Probab. Theory Relat. Fields 126, 289–306 (2003)
Digital Object Identifier (DOI) 10.1007/s00440-003-0268-0

Franco Fagnola · Rolando Rebolledo

Transience and recurrence of quantum Markov semigroups

Received: 27 January 2003 / Revised version: 19 February 2003 /
Published online: 12 May 2003 – © Springer-Verlag 2003

Abstract. This article introduces a concept of transience and recurrence for a Quantum Markov Semigroup and explores its main properties via the associated potential. We show that an irreducible semigroup is either recurrent or transient and characterize transient semigroups by means of the existence of non trivial superharmonic operators.

1. Introduction

Transience and recurrence come to a probabilist mind as the first step in the classification of Markov processes. In classical probability, these two notions have been extensively studied in connection with Semigroup and Potential Theory. In the non commutative framework, quantum Markov semigroups arise as a natural extension of the classical notion. Moreover, they are a fundamental tool in the non commutative version of the theory due to both, the non existence of trajectories and the difficulties to handle a good notion of stopping time. Although the large time behavior of quantum Markov semigroup has been the subject of a number of investigations, a detailed study of recurrence and transience is still unavailable. This paper is aimed at filling that gap.

Quantum Markov Semigroups are usually defined on von Neumann algebras, typical examples of which are L^∞ spaces (the commutative case) and $\mathcal{B}(\mathfrak{h})$, the non commutative algebra of all bounded linear operators on a complex Hilbert space $\mathfrak{h}$. In general, any von Neumann algebra $\mathcal{A}$ may be considered as a subalgebra of $\mathcal{B}(\mathfrak{h})$ for a suitable space $\mathfrak{h}$ and it is the dual of a Banach space $\mathcal{A}_*$. Within this framework a *Quantum Markov Semigroup*, denoted $\mathcal{T}$ and abbreviated QMS, is a weak*-continuous semigroup of normal completely positive maps $\mathcal{T} = (\mathcal{T}_t(\cdot))_{t\geq 0}$ on $\mathcal{A}$ such that $\mathcal{T}_t(\mathbf{1}) = \mathbf{1}$ for all $t \geq 0$, where $\mathbf{1}$ is the unit of $\mathcal{A}$. This mathematical

F. Fagnola: Università degli Studi di Genova, Dipartimento di Matematica, via Dodecaneso 35, I-16146 Genova

R. Rebolledo: Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile. e-mail: `rrebolle@puc.cl`

This research has been partially supported by the "Cátedra Presidencial en Análisis Cualitativo de Sistemas Dinámicos Cuánticos", DIPUC, FONDECYT project 1030552, MIUR program "Probabilità Quantistica e Applicazioni", 2003-2004 and MECESUP PUC 0103.

Mathematics Subject Classification (2000): 60J45, 81S25, 60J99, 37A50, 47A40

Key words or phrases: Quantum Markov semigroups – Potential theory – Markov processes

object provides both, a good model for the description of open quantum systems and a suitable non commutative extension of classical Markov semigroups. This is a rich interplay between Probability and Physics which allows us to investigate new mathematical problems with physical significance.

Here we introduce a potential associated to a QMS, $\mathcal{U}(a) = \int_0^\infty \mathcal{T}_t(a)\,dt$, for $a \in \mathcal{A}$ (see the precise definition in section 3). According to the nature of $\mathcal{U}(a)$ we define transience or recurrence. Then we explore its main related properties as, for instance, the relation with sub (or super) harmonic operators and the dichotomy transience-recurrence for irreducible semigroups. We end the paper by some applications to QMS arising in Quantum Optics.

It is worth noticing here that the scope of the present investigation is not merely mathematical. Its results apply to the classification of an overwhelming variety of open quantum system evolutions described through master equations (see for instance [1], [11], [17], [21]), whose solution is a QMS. Thus, a QMS is the most rigorous and successful model of the irreversible evolution of quantum systems in interaction with the environment, which is influenced by effects such as dissipation and decoherence. In this paper we show that a suitable probabilistic view on these problems provides powerful tools to attack difficult operator equations that are hard to handle otherwise.

2. Markov semigroups on von Neumann algebras

Throughout the paper we keep von Neuman algebras as the basic structure supporting our investigation. If $\mathcal{A}$ is a von Neumann algebra, its predual is denoted $\mathcal{A}_*$. A *state* φ is an element of $\mathcal{A}_*$ such that $\varphi(\mathbf{1}) = 1$, where $\mathbf{1}$ is the unit of $\mathcal{A}$. As a rule, we will only deal with *normal states* φ for which there exists a density matrix ρ, that is, a positive trace-class operator of $\mathfrak{h}$ with unit trace, such that $\varphi(a) = \mathrm{tr}(\rho\, a)$ for all $a \in \mathcal{A}$.

We say that an operator is *non trivial* if it is not a multiple of $\mathbf{1}$.

The concept of *irreducible* Quantum Markov semigroups is oftenly used in the paper. It refers to the probabilistic notion of irreducibility as defined in our paper [13]. This simply means that a QMS has no non-trivial subharmonic projections. A positive operator a is subharmonic (respectively superharmonic, resp. harmonic), if $\mathcal{T}_t(a) \geq a$ (resp. $\mathcal{T}_t(a) \leq a$, resp. $\mathcal{T}_t(a) = a$), for all $t \geq 0$.

We shall often make use of the following elementary remark. Given a positive $x \in \mathcal{A}$ and a projection p, $pxp = 0$ implies $p^\perp x p = p x p^\perp = 0$, (see Lemma II.1 in [13]).

We recall that for any normal state φ on the von Neumann algebra $\mathcal{A}$, the *support projection* $S(\varphi)$ is the smallest projection in $\mathcal{A}$ such that $\varphi(aS(\varphi)) = \varphi(S(\varphi)a) = \varphi(a)$, for all $a \in \mathcal{A}$ (c.f. [9], Prop.3, p. 63). If $S(\varphi) = \mathbf{1}$, we say that the state is *faithful*.

Proposition 1. *Let $\mathcal{T}$ be a QMS defined on the von Neumann algebra $\mathcal{A}$. Assume that there is an invariant state φ, then its support projection is subharmonic.*

Proof. Let $p = S(\varphi)$. The invariance of φ yields $\varphi(p - p\mathcal{T}_t(p)\,p) = \varphi(p - \mathcal{T}_t(p)) = 0$. Moreover $p\mathcal{T}_t(p)\,p \leq p$, whence $p = p\mathcal{T}_t(p)\,p$. Therefore, the projection $p^\perp = \mathbf{1} - p$ satisfies $p\mathcal{T}_t\left(p^\perp\right)p = 0$. It follows that $p^\perp\mathcal{T}_t\left(p^\perp\right)p = 0$ and $p\mathcal{T}_t\left(p^\perp\right)p^\perp = 0$. Thus, $\mathcal{T}_t\left(p^\perp\right) = p^\perp\mathcal{T}_t\left(p^\perp\right)p^\perp \leq p^\perp$ and $p \leq \mathcal{T}_t(p)$ for all $t \geq 0$. □

3. Potential

Let $\mathcal{T}$ be a Quantum Markov Semigroup (QMS) on a von Neumann algebra $\mathcal{A}$ of operators on a complex Hilbert space $\mathfrak{h}$.

Inspired by the classical theory of Markov processes [8], this section introduces the non commutative version of *potential* and discusses its main properties. This is the main tool in the study of recurrence and transience.

Throughout this paper, the use of quadratic forms settings will follow the book of Kato (see [18]).

Definition 1. *Given a positive operator $x \in \mathcal{A}$ we define the* form-potential of x *as a quadratic form $\mathfrak{U}(x)$ on the domain*

$$D(\mathfrak{U}(x)) = \left\{u \in \mathfrak{h} : \int_0^\infty \langle u, \mathcal{T}_s(x)\,u\rangle ds < \infty\right\},$$

by

$$\mathfrak{U}(x)[u] = \int_0^\infty \langle u, \mathcal{T}_s(x)\,u\rangle ds, \;\; (u \in D(\mathfrak{U}(x))).$$

This is clearly a symmetric and positive form and by Thm. 3.13a and Lemma 3.14a p.461 of [18] it is also closed. Therefore, when it is densely defined, it is represented by a self-adjoint operator (see Th.2.1, p.322, Th. 2.6, p.323 and Th. 2.23 p.331 of [18]). This motivates the following definition.

Definition 2. *A positive $x \in \mathcal{A}$ such that $D(\mathfrak{U}(x))$ is dense is called $\mathcal{T}$*–integrable *or simply* integrable. *We denote $\mathcal{A}^+_{\text{int}}$ the cone of positive integrable elements of $\mathcal{A}$. For any $x \in \mathcal{A}^+_{\text{int}}$, we call* potential of x *the self-adjoint operator $\mathcal{U}(x)$ which represents $\mathfrak{U}(x)$.*

Note that $D(\mathcal{U}(x)^{1/2}) = D(\mathfrak{U}(x))$ (see Th. 2.23, p.331 in [18]).

We recall that a closed operator A is affiliated with a von Neumann algebra $\mathcal{A}$ if $a'D(A) \subseteq D(A)$ and $a'A \subseteq Aa'$ for all $a' \in \mathcal{A}'$.

Proposition 2. *For all $x \in \mathcal{A}^+_{\text{int}}$, the operator $\mathcal{U}(x)$ is affiliated with $\mathcal{A}$.*

Proof. Fix $y \in \mathcal{A}'$ and define $X_t = \int_0^t \mathcal{T}_s(x)\,ds$, for all $t \geq 0$. Clearly, both X_t and $X_t^{1/2}$ belong to $\mathcal{A}$. Given any $u \in \mathfrak{h}$,

$$\int_0^t \langle yu, \mathcal{T}_s(x)\,yu\rangle ds = \langle yX_t^{1/2}u, yX_t^{1/2}u\rangle \leq \|y\|^2 \langle u, X_t u\rangle.$$

Thus, if $u \in D(\mathfrak{U}(x))$, then

$$\sup_{t\geq 0} \int_0^t \langle yu, \mathcal{T}_s(x)\,yu\rangle ds \leq \|y\|^2 \int_0^\infty \langle u, \mathcal{T}_s(x)\,u\rangle ds = \|y\|^2\,\mathfrak{U}(x)[u].$$

It follows that, if $u \in D(\mathfrak{U}(x)) = D(\mathcal{U}(x)^{1/2})$, then $yu \in D(\mathfrak{U}(x))$.

Now, if $v, u \in D(\mathcal{U}(x))$, then $y^*v,\ yu \in D(\mathfrak{U}(x))$ and

$$\int_0^t \langle y^*v, \mathcal{T}_s(x)\, u\rangle ds = \int_0^t \langle \mathcal{T}_s(x)\, v, yu\rangle ds,$$

so that letting $t \to \infty$ and using complex polarization, we get

$$\langle y^*v, \mathcal{U}(x)u\rangle = \langle \mathcal{U}(x)v, yu\rangle.$$

That is, $\langle v, y\mathcal{U}(x)u\rangle = \langle \mathcal{U}(x)v, yu\rangle$ it follows that $yu \in D(\mathcal{U}(x))$ and $\mathcal{U}(x)yu = y\mathcal{U}(x)u$, hence $y\mathcal{U}(x) \subseteq \mathcal{U}(x)y$. □

Proposition 3. *Let $\mathcal{T}$ be a Quantum Markov Semigroup and let $x \in \mathcal{A}$ positive. Then the orthogonal projection p onto the closure of $D(\mathfrak{U}(x))$ is subharmonic. In particular, if $\mathcal{T}$ is irreducible, then $D(\mathfrak{U}(x))$ is either dense or* $\{0\}$.

Proof. We first notice that $p \in \mathcal{A}$. Indeed, arguing as in the proof before, we can show that for every $u \in D(\mathfrak{U}(x))$ and $y \in \mathcal{A}'$, $yu \in D(\mathfrak{U}(x))$. Hence,

$$pypu = pyu = yu = ypu.$$

In other words, since $D(\mathfrak{U}(x))$ is dense in the range of p, we obtain $pyp = yp$.

On the other hand, $y^* \in \mathcal{A}'$, so that $py^*p = y^*p$. Therefore, $pyp = py$. Hence $yp = py$, so that $p \in \mathcal{A}'' = \mathcal{A}$.

We now show that $\mathcal{T}_t(p) \geq p$ for any $t \geq 0$. Let ρ be a density matrix ρ such that

$$\rho = \sum_k \lambda_k |u_k\rangle\langle u_k|,\ \lambda_k \geq 0,\ \sum_k \lambda_k = 1,\ u_k \in D(\mathfrak{U}(x)).$$

Note that ρ defines a normal linear functional on $\mathcal{A}$. Therefore $\rho \in \mathcal{A}_*$. Moreover, denote $\varphi \in \mathcal{A}_*$ the state given by $\varphi(a) = \mathrm{tr}(\rho a)$, for all $a \in \mathcal{A}$. For any $t \geq 0$ there exists (see [9] Th.1, p.57) a density matrix ρ_t such that $\mathcal{T}_{*t}(\varphi)(a) = \mathrm{tr}(\rho_t a)$ for all $a \in \mathcal{A}$. Notice that for all $s \geq 0$, $\mathrm{tr}(\rho_t \mathcal{T}_s(a)) = \mathcal{T}_{*t}(\varphi)(\mathcal{T}_s(a)) = \mathrm{tr}(\rho \mathcal{T}_{t+s}(a))$. Hence,

$$\int_0^\infty \mathrm{tr}(\rho_t \mathcal{T}_s(a))ds = \int_0^\infty \mathrm{tr}(\rho \mathcal{T}_{t+s}(a))ds = \int_t^\infty \mathrm{tr}(\rho \mathcal{T}_s(a))ds < \infty.$$

It follows that

$$\rho_t = \sum_k \lambda_k(t) |u_k(t)\rangle\langle u_k(t)|,$$

with $u_k(t) \in D(\mathfrak{U}(x))$, for all $k \geq 1$ and $t \geq 0$ such that $\lambda_k(t) > 0$.

As a result, the range of ρ_t is included in $D(\mathfrak{U}(x))$, i.e. $p\rho_t = \rho_t p = p\rho_t p = \rho_t$.

Thus, $\mathrm{tr}(\rho\mathcal{T}_t(p)) = \mathrm{tr}(\rho_t p) = \mathrm{tr}(\rho_t) = 1$, and

$$0 = \mathrm{tr}(\rho(p - \mathcal{T}_t(p))) = \mathrm{tr}(\rho(p - p\mathcal{T}_t(p)\, p)).$$

However, we also have $p\mathcal{T}_t(p)\, p \leq p\mathcal{T}_t(\mathbf{1})\, p \leq p$. Therefore, $p\mathcal{T}_t(p)\, p = p$, i.e. $p\mathcal{T}_t\left(p^\perp\right) p = 0$, (see Lemma II.1 in [13]) so that $\mathcal{T}_t(p) \geq p$.

The second part is a trivial consequence of the above. □

Potentials are a natural source of superharmonic (or excessive) operators. Indeed, heuristically,

$$\mathcal{T}_t\,(\mathcal{U}(x)) = \mathcal{T}_t\left(\int_0^\infty \mathcal{T}_s\,(x)\,ds\right) = \int_t^\infty \mathcal{T}_s\,(x)\,ds \le \mathcal{U}(x),$$

however $\mathcal{U}(x)$ is possibly unbounded. Further on, bounded potentials will be associated with our concept of transience (see Theorems 2 and 5).

Theorem 1. *For any $x \in \mathcal{A}^+_{\mathrm{int}}$, the contraction*

$$y = \mathcal{U}(x)(\mathbf{1} + \mathcal{U}(x))^{-1}, \tag{1}$$

is superharmonic and $\mathcal{T}_t\,(y)$ converges strongly to 0 as $t \to \infty$.

Proof. Fix $x \in \mathcal{A}^+_{\mathrm{int}}$ and define $\mathcal{U}_t(x) = \int_0^t \mathcal{T}_s\,(x)\,ds$ $(t \ge 0)$. For any $s, t \ge 0$,

$$\mathcal{T}_t\,(\mathcal{U}_s(x)) = \int_t^{t+s} \mathcal{T}_r\,(x)\,dr = \mathcal{U}_{t+s}(x) - \mathcal{U}_t(x). \tag{2}$$

It follows:

$$\mathcal{T}_t\,(\mathcal{U}_s(x)) \le \mathcal{U}_{t+s}(x). \tag{3}$$

Since $\mathcal{T}_t\,(\cdot)$ is in particular 2-positive, identity preserving and the function $x \mapsto (1+x)^{-1}$ is operator monotone (see e.g. [6]), we have

$$(\mathbf{1} + \mathcal{T}_t\,(\mathcal{U}_s(x)))^{-1} \le \mathcal{T}_t\left((\mathbf{1} + \mathcal{U}_s(x))^{-1}\right).$$

From (3),

$$(\mathbf{1} + \mathcal{U}_{t+s}(x))^{-1} \le \mathcal{T}_t\left((\mathbf{1} + \mathcal{U}_s(x))^{-1}\right).$$

It follows:

$$\begin{aligned}\mathcal{T}_t\left(\mathcal{U}_s(x)(\mathbf{1} + \mathcal{U}_s(x))^{-1}\right) &= \mathbf{1} - \mathcal{T}_t\left((\mathbf{1} + \mathcal{U}_s(x))^{-1}\right)\\ &\le \mathbf{1} - (\mathbf{1} + \mathcal{U}_{t+s}(x))^{-1}\\ &= \mathcal{U}_{t+s}(x)\,(\mathbf{1} + \mathcal{U}_{t+s}(x))^{-1}.\end{aligned}$$

The map $\mathcal{T}_t\,(\cdot)$ is normal and $\mathcal{U}_{t+s}(x)\,(\mathbf{1} + \mathcal{U}_{t+s}(x))^{-1}$ strongly converges to y as $s \to \infty$. Therefore, letting $s \to \infty$ yields $\mathcal{T}_t\,(y) \le y$.

Finally, (2) implies

$$\mathcal{T}_t\left(\mathcal{U}_s(x)(\mathbf{1} + \mathcal{U}_s(x))^{-1}\right) \le \mathcal{T}_t\,(\mathcal{U}_s(x)) = \mathcal{U}_{t+s}(x) - \mathcal{U}_t(x),$$

so that for all $u \in D(\mathfrak{U}(x))$,

$$\langle u, \mathcal{T}_t\left(\mathcal{U}_s(x)(\mathbf{1} + \mathcal{U}_s(x))^{-1}\right) u\rangle \le \int_t^{t+s} \langle u, \mathcal{T}_r\,(x)\,u\rangle dr.$$

Letting $s \to \infty$ again,

$$\langle u, \mathcal{T}_t(y)u\rangle \le \int_t^\infty \langle u, \mathcal{T}_r(x)u\rangle dr,$$

thus, $\langle u, \mathcal{T}_t(y)u\rangle$ vanishes, as t goes to infinity. Since $D(\mathfrak{U}(x))$ is dense and the operators $\mathcal{T}_t(y)$ are uniformly bounded in norm by $\|y\| \le 1$, the last statement of the theorem follows. □

Proposition 4. *For any $x \in \mathcal{A}^+$, let $\mathcal{K}(x) = \{u \in D(\mathfrak{U}(x)) : \mathfrak{U}(x)[u] = 0\}$. Then the projection p on $\mathcal{K}(x)$ is subharmonic.*

Proof. We use here the notations of the previous proof. Note that for $x \in \mathcal{A}^+$, $\mathfrak{U}(x)[u] = 0$ if and only if $\mathcal{U}_s(x)u = 0$ for each $s \ge 0$. Fix $s > 0$ and let $q_n(s)$ denote the spectral projection of $\mathcal{U}_s(x)$ associated with the interval $]1/n, \|\mathcal{U}_s(x)\|]$, $(n \ge 1)$.

It is worth noticing that $q(s) = \text{l.u.b. } q_n(s)$ is the projection onto the closure of the range of $\mathcal{U}_s(x)$. Equation (3) yields

$$\mathcal{T}_t(q_n(s)) \le n\mathcal{T}_t(\mathcal{U}_s(x)) \le n\,\mathcal{U}_{t+s}(x).$$

Since $\mathcal{T}_t(q_n(s)) \le \mathbf{1}$ we obtain,

$$\mathcal{T}_t(q_n(s))^n \le n\,\mathcal{U}_{t+s}(x),$$

that is

$$\mathcal{T}_t(q_n(s)) \le n^{1/n}\,\mathcal{U}_{t+s}(x)^{1/n}.$$

Therefore, letting $n \to \infty$,

$$\mathcal{T}_t(q(s)) \le q(t+s).$$

Now, notice that the family $q(s)$ is increasing with s and $q = \text{l.u.b. } q(s)$ is equal to $\mathbf{1} - p$, the projection onto the orthogonal of $\mathcal{K}(x)$. The conclusion follows from the previous inequality letting $s \to \infty$. □

4. Recurrent and transient QMS

Recurrence and transience arise from the properties of non commutative potential, as this section shows.

A self-adjoint operator X is *strictly positive* if $\langle u, Xu\rangle > 0$ for any $u \in D(X)$, $u \ne 0$ (we will write simply $X > 0$).

Theorem 2. *The following statements are equivalent:*

1. *There exists a positive $x \in \mathcal{A}$ with $\mathcal{U}(x)$ bounded and $\mathcal{U}(x) > 0$.*
2. *There exists a strictly positive $x \in \mathcal{A}$ with $\mathcal{U}(x)$ bounded.*
3. *There exists a positive $x \in \mathcal{A}$ with $\mathcal{U}(x) > 0$.*
4. *There exists an increasing sequence of projections $(p_n)_{n\ge1}$, with l.u.b. $p_n = \mathbf{1}$ and $\mathcal{U}(p_n)$ bounded for all n.*

Proof. 1⇒2: Let $x_\lambda = \mathcal{R}_\lambda(x)$ $(\lambda > 0)$, where $\mathcal{R}_\lambda(\cdot)$ is the resolvent of the semigroup $\mathcal{T}$. Since $\mathcal{U}(x) > 0$, then $x_\lambda > 0$. Moreover, the resolvent identity implies

$$\mathcal{U}(x_\lambda) = \mathcal{U}(\mathcal{R}_\lambda(x)) = \lambda^{-1}\left(\mathcal{U}(x) - \mathcal{R}_\lambda(x)\right) \le \lambda^{-1}\mathcal{U}(x).$$

Thus, $\mathcal{U}(x_\lambda)$ is bounded.

2⇒1: Clearly if $x > 0$ then $\mathcal{U}(x) > 0$.

1⇒3 is self-evident.

3⇒1: Let $x \in \mathcal{A}$ positive with $\mathcal{U}(x) > 0$ and set y as in Theorem 1. Clearly $0 < y < \mathbf{1}$, and is a superharmonic operator. We may assume y in the domain of the generator $\mathcal{L}(\cdot)$ of $\mathcal{T}$ (otherwise replace y by $\mathcal{R}_\lambda(y)$; $\mathcal{T}_t(\mathcal{R}_\lambda(y))$ still vanishes as $t \to \infty$), then $\mathcal{L}(y) \le 0$ and

$$\int_0^t \mathcal{T}_s\left(-\mathcal{L}(y)\right) ds = y - \mathcal{T}_t(y)\,, \ \ (t \ge 0).$$

Letting $t \to \infty$ yields $\mathcal{U}(-\mathcal{L}(y)) = y$. Thus $-\mathcal{L}(y)$ satisfies condition 1.

4⇒1: Define $c_n = 2^{-n}\,\|\mathcal{U}(p_n)\|^{-1}$, and $x = \sum_{n\ge 0} c_n p_n$. Then x is strictly positive, $\mathcal{U}(x)$ is bounded and $\mathcal{U}(x) > 0$.

1⇒4: It suffices to take p_n as the spectral projection of x associated with the interval $]1/n, \|x\|]$. □

Corollary 1. *If $\mathcal{A} = \mathcal{B}(\mathfrak{h})$ and $\mathfrak{h}$ is separable, then the statements of Theorem 2 are all equivalent to the following:*

There exists an increasing sequence of finite dimensional projections $(p_n)_{n\ge 1}$, with l.u.b. $p_n = \mathbf{1}$ and $\mathcal{U}(p_n)$ bounded for all n.

Proof. Clearly it suffices to prove that the statements of Theorem 2 imply the above condition on finite dimensional projections. Let $(p_m; m \ge 1)$ be an increasing sequence of projections satisfing the statement 4. For each m let $(p_{m,k}; k \ge 1)$ be an increasing sequence of finite dimensional projections on $\mathfrak{h}$ with l.u.b.$_k\, p_{m,k} = p_m$. Note that $0 \le \mathcal{U}(p_{m,k}) \le \mathcal{U}(p_m)$ for all m, k. Therefore we have $\|\mathcal{U}(p_{m,k})\| \le \|\mathcal{U}(p_m)\| < \infty$. Finally, since $\mathfrak{h}$ is separable, a diagonalization argument shows the existence of a subsequence $(p_{m_n,k_n}; n \ge 1)$ with l.u.b.$_n\, p_{m_n,k_n} = \mathbf{1}$ and $\|\mathcal{U}(p_{m_n,k_n})\| < \infty$ for all n. □

Corollary 2. *Let $\mathcal{A} = L^\infty(E, \mathcal{E}, \mu)$ where E is a topological space, $\mathcal{E}$ its Borel σ-field and μ a σ-finite measure. Suppose that each $\mathcal{U}(f)$, with $f \in \mathcal{A}$ is a lower semicontinuous function, then, under the equivalent conditions of Theorem 2, $\mathcal{U}(1_K)$ is bounded for each compact set K. If in addition E is a countable union of compact sets, all the conditions of Theorem 2 are equivalent to: there exists an increasing sequence of $(K_n)_{n\ge 1}$ of compact sets such that E is the union of the K_n's and $\mathcal{U}(1_{K_n})$ is bounded for all $n \ge 1$.*

Proof. We first notice that, within this framework, any projection turns out to be the multiplication operator by the indicator function of an element of $\mathcal{E}$. Thus, condition 4 can be rewritten as follows: there exists an increasing sequence $(A_n)_{n\ge 1} \subseteq \mathcal{E}$ whose union is E and $\mathcal{U}(1_{A_n})$ is bounded for any $n \ge 1$. If each $\mathcal{U}(f)$ is lower semicontinuous, then arguing as in the proof of 1⇒4, we can choose open A_n's. Therefore, for each compact K there exists an m, such that $K \subseteq A_m$, so that $\mathcal{U}(1_K) \leq \mathcal{U}(1_{A_m})$ is bounded.

The last claim follows now immediately. □

Remark. The condition that $\mathcal{U}(f)$ be lower semicontinuous is fulfilled, for instance, by strong Feller semigroups, i.e. $\mathcal{T}_t\,(f)$ is a continuous function for any $t > 0$ and $f \in L^\infty(E, \mathcal{E}, \mu)$.

Theorem 3. *The following are equivalent:*

1. *For each positive $x \in \mathcal{A}$ and $u \in \mathfrak{h}$ either $u \notin D(\mathfrak{U}(x))$ or $u \in D(\mathfrak{U}(x))$ and $\mathcal{U}(x)[u] = 0$.*
2. *For each projection p and $u \in \mathfrak{h}$ either $u \notin D(\mathfrak{U}(p))$ or $u \in D(\mathfrak{U}(p))$ and $\mathcal{U}(p)[u] = 0$.*

Proof. Clearly 1⇒2. We prove then that 2⇒1. Let $x \in \mathcal{A}$ and $u \in \mathfrak{h}$. If $u \in D(\mathfrak{U}(x))$ then, for each spectral projection p of x associated with an interval $]r, \|x\|]$, $u \in D(\mathfrak{U}(p))$. Therefore, by condition 2, we have $\mathfrak{U}(p)[u] = 0$ i.e. $\langle u, \mathcal{T}_t(p)u\rangle = 0$ for all $t \geq 0$. It follows then that $\langle u, \mathcal{T}_t(x)u\rangle = 0$ for all $t \geq 0$. As a consequence $\mathfrak{U}(x)[u] = 0$. □

Corollary 3. *If $\mathcal{A} = \mathcal{B}(\mathfrak{h})$, with $\mathfrak{h}$ separable, the statements of Theorem 3 are all equivalent to: for each finite dimensional projection p either $\mathcal{U}(p) = 0$ or $D(\mathfrak{U}(p)) = \{0\}$.*

Proof. Suppose that the above condition on finite dimensional projections holds and let p be any projection in $\mathcal{A}$. Let $(p_n; n \geq 1)$ be an increasing sequence of finite dimensional such that l.u.b.$p_n = p$. If $u \in D(\mathfrak{U}(p))$ then $u \in D(\mathfrak{U}(p_n))$ and $\mathfrak{U}(p_n)[u] = 0$ for all $n \geq 1$. This implies clearly $\langle u, \mathcal{T}_t(p_n)u\rangle = 0$ for all $n \geq 1$ and all $t \geq 0$ and, letting n tend to infinity, $\langle u, \mathcal{T}_t(p)u\rangle = 0$ for all $t \geq 0$. It follows that $\mathfrak{U}(p)[u] = 0$. □

Corollary 4. *Let $\mathcal{A} = L^\infty(E, \mathcal{E}, \mu)$ where E is a locally compact space with countable basis, $\mathcal{E}$ its Borel σ-field and μ a σ-finite measure. Then all the conditions of Theorem 3 are equivalent to: for all compact set K either $\mathcal{U}(1_K) = 0$ or $D(\mathfrak{U}(1_K)) = \{0\}$.*

Proof. Here, again, each projection is identified with the indicator function of a Borel set. The hypothesis that E is locally compact with countable basis implies the inner regularity of the measure μ, so that inside any Borel set A, such that $\mu(A) > 0$ we can find a compact K with $\mu(K) > 0$. The conclusion follows readily. □

Corollaries 2 and 4, which recover well-known properties of recurrent and transient classical Markov semigroups (see [4], [16]), inspire the following definition.

Definition 3. *A QMS is* transient *(resp.* recurrent*) if any of the equivalent conditions of Theorem 2 (resp. Theorem 3) holds.*

Proposition 5. *If a QMS is irreducible, then it is either recurrent or transient.*

Proof. Indeed, if a QMS is irreducible, then the domain of the form-potential $\mathfrak{U}(x)$ is either $\{0\}$ or dense by Proposition 3. □

We end this section by a last proposition on the large time behaviour of a QMS, derived from the definition of potential.

Proposition 6. *Let $x \in \mathcal{A}^+_{\text{int}}$. Then, for any state φ, $\lim_{t\to\infty} \varphi(\mathcal{T}_t(x)) = 0$. In particular, a transient semigroup has no invariant state.*

Proof. The predual semigroup $\mathcal{T}_*$ on $\mathcal{A}_*$ is a C_0-strongly continuous contraction semigroup and the function $t \mapsto \mathcal{T}_{*t}(\varphi)$ is uniformly continuous in the norm of $\mathcal{A}_*$. Thus, the function $f(t) = \varphi(\mathcal{T}_t(x)) = \mathcal{T}_{*t}(\varphi)(x)$ is uniformly continuous.

Suppose first that the eigenvectors of the density matrix of φ belong to the domain of $\mathcal{U}(x)$, then f is also integrable on $[0, \infty[$. If $\limsup_{t\to\infty} f(t) > 2c > 0$, then, by uniform continuity, there exists $\delta = \delta(c) > 0$, such that for any $t \geq 0$, $\int_t^\infty f(s)ds > c\delta$. But this contradicts the integrability of f, so that $\limsup_{t\to\infty} f(t) = 0$.

The same conclusion also holds for an arbitrary state φ by an approximation based on the density of $D(\mathcal{U}(x))$.

If the QMS is transient, take an increasing sequence of projections $(p_n)_{n\geq 1}$ with $\mathcal{U}(p_n)$ bounded. Suppose that φ is an invariant state, we have then $\varphi(p_n) = \varphi(\mathcal{T}_t(p_n)) \to 0$ as $t \to \infty$ and we obtain $\varphi(p_n) = 0$, for each n. Letting $n \to \infty$, we end up with the contradiction $\varphi(\mathbf{1}) = 0$. □

5. Transience and superharmonic operators

The existence of non trivial superharmonic operators turns out to be intimately connected with transience.

Definition 4. *We say that a positive $y \in \mathcal{A}$ is a* potential *if there exists $x \in \mathcal{A}^+_{\text{int}}$, such that $\mathcal{U}(x) = y$.*

For a family of positive operators $(y_t)_{t\geq 0}$ we write $y_t \downarrow 0$ if $y_t \leq y_s$ for each $s \leq t$, and moreover $y_t \to 0$ strongly.

Theorem 4. *A positive $y \in \mathcal{A}$ is a potential if and only if y is a superharmonic operator and $\mathcal{T}_t(y) \downarrow 0$ as $t \uparrow \infty$.*

Proof. If y is a potential, it is superharmonic since, like in Theorem 1,

$$\mathcal{T}_t(y) = \mathcal{T}_t(\mathcal{U}(x)) = \int_t^\infty \mathcal{T}_s(x)\, ds \leq \mathcal{U}(x) = y,$$

and $\mathcal{T}_t(y)$ converges strongly to 0.

Conversely, assume y to be superharmonic and $\mathcal{T}_t(y) \downarrow 0$ as $t \uparrow \infty$. Replacing y by $\mathcal{R}_\lambda(y)$, if needed, one may also assume that y belongs to the domain of the generator $\mathcal{L}$ of $\mathcal{T}$. Since $y - \mathcal{T}_t(y) \geq 0$ for all $t > 0$, it turns out that $-\mathcal{L}(y)$ is in $\mathcal{A}^+$. Define $x = -\mathcal{L}(y)$. Then, given any $t \geq 0$,

$$\mathcal{U}_t(x) = \int_0^t \mathcal{T}_s(x)\, ds = -\int_0^t \mathcal{T}_s(\mathcal{L}(y))\, ds = y - \mathcal{T}_t(y)\,.$$

Since $\mathcal{T}_t(y) \downarrow 0$, the family of positive operators $\mathcal{U}_t(x)$ converges strongly to y as $t \to \infty$. So that $y = \mathcal{U}(x)$. □

Theorem 5. *An irreducible QMS $\mathcal{T}$ is transient if and only if there exists a non-trivial $\mathcal{T}$-superharmonic operator in $\mathcal{A}$.*

Proof. If $\mathcal{T}$ is transient then, by Theorem 2, part 1, there exists a $x \in \mathcal{A}^+_{\text{int}}$ with $y = \mathcal{U}(x)$ bounded and $y > 0$. Then y is a superharmonic operator by Theorem 4. Moreover $\mathcal{T}_t(y) \downarrow 0$, thus y is non trivial.

Conversely, if there exists a non-trivial $\mathcal{T}$-superharmonic operator y in $\mathcal{A}$ by adding a multiple of $\mathbf{1}$ we can assume that y is also positive. Since $\mathcal{T}$ is irreducible, as we shall prove in the forthcoming lemma, we may suppose in addition that $\mathcal{T}_t(y) < y$ for some $t > 0$. Note that $\mathcal{R}_\lambda(y)$ is also non-trivial and satisfies $\mathcal{T}_t(\mathcal{R}_\lambda(y)) \le \mathcal{R}_\lambda(y)$, for some $t \ge 0$ and $\mathcal{T}_t(\mathcal{R}_\lambda(y)) < \mathcal{R}_\lambda(y)$ for some $t > 0$ and $\lambda > 0$. Therefore, replacing y by $\mathcal{R}_\lambda(y)$ if necessary, we can assume that y belongs to the domain of the generator $\mathcal{L}$ too. Clearly $\mathcal{L}(y) < 0$ and, as in the proof of Theorem 4, $\mathcal{U}(-\mathcal{L}(y)) \le y$ and $\mathcal{T}$ is transient by Theorem 2, part 2. □

Lemma 1. *Let $\mathcal{T}$ be a QMS on a von Neumann algebra $\mathcal{A}$ and let y be a strictly positive $\mathcal{T}$-harmonic operator in $\mathcal{A}$. Then,*

- *either there exists $n > 1$, such that for all $k < n$, the operators y^{2^k} are $\mathcal{T}$-harmonic, and $\|y^{2^n}\|\mathbf{1} - y^{2^n}$ is superharmonic but not harmonic,*
- *or every spectral projection of y associated with an interval $]r, +\infty[$ is $\mathcal{T}$-superharmonic. In particular, if y is non trivial, then $\mathcal{T}$ is not irreducible.*

Proof. Since y is harmonic, the Schwarz inequality yields,

$$y^2 = \mathcal{T}_t(y^*)\mathcal{T}_t(y) \le \mathcal{T}_t(y^2).$$

Now, if y^2 is not harmonic the proof is finished, otherwise, we can repeat the above argument with y^2 (instead of y), $y^4, \ldots$ and so on until we find an n such that y^{2^n} is subharmonic but not harmonic. If such an n does not exist then, arguing as in the proof of Theorem 1, we can show that the operators $y^{2^n}(s + y^{2^n})^{-1}$ $(n \ge 1, s > 0)$ are $\mathcal{T}$-superharmonic.

Note that, for each $r > 0$, the operator

$$\lim_n (r^{-1}y)^{2^n}(s + (r^{-1}y)^{2^n})^{-1} = (s+1)^{-1}E\{r\} + E]r, +\infty[$$

where $E\{r\}$ denotes the orthogonal projection on the (possibly empty) eigenspace of y corresponding to r and $E]r, +\infty[$ the spectral projection of y associated with the interval $]r, +\infty[$ and the limit exists in the strong operator topology. It follows that

$$\mathcal{T}_t\left((s+1)^{-1}E\{r\} + E]r, +\infty[\right) \le (s+1)^{-1}E\{r\} + E]r, +\infty[.$$

The conclusion follows letting s tend to infinity. □

6. Applications

In this section our classification of QMS will be illustrated through a couple of applications which show a nice interplay between classical and quantum probability. Moreover, we discuss a connection with Scattering Theory.

A quantum Markov *process* is determined by a quantum Markov semigroup and an initial law and, as far as we are concerned with recurrence or transience, only the semigroup matters.

It is worth noticing here that a quantum Markov semigroup $\mathcal{T}$ on a von Neumann algebra $\mathcal{A}$ might leave invariant some abelian subalgebra $\mathcal{A}_c$ which turns out to be isomorphic to $L^\infty(E, \mathcal{E}, \mu)$, with μ σ-finite. In this case the maps $T_t = \mathcal{T}_{t|\mathcal{A}_c}$ determine a classical Markov semigroup T which allows us to apply all the rich classical theory. For example, we have the following fact.

Proposition 7. *Under the above assumptions and notations, if T is transient, then so is $\mathcal{T}$.*

Proof. Since T is transient, there exists a sequence $A_n \in \mathcal{E}$, such that $p_n = 1_{A_n}$ satisfies condition 4 of Theorem 2. □

Notice that, on the contrary, if T is recurrent this by no means implies the recurrence of $\mathcal{T}$ as the following example shows.

6.1. *Quantum Brownian Motion*

Let $\mathfrak{h} = L^2(\mathbb{R}^d; \mathbb{C})$ and let $\mathcal{A} = \mathcal{B}(\mathfrak{h})$. Our framework here is the same as that of the harmonic oscillator in [19], Ch. III. By a Quantum Brownian Motion we mean a quantum Markov process with associated semigroup $\mathcal{T}$ on $\mathcal{A}$ which is the minimal semigroup (see [7], [10] and the references therein) with form generator

$$\pounds(x) = -\frac{1}{2}\sum_{j=1}^{d}\left(a_j a_j^* x - 2a_j x a_j^* + x a_j a_j^*\right) - \frac{1}{2}\sum_{j=1}^{d}\left(a_j^* a_j x - 2a_j^* x a_j + x a_j^* a_j\right),$$

where a_j^*, a_j are the creation and annihilation operators

$$a_j = \left(q_j + \partial_j\right)/\sqrt{2}, \qquad a_j^* = \left(q_j - \partial_j\right)/\sqrt{2},$$

∂_j being the partial derivative with respect to the j^{th} coordinate q_j.

The commutative von Neumann subalgebra $\mathcal{A}_q$ of $\mathcal{A}$ whose elements are multiplication operators M_f by a function $f \in L^\infty(\mathbb{R}^d; \mathbb{C})$ is $\mathcal{T}$-invariant and $\mathcal{T}_t(M_f) = M_{T_t f}$ where

$$(T_t f)(x) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} f(y) \mathrm{e}^{-|x-y|^2/2t} dy. \tag{4}$$

The same conclusion holds for the commutative algebra $\mathcal{A}_p = F^* \mathcal{A}_q F$, where F denotes the Fourier transform. Therefore, our process deserves the name of *quantum Brownian motion* since its contains a couple of non commuting classical Brownian motions.

Moreover, notice that the von Neumann algebra $\mathcal{A}_N$ generated by the number operator $N = \sum_j a_j^* a_j$ is also $\mathcal{T}$ invariant and the classical semigroup obtained by restriction of $\mathcal{T}$ to $\mathcal{A}_N$ is a birth and death on $\mathbb{N}$ with birth rates $(n+1)_{n\geq 0}$ and death rates $(n)_{n\geq 0}$.

It can be shown by the methods of [13] that $\mathcal{T}$ is irreducible.

The unit vector $e_0(q) = \pi^{-d/4}\exp(-|q|^2/2)$, called the *vacuum vector*, satisfies $a_j e_0 = 0$ for all j. The rank-one projection $|e_0\rangle\langle e_0|$ onto e_0 belongs to $\mathcal{A}_N$ and satisfies

$$\mathcal{T}_t(|e_0\rangle\langle e_0|) = \frac{1}{(1+t)^d}\,(1+1/t)^{-N} \tag{5}$$

This formula can be checked as follows. Notice first that each Weyl operator $W(z)$ (see [19], III.4) belongs to the domain of $\mathcal{L}$ and $\mathcal{L}(W(z)) = -|z|^2 W(z)$ (e.g. by [12], Lemma 1.1) Therefore, we have $\mathcal{T}_t(x)(W(z)) = \exp(-t|z|^2)W(z)$ and the canonical commutation relation

$$W(-\zeta/\sqrt{2})W(z)W(\zeta/\sqrt{2}) = \exp(-i\sqrt{2}\Im\langle z,\zeta\rangle)W(z)$$

leads to the explicit formula (see [3]) for $x = W(z) \in \mathcal{B}(\mathfrak{h})$

$$\mathcal{T}_t(x) = \frac{1}{(2\pi t)^d}\int_{\mathbb{R}^{2d}} W(-\zeta/\sqrt{2})xW(\zeta/\sqrt{2})\exp(-|\zeta|^2/2t)\,d\zeta \tag{6}$$

where $\zeta = r + is$ with $r, s \in \mathbb{R}^d$ and $d\zeta$ means $drds$. By normality this formula also holds for an arbitrary $x \in \mathcal{B}(\mathfrak{h})$.

We now check (5). Indeed, for each unit vector e_α ($\alpha \in \mathbb{N}^d$) of the canonical orthonormal basis of $\mathfrak{h}$ given by d dimensional Hermite polynomials multiplied by the function e_0, we have

$$\begin{aligned}\langle e_\alpha, \mathcal{T}_t(|e_0\rangle\langle e_0|)e_\alpha\rangle &= \frac{1}{(2\pi t)^d}\int_{\mathbb{R}^{2d}}\left|\langle e_\alpha, W(\zeta/\sqrt{2})e_0\rangle\right|^2\exp(-|\zeta|^2/2t)\,d\zeta\\ &= \frac{1}{(2\pi t)^d}\int_{\mathbb{R}^{2d}}\frac{|\zeta_1|^{2\alpha_1}\cdots|\zeta_d|^{2\alpha_d}}{2^{|\alpha|}\alpha_1!\cdots\alpha_d!}\exp(-(1+1/t)|\zeta|^2/2)\,d\zeta\end{aligned}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. By the change of variables $\zeta = \xi/(1+1/t)^{1/2}$ we find

$$\langle e_\alpha, \mathcal{T}_t(|e_0\rangle\langle e_0|)e_\alpha\rangle = c_\alpha\frac{(1+1/t)^{-|\alpha|}}{(1+t)^d}$$

where c_α is a strictly positive constant that can be evaluated by computing a Gaussian integral and shown to be equal to 1.

By means of (5), for each $d \geq 2$, we can compute

$$\int_0^\infty \langle e_\alpha, \mathcal{T}_t(|e_0\rangle\langle e_0|)e_\alpha\rangle dt = \int_0^\infty \frac{(1+1/t)^{-|\alpha|}}{(1+t)^d}\,dt < +\infty.$$

Moreover, since the restriction of $\mathcal{T}$ to $\mathcal{A}$ is also irreducible, for each β, we have $\mathcal{T}_{t_\beta}(|e_0\rangle\langle e_0|) \geq \kappa(\beta, t_\beta)|e_\beta\rangle\langle e_\beta|$ for some $t_\beta > 0$ and some constant $\kappa(\beta, t_\beta) > 0$. It follows that, for each $d \geq 2$, our QMS is transient.

On the other hand, when $d = 1$, suppose that $\mathcal{T}$ is again transient and let $(p_n)_{n\geq 1}$ be an increasing sequence of projections with l.u.b.$p_n = \mathbf{1}$ and $\mathcal{U}(p_n)$ bounded. We have then

$$\int_0^\infty \langle e_0, \mathcal{T}_t(p_n)e_0\rangle dt = |\langle e_0, p_n e_0\rangle|^2 \int_0^\infty \frac{dt}{1+t}$$

which diverges whenever $|\langle e_0, p_n e_0\rangle|^2$ is nonzero. This contradicts the fact that e_0 belongs to the domain of $\mathcal{U}(p_n)$ for all $n \geq 0$. Therefore, $\mathcal{T}$ being irreducible, it must be recurrent.

Thus we have proved the following

Corollary 5. *The QBM is recurrent for $d = 1$ and transient for $d \geq 2$.*

This apparently surprising result turns out to be natural if we recall the "principle": a d-dimensional QBM is a pair of non-commuting d-dimensional classical Brownian motions.

6.2. The micromaser model

We follow our article [11] to introduce the Jaynes-Cummings model of the micromaser. Let $\mathfrak{h} = l^2(\mathbb{N})$, we use the customary notations a^*, a, N for creation, annihilation and number operators respectively. Furthermore, $(e_n)_{n\geq 0}$ stands for the canonical orthonormal basis in $l^2(\mathbb{N})$.

The form generator is given by

$$\begin{aligned}\pounds(x) = &-\frac{\mu^2}{2}\left(a^*ax - 2a^*xa + xa^*a\right) - \frac{\lambda^2}{2}(aa^*x - 2axa^* + xaa^*)\\ &+R^2(\cos(\phi\sqrt{aa^*})x\cos(\phi\sqrt{aa^*}) + \sin(\phi\sqrt{aa^*})S^*xS\sin(\phi\sqrt{aa^*}) - x),\end{aligned}$$

where $\lambda > 0$, $\mu > 0$, R and ϕ are real constants. In [11] the rigorous construction of the minimal QDS was done showing also that it is identity preserving. Moreover it can be shown by the methods of [13] that it is irreducible.

The above Jaynes-Cummings generator has a faithful invariant state if and only if $\mu > \lambda$. This state can be computed explicitly (the interested reader is referred to [11]). Indeed, the procedure consists of noticing first that the von Neumann algebra $\mathcal{A}_N$ generated by the number operator N is invariant for the QMS. The operators $\mathcal{T}_{t|\mathcal{A}_N}$ constitute the semigroup of a birth and death process on $\mathbb{N}$, with birth rates and death rates, respectively

$$\lambda_k = \lambda^2(k+1) + R^2\sin^2(\phi\sqrt{k+1}), \quad \mu_k = \mu^2 k. \tag{7}$$

So that if $\mu > \lambda$, by a well-known classical result there exists a stationary probability distribution for the birth and death process given by

$$\pi_0 = c, \qquad \pi_n = c\prod_{k=1}^{n}\frac{\lambda^2 k + R^2\sin^2(\phi\sqrt{k})}{\mu^2 k} \quad (n \geq 1). \tag{8}$$

where c is a suitable normalization constant. The corresponding stationary state is given by the density matrix

$$\rho_\infty = \sum_{n\geq 0} \pi_n |e_n\rangle\langle e_n|. \tag{9}$$

Thus, the existence of the stationary distribution for the classical process allows to prove that the irreducible quantum semigroup has a stationary state, and it is recurrent by Proposition 7. By the same proposition, if the classical process is transient, then the QMS is transient in $\mathcal{B}(\mathfrak{h})$ as well. We now show that this happens if $\lambda > \mu$. Call

$$\gamma_n = \prod_{k=1}^{n} \frac{\mu_k}{\lambda_k} = \prod_{k=1}^{n} \frac{\mu^2 k}{\lambda^2(k+1) + R^2 \sin^2(\phi\sqrt{k+1})}. \tag{10}$$

The classical birth and death process is transient if and only if $\sum_n \gamma_n < \infty$. When $\lambda > \mu$ this condition is easily checked since $\gamma_n \leq (\mu/\lambda)^{2n}$.

To decide whether the birth and death process is transient or not when $\lambda = \mu$ requires a more subtle study, which depends on R and ϕ as the next lemma (where we take $\lambda = \mu = 1$ to simplify notations) shows.

Lemma 2. *For each $n \geq 1$ let γ_n be as in (10) with $\lambda = \mu = 1$). Then the series $\sum_{n\geq 1} \gamma_n$ is divergent if $R\phi = 0$, convergent if $R\phi \neq 0$.*

We postpone the proof to the appendix. The case $\lambda = \mu$ corresponds, up to the scaling parameter, to the 1-dimensional Quantum Brownian Motion already discussed in the first example. To summarize,

Corollary 6. *If $\lambda < \mu$, the QMS of the micromaser is recurrent and has a faithful stationary state ρ_∞ given by* (9) *and* (8).

If $\lambda > \mu$, the QMS above is transient.

If $\lambda = \mu$ we have two possibilities: either $R\phi = 0$, which implies that the QMS is recurrent or $R\phi \neq 0$ and the QMS is transient.

6.3. A connection with Scattering Theory

Large-time behaviour of quantum dynamics is the main concern of Scattering Theory. In [24], Sinha introduced the concept of *time of sojourn in a given region* of the evolution of a pure state. More precisely, let $\mathfrak{h} = L^2(\mathbb{R}^d)$, $B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$, H a self-adjoint operator in $\mathfrak{h}$, and $\psi \in \mathfrak{h}$ with $\|\psi\| = 1$, consider

$$J(R,\psi) = \int_{-\infty}^{+\infty} \left\| 1_{B_R} e^{-itH}\psi \right\|^2 dt.$$

Sinha calls $J(R,\psi)$ the *time of sojourn* of ψ in the ball B_R. Under a suitable hypothesis on the Hamiltonian, Sinha has proved that the set of vectors ψ for which $J(R,\psi) < \infty$ for all $R > 0$, is dense in the space of H-absolutely continuous elements of $\mathfrak{h}$, which is usually associated with "scattering" pure states (see for instance the book of Perry [22] for an account of this research in Scattering Theory).

Here below, we give a definition of time of sojourn for a QMS and analyze its use in the large time analysis of the semigroup. This generalization is a natural consequence of our concept of potential operator.

Consider a QMS $\mathcal{T}$ on a von Neumann algebra $\mathcal{A}$.

Definition 5. *Let φ be a state and $p \in \mathcal{A}$ a projection. The* **time of sojourn** *of φ in p is defined by*

$$\tau(\varphi, p) = \int_0^\infty \varphi\left(\mathcal{T}_t(p)\right) dt = \int_0^\infty \mathcal{T}_{*t}(\varphi)(p) dt. \tag{11}$$

Given a pure state φ_u defined by a density matrix $\rho = |u\rangle\langle u|$ with $u \in \mathfrak{h}$ of unitary norm, the time of sojourn $\tau(\varphi_u, p)$ coincides with the potential form $\mathfrak{U}(p)[u]$, for all projection $p \in \mathcal{A}$, which implicitly means that it takes the value $+\infty$ if $u \notin D(\mathfrak{U}(p))$.

Definition 6. *A state φ is* scattering *if for all finite-dimensional projection p, it holds* $\lim_{t\to\infty} \varphi\mathcal{T}_t(p) = \lim_{t\to\infty} \mathcal{T}_{*t}(\varphi)(p) = 0$.

A state φ is bound *if the orbit $(\mathcal{T}_{*t}(\varphi),\ t \geq 0)$ is tight. This means that for each $\epsilon > 0$ there exists a finite-dimensional projection p such that $\mathcal{T}_{*t}(\varphi)(p) \geq 1 - \epsilon$, for all $t \geq 0$.*

The set of scattering states $\mathfrak{S}_s(\mathcal{T})$ and that of bound states $\mathfrak{S}_b(\mathcal{T})$ are disjoints. In the remain of this subsection we consider $\mathcal{A} = \mathcal{B}(\mathfrak{h})$.

Proposition 8. *Assume there exists a bound state φ. Then there exists an invariant state φ_∞. Moreover, for all projection $p \in \mathcal{A}$ such that $\varphi(p) \neq 0$, one has $\tau(\varphi_\infty, p) = \infty$.*

Proof. If φ is a bound state, then for all $\epsilon > 0$ there exists a finite dimensional projection p_ϵ such that $\mathcal{T}_{*t}(\varphi)(p_\epsilon) \geq 1 - \epsilon$, for all $t \geq 0$. Therefore, if we choose any net $(t_\alpha)_\alpha$ in $]0, \infty[$, it follows that

$$M_\alpha(\varphi) := \frac{1}{t_\alpha} \int_0^{t_\alpha} \mathcal{T}_{*t}(\varphi)\, dt,$$

is a tight net of states, so that it contains a convergent subnet. We call φ_∞ its weak*-limit. Due to the tightness condition, φ_∞ is a state too and it is invariant under the semigroup $\mathcal{T}_*$.

Moreover, for any projection $p \in \mathcal{A}$,

$$\tau(\varphi_\infty, p) = \int_0^\infty \varphi_\infty(\mathcal{T}_t(p))dt = \int_0^\infty \varphi_\infty(p)dt = \infty.$$

□

Proposition 9. *If the semigroup is transient, then all states are scattering.*

This is clear from Proposition 6.

Final Remark. The last proposition shows that for a transient semigroup $\mathcal{T}$, $\mathfrak{S} = \mathfrak{S}_s(\mathcal{T})$. For a recurrent semigroup $\mathcal{T}$, both possibilities $\tau(\varphi, p) = 0$ or $\tau(\varphi, p) = \infty$ do exist for a $\varphi \in \mathfrak{S}_s(\mathcal{T})$ and a compact projection p, (even though φ cannot be invariant). This suggests that a compact projection p should be called *a region of resonance* whenever $\tau(\varphi, p) = \infty$ for a recurrent semigroup $\mathcal{T}$ and a state $\varphi \in \mathfrak{S}_s(\mathcal{T})$.

Appendix

Proof (of Lemma 2). If $R\phi = 0$, then $\gamma_n = 1/(n+1)$ and the conclusion is immediate.

We concentrate on the case $R\phi \neq 0$. Clearly

$$-\log(\gamma_n) = \sum_{k=1}^{n} \log\left(1 + \frac{1 + R^2\sin^2(\phi\sqrt{k+1})}{k}\right).$$

By the elementary inequality $\log(1+x) \geq x - x^2/2$ $(x \geq 0)$, for all $n \geq 1$, we have

$$-\log(\gamma_n) \geq \sum_{k=1}^{n} \frac{1 + R^2\sin^2(\phi\sqrt{k+1})}{k} - \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1 + R^2\sin^2(\phi\sqrt{k+1})}{k}\right)^2.$$

The last term is easily estimated by

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{1 + R^2\sin^2(\phi\sqrt{k+1})}{k}\right)^2 &\leq (1+R^2)^2\sum_{k=1}^{n}\frac{1}{k^2}\\ &\leq (1+R^2)^2\left(1 + \int_1^{+\infty} x^{-2}dx\right)\\ &= 2(1+R^2)^2.\end{aligned}$$

Then the inequality

$$\sum_{k=1}^{n}\frac{1}{k} \geq \int_0^n (1+x)^{-1}dx = \log(n+1)$$

yields

$$-\log(\gamma_n) \geq \log(n+1) + R^2\sum_{k=1}^{n}\frac{\sin^2(\phi\sqrt{k+1})}{k} - (1+R^2)^2. \tag{12}$$

In order to estimate the sum over the index k notice that $\sin^2(\phi\sqrt{k+1}) \geq 1/2$ whenever $\pi(m+1/4) \leq \phi\sqrt{k+1} \leq \pi(m+3/4)$ for $m = 0, 1, \ldots$ Thus

$$\sum_{k=\pi^2\phi^{-2}(m+1/4)^2-1}^{\pi^2\phi^{-2}(m+3/4)^2-1} \frac{\sin^2(\phi\sqrt{k+1})}{k} \geq \frac{1}{2}\sum_{k=\pi^2\phi^{-2}(m+1/4)^2-1}^{\pi^2\phi^{-2}(m+3/4)^2-1}\frac{1}{k}.$$

Then the inequality

$$\sum_{r\le k\le s}\frac{1}{k}\ge\int_{r+1}^{s+1}\frac{1}{x}\,dx=\log\left(\frac{s+1}{r+1}\right)$$

for $r,s\in\mathbb{R}$, $r<s+1$ yields

$$\sum_{k=\pi^2\phi^{-2}(m+1/4)^2-1}^{\pi^2\phi^{-2}(m+3/4)^2-1}\frac{\sin^2(\phi\sqrt{k+1})}{k}\ge\log\left(\frac{m+3/4}{m+1/4}\right)$$

for all m such that there exists at least an integer k in the interval $[\pi^2\phi^{-2}(m+1/4)^2-1,\pi^2\phi^{-2}(m+3/4)^2-1]$ i.e. for all m such that $m+1/2\ge\pi^{-2}\phi^2$.

Using again the inequality $\log(1+x)\ge x-x^2/2$ $(x\ge 0)$, we have

$$\sum_{k=\pi^2\phi^{-2}(m+1/4)^2-1}^{\pi^2\phi^{-2}(m+3/4)^2-1}\frac{\sin^2(\phi\sqrt{k+1})}{k}\ge\frac{2}{4m+1}-\frac{1}{(4m+1)^2}.$$

Thus, finally,

$$\begin{aligned}\sum_{k=1}^{n}\frac{\sin^2(\phi\sqrt{k+1})}{k}&\ge\sum_{\pi^{-2}\phi^2-1/2\le m\le\pi^{-1}\phi\sqrt{n+1}-3/4}\left(\frac{2}{4m+1}-\frac{1}{(4m+1)^2}\right)\\&\ge\sum_{\pi^{-2}\phi^2-1/2\le m\le\pi^{-1}\phi\sqrt{n+1}-3/4}\frac{2}{4m+1}-\sum_{m=1}^{\infty}\frac{1}{(4m+1)^2}\\&\ge-\frac{5}{4}+\sum_{\pi^{-2}\phi^2-1/2\le m\le\pi^{-1}\phi\sqrt{n+1}-3/4}\frac{2}{4m+1}\\&\ge-\frac{5}{4}+\int_{\pi^{-2}\phi^2+1/2}^{\pi^{-1}\phi\sqrt{n+1}+1/4}\frac{2\,dx}{4x+1}\\&=\frac{1}{4}\log(n+1)+c(\phi)\end{aligned}$$

where $c(\phi)$ is a real constant depending only on ϕ.

The above inequality, together with (12), yields

$$-\log(\gamma_n)\ge\left(1+R^2/4\right)\log(n+1)+R^2c(\phi)-(1+R^2)^2,$$

i.e.

$$\gamma_n\le\left(\exp\left((1+R^2)^2-R^2c(\phi)\right)\right)(n+1)^{-(1+R^2/4)}.$$

Since $R\ne 0$ the conclusion follows. □

References

1. Accardi, L., Lu, Y.G., Volovich, I.: Quantum Theory and its Stochastic Limit. Springer-Verlag, 2002
2. Accardi, L., Koroliuk, D.: Quantum Markov chains: The recurrence problem. Quantum Probability and Related Topics **VII**, 63–73 (1991)
3. Arveson, W.: The heat flow of the CCR algebra. Bull. London Math. Soc. **34**(1), 73–83 (2002)
4. Azéma, J., Duflo, M., Revuz, D.: Propriétés relatives des processus de Markov récurrents. Z. Wahr. und Verw. Gebiete **13**, 286–314 (1969)
5. Biane, Ph.: Quelques propriétés du mouvement Brownien non-commutatif. Hommage à P.A. Meyer et J. Neveu. Astérisque **236**, 73–102 (1996)
6. Carbone, R.: Exponential ergodicity of a class of quantum Markov semigroups. Tesi di Dottorato. Università di Milano, 2000
7. Chebotarev, A.M., Fagnola, F.: Sufficient conditions for conservativity of minimal quantum dynamical semigroups. J. Funct. Anal. **153**, 382–404 (1998)
8. Dellacherie, C., Meyer, P.-A.: Probabilités et potentiel. Chapitres XII–XVI. Second edition, Hermann, Paris, 1987
9. Dixmier, J.: Von Neumann Algebras. North-Holland, 1981
10. Fagnola, F.: Quantum markov semigroups and quantum flows. Proyecciones, Journal of Math. **18**(3), 1–144 (1999)
11. Fagnola, F., Rebolledo, R., Saavedra, C.: Quantum flows associated to master equations in quantum optics. J. Math. Phys. **35**(1), 1–12 (1994)
12. Fagnola, F., Rebolledo, R.: The approach to equilibrium of a class of quantum dynamical semigroups. Inf. Dim. Anal. Q. Prob. and Rel. Topics **1**(4), 1–12 (1998)
13. Fagnola, F., Rebolledo, R.: Subharmonic projections for a Quantum Markov Semigroup. J. Math. Phys. **43**, 1074–1082 (2002)
14. Fagnola, F., Rebolledo, R.: Lectures on the Qualitative Analysis of Quantum Markov Semigroups. Quantum Probability and White Noise Analysis. World Scientific **XIV**, 197–240 (2002)
15. Frigerio, A., Verri, M.: Long–time asymptotic properties of dynamical semigroups on w^*–algebras. Math. Zeitschrift **180**, 275–286 (1982)
16. Getoor, R.K.: Transience and recurrence of Markov processes. Séminaire de Probabilités, XIV (Paris, 1978/1979), pp. 397–409, Lecture Notes in Math., 784, Springer, Berlin, 1980
17. Gisin, N., Percival, I.: The quantum-state diffusion model applied to open systems. J. Phys. A **25**, 5677 (1992)
18. Kato, T.: Perturbation theory for linear operators. Corr. printing of the 2nd ed. Springer-Verlag, N.Y., 1980
19. Meyer, P.-A.: Quantum Probability for Probabilists, volume 1538 of Lect. Notes in Math. Springer-Verlag, Berlin, Heidelberg, New York, 1993
20. Parthasarathy, K.R.: An Introduction to Quantum Stochastic Calculus, volume 85 of Monographs in Mathematics. Birkhaüser–Verlag, Basel-Boston-Berlin, 1992
21. Percival, I.: Quantum state diffusion, Cambridge Univ. Press, Cambridge, 1998
22. Perry, P.A.: Scattering Theory by the Enss Method. Harwood Academic Publishers
23. Rebolledo, R.: On the recurrence of Quantum Dynamical Semigroups. Proc. ANESTOC'96. World Scientific Pub., 130–141 (1997)
24. Sinha, K.: On the absolutely and singularly continuous subspaces in Scattering Theory. Ann. I.H.P. ser. A **26**, 263–277