansion we get

$$K_1^\varepsilon \le \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{2pC|\widetilde{y}|^\beta y_n^{p-1}}{|y-x|^{n+\alpha}}\, dy \le 2pC \int_{B(0,2)\cap\mathbf{R}^n_+} \frac{|\widetilde{y}|^\beta y_n^{p-1}}{|y-x|^{n+\alpha}}\, dy .$$

We note that $|\widetilde{y}|^\beta y_n^{p-1}/|y-x|^{n+\alpha} \le y_n^{p-1}/|y-x|^{n+\alpha-\beta}$, so this function of y is integrable in a neighborhood of x and also in a neighborhood of $\partial\mathbf{R}^n_+$. Using a change of variable, polar coordinates and the observation that $x_n \le 1/2$ and

$$p - \alpha + \beta - 2 < -1,$$

we obtain

$$\begin{aligned} K_1^\varepsilon &\le 2pCx_n^{p-\alpha+\beta-1} \int_{B(0,2/x_n)\cap\mathbf{R}^n_+} \frac{|\widetilde{z}|^\beta z_n^{p-1}}{|z-e_n|^{n+\alpha}}\, dz \\ &\le 2pCx_n^{p-\alpha+\beta-1}\Bigg[\int_{B(0,2)\cap\mathbf{R}^n_+} \frac{|\widetilde{z}|^\beta z_n^{p-1}}{|z-e_n|^{n+\alpha}}\, dz + 2^{n+\alpha} \\ &\quad \times \int_{[B(0,2/x_n)\setminus B(0,2)]\cap\mathbf{R}^n_+} \frac{|\widetilde{z}|^\beta z_n^{p-1}}{|z|^{n+\alpha}}\, dz\Bigg] \\ &\le c_1 x_n^{p-\alpha+\beta-1} , \end{aligned} \tag{5.28}$$

where $c_1 = c_1(n, \alpha, \beta, p)$. It is now clear that the Lebesgue integral

$$\int_{\mathcal{P}'} \frac{v(y) - w(y)}{|y - x|^{n+\alpha}} dy$$

exists. Moreover,

$$\int_{\mathcal{P}'} \frac{v(y) - w(y)}{|y - x|^{n+\alpha}} dy = K_1 = \lim_{\varepsilon \to 0^+} K_1^{\varepsilon} \le c_1 x_n^{p-\alpha+\beta-1} < \infty, \tag{5.29}$$

although the quantity on the left hand side may be equal to $-\infty$.

To estimate K_3 we continue to assume that $Q = 0$ and $b = e_n$ and we reuse the sets T, T_1, T_2 from the proof of Lemma 5.6. We have

$$K_3 = \int_{\mathbf{R}^n \setminus (\mathcal{P} \cup \mathcal{P})} \frac{v(y) - w(y)}{|y - x|^{n+\alpha}} dy = \int_{T^c} + \int_{T_1} + \int_{T_2} .$$

For $y \in T^c$ we have $|y - x| \ge C^{-2/\beta}/2 \ge C^{1-2/\beta}|x| \ge C^{-2/\beta}|x|/2$, hence

$$v(y) \le |y|^p \le [|y - x| + |x|]^p \le |y - x|^p [1 + 2C^{2/\beta}]^p \tag{5.30}$$

and

$$w(y) \le |y|^p \le |y - x|^p [1 + 2C^{2/\beta}]^p . \tag{5.31}$$

It follows that there is $c_2 = c_2(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1}, p)$ such that

$$\int_{T^c} \frac{|v(y) - w(y)|}{|y - x|^{n+\alpha}} dy \le 2[1 + 2C^{2/\beta}]^p \int_{T^c} |y - x|^{-n-\alpha+p} dy \le c_2 . \tag{5.32}$$

If $y \in T_1$ then $|v(y) - w(y)| \le 2|y|^p \le 2x_n^p$ and by (5.19)

$$\int_{T_1} \frac{|v(y) - w(y)|}{|y - x|^{n+\alpha}} dy \le 2x_n^p \int_{T_1} \frac{dy}{|y - x|^{n+\alpha}} \tag{5.33}$$

$$\le \frac{8C}{n + \beta - 1} \omega_{n-1} x_n^{p-\alpha+\beta-1} . \tag{5.34}$$

If $y \in T_2$ then $|y - x| > C^{-2/\beta}|x|/2$ so in view of (5.30)–(5.31),

$$\int_{T_2} \frac{|v(y) - w(y)|}{|y - x|^{n+\alpha}} dy \le 2[1 + 2C^{2/\beta}]^p \int_{T_2} |y - x|^{-n-\alpha+p} dy$$

$$\le c_3 x_n^{p-\alpha+\beta-1} , \tag{5.35}$$

where $c_3 = c_3(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1}, p)$, cf. (5.20).

By (5.22)–(5.35),

$$(\Delta^{\alpha/2} + \kappa_D(x))v(x) \le \mathcal{A}(n, -\alpha)\left[c_1 + c_2 + \frac{8C}{n + \beta - 1}\omega_{n-1} + c_3\right]\rho(x)^{p-\alpha+\beta-1}$$
$$+ \rho(x)^p \kappa_{\Pi_+}(x) \alpha \gamma(\alpha, p) + C_2 \rho(x)^{p-\alpha+\beta-1},$$

provided $\rho(x) \le 1/(2C)$. It follows from (5.18) and (5.3) that for small $\rho(x)$ the second term on the right hand side is negative and dominates the remaining terms as stated in the theorem. □

For $\eta > 0$ we will write $\overline{D}_\eta = \{x \in D : \rho(x) \geq \eta\}$. We first give a simple consequence of Theorem 5.7.

Proposition 5.8. *Let* $1 < \alpha < \beta \leq 2$ *and let* $0 < p < \alpha - 1$. *There is a constant* $C_4 = C_4(n, \alpha, \beta, p, \|\Gamma\|_{1,\beta-1})$ *such that for any special* $C^{1,\beta-1}$ *domain* $D = D_\Gamma$ *we have*

$$\mathbf{P}_x\left(Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right) \leq C_4\rho(x)^p\,, \quad x \in D\,. \tag{5.36}$$

Proof. Let $v(x) = \rho(x)^p$. By Theorem 5.7, v is superharmonic in $D \setminus \overline{D}_{C_3}$, where C_3 is the constant of Theorem 5.7. By Lemma 3.3 we have $v(x) \geq \mathbf{E}_x v(Y_{\tau_B})$ whenever B is an open precompact subset of $D \setminus \overline{D}_{C_3}$. If we let $B_n = \{x \in D : 1/n < \rho(x) < C_3 - 1/n\,,\ |x| < n\}$ and $n \to \infty$ then by the quasi-left continuity of Y we have that $Y_{\tau_{B_n}} \to Y_{\tau_{D\setminus\overline{D}_{C_3}}}$. Therefore

$$\begin{aligned}\rho(x)^p = v(x) &\geq \mathbf{E}_x\left[\liminf_{n\to\infty} v(Y_{\tau_{B_n}})\right] = \mathbf{E}_x\left[v(Y_{\tau_{D\setminus\overline{D}_{C_3}}})\right]\\ &\geq C_3^p\mathbf{P}_x\left(Y_{\tau_{D\setminus\overline{D}_{C_3}}} \in \overline{D}_{C_3}\right) \geq C_3^p\mathbf{P}_x\left(Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\right).\end{aligned}$$

We can take $C_4 = C_3^{-p}$ in (5.36). □

Lemma 5.9. *Let* $v(x) = \rho(x)^{\alpha-1}, x \in \mathbf{R}^n$. *There is* $C_5 = C_5(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1}) > 1$ *such that for every* $x \in D$ *satisfying* $\rho(x) \leq C_5^{-1}$,

$$\left(\Delta^{\alpha/2} + \kappa_D(x)\right)v(x) \leq \begin{cases} C_5\rho(x)^{\beta-2} & \text{if } \beta < 2\,,\\ C_5\log\frac{1}{\rho(x)} & \text{if } \beta = 2\,.\end{cases} \tag{5.37}$$

Proof. We adopt the notation of the proof of Theorem 5.7. To prove (5.37) we only need to estimate J_1, J_2, J_3 in (5.22). It follows from (5.17) and (5.18) that for $x \in D$ with $\rho(x)$ small enough we have $|J_3| \leq C_2\rho(x)^{\beta-2}$, where $C_2 = C_2(n, \alpha, 2, \|\Gamma\|_{1,1})$ is the constant of Lemma 5.6. By (5.3) and (5.4), $J_2 = 0$.

Let us consider $J_1 = \mathcal{A}(n, -\alpha)[K_1 + K_2 + K_3]$ as in (5.25). We have $K_2 = 0$ and by a calculation in the proof of Theorem 5.7 we have

$$\begin{aligned}K_1^\varepsilon &= \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{v(y) - w(y)}{|y - x|^{n+\alpha}}\,dy\\ &\leq 2(\alpha-1)Cx_n^{\beta-2}\left[\int_{B(0,2)\cap\mathbf{R}^n_+} \frac{|\widetilde{z}|^\beta z_n^{\alpha-2}}{|z - e_n|^{n+\alpha}}\,dz + 2^{n+\alpha}\right.\\ &\quad\left.\times\int_{[B(0,2/x_n)\setminus B(0,2)]\cap\mathbf{R}^n_+} \frac{|\widetilde{z}|^\beta z_n^{\alpha-2}}{|z|^{n+\alpha}}\,dz\right].\end{aligned} \tag{5.38}$$

We assume here that $Q = 0$ and $b = e_n$, as in the proof of Theorem 5.7. Using polar coordinates, we obtain

$$K_1 = \lim_{\varepsilon\to 0^+} K_1^\varepsilon \leq \begin{cases} c_1x_n^{\beta-2} & \text{if } \beta < 2\,,\\ c_1\log\frac{1}{x_n} & \text{if } \beta = 2\,,\end{cases}$$

where $c_1 = c_1(n, \alpha, \beta)$. To estimate K_3 we can use (5.32), (5.34) and (5.35) except that we need to replace (5.35) by

$$\int_{T_2} \frac{|v(y) - w(y)|}{|y - x|^{n+\alpha}} \, dy \le c_2 \log(1/x_n) ,$$

if $\beta = 2$, see also (5.20). Here $c_2 = c_2(n, \alpha, \|\Gamma\|_{1,1})$. This completes the proof. □

We already know from Theorem 2.9 that if $1 < \alpha < 2$, then the censored process Y will approach the boundary of a $C^{1,\beta-1}$ domain. The main result of this section, Theorem 5.10 below, provides a quantitative version of this statement in the form of a "gambler's ruin" estimate.

Recall that $\overline{D}_1 = \{x \in D : \rho(x) \ge 1\}$.

Theorem 5.10. *Let* $1 < \alpha < 2$ *and let* $2 \ge \beta \ge \alpha \vee (1 - \alpha/2 + \sqrt{1 + \alpha^2/4})$. *There is a constant* $C_6 = C_6(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1})$ *such that for any special* $C^{1,\beta-1}$ *domain* $D = D_\Gamma$ *we have*

$$\mathbf{P}_x\{Y_{\tau_{D \setminus \overline{D}_1}} \in \overline{D}_1\} \le C_6 \rho(x)^{\alpha-1} , \quad x \in D . \tag{5.39}$$

Proof. Let $u(x) = \mathbf{P}_x\{X_{\tau_{D \setminus \overline{D}_1}} \in \overline{D}_1\}$, $x \in \mathbf{R}^n$. Here X denotes the (uncensored) symmetric α-stable process on $\mathbf{R}^n$. By Propositions 7.4 and 7.6 in the Appendix there is $c = c(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1})$ such that

$$c^{-1}[\rho(x)^{\alpha/\beta} \wedge 1] \le u(x) \le c[\rho(x)^{\alpha-(\alpha/\beta)} \wedge 1] , \quad x \in \mathbf{R}^n . \tag{5.40}$$

Let $v(x) = \rho(x)^{\alpha-1} - u(x)/(2c)$, $x \in \mathbf{R}^n$. Note that $v = 0$ on D^c. Given that $1 < \alpha < \beta \le 2$ we have $\rho(x)^{\alpha-(\alpha/\beta)} \le \rho(x)^{\alpha-1}$ for $\rho(x) \le 1$ because $\alpha - (\alpha/\beta) > \alpha - 1$. By the right hand side of (5.40),

$$v(x) \ge \rho(x)^{\alpha-1}/2 , \quad x \in \mathbf{R}^n .$$

Furthermore, by Lemma 5.9, the left hand side of (5.40) and (5.17),

$$\begin{aligned}(\Delta^{\alpha/2} + \kappa_D(x))v(x) &= (\Delta^{\alpha/2} + \kappa_D(x))\rho(x)^{\alpha-1} - \kappa_D(x)u(x)/(2c) \\ &\le \begin{cases} C_5\rho(x)^{\beta-2} - \rho(x)^{\alpha/\beta-\alpha}/(2c^2 \cdot c_1) , & \text{if } \beta < 2 , \\ -C_5 \log \rho(x) - \rho(x)^{-\alpha/2}/(2c^2 \cdot c_1) , & \text{if } \beta = 2 , \end{cases}\end{aligned}$$

provided $\rho(x)$ is small enough. Here $c_1 = c_1(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1})$. We have $0 > \beta - 2 > \alpha/\beta - \alpha$ if $2 > \beta > 1 - \alpha/2 + \sqrt{1 + \alpha^2/4}$, so

$$(\Delta^{\alpha/2} + \kappa_D(x))v(x) < 0 , \quad 0 < \rho(x) \le c_2 ,$$

where $c_2 = c_2(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1}) < 1$, that is, v is superharmonic on $D \setminus \overline{D}_{c_2}$. We now obtain (5.39) as in the proof of Proposition 5.8. □

Note that for all $\alpha \in (1, 2)$ we have $1 - \alpha/2 + \sqrt{1 + \alpha^2/4} < 2$, so Theorem 5.10 always applies to $\beta = 2$. We conjecture that the theorem holds if we replace the "technical" assumption $2 \ge \beta \ge \alpha \vee (1 - \alpha/2 + \sqrt{1 + \alpha^2/4})$ by $1 < \alpha < \beta \le 2$. The conjecture holds trivially for $\alpha \ge 3/2$ because $\alpha \ge 1 - \alpha/2 + \sqrt{1 + \alpha^2/4}$ for such α.

6. Boundary Harnack principle in $C^{1,1}$ open sets

In the present section we will prove the boundary Harnack principle in $C^{1,1}$ open sets. This essentially amounts to verifying that for $\alpha > 1$, the rate of decay of a harmonic function of the censored process near the boundary of a $C^{1,1}$ open set is precisely $p = \alpha - 1$, i.e., the opposite inequality to that in Theorem 5.10 is also true.

We state without proof two geometric results on $C^{1,1}$ open sets.

Lemma 6.1. *Let $b \in \mathbf{R}^n$, $|b| = 1$. If $\beta = 2$ then the region of 2-tangential approach $\mathcal{P}_b$ defined by (5.12) satisfies $\mathcal{P}_b \supset B(Rb, R)$, where $R = 1/[(4\|\Gamma\|_{1,1})\vee 2]$.*

The inclusion in Lemma 6.1 follows from the proof of Lemma 5.5. By Lemma 6.1 and Lemma 5.5 we obtain the following result.

Lemma 6.2. *Assume that $\Gamma : \mathbf{R}^{n-1} \to \mathbf{R}$ is a $C^{1,1}$ function and let $D = D_\Gamma$. Then $B(Q + Rb, R) \subset D$ and $B(Q - Rb, R) \subset D^c$ for every $Q \in \partial D$. Here b is the unit inward vector at Q (see (5.13)) and $R = 1/[(4\|\Gamma\|_{1,1}) \vee 2]$.*

Lemma 6.2 states a well known geometric fact that for a $C^{1,1}$ open set, there exist *inner* and *outer* tangent balls of fixed diameter at every point on the boundary of the domain. This geometric result yields sharp estimates for the Green function of symmetric stable processes in a $C^{1,1}$ domain, using explicit formulas for the Green function for a ball and for the complement of a ball. Details of this argument may be found in [23] in the case of the Brownian motion and in [13], [18] and [40] in the case of symmetric stable processes; see also our Appendix. For censored stable processes we do not have an explicit formula for the Green function in a ball. Moreover, the Green function of the censored process in a domain is not related in a simple way to the Green function in a subdomain. To find the exact rate of decay of harmonic functions of censored processes in $C^{1,1}$ open sets, we will use a different approach based on explicit formulas for half-spaces and the approximation technique introduced in the previous section.

Throughout the section, unless stated otherwise, $\alpha \in (1, 2)$ and $D = D_\Gamma$ is a special $C^{1,1}$ domain in $\mathbf{R}^n$, $n \geq 2$.

Lemma 6.3. *Let $v(x) = \rho(x)^{\alpha-1}$, $x \in \mathbf{R}^n$. There is $C_7 = C_7(n, \alpha, \|\Gamma\|_{1,1}) < 1$ such that for every $x \in D$ satisfying $\rho(x) \leq C_7$,*

$$\log(C_7\rho(x)) \leq \Delta^{\alpha/2}v(x) + \kappa(x)v(x) \leq \log\frac{1}{C_7\rho(x)}\,. \tag{6.1}$$

Proof. The upper bound in (6.1) follows from a more general inequality (5.37).

The proof of the lower bound proceeds along the same lines as the proofs of Theorem 5.7 and Lemma 5.9. We will outline only those steps which require modifications. Take $\beta = 2$ in those calculations. A direct examination of the proofs of Theorem 5.7 and Lemma 5.9 reveals that the only term out of J_1, J_2, J_3, K_1, K_2 and K_3 which requires new bounds in the present context is K_1.

To estimate K_1^ε, and, consequently, $\Delta^{\alpha/2}v(x) + \kappa(x)v(x)$, from below we use the inner tangent ball $B = B(Q + Rb, R) = B(Re_n, R) \subset D$. Here $R =$

$1/[(4\|\Gamma\|_{1,1}) \vee 2]$, see Lemma 6.2. We assume that $Q = 0$ and $b = e_n$. We will also use the flexibility of choice of the constant $C \geq (2\|\Gamma\|_{1,1}) \vee 1$ defining $\mathcal{P}$ in (5.11), see the beginning of the proof of Lemma 5.5. We have

$$K_1^\varepsilon \geq \int_{\mathcal{P}\setminus B(x,\varepsilon)} \frac{\mathrm{dist}(y, B^c)^{\alpha-1} - y_n^{\alpha-1}}{|y-x|^{n+\alpha}}\, dy\,, \tag{6.2}$$

and $\mathrm{dist}(y, B^c) = R - \sqrt{(R-y_n)^2 + |\widetilde{y}|^2}$ provided $y \in \mathcal{P} \subset B$. The inclusion $\mathcal{P} \subset B$ takes place if $C \geq 1/R$. Our actual choice will be however $C = 2/R = (8\|\Gamma\|_{1,1}) \vee 4$. By (5.11) we have $y_n < 1/C = R/2$ for $y \in \mathcal{P}$. We consider the function $f(s) = \left(R - \sqrt{(R-y_n)^2 + s}\right)^{\alpha-1} - y_n^{\alpha-1}$. By the mean value theorem applied to f, we see that for $y \in \mathcal{P}$ there is $\theta \in (0,1)$ such that

$$\begin{aligned}
&\left(R - \sqrt{(R-y_n)^2 + |\widetilde{y}|^2}\right)^{\alpha-1} - y_n^{\alpha-1} \\
&= -|\widetilde{y}|^2 \frac{\alpha-1}{2} \frac{\left(R - \sqrt{(R-y_n)^2 + \theta|\widetilde{y}|^2}\right)^{\alpha-2}}{\sqrt{(R-y_n)^2 + \theta|\widetilde{y}|^2}} \\
&\geq -\frac{\alpha-1}{R}|\widetilde{y}|^2 \left(R - \sqrt{(R-y_n)^2 + \theta|\widetilde{y}|^2}\right)^{\alpha-2} \geq -\frac{2^\alpha(\alpha-1)}{4R}|\widetilde{y}|^2 y_n^{\alpha-2}\,.
\end{aligned}$$

Substituting this bound into (6.2), we obtain an integral similar to that in (5.38) with $\beta = 2$, and thus the logarithmic estimate for K_1 follows. In particular, the integral defining K_1 is absolutely convergent and we have (6.1). □

Recall that for $r > 0$, $\overline{D}_r = \{x \in D :\ \rho(x) \geq r\}$.

Theorem 6.4. *There is* $C_8 = C_8(n, \alpha, \|\Gamma\|_{1,1})$ *such that*

$$C_8^{-1}\rho(x)^{\alpha-1} \leq \mathbf{P}_x\{Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\} \leq C_8\rho(x)^{\alpha-1}\,, \quad x \in D \setminus \overline{D}_1\,. \tag{6.3}$$

Proof. The right hand side of (6.3) is a special case of (5.39). To prove the left hand side let

$$u(x) = \mathbf{P}_x\{X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\}\,, \quad x \in \mathbf{R}^n\,.$$

We will show that the function $v(x) = \rho(x)^{\alpha-1} + u(x)$ is subharmonic with respect to Y_t on $D \setminus \overline{D}_c$, for some $c = c(n, \alpha, \|\Gamma\|_{1,1}) < 1$. Indeed, by Lemma 6.3, Lemma 5.6 and Proposition 7.6 of the Appendix with $\beta = 2$,

$$\begin{aligned}
(\Delta^{\alpha/2} + \kappa_D)v(x) &= (\Delta^{\alpha/2} + \kappa_D)\rho(x)^{\alpha-1} + \kappa_D(x)u(x) \\
&\geq \log(C_7\rho(x)) + \mathrm{const}\cdot\ \rho(x)^{-\alpha+\alpha/2}\,,
\end{aligned} \tag{6.4}$$

which is positive if $\rho(x)$ is small enough. Note that $v = 0$ on D^c and $v(x) \leq (A_3+1)\rho(x)^{\alpha-1}$, $x \in \mathbf{R}^n$, by Proposition 7.4 of the Appendix with $\beta = 2$, because $\alpha - 1 < \alpha/2$. Let $v_1 = v \wedge 1$. Taking a smaller $c = c(n, \alpha, \|\Gamma\|_{1,1})$ if necessary, we can assume that $v_1 = v$ on $D \setminus \overline{D}_c$ and v_1 is subharmonic on $D \setminus \overline{D}_c$. Indeed, let $f = v_1 - v$. Clearly, the support of f is contained in $\overline{D}_c$; thus for $x \in D \setminus \overline{D}_c$,

$$(\Delta^{\alpha/2} + \kappa_D) f(x) = \Delta^{\alpha/2} f(x) = \mathcal{A}(n, -\alpha) \int_{\overline{D}_c} \frac{f(y)}{|y-x|^{n+\alpha}} dy \,,$$

which is bounded if $\rho(x)$ is small enough. So by (6.4), $(\Delta^{\alpha/2} + \kappa_D) v_1(x) > 0$ if $\rho(x)$ is small enough. We obtain

$$\rho(x)^{\alpha-1} \le v(x) = v_1(x) \le \mathbf{E}_x v_1(Y_{\tau_{D\setminus\overline{D}_c}}) \le \mathbf{P}_x\{Y_{\tau_{D\setminus\overline{D}_c}} \in \overline{D}_c\} \quad \text{when} \quad \rho(x) \le c \,.$$

To finish the proof of Theorem 6.4 we only need to replace $\overline{D}_c$ by $\overline{D}_1$ above. For $y = Y_{\tau_{D\setminus\overline{D}_c}} \in \overline{D}_c$ we have

$$\begin{aligned} \mathbf{P}_y\{Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\} &\ge \mathbf{P}_y\{Y_{\tau_{B(y,c/2)}} \in \overline{D}_1\} \\ &\ge \mathbf{P}_y\{X_{\tau_{B(y,c/2)}} \in \overline{D}_1\} \ge c_1 = c_1(n, \alpha, \|\Gamma\|_{1,1}) \,, \end{aligned}$$

as follows from the explicit formula (3.4) for the Poisson kernel for the ball and Lemma 7.5. Then

$$\begin{aligned} \mathbf{P}_x\{Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\} &= \mathbf{E}_x\{Y_{\tau_{D\setminus\overline{D}_c}} \in \overline{D}_c \,;\, \mathbf{P}_{Y_{\tau_{D\setminus\overline{D}_c}}}\{Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\}\} \\ &\ge c_1 \mathbf{P}_x\{Y_{\tau_{D\setminus\overline{D}_c}} \in \overline{D}_c\} \,. \end{aligned}$$

This completes the proof. □

In the remainder of the section we will prove the Carleson estimate and the boundary Harnack principle for nonnegative harmonic functions of the censored process Y on $C^{1,1}$ domains. As before, we will assume that $1 < \alpha < 2$. However we will now put an additional constraint on the domain $D = D_\Gamma$. Namely, we will assume that Γ is a Lipschitz function with Lipschitz constant λ:

$$|\Gamma(\widetilde{x}) - \Gamma(\widetilde{y})| \le \lambda |\widetilde{x} - \widetilde{y}| \,, \quad \widetilde{x}, \widetilde{y} \in \mathbf{R}^{n-1} \,, \tag{6.5}$$

as well as a $C^{1,1}$ function. Obviously, every $C^{1,1}$ function is Lipschitz on every compact set, so (6.5) imposes a constraint only on the global shape of D—this is a technically convenient but inessential assumption.

We will also assume that $\Gamma(\widetilde{0}) = 0$. From now on, $D = D_\Gamma$ will denote a special Lipschitz and $C^{1,1}$ domain. We let Y be the censored process in D.

Recall that for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, we write $x = (\widetilde{x}, x_n)$, where $\widetilde{x} = (x_1, \ldots, x_{n-1})$. We will use the following notation: $\eta(x) = [x_n - \Gamma(x)] \vee 0$, $x \in \mathbf{R}^n$. The function $\eta(x)$ represents the distance from x to the complement of D along the vertical line (in the direction of x_n). By the Pythagorean theorem,

$$\rho(x) \le \eta(x) \le \sqrt{\lambda^2 + 1}\, \rho(x) \,, \quad x \in \mathbf{R}^n \,. \tag{6.6}$$

Let $\Delta(x, a, r)$ be a "box" with bottom on ∂D, defined as follows:

$$\Delta(x, a, r) = \{y \in D :\ 0 < \eta(y) < a,\ |\widetilde{y} - \widetilde{x}| < r\} \,,$$

where $x \in \mathbf{R}^n$, and $a, r \in (0, \infty]$. We note that $\Delta(x, a, r)$ depends on x only through $\widetilde{x}$.

The following result is a preliminary version of the boundary Harnack principle (cf. [3]).

Theorem 6.5. *There is a constant $C_9 = C_9(n, \alpha, \lambda, \|\Gamma\|_{1,1})$ such that*

$$\mathbf{P}_x[Y_{\tau_{\Delta(0,1,1)}} \in D] \le C_9 \mathbf{P}_x[Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0, 2, 1)] , \quad \textit{for } x \in \Delta(0, 1, 1) \textit{ with } \widetilde{x} = 0 . \tag{6.7}$$

Proof. Let

$$v(x) = \mathbf{P}_x\{Y_{\tau_{\Delta(0,1,\infty)}} \in D\} , \quad x \in D .$$

By (6.6) and Theorem 6.4 there is $c_1 = c_1(n, \alpha, \lambda, \|\Gamma\|_{1,1}) \ge 1$ such that

$$c_1^{-1}[\eta(x)^{\alpha-1} \wedge 1] \le v(x) \le c_1[\eta(x)^{\alpha-1} \wedge 1] , \quad x \in D . \tag{6.8}$$

We also define

$$u(x) = \mathbf{P}_x\{X_{\tau_{\Delta(0,1,\infty)}} \in D\} , \quad x \in \mathbf{R}^n .$$

Then there is $c_2 = c_2(n, \alpha, \lambda, \|\Gamma\|_{1,1}) \ge 1$ such that

$$c_2^{-1}[\eta(x)^{\alpha/2} \wedge 1] \le u(x) \le c_2[\eta(x)^{\alpha/2} \wedge 1] , \quad x \in D , \tag{6.9}$$

by Propositions 7.4 and 7.6 in the Appendix. Let ϕ be a C^2 function with $\|\phi\|_{C^2} < \infty$ such that $\phi(x) = |\widetilde{x}|^2 = x_1^2 + \ldots + x_{n-1}^2$ if $|\widetilde{x}| < 1$ and $\phi(x) \ge 1$ if $|\widetilde{x}| \ge 1$. We put

$$v_1(x) = v(x) - u(x)/(2c_2c_1) + 8c_1^2\phi(x) , \quad x \in D .$$

Here the coefficient $1/(2c_2c_1)$ is chosen so that $v(x) - u(x)/(2c_1c_2) \ge [\eta(x)^{\alpha-1} \wedge 1]/(2c_1)$ for $x \in D$ (recall that $\alpha - 1 < \alpha/2$). By (5.17), (3.28) and (6.9), for small $\eta(x)$,

$$\begin{aligned} A_D^\alpha v_1(x) &= -\kappa_D(x)u(x)/(2c_2c_1) + 8c_1^2(\Delta^{\alpha/2} + \kappa_D)\phi(x) \\ &\approx \ -\text{const} \cdot \eta(x)^{-\alpha/2} + \text{const} \cdot \eta(x)^{1-\alpha} , \end{aligned}$$

which is negative provided $\eta(x) > 0$ is small enough. Thus there is $m = m(n, \alpha, \lambda, \|\Gamma\|_{1,1}) \le 1$ such that v_1 is superharmonic in $\Delta(0, m, \infty)$. By the super-mean value property (i.e., (3.1) with the equality sign replaced by $\ge$) we have for every $x = (\widetilde{0}, x_n) \in D$ that

$$c_1[\eta(x)^{\alpha-1} \wedge 1] \ge v_1(x) \ge 2c_1^2\,\mathbf{P}_x[Y_{\tau_{\Delta(0,m,1/2)}} \in D \setminus \Delta(0, \infty, 1/2)],$$

and, using (6.8),

$$\mathbf{P}_x[Y_{\tau_{\Delta(0,m,1/2)}} \in D \setminus \Delta(0, \infty, 1/2)] \le [\eta(x)^{\alpha-1} \wedge 1]/(2c_1) \le \frac{1}{2}v(x). \tag{6.10}$$

As

$$
\begin{aligned}
&\mathbf{P}_x[Y_{\tau_{\Delta(0,m,1/2)}} \in D] \\
&\quad = \mathbf{P}_x[Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2)] + \mathbf{P}_x[Y_{\tau_{\Delta(0,m,1/2)}} \in D \setminus \Delta(0,\infty,1/2)] \\
&\quad \le \mathbf{P}_x[Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2)] + (1/2)v(x) \\
&\quad = \mathbf{P}_x[Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2)] + (1/2)\mathbf{P}_x[Y_{\tau_{\Delta(0,1,\infty)}} \in D] \\
&\quad \le \mathbf{P}_x[Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2)] + (1/2)\mathbf{P}_x[Y_{\tau_{\Delta(0,m,1/2)}} \in D],
\end{aligned}
$$

we have

$$
\mathbf{P}_x(Y_{\tau_{\Delta(0,m,1/2)}} \in D) \le 2\mathbf{P}_x(Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,\infty,1/2))\,. \tag{6.11}
$$

On the other hand, there is $c_3 = c_3(n, \alpha, \lambda, \|\Gamma\|_{1,1})$ such that

$$
\begin{aligned}
&\int_{\Delta(0,\infty,1/2)\setminus\Delta(0,m,1/2)} \frac{\mathcal{A}(n,-\alpha)}{|z-y|^{n+\alpha}} dy \\
&\quad \le c_3 \int_{\Delta(0,3m/2,1/2)\setminus\Delta(0,m,1/2)} \frac{\mathcal{A}(n,-\alpha)}{|z-y|^{n+\alpha}} dy\,, \quad z \in \Delta(0,m,1/2)\,.
\end{aligned}
$$

By (3.23) and (6.11) we have

$$
\mathbf{P}_x(Y_{\tau_{\Delta(0,m,1/2)}} \in D) \le 2c_3\mathbf{P}_x(Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,3m/2,1/2)),
$$

provided $\widetilde{x} = 0$.

There is $c_4 = c_4(n, \alpha, \lambda, \|\Gamma\|_{1,1})$ such that the process Y starting from any point of $\Delta(0, 3m/2, 1/2) \setminus \Delta(0, m, 1/2)$ can hit $\Delta(0, 2, 1) \setminus \Delta(0, 1, 1)$ before leaving $\Delta(0, 2, 1)$ with probability greater than c_4, because this is true for X^D. It follows that

$$
\begin{aligned}
\mathbf{P}_x(Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,2,1)) &\ge c_4\mathbf{P}_x(Y_{\tau_{\Delta(0,m,1/2)}} \in \Delta(0,3m/2,1/2)) \\
&\ge \frac{c_4}{2c_3}\mathbf{P}_x(Y_{\tau_{\Delta(0,m,1/2)}} \in D) \\
&\ge \frac{c_4}{2c_3}\mathbf{P}_x(Y_{\tau_{\Delta(0,1,1)}} \in D),
\end{aligned}
$$

for x with $\widetilde{x} = 0$. The proof is complete. □

For later reference we note that

$$
\mathbf{P}_x[Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,2,1)] \le \mathbf{P}_x[Y_{\tau_{\Delta(0,1,\infty)}} \in D] \le C_{10}[\eta(x)^{\alpha-1} \wedge 1]\,, \quad x \in D\,, \tag{6.12}
$$

where C_{10} is the constant c_1 in (6.8) above, so that $C_{10} = C_{10}(n, \alpha, \lambda, \|\Gamma\|_{1,1})$.

To prove the next result we combine arguments used for the proof of the Carleson estimate for classical harmonic functions of Brownian motion ([3], [36]) with those used for harmonic functions of the symmetric stable processes X_t ([8]).

Proposition 6.6 (Carleson estimate). *Let* $\Delta_1 = \Delta(0, 1, 1)$*,* $\Delta_2 = \Delta(0, 2, 2)$*,* $F = \{x \in \mathbf{R}^n : |\widetilde{x}| < 2\,,\ \eta(x) = 0\} \subset \partial D$*, and* $A = (\widetilde{0}, 1/2) \in \Delta_1$*. Assume that a function u is nonnegative on* $\mathbf{R}^n$*, vanishes continuously at every point of F and is harmonic on* Δ_2 *for the censored process Y on D. Then there is a constant* $C_{11} = C_{11}(n, \alpha, \lambda, \|\Gamma\|_{1,1})$ *such that*

$$u(x) \le C_{11} u(A) \quad \textit{for } x \in \Delta_1\,. \tag{6.13}$$

Proof. By the Harnack inequality (Theorem 3.2), there is $M = M(n, \alpha, \lambda)$ such that

$$u(x) \le 2^{kM} u(A) \quad \text{for } x \in \Delta_1 \text{ with } \eta(x) \ge 2^{-k}\,. \tag{6.14}$$

Here k is a positive integer. Assume that at some $x_0 \in \Delta_1$ we have $u(x_0) > 2^{k_0 M} u(A)$. We will show below that if a positive integer $k_0 = k_0(n, \alpha, \lambda, \|\Gamma\|_{1,1})$ is large enough then this assumption contradicts the continuous decay of u at F.

In view of (6.14), $\eta(x_0) \le 2^{-k_0}$. Let $\varepsilon = (\alpha - 1)/(2\alpha)$ so that in particular $0 < \varepsilon < 1$. We define $\Delta^0 = \Delta(x_0, 2^{-\varepsilon k_0}, 2^{-\varepsilon k_0})$ and $2\Delta^0 = \Delta(x_0, 2^{1-\varepsilon k_0}, 2^{1-\varepsilon k_0})$. We will argue that

$$u(x_0) = \mathbf{E}_{x_0} u(Y_{\tau_{\Delta^0}})\,. \tag{6.15}$$

By harmonicity, $u(x_0) = \mathbf{E}_{x_0} u(Y_{\tau_{B_l}})$, where $B_l = \Delta^0 \cap \{\eta(x) > 1/l\}, l = 1, 2, \ldots$ By the continuity of u on F and quasi-left continuity of Y we obtain (6.15). We further have

$$u(x_0) = \mathbf{E}_{x_0}[Y_{\tau_{\Delta^0}} \in 2\Delta^0\,;\ u(Y_{\tau_{\Delta^0}})] + \mathbf{E}_{x_0}[Y_{\tau_{\Delta^0}} \in D \setminus 2\Delta^0\,;\ u(Y_{\tau_{\Delta^0}})] = E' + E''\,.$$

We will find a point x_1 near x_0 such that $u(x_1)$ is substantially larger than $u(x_0)$. First consider the case when $E'' \ge E'$. Then $E'' \ge u(x_0)/2$. By (3.23) and the second sentence following (3.23),

$$\begin{aligned}
\frac{1}{2} u(x_0) &\le \int_{D \setminus 2\Delta^0} \int_{\Delta^0} \mathcal{A}(n, -\alpha) \frac{G^X_{\Delta^0}(x_0, v) \mathbf{E}^v_{x_0} e_\kappa(\tau_{\Delta^0})}{|y - v|^{n+\alpha}} dv\, u(y)\, dy \\
&\le \int_{D \setminus 2\Delta^0} \int_{\Delta^0} \mathcal{A}(n, -\alpha) \frac{G^X_{\Delta^0}(x_0, v) \mathbf{E}^v_{x_0} e_\kappa(\tau_{\Delta^0})}{(|y - (\widetilde{x}_0, 0)|/4)^{n+\alpha}} dv\, u(y)\, dy \\
&= \mathcal{A}(n, -\alpha) 4^{n+\alpha} \int_{\Delta^0} G^X_{\Delta^0}(x_0, v) \mathbf{E}^v_{x_0} e_\kappa(\tau_{\Delta^0}) dv \int_{D \setminus 2\Delta^0} \frac{u(y)}{|y - (\widetilde{x}_0, 0)|^{n+\alpha}}\, dy \\
&= \mathcal{A}(n, -\alpha) 4^{n+\alpha} \mathbf{E}_{x_0} \tau^Y_{\Delta^0} \int_{D \setminus 2\Delta^0} \frac{u(y)}{|y - (\widetilde{x}_0, 0)|^{n+\alpha}}\, dy\,.
\end{aligned}$$

Here superscripts X and Y indicate with respect to which process a quantity is calculated. Let x_1 be such that $\widetilde{x}_1 = \widetilde{x}_0$ and $\eta(x_1) = 2^{-\varepsilon k_0}/2$. We have

$$\begin{aligned} u(x_1) &\geq \int_{D\setminus 2\Delta^0} \int_{\Delta^0} \mathcal{A}(n,-\alpha) \frac{G^X_{\Delta^0}(x_1,v)\mathbf{E}^v_{x_1} e_\kappa(\tau_{\Delta^0})}{|y-v|^{n+\alpha}} dv\, u(y)\, dy \\ &\geq \int_{D\setminus 2\Delta^0} \int_{\Delta^0} \mathcal{A}(n,-\alpha) \frac{G^X_{\Delta^0}(x_1,v)\mathbf{E}^v_{x_1} e_\kappa(\tau_{\Delta^0})}{(2|y-(\widetilde{x}_0,0)|)^{n+\alpha}} dv\, u(y)\, dy \\ &= \mathcal{A}(n,-\alpha) 2^{-n-\alpha} \mathbf{E}_{x_1} \tau^Y_{\Delta^0} \int_{D\setminus 2\Delta^0} \frac{u(y)}{|y-(\widetilde{x}_0,0)|^{n+\alpha}} dy\,. \end{aligned}$$

Therefore

$$\frac{u(x_1)}{u(x_0)} \geq 2^{-3(n+\alpha)} \frac{\mathbf{E}_{x_1}\tau^Y_{\Delta^0}}{\mathbf{E}_{x_0}\tau^Y_{\Delta^0}}\,. \tag{6.16}$$

By Theorem 6.4,

$$\begin{aligned} C_8 \eta(x_0)^{\alpha-1} &\geq C_8 \rho(x_0)^{\alpha-1} \geq \mathbf{P}_{x_0}\{Y_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1\} \\ &= \int_{\overline{D}_1} \int_{D\setminus\overline{D}_1} \mathcal{A}(n,-\alpha) \frac{G^X_{D\setminus\overline{D}_1}(x_0,v)\mathbf{E}^v_{x_0} e_\kappa(\tau_{D\setminus\overline{D}_1})}{|y-v|^{n+\alpha}} dv\, dy\,. \end{aligned}$$

There exists $S_1 = S_1(n,\lambda)$ such that for every $v \in D\setminus\overline{D}_1$ we have $|B(v,S_1)\cap\overline{D}_1| > 1$, see Lemma 7.5 in the Appendix. Thus

$$C_8 \eta(x_0)^{\alpha-1} \geq S_1^{-n-\alpha} \mathcal{A}(n,-\alpha) \mathbf{E}_{x_0} \tau^Y_{D\setminus\overline{D}_1} \geq S_1^{-n-\alpha} \mathcal{A}(n,-\alpha) \mathbf{E}_{x_0} \tau^Y_{\Delta^0}\,. \tag{6.17}$$

On the other hand we note that $B_1 = B(x_1, \eta(x_1)/\sqrt{\lambda^2+1}) = B(x_1, 2^{-\varepsilon k_0}/(2\sqrt{\lambda^2+1}))$ is a subset of Δ^0. Thus, by (3.6),

$$\mathbf{E}_{x_1}\tau^Y_{\Delta^0} \geq \mathbf{E}_{x_1}\tau^X_{B_1} = \frac{C^n_\alpha}{\mathcal{A}(n,-\alpha)} \left(\frac{2^{-\varepsilon k_0}}{2\sqrt{\lambda^2+1}}\right)^\alpha\,. \tag{6.18}$$

By (6.16), (6.17) and (6.18) there is $c_2 = c_2(n,\alpha,\lambda,\|\Gamma\|_{1,1})$ such that

$$u(x_1) \geq c_2 2^{-k_0(\alpha\varepsilon+1-\alpha)} u(x_0)\,.$$

Our choice of $\varepsilon = (\alpha-1)/(2\alpha)$ yields $\alpha\varepsilon + 1 - \alpha = (1-\alpha)/2$, and so

$$u(x_1) \geq c_2 2^{k_0(\alpha-1)/2} u(x_0)\,. \tag{6.19}$$

Next consider the case when $E'' \leq E'$. We have

$$u(x_0) \leq 2\mathbf{E}_{x_0}[Y_{\tau_{\Delta^0}} \in 2\Delta^0;\ u(Y_{\tau_{\Delta^0}})] \leq 2 \sup_{y\in 2\Delta^0} u(y)\,\mathbf{P}_{x_0}[Y_{\tau_{\Delta^0}} \in 2\Delta^0]\,.$$

We would like to apply properly rescaled version of (6.12), so we will make a digression on scaling. Suppose that we want to apply a result proved for D to a domain $kD = \{kx : x \in D\} = D_{\Gamma_k}$, where $\Gamma_k(x) = k\Gamma(x/k)$. The Lipschitz constants of Γ and Γ_k are clearly the same. On the other hand, $\|\Gamma_k\|_{1,1} = k^{-1}\|\Gamma\|_{1,1}$. In particular $\|\Gamma_k\|_{1,1} \le \|\Gamma\|_{1,1}$ if and only if $k \ge 1$. It follows that we can apply (6.12), Theorems 6.4 and 6.5 to D_{Γ_k} (without having to change the constants in these results) provided $k \ge 1$. By (6.12) and scaling,

$$\begin{aligned}\mathbf{P}_{x_0}[Y_{\tau_{\Delta^0}} \in 2\Delta^0] \le \mathbf{P}_{x_0}[Y_{\tau_{\Delta^0}} \in D] &\le C_9 C_{10}[\eta(x_0)/2^{-\varepsilon k_0}]^{\alpha-1} \\ &\le C_9 C_{10} 2^{-k_0(\alpha^2-1)/(2\alpha)} .\end{aligned}$$

Thus $u(x_0) \le 2C_9C_{10}2^{-k_0(\alpha^2-1)/(2\alpha)} \sup_{y\in 2\Delta^0} u(y)$, and so there is $x_1 \in 2\Delta^0$ such that

$$u(x_1) \ge \frac{2^{k_0(\alpha^2-1)/(2\alpha)}}{4C_9C_{10}} u(x_0) . \tag{6.20}$$

The distance from x_0 to any point in $2\Delta^0$ is bounded by $2^{-\varepsilon k_0} 2\sqrt{1+(1+\lambda)^2}$ because of the Lipschitz character of ∂D. Hence in both cases $E'' \ge E'$ and $E'' \le E'$ we have for our choice of x_1,

$$|x_1 - x_0| \le 2^{-\varepsilon k_0} 2\sqrt{1+(1+\lambda)^2} .$$

Also, if k_0 is large enough then both (6.19) and (6.20) can be combined into the following weaker but simpler inequality

$$u(x_1) \ge 2^{k_0(\alpha-1)/4} u(x_0) > 2^{M[k_0+k_0(\alpha-1)/(4M)]}\, u(A) ,$$

or

$$u(x_1) \ge 2^{k_1 M} u(A) , \tag{6.21}$$

where k_1 is the smallest integer larger than $k_0 + k_0(\alpha-1)/(4M) - 1$. We may and do choose k_0 so large that $k_1 \ge k_0 + 1$.

We proceed by induction. We find a point $x_2 \in \Delta_2$ and an integer $k_2 > k_1$ using x_1 and k_1, then a point $x_3 \in \Delta_2$ and an integer $k_3 > k_2$ using x_2 and k_2, etc., with the following properties. First of all,

$$u(x_i) \ge 2^{Mk_i} u(A), \tag{6.22}$$

where $k_i \ge k_{i-1} + 1 \ge k_0 + i$. Also,

$$\begin{aligned}|x_i - x_{i-1}| &\le 2^{-k_{i-1}(\alpha-1)/2} 2\sqrt{1+(1+\lambda)^2} \\ &\le 2^{-k_0(\alpha-1)/2} 2^{-i(\alpha-1)/2} 2\sqrt{1+(1+\lambda)^2} .\end{aligned} \tag{6.23}$$

We have $|x_i - x_0| \le \sum_{j=1}^{\infty} |x_j - x_{j-1}|$. If k_0 is large enough then the sum of the series is smaller than $1/(2\sqrt{1+\lambda^2})$ and so the points x_i do not leave $\Delta(0, 3/2, 3/2)$. By (6.22) and (6.14), $\eta(x_i) \le 2^{-k_i}$, $i = 1, 2, \ldots$ This contradicts the continuous decay of u at F. The contradiction proves that $u(x) \le 2^{Mk_0}$ for $x \in \Delta_1$. □

The boundary Harnack principle for classical harmonic functions was first proved in [2], [25] and [47]; see also [3] and [7] for a more recent exposition. The following theorem gives the boundary Harnack inequality in special $C^{1,1}$ domains. We would like to remind the reader that the harmonicity of a function in an open subset of D is characterized by the underlying censored stable process, which is dependent on the global geometry of the domain D.

The assumptions on the domain in Theorem 6.7 can be relaxed—see Theorem 1.2 and its proof below and Remark 6.3.

Theorem 6.7. *Assume that $u \geq 0$ on $\mathbf{R}^n$ and $u = 0$ on D^c. Furthermore assume that u is regular harmonic on $\Delta(0, 4, 4)$ for the censored process Y on D, i.e.,*

$$u(x) = \mathbf{E}_x[u(Y_{\tau_{\Delta(0,4,4)}})\,;\ Y_{\tau_{\Delta(0,4,4)}} \in D\,], \quad x \in \Delta(0, 4, 4)\,. \tag{6.24}$$

Let $A = (\widetilde{0}, 1/2) \in \Delta(0, 1, 1)$. There is $C_{12} = C_{12}(n, \alpha, \lambda, \|\Gamma\|_{1,1})$ such that

$$C_{12}^{-1} u(A)\eta(x)^{\alpha-1} \leq u(x) \leq C_{12} u(A)\eta(x)^{\alpha-1}\,, \quad x \in \Delta(0, 1, 1)\,. \tag{6.25}$$

Proof. We would like to point out that the assumption that $u = 0$ on D^c is a normalizing convention and not an essential restriction, see Section 3. First assume that $u(x) = \mathbf{P}_x[Y_{\tau_{\Delta(0,3,3)}} \in S]$ for $x \in D$ and $u(x) = 0$ for $x \in D^c$, where $S \subset D \setminus \Delta(0, 3, 3)$. For $x \in \Delta(0, 2, 2)$, using Theorem 6.5 and (6.12),

$$\begin{aligned} u(x) &= \mathbf{P}_x[Y_{\tau_{\Delta(0,3,3)}} \in S] \leq \mathbf{P}_x[Y_{\tau_{\Delta(x,1,1)}} \in D] \leq C_9 \mathbf{P}_x[Y_{\tau_{\Delta(x,1,1)}} \in \Delta(x, 2, 1)] \\ &\leq C_9 \mathbf{P}_x[Y_{\tau_{\Delta(x,1,\infty)}} \in D] \leq C_9 C_{10}[\eta(x)^{\alpha-1} \wedge 1]. \end{aligned}$$

Thus $u(x)$ decays continuously at the bottom part of the boundary of $\Delta(0, 2, 2)$. Using the strong Markov property of Y we have

$$\begin{aligned} u(x) &= \mathbf{E}_x[u(Y_{\tau_{\Delta(0,1,1)}})\,;\ Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0, 2, 1)] \\ &\quad + \mathbf{E}_x[u(Y_{\tau_{\Delta(0,1,1)}})\,;\ Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0, 3, 2) \setminus \Delta(0, 2, 1)] \\ &\quad + \mathbf{E}_x[u(Y_{\tau_{\Delta(0,1,1)}})\,;\ Y_{\tau_{\Delta(0,1,1)}} \in D \setminus \Delta(0, 3, 2)] \\ &= E_1(x) + E_2(x) + E_3(x)\,. \end{aligned}$$

We will assume for now that $x = (\widetilde{0}, x_n)$ with $0 < x_n < 1$; we will remove this assumption at the end of the proof. By the Harnack inequality (Theorem 3.2) there is $c_1 = c_1(n, \alpha, \lambda)$ such that

$$c_1^{-1} u(A) \leq u(y) \leq c_1 u(A)\,, \quad y \in \Delta(0, 2, 1) \setminus \Delta(0, 1, 1)\,.$$

Therefore, by (6.7) and (6.8),

$$\begin{aligned} u(x) \geq E_1(x) &\geq c_1^{-1} \mathbf{P}_x[Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0, 2, 1)] u(A) \\ &\geq c_1^{-1} C_9^{-1} \mathbf{P}_x[Y_{\tau_{\Delta(0,1,1)}} \in D] u(A) \\ &\geq c_1^{-1} C_9^{-1} \mathbf{P}_x[Y_{\tau_{\Delta(0,1,\infty)}} \in D] u(A) \geq c_2[\eta(x)^{\alpha-1} \wedge 1] u(A)\,, \end{aligned} \tag{6.26}$$

where $c_2 = c_2(n, \alpha, \lambda, \|\Gamma\|_{1,1})$.

Next we will prove the opposite inequality. First we note that by (6.8),

$$\begin{aligned} E_1(x) &\le c_1 \mathbf{P}_x[Y_{\tau_{\Delta(0,1,1)}} \in \Delta(0,2,1)]u(A) \\ &\le c_3[\eta(x)^{\alpha-1} \wedge 1]u(A)\,, \end{aligned} \tag{6.27}$$

where $c_3 = c_3(n, \alpha, \lambda, \|\Gamma\|_{1,2})$. By the Carleson estimate (Proposition 6.6) and scaling, and by (6.7) and (6.12),

$$\begin{aligned} E_2(x) &\le c_4 \mathbf{P}_x[Y_{\tau_{\Delta(0,1,1)}} \in D]u(A) \\ &\le c_4 C_9 C_{10}[\eta(x)^{\alpha-1} \wedge 1]u(A)\,, \end{aligned} \tag{6.28}$$

where $c_4 = c_4(n, \alpha, \lambda, \|\Gamma\|_{1,1})$.

Recall that $S \subset D \setminus \Delta(0,3,3)$. To estimate $E_3(x)$ let

$$I(v) = \int_{D\setminus\Delta(0,3,2)} \frac{\mathcal{A}(n,-\alpha)u(y)\,dy}{|y-v|^{n+\alpha}}\,, \quad v \in \Delta(0,3,2)\,.$$

It is straightforward to check that for some $c_5 = c_5(n, \alpha, \lambda)$,

$$c_5^{-1} I(v) \le I(A) \le c_5 I(v)\,, \quad v \in \Delta(0,\ 5/2, 3/2)\,. \tag{6.29}$$

Thus by (3.23) we have,

$$E_3(x) \le c_5 \mathbf{E}_x \tau^Y_{\Delta(0,1,1)} I(A)\,. \tag{6.30}$$

For every $z \in \Delta(0,2,1) \setminus \Delta(0,1,1)$ let $B = B(z, 1/(2\sqrt{\lambda^2+1})) \subset \Delta(0,\ 5/2,\ 3/2)$. By (6.29), (3.23) and (3.6) we have

$$u(z) \ge c_5^{-1} I(A) \mathbf{E}_z \tau^Y_B \ge c_5^{-1} I(A) \mathbf{E}_z \tau^X_B = c_5^{-1} \frac{C_{n,\alpha}}{\mathcal{A}(n,-\alpha) 2^\alpha (\lambda^2+1)^{\alpha/2}} I(A)\,. \tag{6.31}$$

There is $c_6 = c_6(n, \alpha, \lambda)$ such that

$$J(v) = \int_{\Delta(0,2,1)\setminus\Delta(0,1,1)} \frac{\mathcal{A}(n,-\alpha)\,dy}{|y-v|^{n+\alpha}} \ge c_6\,, \quad v \in \Delta(0,1,1)\,.$$

By (6.30), (3.23) and (6.31),

$$E_1(x) \ge \mathbf{E}_x \tau^Y_{\Delta(0,1,1)} c_6 c_5^{-1} \frac{C_{n,\alpha}}{\mathcal{A}(n,-\alpha) 2^\alpha (\lambda^2+1)^{\alpha/2}} I(A) \ge \text{const} \cdot\ E_3(x)\,,$$

so, combining this with (6.27) and (6.28),

$$u(x) = E_1(x) + E_2(x) + E_3(x) \le \text{const} \cdot (\eta(x)^{\alpha-1} \wedge 1)u(A).$$

This and (6.26) yield for some $C_{12} = C_{12}(n, \alpha, \lambda, \|\Gamma\|_{1,1})$ and all $x = (\widetilde{0}, x_n)$ with $0 < x_n < 1$,

$$C_{12}^{-1} u(A)(\eta(x)^{\alpha-1} \wedge 1) \le u(x) \le C_{12} u(A)(\eta(x)^{\alpha-1} \wedge 1). \tag{6.32}$$

The inequality holds also for positive finite linear combinations of functions of the form $u_i(x) = \mathbf{P}_x[Y_{\tau_{\Delta(0,3,3)}} \in S_i]$, for arbitrary measurable sets $S_i \subset D \setminus \Delta(0,3,3)$. An approximation argument then extends (6.32) to all u which are positive and regular harmonic for Y in $\Delta(0,3,3)$. Finally the Harnack inequality (Theorem 3.2) can be used to replace A in (6.32) with any point y such that $|\tilde{y}| \le 1$ and $\eta(y) = 1/2$ (the constant C_{12} may have to be adjusted). Thus, (6.25) may be obtained by applying (6.32) in $\Delta(x,3,3)$. Note that the assumption made in the statement of the theorem that u is harmonic in $\Delta(0,4,4)$ and not only in $\Delta(0,3,3)$ enables us to apply (6.32) in $\Delta(x,3,3)$ with $x \in \Delta(0,1,1)$. On the technical side, this argument relies on invariance of the norm $\|\Gamma\|_{1,1}$ upon translations of Γ. □

Remark 6.1. Consider two functions u_1, u_2 satisfying the assumptions of Theorem 6.7 and such that $u_1(x) = u_2(x) > 0$ for some $x \in \Delta(0,1,1)$. We will sketch an argument showing that

$$\lim_{D \ni x \to 0} \frac{u_1(x)}{u_2(x)} := q \tag{6.33}$$

exists. In fact, one can show that

$$\left| \frac{u_1(x)}{u_2(x)} - q \right| \le c|x|^\sigma , \quad x \in \Delta(0,1,1), \tag{6.34}$$

where c and σ depend only on n, α, λ and $||\Gamma||_{1,1}$, but the proof of this result is rather technical and we refer the reader to the proof of Lemma 16 in [7], whose arguments can be easily adapted to censored processes.

It can be also proved that the limit

$$\lim_{D \ni x \to 0} \frac{u_1(x)}{\rho(x)^{\alpha-1}} \quad \text{exists,} \tag{6.35}$$

which is a result of independent interest. This can be done by analyzing, at ∂D, super- and subharmonic functions similar to those in the proofs of Theorems 5.10 and 6.4. Namely, the functions v and v_1 used in the proofs of Theorems 5.10 and 6.4 should be replaced by $v\mathbf{1}_{D\setminus \overline{D}_\varepsilon}$ and $v_1\mathbf{1}_{D\setminus \overline{D}_\varepsilon}$ for small $\varepsilon > 0$. As (6.35) is outside of the scope of the paper, we do not give the proof here. Certain details of our calculation suggest that $\sigma = 1 - \alpha/2$ is a possible choice for the Hölder exponent in (6.34), and also that a modification of the calculation should give a better exponent σ. We conjecture that σ may be arbitrarily close to 1.

Going back to (6.33), we first note that

$$C_{12}^{-4} \le \frac{u_1(x)}{u_2(x)} \le C_{12}^4 , \quad \text{for all } x \in \Delta(0,1,1). \tag{6.36}$$

This version of the boundary Harnack principle and an argument from [3] imply the following. If functions u_1, u_2 satisfy the assumptions of Theorem 6.7 and for some $M \ge m \ge 0$ we have $mu_1(x) \le u_2(x) \le Mu_1(x)$ on $\Delta(0,4,4)^c$, then

$$m'u_1(x) \le u_2(x) \le M'u_1(x), \quad x \in \Delta(0,1,1)^c, \tag{6.37}$$

where $M \geq M' \geq m' \geq m$ and

$$M' - m' \leq (1 - C_{12}^{-4})(M - m)\,.$$

This rather easily yields that

$$\sup_{\Delta(0,\varepsilon,\varepsilon)} \frac{u_1(x)}{u_2(x)} - \inf_{\Delta(0,\varepsilon,\varepsilon)} \frac{u_1(x)}{u_2(x)} \to 0\,, \quad \text{as } \varepsilon \to 0\,,$$

which is the same as (6.33).

Proof of Theorem 1.2. We first note that u is *bounded* and *regular* harmonic for Y in, say, $D \cap B(Q, 3r/4)$ because of the assumption of continuous decay of u at $\partial D \cap B(Q, r)$, see the proof of (6.15). This is the only place where the present proof uses the property of continuous decay of u at $\partial D \cap B(Q, r)$.

Let $n \geq 2$. To simplify the notation we may and do assume that $Q = 0$ and $r = 1$ (see the discussion of scaling by a factor $k \geq 1$ in the proof of Proposition 6.6), and, by an isometric mapping of D, that

$$D \cap B(0,1) = D_0 \cap B(0,1)\,, \tag{6.38}$$

where D_0 is a special $C^{1,1}$ domain with defining function Γ satisfying $\|\nabla\Gamma\|_\infty \leq \Lambda$ and $\|\Gamma\|_{1,1} \leq \Lambda$. In what follows, all the boxes $\Delta(x, a, r)$ are defined relative to Γ.

Note that even though $D \cap B(0, r)$ may be disconnected, our version of the Harnack inequality (Theorem 3.2) shows that (1.1) holds if $\rho(x)$, $\rho(y)$ are not too small compared to r. Thus, by Theorem 3.2, to prove Theorem 1.2 we only need to verify that there are constants $a = a(n, \alpha, \Lambda)$ and $c = c(n, \alpha, \Lambda)$ such that

$$c^{-1}u(A)\rho(x)^{\alpha-1} \leq u(x) \leq c\,u(A)\rho(x)^{\alpha-1}\,, \quad x \in \Delta(0, a, a)\,, \tag{6.39}$$

where $A = (\tilde{0}, a/2)$, provided $u \geq 0$ on $\mathbf{R}^n$, $u = 0$ on D^c and

$$u(x) = \mathbf{E}_x u(Y_{\tau_{\Delta(0,17a,17a)}})\,, \quad x \in D\,.$$

To prove (6.39) we will introduce two auxiliary functions Γ_-, $\Gamma_+ : \mathbf{R}^{n-1} \to \mathbf{R}$, and sets $D_- = D_{\Gamma_-}$, $D_+ = D_{\Gamma_+}$ such that $D_+ \subset D \subset D_-$ and

$$\kappa_D(x) - c_1 \leq \kappa_{D_-}(x) \leq \kappa_D(x) \leq \kappa_{D_+}(x) \leq \kappa_D(x) + c_1\,, \quad x \in \Delta(0, b, b)\,, \tag{6.40}$$

where $c_1 = c_1(n, \alpha, \Lambda)$, $b = b(n, \alpha, \Lambda)$ are two positive constants. To this end we fix a function $\phi \in C^\infty(\mathbf{R}^{n-1})$ which is nonnegative, supported in $\{\tilde{x} \in \mathbf{R}^{n-1} : 3/4 < |\tilde{x}| < 1\}$ with $\int_{\mathbf{R}^{n-1}} \phi(\tilde{x})d\tilde{x} = 1$.

For a positive integer j, define

$$\phi_j(\tilde{x}) = 2^{-2j}\phi(2^j\tilde{x})\,, \quad \tilde{x} \in \mathbf{R}^{n-1}\,.$$

Clearly $\|\phi_j\|_{1,1} = \|\phi\|_{1,1}$ and

$$\int_{\mathbf{R}^{n-1}} \phi_j(\tilde{x})d\tilde{x} = 2^{-j(n+1)}\,. \tag{6.41}$$

We put $\Gamma_j = \Gamma - \phi_j$ and $D_j = D_{\Gamma_j}$. Obviously, $D_0 \subset D_{\Gamma_j}$ or, equivalently, $D_0^c \supset D_{\Gamma_j}^c$. Consider $x \in D \cap B(0, 2^{-j-1})$. We have

$$\kappa_D(x) - \kappa_{D_j}(x) = [\kappa_D(x) - \kappa_{D_0}(x)] + [\kappa_{D_0}(x) - \kappa_{D_j}(x)] .$$

By (6.38) there is $c_2 = c_2(n, \alpha)$ such that

$$-c_2 \le \kappa_D(x) - \kappa_{D_0}(x) \le c_2 . \tag{6.42}$$

By (6.41) we have

$$\kappa_{D_0}(x) - \kappa_{D_j}(x) \\ = \int_{D_0^c \setminus D_j^c} \frac{\mathcal{A}(n, -\alpha)}{|y - x|^{n+\alpha}} dy \ \ge c_3^{-1} 2^{-j(n+1)} 2^{-j(-n-\alpha)} = c_3^{-1} 2^{j(\alpha-1)} ,$$

and

$$\kappa_{D_0}(x) - \kappa_{D_j}(x) \le c_3 2^{j(\alpha-1)} ,$$

where $c_3 = c_3(n, \alpha, \Lambda)$. By (6.42) we can choose $j = j(n, \alpha, \Lambda)$ so that the first two inequalities in (6.40) are valid in $D \cap B(0, 2^{-j-1})$ for $\Gamma_- := \Gamma_j = \Gamma - \phi_j$.

By a similar calculation, we can choose j so that if $\Gamma_+ := \Gamma + \phi_j$ then the last two inequalities in (6.40) hold for $x \in D \cap B(0, 2^{-j-1})$. Note that $\|\Gamma_-\|_{1,1} \le \|\Gamma\|_{1,1} + \|\phi\|_{1,1}$ and $\|\nabla \Gamma_-\|_\infty \le \|\nabla \Gamma\|_\infty + \|\nabla \phi\|_\infty$ (similarly for Γ_+), which gives us a control of characteristics of D_- (and D_+) by those of D. Clearly, $\Gamma_- \le \Gamma \le \Gamma_+$ and $\Gamma_-(\tilde{x}) = \Gamma(\tilde{x}) = \Gamma_+(\tilde{x})$ if $|\tilde{x}| \le 2^{-j-1}$. In particular, the domains D_-, D, D_+ coincide locally at 0 or, more precisely, the boxes $\Delta(0, s, s)$ defined by Γ_-, Γ, Γ_+ are the same provided $s \le 2^{-j-1}$. We choose $b = 2^{-j-1}/(2\sqrt{\Lambda^2 + 1})$ to have $\Delta(0, b, b) \subset D \cap B(0, 2^{-j-1})$, which yields (6.40).

Let $Y^{(1)}$ and $Y^{(2)}$ denote the censored processes on D_- and D_+, respectively.

Case 1. Take $a = \min\{b/19, r/26\}$ and assume additionally that $u = 0$ on $D \setminus \Delta(0, 19a, 19a)$. Note that u is bounded in $\Delta(0, 19a, 19a)$. Let

$$U(x) = \mathbf{E}_x u(Y^{(2)}_{\tau_{\Delta(0,17a,17a)}}) , \quad x \in D_+ ; \quad U(x) = 0 , \quad x \in D_+^c .$$

By (6.40) and Theorem 2.1(3),

$$u(x) \le U(x) , \quad x \in \mathbf{R}^n .$$

(In fact, U is superharmonic on $\Delta(0, 17a, 17a)$ for the censored process Y on D.) By Theorem 6.7 and scaling applied to U we see that U and so u decay continuously at ∂D in a neighborhood of the closure of $\Delta(0, 4a, 4a)$. Thus, $u(x)$ is continuous in a neighborhood of the closure of $\Delta(0, 4a, 4a)$.

Let

$$u_-(x) = \mathbf{E}_x u(Y^{(1)}_{\tau_{\Delta(0,4a,4a)}}) , \quad x \in D_- ; \qquad u_-(x) = 0 , \quad x \in D_-^c ,$$

and

$$u_+(x) = \mathbf{E}_x u(Y^{(2)}_{\tau_{\Delta(0,4a,4a)}}) , \quad x \in D_+ ; \qquad u_+(x) = 0 , \quad x \in D_+^c .$$

The above continuity result for u implies that u_- and u_+ are continuous at all the points of the closure of $\Delta(0, 4a, 4a)$. We note here that the regularity of points in the bottom part of $\partial\Delta(0, 4a, 4a)$ follows from Theorem 6.7.

By (6.40) and Theorem 2.1(3),

$$u_-(x) \le u(x) \le u_+(x)\,, \quad x \in \mathbf{R}^n\,. \tag{6.43}$$

Thus, by Theorem 6.7 and scaling,

$$c_4^{-1} u_-(A)\rho(x)^{\alpha-1} \le u(x) \le c_4 u_+(A)\rho(x)^{\alpha-1}\,, \quad x \in \Delta(0, a, a)\,,$$

where $A = (\tilde{0}, a/2)$ and $c_4 = c_4(n, \alpha, \Lambda)$. Now, to prove (6.39) we only need to verify that there is $c_5 = c_5(n, \alpha, \Lambda)$ such that

$$u_+(A) \le c_5 u_-(A)\,. \tag{6.44}$$

Let $f = u_+ - u_-$. Clearly, $f \in C_0(\Delta(0, 4a, 4a))$ and $f \ge 0$. Let $f(x_0) = \max_{x\in\Delta(0,4a,4a)} f(x)$. We have

$$\begin{aligned} A_D^\alpha f(x_0) &= \mathcal{A}(n, -\alpha) P.V. \int_D \frac{f(y) - f(x_0)}{|y - x_0|^{n+\alpha}} dy \\ &\le -f(x_0)\mathcal{A}(n, -\alpha) \int_{D\setminus\Delta(0,4a,4a)} \frac{dy}{|y - x_0|^{n+\alpha}} \le -f(x_0)c_6 a^{-\alpha}\,, \end{aligned} \tag{6.45}$$

where $c_6 = c_6(n, \alpha, \Lambda)$. On the other hand

$$A_D^\alpha f(x_0) = [\kappa_D(x_0) - \kappa_{D_+}(x_0)]u_+(x_0) - [\kappa_D(x_0) - \kappa_{D_-}(x_0)]u_-(x_0)\,.$$

By (6.40), (6.43) and Proposition 6.6 we have that $A_D^\alpha f(x_0) \ge -2c_1 u_+(x_0) \ge -c_7 u_+(A)$, where $c_7 = c_7(n, \alpha, \Lambda) > 0$. Thus, by (6.45),

$$f(x_0) \le c_7 c_6^{-1} u_+(A) a^\alpha\ . \tag{6.46}$$

Choosing $a = a(n, \alpha, \Lambda)$ small enough we have $f(A) \le f(x_0) \le u_+(A)/2$, thus $u_-(A) = u_+(A) - f(A) \ge u_+(A)/2$. We have obtained (6.44), which finishes the proof of (6.39) in Case 1.

Case 2. We now assume that $u = 0$ on $\Delta(0, 18a, 18a) \setminus \Delta(0, 17a, 17a)$ and, as before, that $u \ge 0$ on $\mathbf{R}^n$, $u(x) = \mathbf{E}_x u(Y_{\tau_{\Delta(0,17a,17a)}})$, $x \in D$, and $u = 0$ on D^c. Here $a = a(n, \alpha, \Lambda)$ is the constant determined in Case 1.

By (3.23) we easily obtain that there is $c_8 = c_8(n, \alpha, \Lambda) > 1$ such that

$$c_8^{-1} u(x) \le \mathbf{E}_x \tau^Y_{\Delta(0,17a,17a)} \int_{\Delta(0,17a,17a)^c} \frac{u(y)}{(1 + |y|)^{n+\alpha}} dy \le c_8 u(x)\,; \tag{6.47}$$

see the proof of Proposition 6.6 for a similar calculation. In particular, there is $c_9 = c_9(n, \alpha, \Lambda)$ such that

$$\begin{aligned} c_9^{-1} \mathbf{E}_x \tau^Y_{\Delta(0,17a,17a)} &\le \mathbf{P}_x[Y_{\tau_{\Delta(0,17a,17a)}} \in \Delta(0, 19a, 19a) \setminus \Delta(0, 18a, 18a)] \\ &\le c_9 \mathbf{E}_x \tau^Y_{\Delta(0,17a,17a)}\,. \end{aligned} \tag{6.48}$$

Case 1 shows that (6.39) applies to the function $x \mapsto \mathbf{P}_x[Y_{\tau_{\Delta(0,17a,17a)}} \in \Delta(0, 19a, 19a) \setminus \Delta(0, 18a, 18a)]$. From this, (6.47) and (6.48) we obtain

$$u(x) \approx \rho(x)^{\alpha-1} \int_{\Delta(0,17a,17a)^c} \frac{u(y)}{(1+|y|)^{n+\alpha}} dy \,. \tag{6.49}$$

In particular $u(A) \approx \int_{\Delta(0,17a,17a)^c} u(y)(1+|y|)^{-n-\alpha} dy$; this and (6.49) imply (6.39) in Case 2. The case of general u in (6.39) follows easily from Cases 1 and 2.

We now prove Theorem 1.2 when dimension $n = 1$. Recall that in dimension $n = 1$, a $C^{1,1}$ open set is any union of open intervals with lengths and distances between distinct intervals bounded away from zero. By scaling we may and do assume that u is harmonic in $D \cap (-1, 1) = (0, 1)$. Let $A = 1/2$. By (5.5) it is easy to see that there is $b = b(\alpha) < 1/3$ such that the functions

$$u_+(x) = 2[w_{\alpha-1}(x) \wedge 1] - \frac{1}{2}[w_{\alpha/2}(x) \wedge 1] \,, \quad x \in \mathbf{R}^n \,,$$
$$u_-(x) = [w_{\alpha-1}(x) \wedge 1] + \frac{1}{2}[w_{\alpha/2}(x) \wedge 1] \,, \quad x \in \mathbf{R}^n \,,$$

are non-negative superharmonic and subharmonic, respectively, on $(0, b)$ for the censored process Y on D. We conclude using the proof of Proposition 5.8 and Theorem 6.4 that there is a constant $c_1 = c_1(n, \alpha) > 0$ such that

$$c_1^{-1} \rho(x)^{\alpha-1} \le \mathbf{P}_x[Y_{\tau_{(0,b)}} \in (0, 2/3)] \le c_1 \rho(x)^{\alpha-1} \,, \quad x \in (0, b) \,. \tag{6.50}$$

By the Harnack inequality (Theorem 3.2), the mean value property of u and (6.50), we obtain the lower bound in the following inequality:

$$c_2^{-1} \rho(x)^{\alpha-1} u(A) \le u(x) \le c_2 \rho(x)^{\alpha-1} u(A) \,, \quad x \in (0, b) \,, \tag{6.51}$$

where $c_2 = c_2(\alpha)$. For the upper bound in (6.51) we decompose u as

$$\begin{aligned} u(x) &= \mathbf{E}_x[Y_{\tau_{(0,b)}} \in (0, 2/3) \,;\; u(Y_{\tau_{(0,b)}})] + \mathbf{E}_x[Y_{\tau_{(0,b)}} \in [2/3, \infty) \,;\; u(Y_{\tau_{(0,b)}})] \\ &= u_1(x) + u_2(x). \end{aligned}$$

The following upper bound for u_1,

$$u_1(x) \le \text{const.} \cdot \rho(x)^{\alpha-1} u(A) \,, \quad x \in (0, b) \,,$$

follows easily from (6.50) and the Harnack inequality for u. A similar upper bound for u_2 is obtained as in Case 2 above for $n \ge 2$. The proof is now complete. □

Remark 6.2. As we see at the beginning of the proof above, if a function $u \ge 0$ on D vanishes continuously on $\partial D \cap B(Q, r)$, and is harmonic in $D \cap B(Q, r)$ for Y then u is bounded and regular harmonic in $D \cap B(Q, \delta r)$ for Y for any $\delta \in (0, 1)$. Conversely, the above proof shows that a nonnegative function on D that is *bounded* and regular harmonic on $D \cap B(Q, r)$ for Y vanishes continuously on $\partial D \cap B(Q, r/2)$. For a general non-negative function u on D that is regular harmonic on $D \cap B(Q,r)$, note that $u_k(x) := \mathbf{E}_x\left[u(Y_{\tau_{D\cap B(Q,r)}}) \wedge k\right]$ is bounded regular harmonic in $D \cap B(Q,r)$. So by Theorem 1.2,

$$\frac{u_k(x)}{u_k(y)} \le C\,\frac{\rho(x)^{\alpha-1}}{\rho(y)^{\alpha-1}}, \qquad x,\ y \in D \cap B(Q,\ r/2).$$

As $u(x) = \lim_{k\to\infty} u_k(x)$, it follows that

$$\frac{u(x)}{u(y)} \le C\,\frac{\rho(x)^{\alpha-1}}{\rho(y)^{\alpha-1}}, \qquad x,\ y \in D \cap B(Q,\ r/2).$$

This implies in particular that such u vanishes continuously on $\partial D \cap B(Q, r/2)$.

Remark 6.3. Using a finite open covering of the boundary ∂D and the Harnack inequality (Theorem 3.2), one can easily show that Theorem 1.2 holds with balls $B(Q,r)$ and $B(Q,r/2)$ replaced by an open set U and a compact set $F \subset U$, but then the constant C would depend on $D,\ U$, and F.

Remark 6.4. For completeness we note that for any two functions u_1 and u_2 satisfying the assumptions of Theorem 1.2,

$$\lim_{D\ni x\to Q} \frac{u_1(x)}{u_2(x)} := q \quad \text{exists and} \quad \left|\frac{u_1(x)}{u_2(x)} - q\right| \le c\,\rho(x)^{\sigma}, \quad x \in D \cap B(Q, r/2), \tag{6.52}$$

The inequality in (6.52) follows easily from (6.34) and the estimates in the proof of Theorem 1.2, in particular (6.46). In consequence, under the assumptions of Theorem 1.2 one obtains the existence and finiteness of the limit

$$\lim_{D\ni x\to Q} \frac{u(x)}{\rho^{\alpha-1}(x)},$$

see Remark 6.1.

7. Appendix

The appendix contains several auxiliary lemmas on the *uncensored* stable process X, i.e., the rotation invariant symmetric α-stable Lévy process in $\mathbf{R}^n$. The results are needed in the main sections of this paper and may be also of independent interest.

We first prove a geometric result related to $C^{1,\beta-1}$ domains.

Lemma 7.1. *Let $C \ge 1$, $1 < \beta \le 2$ and $\mathcal{P} = \left\{x = (\widetilde{x}, x_n) \in \mathbf{R}^n :\ C|\widetilde{x}|^{\beta} < x_n < C^{-1}\right\}$. Let $A = (\widetilde{0}, a)$ with $0 < a < (2^{3-\beta}C)^{1/(1-\beta)}$. Then*

$$a/2 < a - 2(2C)^{\frac{2}{2-\beta}}\, a^{\frac{\beta}{2-\beta}} \le \operatorname{dist}(A, \mathcal{P}^c) \le a \quad \text{if} \quad \beta < 2, \tag{7.1}$$

and

$$\operatorname{dist}(A, \mathcal{P}^c) = a \quad \text{if} \quad \beta = 2. \tag{7.2}$$

Proof. We first assume that $1 < \beta < 2$. The rightmost inequality in (7.1) is obvious. The assumption on a implies that $a < 1/(2C)$, thus $\text{dist}(A, \mathcal{P}^c) = \inf_{t\in[0,\infty)} \sqrt{t^2 + (a - Ct^\beta)^2}$. Every $t \geq 0$ can be written in the form $t = ca^{1/(2-\beta)}$ with some $c \geq 0$. We have that

$$\begin{aligned} W = t^2 + (a - Ct^\beta)^2 &= a^2 + a^{2/(2-\beta)}(c^2 - 2Cc^\beta) + c^{2\beta}C^2 a^{2\beta/(2-\beta)} \\ &\geq a^2 + a^{2/(2-\beta)}(c^2 - 2Cc^\beta)\,. \end{aligned}$$

If $c^2 \geq 2Cc^\beta$, or $c \geq (2C)^{1/(2-\beta)}$ then $W \geq a^2$. If $c < (2C)^{1/(2-\beta)}$ then we have

$$W \geq a^2 - a^{2/(2-\beta)}2Cc^\beta > a^2 - 2(2C)^{2/(2-\beta)}a^{2/(2-\beta)}\,.$$

It follows that

$$\begin{aligned} \text{dist}(A, \mathcal{P}^c) &\geq \sqrt{[a^2 - 2(2C)^{2/(2-\beta)}a^{2/(2-\beta)}] \vee 0} \\ &= a\sqrt{[1 - 2(2C)^{2/(2-\beta)}a^{(2\beta-2)/(2-\beta)}] \vee 0} \\ &\geq [a - 2(2C)^{2/(2-\beta)}a^{\beta/(2-\beta)}] \vee 0\,. \end{aligned}$$

Note that $2(2C)^{2/(2-\beta)}a^{\beta/(2-\beta)} \leq a/2$ if $a \leq (2^{3-\beta}C)^{1/(1-\beta)}$. The proof of (7.1) is complete. Equality (7.2) is obtained by taking $\beta \to 2^-$ in (7.1), see also Lemma 6.2. □

Lemma 7.2. *Let $B = B(0, 1)$ be the unit ball in $\mathbf{R}^n$, $n \geq 1$, and assume that $0 < \alpha < 2 \wedge n$. Let T_B be the first entrance time of B by the symmetric α-stable process X_t. There is a constant $A_1 = A_1(n, \alpha)$ such that*

$$\mathbf{P}_x[T_B = \infty] \leq A_1(|x| - 1)^{\alpha/2} \quad \textit{for } x \in \mathbf{R}^n \textit{ with } |x| > 1. \tag{7.3}$$

Proof. For $|x| \in (1, 2)$, by Corollary 2 of [6] and a change of variable $u = (|x|^2 - 1)v$,

$$\begin{aligned} \mathbf{P}_x(\tau_{B^c} = \infty) &= \frac{\Gamma(n/2)}{\Gamma((n-\alpha)/2)\Gamma(\alpha/2)} \int_0^{|x|^2-1} (u+1)^{-\frac{n}{2}} u^{\frac{\alpha}{2}-1} du \\ &\leq \frac{\Gamma(n/2)}{\Gamma((n-\alpha)/2)\Gamma(\alpha/2)} \left(|x|^2 - 1\right)^{\alpha/2} \int_0^1 v^{\frac{\alpha}{2}-1} dv \\ &\leq \frac{6\Gamma(n/2)}{\alpha\Gamma((n-\alpha)/2)\Gamma(\alpha/2)} (|x| - 1)^{\alpha/2}\,. \end{aligned}$$

This proves the lemma as probability is always bounded by 1. □

Note the (7.3) follows from the more general results given for $C^{1,1}$ domains in [18] and [19], but the present derivation is more explicit. The same remark applies to the next estimate.

Lemma 7.3. *Suppose $0 < \alpha < 2$. Consider points $x, y \in B(0, 2) \subset \mathbf{R}^n$ such that $|x| < 1/4$ and $|y| < 1/4$. Let $0 < r < 1/4$ and $U = B(0, 2) \setminus B(y, r)$. There is $A_2 = A_2(n, \alpha)$ such that*

$$\mathbf{P}_x[X_{\tau_U} \in B(0, 2)^c] \leq A_2 \left(\frac{\text{dist}(x, B(y, r))}{r}\right)^{\alpha/2}. \tag{7.4}$$

Proof. Consider the case $\alpha < n$. We need to consider only $x \notin \overline{B(y,r)}$, or $|x-y| > r$. Let $z = x - y$ and $B = B(0,1) \subset \mathbf{R}^n$. We have

$$\mathbf{P}_x[X_{\tau_U} \in B(0,2)^c] \le \mathbf{P}_x[\tau_{B(y,1)} < T_{B(y,r)}] = \mathbf{P}_z[\tau_B < T_{B(0,r)}] .$$

By the strong Markov property,

$$\mathbf{P}_z[T_{B(0,r)} = \infty] \ge \mathbf{P}_z[\tau_B < T_{B(0,r)} \,;\, \mathbf{P}_{X_{\tau_B}}\{T_{B(0,r)} = \infty\}] . \tag{7.5}$$

We note that $K(v) = |v|^{\alpha-n}$ is harmonic in $\mathbf{R}^n \setminus \{0\}$ with respect to the symmetric α-stable process X, as it is the Green function $G(v,0)$ of X modulo a constant multiple (see (3.7)). So for $|w| \ge 1$ we obtain

$$1 \ge K(w) = \mathbf{E}_w[K(X_{T_{B(0,r)}}) \,;\, T_{B(0,r)} < \infty] \ge r^{\alpha-n}\mathbf{P}_w[T_{B(0,r)} < \infty] .$$

As a consequence, a.s.,

$$\mathbf{P}_{X_{\tau_B}}[T_{B(0,r)} = \infty] \ge 1 - r^{n-\alpha} \ge 1 - 4^{\alpha-n} ,$$

and by (7.5),

$$\mathbf{P}_z[\tau_B < T_{B(0,r)}] \le \mathbf{P}_z[T_{B(0,r)} = \infty]/(1 - 4^{\alpha-n}) .$$

By scaling and Lemma 7.2 we obtain,

$$\begin{aligned}\mathbf{P}_x[X_{\tau_U} \in B(0,2)^c] &\le (1 - 4^{\alpha-n})^{-1}\mathbf{P}_{x-y}[T_{B(0,r)} = \infty] \\ &\le A_1(1 - 4^{\alpha-n})^{-1} \left(|(x-y)/r| - 1\right)^{\alpha/2} \\ &= A_1(1 - 4^{\alpha-n})^{-1} \left(\frac{|x-y| - r}{r}\right)^{\alpha/2} .\end{aligned}$$

The case $n = 1 \le \alpha$ follows easily from (3.4) and is left to the reader. □

Proposition 7.4. *Let* $n \in \{2, 3, \dots\}$ *and* $0 < \alpha < 2$. *Let* $1 < \beta \le 2$ *and* $\beta > \alpha$. *Consider a* $C^{1,\beta-1}$ *function* $\Gamma : \mathbf{R}^{n-1} \to \mathbf{R}$ *and let* $D = D_\Gamma$, $\rho(x) = \mathrm{dist}(x, D^c)$, $\overline{D}_1 = \{x \in D : \rho(x) \ge 1\}$. *There is* $A_3 = A_3(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1})$ *such that*

$$\mathbf{P}_x[X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1] \le A_3[\rho(x)^{\alpha-\alpha/\beta} \wedge 1] , \quad x \in \mathbf{R}^n . \tag{7.6}$$

Proof. Consider the case when $\beta < 2$. Suppose $x \in D$ and let $Q \in \partial D$ be such that $|x - Q| = \rho(x)$. As in Lemma 5.5 we write $\mathcal{P}_{-b} + Q$ for the *outer* tangential region at Q. Note that

$$\mathbf{P}_x[X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1] \le \mathbf{P}_x[\tau_{B(Q,1)} < T_{\mathcal{P}_{-b}+Q}] .$$

To estimate the latter probability we will assume without loss of generality that $Q = 0$ and $b = e_n = (0, \dots, 0, 1) \in \mathbf{R}^n$. Let $y = -ae_n$ where $0 < a < (2^{3-\beta}C)^{1/(1-\beta)}$ (comp. Lemma 7.1) and $C = C(\beta, \|\Gamma\|_{1,\beta-1})$ is the constant of Lemma 5.5. We define r by $r/2 = a - 2(2C)^{2/(2-\beta)}\, a^{\beta/(2-\beta)}$. By Lemma 7.1,

$$B(y, r/2) \subset \mathcal{P}_{-e_n} .$$

By Lemma 7.3 and scaling

$$\mathbf{P}_x[X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1] \le \mathbf{P}_x[\tau_{B(0,1)} < T_{B(y,r/2)}] \le A_2 \left(\frac{\operatorname{dist}(x, B(y, r/2))}{r/2}\right)^{\alpha/2} .$$

Assume that $\rho(x)^{(2-\beta)/\beta} < (2^{3-\beta}C)^{1/(1-\beta)}$ and let $a = \rho(x)^{(2-\beta)/\beta} = |x|^{(2-\beta)/\beta}$. We have

$$r/2 = |x|^{(2-\beta)/\beta} - 2(2C)^{2/(2-\beta)}|x| \ge \frac{a}{2} = \frac{1}{2}|x|^{(2-\beta)/\beta} ,$$

and $\operatorname{dist}(x, B(y, r/2)) = |x|[1 + 2(2C)^{2/(2-\beta)}]$. We conclude that

$$\mathbf{P}_x[X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1] \le A_2 \left(2[1 + 2(2C)^{2/(2-\beta)}]|x|^{1-(2-\beta)/\beta}\right)^{\alpha/2} = \text{const}\cdot |x|^{\alpha[1-1/\beta]} .$$

This proves (7.6) for small values of $\rho(x)$. For values of $\rho(x)$ which are greater than a positive constant, (7.6) is trivial.

The remaining case $\beta = 2$ follows even more easily by an appropriate choice of r independent of x in the proof above. It can also be found in [18] and [19]. □

Lemma 7.5. *Let* $1 < \beta \le 2$ *and let* $\Gamma : \mathbf{R}^{n-1} \to \mathbf{R}$ *be a* $C^{1,\beta-1}$ *function. Let* $D = D_\Gamma \subset \mathbf{R}^n$ *and* $\overline{D}_1 = \{x \in D : \rho(x) \ge 1\}$. *There is* $S = S(n, \beta, \|\Gamma\|_{1,\beta-1})$ *such that for every* $Q \in \partial D$ *we have*

$$|B(Q, S) \cap \overline{D}_1| \ge 1 , \tag{7.7}$$

where $|B(Q, S) \cap \overline{D}_1|$ *is the Lebesgue measure of* $B(Q, S) \cap \overline{D}_1$.

The proof is somewhat tedious (because the seminorm $\|\Gamma\|_{1,\beta-1}$ does not dominate $\nabla\Gamma(\tilde{0})$), but completely elementary so it is left to the reader.

Proposition 7.6. *Under the assumptions of* Proposition 7.4 *there is* $A_4 = A_4(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1})$ *such that*

$$\mathbf{P}_x[X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1] \ge A_4^{-1}[\rho(x)^{\alpha/\beta} \wedge 1] , \quad x \in \mathbf{R}^n . \tag{7.8}$$

Proof. Consider the case of $\beta < 2$. Let $\rho(x) \le 1$ and let $Q \in \partial D$ be such that $|x - Q| = \rho(x)$. As in Lemma 5.5 we write $\mathcal{P}_b + Q$ for the inner tangential region at Q. Consider a ball $B = B(A, r)$ in $\mathcal{P}_b + Q$ such that $x \in B \subset D \setminus \overline{D}_1$. By (3.2), Lemma 7.5 and (3.6)

$$\begin{aligned}\mathbf{P}_x[X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1] \ge \mathbf{P}_x[X_{\tau_B} \in \overline{D}_1] &= \mathcal{A}(n, -\alpha) \int_{\overline{D}_1} \int_B \frac{G_B(x, v)}{|y - v|^{n+\alpha}} dv\, dy \\ &\ge \mathcal{A}(n, -\alpha)(S + \sqrt{2})^{-(n+\alpha)} \int_B G_B(x, v)\, dv \\ &= C_\alpha^n (S + \sqrt{2})^{-(n+\alpha)}[r^2 - |x - A|^2]^{\alpha/2} . \end{aligned} \tag{7.9}$$

Here $S = S(n, \beta, \|\Gamma\|_{1,\beta-1})$ is the constant of Lemma 7.5. To obtain the desired lower bound for the last expression we need to choose A and r appropriately.

We will assume without loss of generality that $Q = 0$, $b = e_n = (0, \dots, 0, 1)$, and thus $\mathcal{P}_b + Q = \mathcal{P}$ and $x = |x|e_n$. We put $A = (\varepsilon|x|)^{(2-\beta)/\beta} e_n$, where $\varepsilon = \varepsilon(n, \alpha, \beta, \|\Gamma\|_{1,\beta-1})$ is a small positive constant to be determined in the course of the following calculation. Note that if $|x|$ is small enough, then A is above x; $|A| > |x|$. We then take $r = |A| - 2(2C)^{2/(2-\beta)}|A|^{\beta/(2-\beta)}$. Here C is the constant defining $\mathcal{P}$, see Lemma 5.5. If $|A|$ is small enough then by Lemma 7.1 we have that $r > |A|/2$, $B = B(A, r) \subset \mathcal{P}$ and $B \subset D \setminus \overline{D}_1$. Note that $r = (\varepsilon|x|)^{(2-\beta)/\beta} - 2(2C)^{2/(2-\beta)}\varepsilon|x|$. Thus

$$\begin{aligned}
&[r^2 - |x - A|^2]^{\alpha/2} \\
&\quad = [r^2 - (|A| - |x|)^2]^{\alpha/2} = [r + |A| - |x|]^{\alpha/2}[r - |A| + |x|]^{\alpha/2} \\
&\quad = [2(\varepsilon|x|)^{(2-\beta)/\beta} - 2(2C)^{2/(2-\beta)}\varepsilon|x| - |x|]^{\alpha/2}[|x|\{1 - 2(2C)^{2/(2-\beta)}\varepsilon\}]^{\alpha/2} .
\end{aligned}$$

For ε small enough

$$[r^2 - |x - A|^2]^{\alpha/2} \ge \left[(\varepsilon|x|)^{(2-\beta)/\beta}|x|\frac{1}{2}\right]^{\alpha/2} = \text{const} \cdot |x|^{\alpha/\beta} ,$$

which proves (7.8) for small $\rho(x)$. When $\rho(x)$ is greater than a positive constant, say, η, we put $B = B(x, \eta)$. We can assume that $|B| \le 1/2$. Using (7.7) and an analogue of (7.9) we see that

$$\mathbf{P}_x[X_{\tau_{D\setminus\overline{D}_1}} \in \overline{D}_1] \ge \mathbf{P}_x[X_{\tau_B} \in \overline{D}_1 \setminus B]$$

is bounded below by a positive constant for such x. The proof is complete for $1 < \beta < 2$.

For the case of $\beta = 2$ we refer the reader to [18] and [19]. This case can also be obtained by an appropriate choice of r independent of x in the proof above. □

The exponents $\alpha[1 - 1/\beta]$ and α/β in (7.6) and (7.8) suffice for the application in the proof of (5.39) above, but it is an open problem if they can actually be replaced by $\alpha/2$. We conjecture it is true for $\beta > \alpha$. This motivates in part our interest in $C^{1,\beta-1}$ domains.

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