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A characterization of the Gaussian distribution on Abelian groups

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Abstract. It is well known that the independence of two linear forms with nonzero coefficients of independent random variables implies that the random variables are Gaussian (the Skitovich-Darmois theorem). The analogous result holds true for two linear forms of independent random vectors with nonsingular matrices as coefficients (the Ghurye-Olkin theorem). In this paper we give the complete description of locally compact Abelian groups X for which the independence of two linear forms of independent random variables with values in X having distributions with nonvanishing characteristic functions (coefficients of the forms are topological automorphisms of X) implies that the random variables are Gaussian.

1. Introduction

Let X be a locally compact Abelian separable metric group, $Y = X^*$ be its character group, (x, y) be the value of a character $y \in Y$ on an element $x \in X$. Denote by $M^1(X)$ the convolution semigroup of probability distributions on X. For $\mu \in M^1(X)$ denote by $\hat{\mu}(y) = \int_X (x, y) d\mu(x)$ its characteristic function.

A distribution $\mu \in M^1(X)$ is called Gaussian ([15]) if its characteristic function can be represented in the form

$$\hat{\mu}(y) = (x, y) \exp\{-\varphi(y)\},$$

where $x \in X$ and $\varphi(y)$ is a continuous nonnegative function satisfying the equation

$$\varphi(y_1 + y_2) + \varphi(y_1 - y_2) = 2[\varphi(y_1) + \varphi(y_2)], \quad y_1, y_2 \in Y. \tag{1}$$

Denote by $\Gamma(X)$ the set of Gaussian distributions on X. Let $\mathrm{Aut}(X)$ be the set of topological automorphisms of X. Consider the linear forms $L_1 = \alpha_1(\xi_1) + \cdots + \alpha_n(\xi_n)$ and $L_2 = \beta_1(\xi_1) + \cdots + \beta_n(\xi_n)$, $n \geq 2$, where $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, ξ_j are independent random variables with values in X and with distributions μ_j such that their characteristic functions $\hat{\mu}_j(y)$ do not vanish. We shall suppose that this condition on $\hat{\mu}_j(y)$ holds true in the course of the whole article.

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By the classical theorem of Skitovich-Darmois on the real line $X = \mathbb{R}$ the independence of L_1 and L_2 implies that all $\mu_j \in \Gamma(X)$ ([1], [16]). The analogous result for the group $X = \mathbb{R}^m, m \geq 2$ was proved by Ghurye and Olkin ([9]). Their proof was simplified by Zinger (see [11, Ch. 3], where one can find various generalizations). The corresponding result for the **a**-adic solenoids $\Sigma_{\mathbf{a}}$ was proved by the author in ([6]). Note that some analogs of the Bernstein theorem ($L_1 = \xi_1 + \xi_2,\ L_2 = \xi_1 - \xi_2$) and the Skitovich-Darmois theorem were considered in non-Abelian case too. Namely, for Lie groups, quantum groups, symmetric spaces (see [8], [12]–[14]).

It should be observed that for the group $X = \mathbb{R}^m,\ m \geq 1$ the independence of L_1 and L_2 implies that all characteristic functions $\hat{\mu}_j(y)$ do not vanish ([11, Ch. 3]). For an arbitrary group X this implication, generally, not necessary is true.

The aim of this article is to find the solution of the following problems.

Problem 1. To describe all locally compact Abelian groups X possessing the property: if ξ_j are independent random variables with values in X having the distributions μ_j with nonvanishing characteristic functions and $\alpha_j,\ \beta_j$ are arbitrary topological automorphisms of X, then the independence of $L_1 = \alpha_1(\xi_1) + \cdots + \alpha_n(\xi_n)$ and $L_2 = \beta_1(\xi_1) + \cdots + \beta_n(\xi_n)$ implies that all $\mu_j \in \Gamma(X)$.

Problem 2. To describe all locally compact Abelian groups X possessing the property: there exist $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ (not all $\alpha_j\beta_j^{-1}$ are equal) such that if ξ_j are independent random variables with values in X having the distributions μ_j with nonvanishing characteristic functions, then the independence of $L_1 = \alpha_1(\xi_1) + \cdots + \alpha_n(\xi_n)$ and $L_2 = \beta_1(\xi_1) + \cdots + \beta_n(\xi_n)$ implies that all $\mu_j \in \Gamma(X)$.

At first agree about the notation. If H is a subgroup of Y, then denote by $A(X, H) = \{x \in X : (x, y) = 1 \text{ for all } y \in H\}$ its annihilator. Denote by C_X the connected component of zero of X. Denote by $X_1 \approx X_2$ a topological isomorphism of groups X_1 and X_2. If $\delta : X_1 \to X_2$ is a continuous homomorphism, then the conjugate homomorphism $\tilde{\delta} : Y_2 \to Y_1$ is defined by the formula $(x_1, \tilde{\delta}(y_2)) = (\delta(x_1), y_2)$ for all $x_1 \in X_1,\ y_2 \in Y_2$. Denote by I_X the identity automorphism of X. Denote by T, resp. $\mathbb{Z}$, resp. $\mathbb{Z}(2)$ the circle group, resp. the group of integers, resp. the residue group modulo 2.

We shall use some results of the structure theory for locally compact Abelian groups and the duality theory (see [10]).

Denote by E_x the degenerate distribution concentrated at a point $x \in X$. For $\mu \in M^1(X)$, we define the distribution $\bar{\mu} \in M^1(X)$ by the formula $\bar{\mu}(E) = \mu(-E)$ for all Borel sets $E \subset X$. Observe that $\hat{\bar{\mu}}(y) = \overline{\hat{\mu}(y)}$. Denote by $\sigma(\mu)$ the support of $\mu \in M^1(X)$. It is useful to remark that if H is a closed subgroup of Y and $\hat{\mu}(y) \equiv 1,\ y \in H$, then $\sigma(\mu) \subset A(X, H)$.

Let $\psi(y)$ be an arbitrary function on Y and $h \in Y$. Denote by Δ_h the difference operator

$$\Delta_h\psi(y) = \psi(y + h) - \psi(y), \quad y \in Y.$$

It is convenient for us to formulate the following simple and well-known result as a lemma.

Lemma 1. *Let G be a Borel subgroup of X, $\mu \in M^1(X)$, $\mu = \mu_1 * \mu_2$, $\mu_j \in M^1(X)$ and μ concentrated on G. Then one can choose shifts $\mu'_j = \mu_j * E_{x_j}$ of μ_j such that $\mu = \mu'_1 * \mu'_2$ and μ'_j concentrated on G.*

2. Solution of Problem 1

Pass to the solution of Problem 1. We need 2 lemmas for this.

Lemma 2 (The group analog of the Cramer theorem ([2], see also [7, § 5]))**.** *If a group X contains no subgroup topologically isomorphic to T, $\mu \in \Gamma(X)$ and $\mu = \mu_1 * \mu_2$, $\mu_j \in M^1(X)$, then $\mu_j \in \Gamma(X)$, $j = 1, 2$.*

Lemma 3 (The group analog of the Marcinkiewicz theorem ([4], see also [7, Appendix 1]))**.** *If a group X contains no subgroup topologically isomorphic to T, $\mu \in M^1(X)$ and the characteristic function $\hat{\mu}(y)$ can be represented in the form $\hat{\mu}(y) = \exp\{-\psi(y)\}$, where $\psi(y)$ is a continuous nonnegative function satisfying for some m the equation*

$$\Delta_h^{m+1}\psi(y) = 0$$

for all y, $h \in Y$ and $\psi(0) = 0$, then $\mu \in \Gamma(X)$.

Theorem 1. *Let X be a locally compact Abelian group X containing no subgroup topologically isomorphic to T. Let ξ_j, $j = 1, 2, ..., n$, $n \geq 2$ be independent random variables with values in X and with distributions μ_j such that their characteristic functions $\hat{\mu}_j(y)$ do not vanish. Let α_j, β_j be arbitrary topological automorphisms of X. Then the independence of linear forms $L_1 = \alpha_1(\xi_1) + \cdots + \alpha_n(\xi_n)$ and $L_2 = \beta_1(\xi_1) + \cdots + \beta_n(\xi_n)$ implies that all $\mu_j \in \Gamma(X)$.*

The results of the articles [1], [16], [9] and [6] follow immediately from this theorem.

Proof of Theorem 1. Observe first that if μ is the distribution of a random variable ξ with values in X and $\alpha \in \mathrm{Aut}(X)$, then the characteristic function of the distribution $\alpha(\xi)$ is equal to $\hat{\mu}(\tilde{\alpha}(y))$. Therefore putting $\zeta_j = \alpha_j(\xi_j)$ and taking into account (1) we reduce the proof to the case when L_1 and L_2 have the form $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \delta_1(\xi_1) + \cdots + \delta_n(\xi_n)$, $\delta_j \in \mathrm{Aut}(X)$, $j = 1, 2, ..., n$, $n \geq 2$.

The condition of the independence of L_1 and L_2 can be written in the form

$$\mathbf{E}[(L_1, u)(L_2, v)] = \mathbf{E}[(L_1, u)]\mathbf{E}[(L_2, v)], \quad u, v \in Y,$$

or

$$\prod_{j=1}^{n} \hat{\mu}_j(u + \tilde{\delta}_j(v)) = \prod_{j=1}^{n} \hat{\mu}_j(u) \prod_{j=1}^{n} \hat{\mu}_j(\tilde{\delta}_j(v)), \quad u, v \in Y, \tag{2}$$

where $\hat{\mu}_j(y) = \mathbf{E}[(\xi_j, y)]$.

Observe now that the characteristic functions of distributions $\nu_j = \mu_j * \bar{\mu}_j$ also satisfy equation (2) and $\hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 > 0$ for all $y \in Y$. If we prove that all $\nu_j \in \Gamma(X)$, then $\mu_j \in \Gamma(X)$ by Lemma 2. Therefore we can assume that $\hat{\mu}_j(y) > 0$ for all $y \in Y,\ j = 1, 2, ..., n$.

We shall modify slightly the classical proof of the Skitovich-Darmois theorem. Put $\psi_j(y) = -\ln \hat{\mu}_j(y)$. It follows from (2) that

$$\sum_{j=1}^{n} \psi_j(u + \tilde{\delta}_j(v)) = A(u) + B(v), \quad u, v \in Y, \tag{3}$$

where

$$A(u) = \sum_{j=1}^{n} \psi_j(u), \quad B(v) = \sum_{j=1}^{n} \psi_j(\tilde{\delta}_j(v)).$$

Let k_1 be an arbitrary element of Y. Set $h_1 = -\tilde{\delta}_n(k_1)$, then $h_1 + \tilde{\delta}_n(k_1) = 0$. Give in (3) the increments h_1 and k_1 of u and v respectively. Subtracting (3) from the received equation we obtain

$$\sum_{j=1}^{n-1} \Delta_{l_{1j}} \psi_j(u + \tilde{\delta}_j(v)) = \Delta_{h_1} A(u) + \Delta_{k_1} B(v), \quad u, v \in Y, \tag{4}$$

where $l_{1j} = h_1 + \tilde{\delta}_j(k_1) = (\tilde{\delta}_j - \tilde{\delta}_n)(k_1),\ j = 1, 2, ..., n-1$. Observe that the left-hand side of this equation does not contain the function ψ_n. Let k_2 be an arbitrary element of Y. Set $h_2 = -\tilde{\delta}_{n-1}(k_2)$, then $h_2 + \tilde{\delta}_{n-1}(k_2) = 0$. Let us give u and v in (4) the increments h_2 and k_2 respectively. Subtracting (4) from the received equation we obtain

$$\sum_{j=1}^{n-2} \Delta_{l_{2j}} \Delta_{l_{1j}} \psi_j(u + \tilde{\delta}_j(v)) = \Delta_{h_2}\Delta_{h_1} A(u) + \Delta_{k_2}\Delta_{k_1} B(v), \quad u, v \in Y,$$

where $l_{2j} = h_2 + \tilde{\delta}_j(k_2) = (\tilde{\delta}_j - \tilde{\delta}_{n-1})(k_2),\ j = 1, 2, ..., n-2$. The left-hand side of this equation contains neither the function ψ_n nor ψ_{n-1}. Reasoning similarly we sequentially exclude the functions $\psi_n,\ \psi_{n-1}, ..., \psi_2$ from the left-hand side of equation (3) and come to the equation

$$\begin{aligned} &\Delta_{l_{n-1,1}} \Delta_{l_{n-2,1}} ... \Delta_{l_{11}} \psi_1(u + \tilde{\delta}_1(v)) \\ &\quad = \Delta_{h_{n-1}} \Delta_{h_{n-2}} ... \Delta_{h_1} A(u) + \Delta_{k_{n-1}} \Delta_{k_{n-2}} ... \Delta_{k_1} B(v), \quad u, v \in Y, \end{aligned} \tag{5}$$

where k_m are arbitrary elements of $Y,\ h_m = -\tilde{\delta}_{n-m+1}(k_m),\ m = 1, 2, ..., n-1$ and $l_{mj} = h_m + \tilde{\delta}_j(k_m) = (\tilde{\delta}_j - \tilde{\delta}_{n-m+1})(k_m),\ j = 1, 2, ..., n-m$. Substituting $v = 0$ in (5) and subtracting the received equation from (5) we obtain

$$\begin{aligned} &\Delta_{l_{n-1,1}} \Delta_{l_{n-2,1}} ... \Delta_{l_{11}} [\psi_1(u + \tilde{\delta}_1(v)) - \psi_1(u)] \\ &\quad = \Delta_{k_{n-1}} \Delta_{k_{n-2}} ... \Delta_{k_1} B(v) - \Delta_{k_{n-1}} \Delta_{k_{n-2}} ... \Delta_{k_1} B(0), \quad u, v \in Y. \end{aligned} \tag{6}$$

Give an increment k_n of v in (6), where k_n is an arbitrary element of Y and subtract (6) from the received equation. We have

$$\Delta_{l_{n-1,1}}\Delta_{l_{n-2,1}}...\Delta_{l_{11}}[\psi_1(u+\tilde{\delta}_1(v)+\tilde{\delta}_1(k_n))-\psi_1(u+\tilde{\delta}_1(v))] \\ =\Delta_{k_n}\Delta_{k_{n-1}}...\Delta_{k_1}B(v), \quad u,v\in Y. \tag{7}$$

It should be noted that u, v and $k_j,\ j=1,2,...,n$ in (7) are arbitrary elements of Y. For this reason, substituting $u=-\tilde{\delta}_1(v),\ k_1=...=k_n=k$ in (7) we obtain that $B(v)$ satisfies the equation

$$\Delta_k^n B(v)=d(k), \quad v,k\in Y.$$

It follows from this that

$$\Delta_k^{n+1}B(v)=0, \quad v,k\in Y. \tag{8}$$

Denote by γ_j the distribution on the group X with the characteristic function $\hat{\gamma}_j=\hat{\mu}_j(\tilde{\delta}_j(y))$, i.e. $\gamma_j=\delta_j(\mu_j),\ j=1,2,,...,n$. Set $\gamma=\gamma_1*...*\gamma_n$ and observe that

$$\hat{\gamma}(y)=\exp\{-B(y)\}, \quad y\in Y.$$

It follows from (8) and Lemma 3 that $\gamma\in\Gamma(X)$, hence $\gamma_j\in\Gamma(X)$ by Lemma 2 and then $\mu_j\in\Gamma(X),\ j=1,2,...,n$. Theorem 1 is proved.

Remark 1. If a group X contains a subgroup topologically isomorphic to T, then the statement of Theorem 1 is not valid. This results from the following fact ([3]): there exist independent random variables ξ_1 and ξ_2 with values in the group T and with distributions μ_1 and μ_2 with nonvanishing characteristic functions such that the linear forms $L_1=\xi_1+\xi_2$ and $L_2=\xi_1-\xi_2$ are independent but $\mu_1,\mu_2\notin\Gamma(T)$ (compare below with Proposition 1).

Remark 2. It is interesting to observe that it may be considered another group analog of the Skitovich-Darmois theorem, namely when coefficients of linear forms are integers. For $n\in\mathbb{Z}$ define a homomorphism $f_n:X\to X$ by the formula $f_n(x)=nx$. Put $X^{(n)}=f_n(X)$. A set of integers $\{a_j\}$ is called admissible for X if $X^{(a_j)}\neq\{0\}$ for all j. Let $\xi_j,\ j=1,2,...,n$ be random variables with values in X. The admissibility of the set $\{a_j\}_{j=1}^n$ when considering the linear form $L=a_1\xi_1+\cdots+a_n\xi_n$ is a group analog of the condition $a_j\neq 0,\ j=1,2,...,n$ for the case $X=\mathbb{R}$. In [5] (see also [7, §10]) the following result was proved (compare with Theorem 1).

Let $\xi_j,\ j=1,2,...,n,\ n\geq 2$ be independent random variables with values in X and with distributions μ_j such that their characteristic functions $\hat{\mu}_j(y)$ do not vanish. Let $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$ be admissible sets of integers for X. The independence of linear forms $L_1=a_1\xi_1+\cdots+a_n\xi_n$ and $L_2=b_1\xi_1+\cdots+b_n\xi_n$ implies that all $\mu_j\in\Gamma(X)$ if and only if either X is a torsion free group or $X^{(p)}=\{0\}$, where p is a prime number (in the latter case all μ_j are degenerate distributions).

Remark 3. Let $\xi_j,\ j = 1, 2, ..., n,\ n \geq 2$ be independent random variables with values in a group X and with distributions μ_j such that their characteristic functions $\hat{\mu}_j(y)$ do not vanish, $\alpha_j,\ \beta_j \in \mathrm{Aut}(X)$. The independence of linear forms $L_1 = \alpha_1(\xi_1) + \cdots + \alpha_n(\xi_n)$ and $L_2 = \beta_1(\xi_1) + \cdots + \beta_n(\xi_n)$ implies that one can choose shifts $\mu'_j = \mu_j * E_{x_j}$ of the distributions μ_j such that $\sigma(\mu'_j) \subset C_X,\ j = 1, 2, ..., n$. In order to prove this set $\nu_j = \mu_j * \bar{\mu}_j,\ \psi_j(y) = -\ln \hat{\nu}_j(y),\ B(y) = \sum_{j=1}^{n} \psi_j(\tilde{\alpha}_j^{-1}\tilde{\beta}_j(y))$. Denote by Y_0 the subgroup of compact elements of Y. It follows from the proof of Theorem 1 that the function $B(y)$ satisfies equation (8). This implies that $B(y) \equiv 0$, where $y \in Y_0$ ([4]). Taking into account that the subgroup Y_0 is invariant with respect to any $\tilde{\alpha} \in \mathrm{Aut}(Y)$ it follows from this that all $\psi_j(y) \equiv 0,\ y \in Y_0$. It means that $\hat{\nu}_j(y) \equiv 1,\ y \in Y_0$ and then $\sigma(\nu_j) \subset A(X, Y_0),\ j = 1, 2, ..., n$. The required assertion follows from $A(X, Y_0) = C_X$ ([10, §24]) and Lemma 1.

3. Solution of Problem 2

We pass now to the solution of Problem 2. We need the following lemmas.

Lemma 4. *If $X \neq \{0\}$ and $X \not\approx \mathbb{Z}(2)$, then there exists $\alpha \in \mathrm{Aut}(X)$ such that $\alpha \neq I_X$.*

Proof. If not all nonzero elements of X have order 2, then set $\alpha = -I_X$. If all nonzero elements of X have the order 2 then by the structure theorem for such groups ([10, §25]) we have

$$X \approx (\mathbb{Z}(2))^{\mathbf{M}} + (\mathbb{Z}(2))^{\mathbf{N}^*},$$

where $\mathbf{M}$ and $\mathbf{N}$ are cardinal numbers, the group $(\mathbb{Z}(2))^{\mathbf{M}}$ is considered in Tychonoff topology and the group $(\mathbb{Z}(2))^{\mathbf{N}^*}$ is considered in discrete topology. It follows from this, that X can be represented in the form $X = X_1 + X_2 + X_3$, where $X_1 \approx X_2 \approx \mathbb{Z}(2)$. Set $\alpha(x_1, x_2, x_3) = (x_2, x_1, x_3),\ x_j \in X_j,\ j = 1, 2, 3$. Lemma 4 is proved.

Lemma 5. *Let $X = T^2$. Then there exist the automorphisms $\alpha_1,\ \alpha_2 \in \mathrm{Aut}(X)$, $\alpha_1 \neq \alpha_2$ such that if $\xi_1,\ \xi_2$ are independent random variables with values in X and with distributions $\mu_1,\ \mu_2$ with nonvanishing characteristic functions and the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \alpha_1(\xi_1) + \alpha_2(\xi_2)$ are independent, then $\mu_1,\ \mu_2 \in \Gamma(X)$.*

Proof. Since $Y \approx \mathbb{Z}^2$ we denote elements of Y by $y = (m, n),\ m, n \in \mathbb{Z}$. It is well known that each automorphism $\alpha \in \mathrm{Aut}(X)$ is assigned to a matrix with integer elements $\alpha \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $|ad - bc| = 1$ and α operates on X in the following way

$$\alpha(z, w) = (z^a w^c, z^b w^d), \quad x = (z, w) \in X, \quad |z| = |w| = 1.$$

The conjugate automorphism $\tilde\alpha \in \mathrm{Aut}(Y)$ has the form $\tilde\alpha(m,n) = (am+bn, cm+dn),\ y=(m,n)\in Y$ ([10, §26]). We check that

$$\alpha_1 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \alpha_2 \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9}$$

are the required automorphisms.

Suppose that $L_1 = \xi_1+\xi_2$ and $L_2 = \alpha_1(\xi_1)+\alpha_2(\xi_2)$ are independent. Consider equation (2) which takes the form

$$\begin{aligned}&\hat\mu_1(u+\tilde\alpha_1(v))\hat\mu_2(u+\tilde\alpha_2(v))\\ &\quad = \hat\mu_1(u)\ \hat\mu_2(u)\hat\mu_1(\tilde\alpha_1(v))\hat\mu_2(\tilde\alpha_2(v)), \quad u,v\in Y.\end{aligned} \tag{10}$$

Put $\nu_j = \mu_j * \bar\mu_j$, then $\hat\nu_j(y) = |\hat\mu_j(y)|^2 > 0,\ y\in Y$. The characteristic functions $\hat\nu_j(y)$ satisfy equation (10) too. Set $\psi_j(y) = -\ln\hat\nu_j(y),\ j=1,2$. Reasoning as in the proof of Theorem 1 we obtain that the function $\psi_1(y)$ satisfies equation (6) which has the form

$$\Delta_{l_{11}}[\psi_1(u+\tilde\alpha_1(v)) - \psi_1(u)] = \Delta_{k_1}B(v) - \Delta_{k_1}B(0), \quad u,v\in Y, \tag{11}$$

where k_1 is an arbitrary element of Y and $l_{11} = (\tilde\alpha_1 - \tilde\alpha_2)(k_1)$. Observe that $(\tilde\alpha_1 - \tilde\alpha_2)\in\mathrm{Aut}(Y)$. Let l be an arbitrary element of Y. Put in (11) $v = \tilde\alpha_1^{-1}(l),\ l_{11} = l$. We have

$$\Delta_l^2\psi_1(u) = d(l), \quad u,l\in Y,$$

and hence

$$\Delta_l^3\psi_1(u) = 0, \quad u,l\in Y. \tag{12}$$

It follows from (12) that $\psi_1(y) = \psi_1(m,n)$ is a polynomial, generally, with complex coefficients of degree ≤ 2. Taking into account that $\psi_1(y)\ge 0,\ \psi_1(-y) = \psi_1(y)$ for all $y\in Y$ and $\psi_1(0)=0$, we have the representation

$$\psi_1(m,n) = a_{11}m^2 + 2a_{12}mn + a_{22}n^2 = <Ay, y>,$$

where $y=(m,n)\in Y$, $A = (a_{ij})_{i,j=1}^2$ is a symmetric positive semidefinite matrix and $<.\,,\,.>$ is the scalar product in $\mathbb{R}^2$. Reasoning similarly we obtain $\psi_2(y) = <By, y>$, where $B = (b_{ij})_{i,j=1}^2$ is a symmetric positive semidefinite matrix.

Substituting the obtained expressions for $\psi_j(y)$ into (10) we find

$$\begin{aligned}&<Au, \tilde\alpha_1(v)> + <A\tilde\alpha_1(v), u> + <Bu, \tilde\alpha_2(v)>\\ &\quad + <B\tilde\alpha_2(v), u> = 0, \quad u,v\in Y.\end{aligned}$$

It follows from this that

$$<u, (A\tilde\alpha_1 + B\tilde\alpha_2)v> = 0, \quad u,v\in Y,$$

hence we arrive at the matrix equation

$$A\tilde\alpha_1 + B\tilde\alpha_2 = 0. \tag{13}$$

It follows from (9) and (13) that elements a_{ij}, b_{ij} satisfy the system of equations

$$\begin{cases} a_{11} + b_{12} = 0, \\ a_{11} + a_{12} + b_{11} = 0, \\ a_{12} + b_{22} = 0, \\ a_{12} + a_{22} + b_{12} = 0. \end{cases} \tag{14}$$

On the other hand taking into account that A and B are symmetric positive semidefinite matrices we obtain that elements a_{ij} and b_{ij} satisfy the system of inequalities

$$\begin{cases} a_{11} \geq 0, \quad a_{22} \geq 0, \quad b_{11} \geq 0, \quad b_{22} \geq 0, \\ a_{12}^2 \leq a_{11}a_{22}, \quad b_{12}^2 \leq b_{11}b_{22}. \end{cases} \tag{15}$$

Set $a_{11} = p,\ a_{22} = q$. It follows from (14) and (15) that $\dfrac{p}{q} = \dfrac{3-\sqrt{5}}{2} = t_0$, and (14) finally implies

$$A = q\begin{pmatrix} t_0 & t_0 - 1 \\ t_0 - 1 & 1 \end{pmatrix}, \qquad B = q\begin{pmatrix} 1 - 2t_0 & -t_0 \\ -t_0 & 1 - t_0 \end{pmatrix},$$

where $q \geq 0$.

A future reasoning we shall give for μ_1 only. It is similarly for μ_2. It follows from the form of the matrix A the representation

$$\tilde{\nu}_1(m,n) = \exp\{-q(\sqrt{t_0}m - n)^2\}, \quad y = (m,n) \in Y.$$

(It should be observed that the obtained characteristic functions $\hat{\nu}_1(y)$ and $\hat{\nu}_2(y)$ satisfy equation (10), hence the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \alpha_1(\xi_1) + \alpha_2(\xi_2)$, where $\xi_1,\ \xi_2$ are independent random variables with values in X and with distributions $\nu_1,\ \nu_2$, are independent.)

Consider the Gaussian distribution N on the group $\mathbb{R}$ with the characteristic function $\hat{N}(s) = \exp\{-qs^2\}$. Let $\pi : Y \to \mathbb{R}^* \approx \mathbb{R}$ be the homomorphism $\pi(m,n) = \sqrt{t_0}m - n$. It is easy to see that $\nu_1 = \tilde{\pi}(N)$, where $\tilde{\pi} : \mathbb{R} \to X = T^2$. Since $\sqrt{t_0}$ is an irrational number, the image $\pi(Y)$ is everywhere dense in $\mathbb{R}^*$ and for this reason $\tilde{\pi}$ is a monomorphism ([10, §24]). The distribution ν_1 concentrated on the Borel subgroup $G = \tilde{\pi}(\mathbb{R}) \subset T^2$. By Lemma 1 the distributions μ_1 and $\bar{\mu}_1$ can be replaced on their shifts μ_1' and $\bar{\mu}_1'$ such that

$$\nu_1 = \mu_1' * \bar{\mu}_1' \tag{16}$$

and the distributions μ_1' and $\bar{\mu}_1'$ are concentrated on G. Since $\tilde{\pi}$ is a monomorphism, (16) implies that

$$N = N_1 * N_2,$$

where $N_1 = \tilde{\pi}^{-1}(\mu_1')$. It follows from the classical Cramer theorem of decomposition of the Gaussian distribution on the real line $\mathbb{R}$ that $N_1 \in \Gamma(\mathbb{R})$, but then $\mu_1' = \tilde{\pi}(N_1) \in \Gamma(X)$ and for this reason $\mu_1 \in \Gamma(X)$. (It is not difficult to check that the characteristic function $\hat{\mu}_1(y)$ has the form

$$\hat{\mu}_1(m,n) = \exp\{-(q/2)(\sqrt{t_0}m - n)^2 + i(tm + sn)\}, \quad y = (m,n) \in Y).$$

Lemma 5 is proved.

Theorem 2. *Let $X \not\approx T$ and $X \not\approx T + \mathbb{Z}(2)$. Then exist the automorphisms $\delta_1, \delta_2 \in$* $\mathrm{Aut}(X)$, $\delta_1 \neq \delta_2$ *such that if $\xi_1,\ \xi_2$ are independent random variables with values in X and with distributions $\mu_1,\ \mu_2$ with nonvanishing characteristic functions and the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1(\xi_1) + \delta_2(\xi_2)$ are independent, then $\mu_1,\ \mu_2 \in \Gamma(X)$*

Proof. The following 3 cases are possible.

1. The group X contains no subgroup topologically isomorphic to T.

In this case the independence of any forms L_1 and L_2 implies that all $\mu_j \in \Gamma(X)$ by Theorem 1.

2. The group X contains a subgroup $G_1 \approx T$ and contains no subgroup topologically isomorphic to T^2.

The subgroup G_1 is a topological direct summand in X ([10, §24]) i.e. $X = G_1 + G_2$, where the subgroup G_2 contains no subgroup topologically isomorphic to T. Under the conditions of Theorem 2 $G_2 \neq \{0\}$ and $G_2 \not\approx \mathbb{Z}(2)$. By Lemma 4 there exists an automorphism $\alpha \in \mathrm{Aut}(G_2)$ such that $\alpha \neq I_{G_2}$. Check that $\delta_1 = I_X$ and $\delta_2 \in \mathrm{Aut}(X),\ \delta_2(g_1, g_2) = (g_1, \alpha(g_2))$ are the required automorphisms. Really, let $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \delta_2(\xi_2)$ be independent. Equation (2) in this case has the form

$$\begin{aligned}&\hat{\mu}_1(u+v)\hat{\mu}_2(u+\tilde{\delta}_2(v))\\ &\quad = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(\tilde{\delta}_2(v)), \quad u, v \in Y.\end{aligned} \tag{17}$$

Since $X = G_1 + G_2$, then $Y = H_1 + H_2$, where $H_j \approx G_j^*,\ j = 1, 2$. Taking into account that $\tilde{\delta}_2(y) = y$ for all $y \in H_1$, the restriction of equation (17) on the subgroup H_1 has the form

$$\hat{\mu}_1(u+v)\hat{\mu}_2(u+v) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(v), \quad u, v \in H_1.$$

Hence, the restriction of the function $f(y) = \hat{\mu}_1(y)\hat{\mu}_2(y)$ on H_1 is a character of the group H_1. It means that the restrictions of the functions $\hat{\mu}_j(y)$ on H_1 are also characters of $H_1,\ j = 1, 2$. This implies that there exist elements $x_1,\ x_2 \in G_1$ such that the characteristic functions of the distributions $\mu'_j = \mu_j * E_{x_j}$ satisfy the condition $\hat{\mu}'_j(y) \equiv 1$, $y \in H_1$, $j = 1, 2$. For this reason $\sigma(\mu'_j) \subset A(X, H_1) = G_2$. The characteristic functions $\hat{\mu}'_j(y)$ also satisfy equation (17). Observe now that $\tilde{\delta}_2(y) = \tilde{\alpha}(y)$ for all $y \in H_2$ and consider the restriction of equation (17) for the characteristic functions $\hat{\mu}'_j(y)$ on H_2. Since $H_2 \approx G_2^*$, applying Theorem 1 to the group G_2 and to the linear forms $L_1 = \xi'_1 + \xi'_2$ and $L_2 = \xi'_1 + \alpha(\xi'_2)$, where ξ'_j are independent random variables with values in G_2 and with distributions μ'_j, we obtain $\mu'_j \in \Gamma(G_2)$ and hence $\mu_j \in \Gamma(X),\ j = 1, 2$.

3. The group X contains a subgroup $G_2 \approx T^2$.

The subgroup G_2 is a topologically direct summand in X ([10, §24]), i.e. $X = G_1 + G_2$. Retaining the notation of the case 2 we observe that if α_1 and α_2 are the automorphisms constructed in Lemma 5 (which can be regarded as automorphisms of G_2), then $\delta_j \in \mathrm{Aut}(X),\ \delta_j(g_1, g_2) = (g_1, \alpha_j(g_2)),\ j = 1, 2$ are the required automorphisms. The proof is in the complete analogy of the case 2. We only use Lemma 5 instead of Theorem 1 in the final part of the proof.

Theorem 2 is sharp. Namely, the following result takes place.

Proposition 1. *Let either $X = T$ or $X = T + \mathbb{Z}(2)$ and let $\alpha_j, \beta_j \in \mathrm{Aut}(X),\ j = 1, 2, ..., n,\ n \geq 2$ be arbitrary automorphisms such that not all $\alpha_j \beta_j{}^{-1}$ are equal. Then there exist independent random variables ξ_j with values in X and with distributions μ_j with nonvanishing characteristic functions $\hat{\mu}_j(y)$ such that the linear forms $L_1 = \alpha_1(\xi_1) + \cdots + \alpha_n(\xi_n)$ and $L_2 = \beta_1(\xi_1) + \cdots + \beta_n(\xi_n)$ are independent but all $\mu_j \notin \Gamma(X)$.*

Proof. It is easy to see that $\mathrm{Aut}(X) = \{I_X, -I_X\}$. Hence, we can restrict ourselves to the case $X = T$. Then $Y \approx \mathbb{Z}$. Without loss of generality we can assume that L_1 and L_2 have the form $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \xi_1 + \cdots + \xi_m - \xi_{m+1} - \cdots - \xi_n$, $1 \leq m < n$. Denote elements of Y by k and consider the functions

$$f(k) = \begin{cases} \exp\{-\dfrac{ak^2}{m}\}, & k = 2p, \\ \exp\{-\dfrac{ak^2 - 1}{m}\}, & k = 2p + 1, \end{cases}$$

$$g(k) = \begin{cases} \exp\{-\dfrac{ak^2}{n - m}\}, & k = 2p, \\ \exp\{-\dfrac{ak^2 + 1}{n - m}\}, & k = 2p + 1 \end{cases}$$

on Y. If $a > 0$ is large enough, then

$$\rho(e^{it}) = \sum_{k=-\infty}^{\infty} f(k) e^{-ikt} > 0, \quad \tau(e^{it}) = \sum_{k=-\infty}^{\infty} g(k) e^{-ikt} > 0.$$

Thus $f(k)$ and $g(k)$ are the characteristic functions of some distributions $\mu, \nu \in M^1(X)$. Let ξ_j be independent random variables with values in X and with distributions $\mu_j = \mu$, $j = 1, 2, ...m$ and $\mu_j = \nu$, $j = m + 1, m + 2, ..., n$. We see that the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\prod_{j=1}^{m} \hat{\mu}_j(u + v) \prod_{j=m+1}^{n} \hat{\mu}_j(u - v) = \prod_{j=1}^{n} \hat{\mu}_j(u) \prod_{j=1}^{m} \hat{\mu}_j(v) \times \prod_{j=m+1}^{n} \hat{\mu}_j(-v), \quad u, v \in Y.$$

Taking into account (2) this implies that L_1 and L_2 are independent, but all $\mu_j \notin \Gamma(X)$.

Theorem 2 and Proposition 1 give the complete solution of Problem 2.

Remark 4. Let X be a group the same as in Theorem 2 and suppose that X satisfies the conditions:

(i) $C_X \neq \{0\}$,

(ii) $C_X \not\approx T$.

Then the automorphisms $\delta_1,\ \delta_2 \in \mathrm{Aut}(X)$ which exist by Theorem 2 may be chosen in such a manner that the following statement is true:

(I) There exist nondegenerate distributions $\mu_1, \mu_2 \in \Gamma(X)$ such that if ξ_1 and ξ_2 are independent random variables with values in X and with distributions μ_1 and μ_2, then the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1(\xi_1) + \delta_2(\xi_2)$ are independent.

In order to prove this we use the scheme of the proof of Theorem 2 and make more precisely the choice of the automorphisms δ_1 and δ_2 in Theorem 2 in such a way that (I) is valid. Consider the same 3 cases.

1. The group X contains no subgroup topologically isomorphic to T.

Observe that if $C_X \neq \{0\}$, then there exist a nondegenerate distribution $\gamma \in \Gamma(X)$ ([15]). It follows from (1) that its characteristic function $\hat{\gamma}(y)$ satisfies the equation

$$\hat{\gamma}(u+v)\hat{\gamma}(u-v) = \hat{\gamma}(u)^2\hat{\gamma}(v)\hat{\gamma}(-v), \quad u, v \in Y.$$

Taking into account (2) it means that if ξ_1 and ξ_2 are independent identically distributed random variables with values in X and with distribution γ, then the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - \xi_2$ are independent.

Put $\delta_1 = I_X,\ \delta_2 = -I_X$ and $\mu_1 = \mu_2 = \gamma$, where γ is a nondegenerate Gaussian distribution on X.

2. The group X contains a subgroup $G_1 \approx T$ and contains no subgroup topologically isomorphic to T^2.

The subgroup G_1 is a topological direct summand in X ([10, §24]) i.e. $X = G_1 + G_2$, where the subgroup G_2 contains no subgroup topologically isomorphic to T. It follows from (ii) that $C_{G_2} \neq \{0\}$. Taking into account the reasoning given above in case 1, we can put $\delta_1 = I_X,\ \delta_2 \in \mathrm{Aut}(X),\ \delta_2(g_1, g_2) = (g_1, -g_2)$, and $\mu_1 = \mu_2 = \gamma$, where γ is a nondegenerate Gaussian distribution on G_2.

3. The group X contains a subgroup $G_2 \approx T^2$.

The automorphisms δ_1 and δ_2 which were constructed in case 3 of the proof of Theorem 2 have the required property.

On the other hand if either $C_X = \{0\}$ or $C_X \approx T$, then as appears from Remark 2 and Proposition 1 it is impossible to choose automorphisms $\delta_1, \delta_2 \in \mathrm{Aut}(X)$ in Theorem 2 in such a manner that (I) is fulfilled.

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