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Images of eigenvalue distributions under power maps

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Abstract. In [9], it was shown that if U is a random $n \times n$ unitary matrix, then for any $p \geq n$, the eigenvalues of U^p are i.i.d. uniform; similar results were also shown for general compact Lie groups. We study what happens when $p < n$ instead. For the classical groups, we find that we can describe the eigenvalue distribution of U^p in terms of the eigenvalue distributions of smaller classical groups; the earlier result is then a special case. The proofs rely on the fact that a certain subgroup of the Weyl group is itself a Weyl group. We generalize this fact, and use it to study the power-map problem for general compact Lie groups.

In [9], it was shown that if a (uniformly) random $n \times n$ unitary matrix U is raised to a power $p \geq n$, the eigenvalues of the resulting matrix are (exactly) independently distributed; this despite the rather complicated dependence between the eigenvalues of U itself. Our purpose in the present note is to extend this result to the case $p < n$. We find that the eigenvalue distribution of U^p can in that case be described in terms of a union of p independent distributions, each of which is itself the eigenvalue distribution of a random unitary matrix. More precisely, we have:

$$U(n)^p \sim \bigoplus_{0 \leq i < p} U\left(\left\lceil \frac{n-i}{p} \right\rceil\right). \tag{1}$$

That is, if we take the pth power of a uniformly distributed element of $U(n)$, the resulting eigenvalue distribution is the same as if we took the union of the eigenvalues of p independent matrices. For $p \geq n$, this reduces to the earlier result, since then we have

$$U(n)^p \sim \bigoplus_{0 \leq i < n} U(1) \oplus \bigoplus_{n \leq i < p} U(0), \tag{2}$$

the latter component being trivial. Similarly, the eigenvalue distribution of a power of a random orthogonal or symplectic matrix can also be described as a union of eigenvalue distributions.

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After showing these results, we then consider the case of more general compact Lie groups, especially since the above independence result extends to this case. It turns out in general that the eigenvalue density after raising to a power can be expressed in terms of a certain parabolic subgroup of an associated affine Weyl group. (See [2], [7], and [8] for definitions and results for affine Weyl groups.) As a special case, we find that for a simple compact Lie group G, the eigenvalues of G^p are (essentially) independent whenever p is greater than the Coxeter number of G, thus refining the result of [9].

1. The classical groups

We first consider the general problem of, given an eigenvalue distribution, determining the distribution of powers of the eigenvalues. We let λ_1 through λ_n denote complex numbers of norm 1, and define

$$dT := \prod_{0\le k<n} \frac{d\lambda_k}{2\pi i\lambda_k}, \tag{3}$$

so that dT is the uniform density on the unit torus T. The following lemma is straightforward:

Lemma 1.1. *Let $\pi(\lambda)$ be a Laurent polynomial in the variables λ_j such that*

$$\pi(\lambda)dT \tag{4}$$

is a probability density on the unit torus. Then for any (monic) Laurent monomial $\mu(\lambda)$,

$$\int_T \pi(\lambda)\overline{\mu(\lambda)}dT \tag{5}$$

is equal to the coefficient of $\mu(\lambda)$ in $\pi(\lambda)$.

The following "Main Lemma" is crucial to our later results:

Lemma 1.2. *Fix an integer $p \ge 1$. With $\pi(\lambda)$ as above, the joint density of the random variables λ_j^p is given by*

$$\pi^{(p)}(\lambda^{1/p})dT, \tag{6}$$

where $\pi^{(p)}(\lambda)$ is the sum of the p-divisible monomials of $\pi(\lambda)$; that is, monomials in which each exponent is a multiple of p.

Proof. Since the torus is a compact set, it suffices to show that all joint moments agree. In other words, we need to show that if $\mu(\lambda)$ is a monic Laurent monomial, then

$$\int_T \mu(\lambda)\pi^{(p)}(\lambda^{1/p})dT = \int_T \mu(\lambda^p)\pi(\lambda)dT. \tag{7}$$

But this follows immediately from Lemma 1.1. □

We now obtain the power-map theorem for the unitary group:

Theorem 1.3. *For any integers $n, p \geq 1$,*

$$U(n)^p \sim \bigoplus_{0\leq i<p} U\left(\left\lceil \frac{n-i}{p} \right\rceil\right); \tag{8}$$

that is to say that the following two eigenvalue distributions are the same:

$$\mathrm{eig}(U^p) \sim \cup_{0\leq i<p} \mathrm{eig}(U_i), \tag{9}$$

where U is a uniform random element of $U(n)$, and for $0 \leq i < pU_i$ is an independent uniform random element of $U\left(\left\lceil \frac{n-i}{p} \right\rceil\right)$.

Proof. Recall [10] that the eigenvalue distribution of $U(n)$ is given by the density

$$\frac{1}{n!} \sum_{\pi_1,\pi_2\in S_n} \sigma(\pi_2\pi_1^{-1}) \prod_{0\leq k<n} \lambda_k^{\pi_1(k)-\pi_2(k)} dT. \tag{10}$$

Equivalently, if we view S_n as acting on the eigenvalues as

$$\pi \cdot \lambda_i = \lambda_{\pi^{-1}(i)}, \tag{11}$$

we have the density

$$\frac{1}{n!} \sum_{\pi'\in S_n} \pi' \cdot \left(\sum_{\pi\in S_n} \sigma(\pi) \prod_{0\leq k<n} \lambda_k^{k-\pi(k)} \right). \tag{12}$$

By the Main Lemma, we must extract the p-divisible monomials. Clearly, the action of π' preserves divisibility, so we may restrict our attention to

$$q(\lambda) = \sum_{\pi\in S_n} \sigma(\pi) \prod_{0\leq k<n} \lambda_k^{k-\pi(k)}. \tag{13}$$

A monomial here is p-divisible if and only if

$$\pi(k) \equiv k \pmod{p}, \ \forall 0 \leq k < n. \tag{14}$$

Given such a permutation, and given a congruence class i mod p, we can define a new permutation $\pi^{(i)} \in S_{\lceil \frac{n-i}{p} \rceil}$ by

$$\pi^{(i)}(k) = \frac{\pi(pk+i)-i}{p}. \tag{15}$$

Then the permutation π is determined by the permutations $\pi^{(i)}$, and we have

$$\sigma(\pi) = \prod_{0\leq i<p} \sigma(\pi^{(i)}) \tag{16}$$

$$\prod_{0\leq k<n} \lambda_k^{k-\pi(k)} = \prod_{0\leq i<p} \prod_{0\leq k<\lceil \frac{n-i}{p} \rceil} \lambda_{pk+i}^{pk-p\pi^{(i)}(k)}. \tag{17}$$

It follows that $q^{(p)}(\lambda^{1/p})$ factors:

$$q^{(p)}(\lambda^{1/p}) = \prod_{0\le i<p} \left(\sum_{\pi \in S_{\lceil \frac{n-i}{p} \rceil}} \sigma(\pi) \prod_{0\le k<\lceil \frac{n-i}{p} \rceil} \lambda_{pk+i}^{k-\pi(k)} \right). \tag{18}$$

Since each factor has the form associated to the density of the unitary group of dimension $\lceil \frac{n-i}{p} \rceil$, the result follows. □

Remark 1. This could also be derived from the Toeplitz determinant factorization of [5, Eq. 38], much as in the derivation of Theorem 1.7 below from the results of [11].

Remark 2. Diaconis (personal communication) observes that the above result immediately implies (via the central limit theorem) that if $n, h \to \infty, n/h \to m \ge 1$, the random variable $\mathrm{Tr}(U^h)$ converges in distribution to $N(0, h)$; similar comments apply to the orthogonal and symplectic group cases. For h fixed, such normal convergence was shown in [4].

As a special case, we recover a result of [9]:

Corollary 1.4. *Fix an integer $n \ge 1$, and let U be a random element of $U(n)$. For any integer $p \ge n$, the eigenvalues of U^p are i.i.d. uniform.*

Proof. Indeed, by the theorem,

$$U(n)^p \sim \bigoplus_{0\le i<p} U\left(\left\lceil \frac{n-i}{p} \right\rceil\right) = \bigoplus_{0\le i<n} U(1) \oplus \bigoplus_{n\le i<p} U(0). \tag{19}$$

But this is precisely the desired result. □

For the orthogonal group, the situation is somewhat more complicated, both because the orthogonal group is not connected, and because random orthogonal matrices are sometimes forced to have eigenvalues ± 1. There are essentially four cases, depending on whether the dimension is even or odd, and on whether the determinant is 1 or -1.

Theorem 1.5. *Fix integers $n \ge 1$, $p \ge 1$. Then we have the following identities of eigenvalue distributions. For p odd:*

$$O^{\pm}(2n)^p \sim O^{\pm}(2n_0) \oplus \bigoplus_{0\le i<(p-1)/2} \mathrm{Re}\, U\left(\left\lceil \frac{2(n-n_0-i)}{p-1} \right\rceil\right). \tag{20}$$

$$O^{\pm}(2n+1)^p \sim \bigoplus_{0\le i<(p-1)/2} \mathrm{Re}\, U\left(\left\lceil \frac{2(n-n_1-i)}{p-1} \right\rceil\right) \oplus O^{\pm}(2n_1+1). \tag{21}$$

For p even:

$$O^{\pm}(2n)^p \sim O^{\pm}(2n_0) \oplus \bigoplus_{0\le i<(p-2)/2} \operatorname{Re} U\left(\left\lceil \frac{2(n-n_0-n_1-i)}{p-2} \right\rceil\right)$$
$$\oplus\, O^{\mp}(2n_1+1). \tag{22}$$

$$O^{\pm}(2n+1)^p \sim \bigoplus_{0\le i<p/2} \operatorname{Re} U\left(\left\lceil \frac{2(n-i)}{p} \right\rceil\right). \tag{23}$$

Here $n_0 = \lceil \frac{n}{p} \rceil$, $n_1 = \lceil \frac{n-\lfloor p/2\rfloor}{p} \rceil$, $O^{\pm}(n)$ *represents the coset of* $O(n)$ *with determinant* ± 1, $\operatorname{Re} U(n)$ *represents the image of* $U(n)$ *in the natural representation in* $O(2n)$*, and eigenvalues* ± 1 *should be ignored.*

Proof. Again, referring to [10], the eigenvalue density for a random orthogonal matrix is (up to an overall constant, and ignoring fixed eigenvalues) given by

$$\sum_{\rho'\in B_n} \rho' \cdot \left(\sum_{\rho\in B_n} \sigma(\rho)\lambda^{\delta-\rho\cdot\delta} \right) dT \tag{24}$$

where B_n is the hyperoctahedral group (signed permutations), σ is a certain character of B_n, δ is a vector in $\mathbb{Z}[1/2]^n$, acted on in the obvious way by B_n, and we define

$$\lambda^v = \prod_{0\le k<n} \lambda_k^{v_k}. \tag{25}$$

The ingredients σ and δ are determined as follows:

$$O^{+}(2n)\colon \delta = (0,1,2\dots n-1),\ \sigma=\sigma_1 \tag{26}$$
$$O^{-}(2n+2)\colon \delta = (1,2,3\dots n),\ \sigma=\sigma_2 \tag{27}$$
$$O^{+}(2n+1)\colon \delta = (1/2,3/2,\dots n-1/2),\ \sigma=\sigma_2 \tag{28}$$
$$O^{-}(2n+1)\colon \delta = (1/2,3/2,\dots n-1/2),\ \sigma=\sigma_1 \tag{29}$$

where σ_1 is the composition of the sign character of S_n with the natural projection $B_n \to S_n$, and σ_2 is the natural sign character of B_n.

Consider a signed permutation ρ with $\rho\cdot\delta \equiv \delta \pmod p$. Consider ρ as a permutation of

$$S = \{-n, 1-n, \dots -2, -1, 1, 2, \dots n-1, n\}, \tag{30}$$

and define $\delta(x)$ for $x \in S$ by $\delta(k)=\delta_k$, $\delta(-k)=-\delta_k$. (Thus, for instance, for $O^{+}(2n)$, $\delta(k) = k - \operatorname{sgn}(k)$.) Then ρ must preserve the partition of S induced by $\delta(x)$ mod p. The action of ρ on an individual piece of the partition is either as S_m, if $\delta(x) \not\equiv \delta(-x)$ mod p, or as B_m if $\delta(x)\equiv\delta(-x)$ mod p. Thus, the p-divisible part of the inner sum

$$\sum_{\rho\in B_n} \sigma(\rho)\lambda^{\delta-\rho\cdot\delta} \tag{31}$$

factors; we obtain one S_m-type factor for each pair $\{i \bmod p, -i \bmod p\}$ with $i \neq -i \bmod p$, and one B_m-type factor for each $i \bmod p$ with $i \equiv -i \bmod p$; the corresponding character is simply the restriction of the original character.

The S_m-type factor corresponding to $\pm i \bmod p$ becomes a unitary group density once we replace the eigenvalues corresponding to those x for which $\delta(x) = -i \bmod p$ by their reciprocals; since the eigenvalues come in conjugate pairs, this does not change the effective density. The dimension of the unitary group is then given by the number of values $1 \leq k \leq n$ such that $\delta_k \bmod p \in \{\pm i\}$. Equivalently, these numbers are given by:

$O^{\pm}(2n)$: The number of values $1 \leq k \leq n-1$ such that $k \bmod p \in \{i, p-i\}$ ($i \in \mathbb{Z}$, $1 \leq i < \lceil \frac{p}{2} \rceil$).
$O^{\pm}(2n+1)$: The number of values $1 \leq k \leq n$ such that $k \bmod p \in \{i+1, p-i\}$ ($i \in \mathbb{Z}$, $0 \leq i < \lceil \frac{p-1}{2} \rceil$).

The nature and dimension of the B_n factors follows from a similar computation; we thus deduce the following identities. For p odd:

$$O^{\pm}(2n)^p \sim O^{\pm}\left(2\left\lceil \frac{n}{p} \right\rceil\right) \oplus \bigoplus_{1 \leq i \leq (p-1)/2} \operatorname{Re} U\left(\left\lceil \frac{n-i}{p} \right\rceil + \left\lceil \frac{n+i}{p} \right\rceil - 1\right). \tag{32}$$

$$O^{\pm}(2n+1)^p \sim \bigoplus_{0 \leq i < (p-1)/2} \operatorname{Re} U\left(\left\lceil \frac{n-i}{p} \right\rceil + \left\lceil \frac{n+1+i}{p} \right\rceil - 1\right) \oplus O^{\pm}\left(2\left\lceil \frac{n-(p-1)/2}{p} \right\rceil + 1\right). \tag{33}$$

For p even:

$$O^{\pm}(2n)^p \sim O^{\pm}\left(2\left\lceil \frac{n}{p} \right\rceil\right) \oplus \bigoplus_{1 \leq i < p/2} \operatorname{Re} U\left(\left\lceil \frac{n-i}{p} \right\rceil + \left\lceil \frac{n+i}{p} \right\rceil - 1\right) \oplus O^{\mp}\left(2\left\lceil \frac{n-p/2}{p} \right\rceil + 1\right). \tag{34}$$

$$O^{\pm}(2n+1)^p \sim \bigoplus_{0 \leq i < p/2} \operatorname{Re} U\left(\left\lceil \frac{n-i}{p} \right\rceil + \left\lceil \frac{n+1+i}{p} \right\rceil - 1\right). \tag{35}$$

The theorem then follows by straightforward $\lceil\rceil$-manipulation. □

Remark. Since

$$Sp(2n) \sim O^{-}(2n+2), \tag{36}$$

ignoring the ± 1 eigenvalues, this result also tells us the images of $Sp(2n)$ under power maps.

Corollary 1.6. *Fix an integer $n \geq 1$, and let U be a random element of $O^{\pm}(n)$. For any integer $p \geq n-1$, the eigenvalues other than ± 1 of U^p are i.i.d. uniform conjugate pairs. Similarly, if U is a random element of $Sp(2n)$ and $p \geq 2n+1$, then the eigenvalues of U^p are i.i.d. uniform conjugate pairs.*

Proof. It suffices to observe that the conclusion is true whenever $O^{\pm}(n)^p$ is equivalent to a union of cosets $O^{\pm}(m)$ for $m \leq 2$ and $\mathrm{Re}(U(m))$ for $m \leq 1$. By a case-by-case analysis, this holds whenever $p \geq n-1$. □

Given the appearance of $\mathrm{Re}(U(n))$ above, it is appropriate to mention the following relation:

Theorem 1.7. *For any $n \geq 0$,*

$$\mathrm{Re}(U(n)) \sim O^+(n+1) \oplus O^-(n+1), \tag{37}$$

ignoring eigenvalues ± 1.

Proof. We claim that it suffices to prove

$$\begin{aligned} g(1)g(-1) &\int_{U\in U(n)} \det(g(U))\det(g(\overline{U})) \\ &= \int_{O\in O^+(n+1)} \det(g(O)) \int_{O\in O^-(n+1)} \det(g(O)), \end{aligned} \tag{38}$$

where $g(z)$ is an arbitrary function on the unit circle. Indeed, we may take g to have the form

$$g(z) = \prod_{1\leq i\leq m} (1 - x_i z), \tag{39}$$

at which point comparing coefficients of the x_i tells us that all joint moments of the polynomials $(1-\lambda^2)\det(1-\lambda U)$ and $\det(1-\lambda O_1)\det(1-\lambda O_2)$ agree, where $U \in U(n)$, $O_1 \in O^+(n+1)$, $O_2 \in O^-(n+1)$ are uniform and independent. But this implies the desired result.

To prove (38), one can express the integrals as determinants (see, e.g., Theorems 2.1 and 2.2 of [1]) then use the main lemma of [11]. An alternate proof, given as Corollary 2.4 of [1], involves expressing the integrals in terms of orthogonal polynomials on the unit circle. □

Remark. It is possible to give similar proofs of the other results of this section; in particular, one uses the fact that the orthogonal polynomials with respect to a weight $g(z^p)$ are simply related to the orthogonal polynomials with respect to $g(z)$. However, it is unclear how to apply this approach to nonclassical groups, just as it is unclear how to apply the other approach to prove this result.

2. Congruential subgroups of Weyl groups

For the unitary and orthogonal groups, the key observation was that an appropriate subgroup of the Weyl group turned out to be itself a product of Weyl groups. More precisely, we were given a Weyl group W acting on a real vector space $\mathbb{R}^n$ with a positive semidefinite inner product (that is, W is the reflection group associated to a root system R in that space), a discrete subgroup (lattice) $\Lambda \subset \mathbb{R}^n$ preserved by W, and a vector $v \in \mathbb{R}^n$, and considered the group

$$W^{v+\Lambda} := \{\rho : \rho \in W | \rho(v) - v \in \Lambda\}. \tag{40}$$

For instance, for $O^+(2n+1)$, we had

$$v = (1/2, 3/2, \ldots (n-1)/2), \tag{41}$$
$$\Lambda = p\mathbb{Z}^n. \tag{42}$$

Equivalently, we could divide v and Λ by p, thus making Λ equal to the root lattice of $O^+(2n+1)$. This suggests that we should first study $W^{v+\Lambda}$ when Λ is the root lattice of W (that is, the lattice spanned by the root system of W).

Theorem 2.1. *Let W be a Weyl group acting on $\mathbb{R}^n$, and let Λ_a be the root lattice of W. Then for any vector $v \in \mathbb{R}\Lambda_a$, $W^{v+\Lambda_a}$ is the Weyl group generated by the roots of W it contains.*

Proof. The key observation is that an element $\rho \in W$ is in $W^{v+\Lambda_a}$ if and only if there exists a translation $t_\lambda \in \Lambda_a$ such that

$$t_\lambda^{-1}(\rho(v)) = v. \tag{43}$$

Thus instead of considering the stabilizer in W of the translate $v + \Lambda$, we can consider the stabilizer in $W^+ := W \ltimes \Lambda_a$ of the vector v; we have a canonical isomorphism between $W^{v+\Lambda_a}$ and $(W^+)^v$.

Since Λ_a is the root lattice of W, W^+ is an affine Weyl group. But then we can apply proposition V.3.3.2 of [2] to conclude that $(W^+)^v$ is generated by the reflections of W^+ that fix v. The theorem follows immediately. □

Remark. The reader is cautioned that unless W is simply-laced, this is *not* the usual affine Weyl group associated to W. Indeed, to obtain the usual group, we must replace the root lattice with the lattice spanned by the vectors $2v/\langle v, v\rangle$ for v ranging over the roots of W (i.e., the lattice of vectors $v \in \mathbb{R}\Lambda_a$ such that $\langle v, w\rangle \in \mathbb{Z}$ for all weights w). In particular, the additional simple root of W^+ is the negative of the highest *short* root, not the negative of the highest *long* root as one might expect.

Corollary 2.2. *With W, v, Λ_a and W^+ as above, $W^{v+\Lambda_a}$ is isomorphic to a finite parabolic subgroup of W^+.*

Proof. Indeed, for any choice of chamber of W^+ containing v, the reflections that fix v consist precisely of those boundary hyperplanes containing v, and are thus simple reflections with respect to that chamber. □

It will be helpful to refine the above result somewhat. Fix a fundamental chamber C of W. We then choose the fundamental chamber C^+ of W^+ to be the unique chamber such that $0 \in C^+ \subset C$. For the parabolic subgroups $W^{v+\Lambda_a}$ for $v \in C^+$, we take as fundamental chamber the convex cone generated by $C^+ - v$; note that for $v = 0$, this recovers C. (For general v, we have only a bijection between the chambers of $W^{v+\Lambda_a}$ and those chambers of W^+ containing v) The corresponding simple roots thus form a subset of the simple roots of W^+. We recall the notation of [8]: given an element $\rho \in W^+$, $D\rho$ is the element $t_{-\rho(0)}\rho$ which, since it fixes 0, is in W.

Lemma 2.3. *There exists an element $\rho \in W^+$ such that $\rho(v)$ is in the fundamental chamber of W^+ and $(D\rho)(v)$ is in the fundamental chamber of $W^{\rho(v)+\Lambda_a} \cong W^{v+\Lambda_a}$. The resulting points $\rho(v)$ and $(D\rho)(v)$ are then independent of ρ.*

Proof. Certainly, there exists an element $\rho_0 \in W^+$ such that $\rho_0(v)$ is in the fundamental chamber of W^+; the point $\rho_0(v)$ is then independent of the choice of ρ_0. The remaining freedom in the choice of ρ_0 is simply that we can apply any element of $(W^+)^{\rho_0(v)}$. Since

$$D((W^+)^{\rho_0(v)}) = W^{\rho_0(v)+\Lambda_a} \tag{44}$$

there exists an element ρ_1 of $(W^+)^{\rho_0(v)}$ such that

$$(D\rho_1)(D\rho_0)(v) \tag{45}$$

is in the fundamental chamber of $W^{\rho_0(v)+\Lambda_a}$, and again the resulting point is unique. Taking $\rho = \rho_1\rho_0$, we are done. □

Remark. Note that if the lowest-dimensional facet of C^+ containing $\rho(v)$ also contains 0, then the induced fundamental chamber of $W^{\rho(v)+\Lambda_a}$ is simply the unique chamber containing C. On the other hand, if that facet does not contain 0, then the induced fundamental chamber will in fact be disjoint from C.

Given the point v, we define new points $\overline{v}$ and $\tilde{v}$ by

$$\overline{v} := \rho(v) \text{ and } \tilde{v} := (D\rho)(v), \tag{46}$$

with ρ as above. Note that $\tilde{v} - \overline{v} \in \Lambda_a$.

If Λ is not the root lattice, then $W \ltimes \Lambda$ is in general no longer an affine Weyl group, so the above results do not apply. We can, however, slightly extend the class of lattices we consider. We say that Λ is a "subweight" lattice of W if it contains the roots of W, and satisfies

$$\frac{2\langle \Lambda, r\rangle}{|r|^2} \subset \mathbb{Z} \tag{47}$$

for all roots r; equivalently, W acts trivially on Λ/Λ_a (equation (47) states this for the reflections of W). In the case $\Lambda \subset \mathbb{R}\Lambda_a$, this simply says that Λ contains the root lattice and is contained in the weight lattice of W. We have:

Lemma 2.4. *Let Λ be a subweight lattice of W contained in $\mathbb{R}\Lambda_a$. Then for each coset of $(W \ltimes \Lambda)/W^+$, there exists a unique representative that preserves the fundamental chamber of W^+, giving an injection from Λ/Λ_a into the group of automorphisms of the Dynkin diagram of W^+.*

Proof. Since for $\rho \in W$, $v \in \Lambda$, we have $\rho(v) - v \in \Lambda_a$, it follows that $W \ltimes \Lambda$ normalizes W^+. The lemma then follows from the discussion in Section VI.4.3 of [2]. □

This gives the following result.

Theorem 2.5. *Let Λ be a subweight lattice of W, and define $\Lambda_0 := \Lambda \cap (\mathbb{R}\Lambda_a)$. Then for any vector $v \in \mathbb{R}\Lambda$,*

$$W^{v+\Lambda_a} \trianglelefteq W^{v+\Lambda}; \tag{48}$$

the quotient is isomorphic to the subgroup of Λ_0/Λ_a such that the corresponding transformations of the fundamental chamber preserve $\overline{v}$.

Proof. The main complication is the fact that $\mathbb{R}\Lambda_a$ might not equal $\mathbb{R}\Lambda$. If not, let V be the orthogonal complement of $\mathbb{R}\Lambda_a$ in $\mathbb{R}\Lambda$, and write $v = v_0 + v_1$ with $v_0 \in \mathbb{R}\Lambda_a$ and $v_1 \in V$. Then we can write

$$\rho(v) - v = \rho(v_0) - v_0 \in \mathbb{R}\Lambda_a; \tag{49}$$

W acts trivially on v_1 since v_1 is orthogonal to the roots of W. Thus $\rho(v) - v \in \Lambda$ if and only if $\rho(v) - v \in \Lambda_0$.

Since Λ_0 is itself a subweight lattice, we may therefore assume that $\Lambda \subset \mathbb{R}\Lambda_a$, and thus $\Lambda_0 = \Lambda$. Furthermore, since the desired result is invariant under conjugation by W^+, we may assume that $v = \overline{v}$. Fix a coset of W^+ in $W \ltimes \Lambda$, and let ψ be the representative of that coset that preserves the fundamental chamber. If $\psi(v) = v$, then ψ clearly normalizes $(W^+)^v$, and

$$(W \ltimes \Lambda)^v \cap \psi W^+ = \psi (W^+)^v. \tag{50}$$

On the other hand, if $\psi(v) \neq v$ then

$$(W \times \Lambda)^v \cap \psi W^+ = 0. \tag{51}$$

The result follows. □

Thus given $\overline{v}$, the group $W^{\overline{v}+\Lambda}$ can essentially be read off by inspection.

3. General compact Lie groups

Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$, and choose a maximal torus T with associated Cartan subalgebra $\mathfrak{t}$. Define a lattice Λ^G inside $\mathfrak{t}^*$ to be the dual of the kernel of the exponential map $\exp : \mathfrak{t} \to T$; that is, Λ^G is the set of all $v \in \mathfrak{t}^*$ such that $v(x) = 0$ whenever $\exp(x) = 1$. The irreducible characters of T are thus in one-to-one correspondence with the vectors of Λ^G; in particular, from a basis of Λ^G, we obtain a set of n characters λ_i of T such that for any representation R of G, the eigenvalues of a matrix in $R(T)$ are given by appropriate (monic) Laurent monomials in the λ_i. Since every element of G can be conjugated into T, the characters λ_i can be thought of as "eigenvalue generators". Given $v \in \Lambda^G$, we denote the associated character of T by λ^v.

By proposition 6.6 of [6], $\mathfrak{g}$ is the direct sum of its center $\mathfrak{z}$ and a semisimple Lie subalgebra $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$. Denoting the commutator subgroup of G by G', we thus find that the natural map $Z(G)_0 \times G' \to G$ is surjective, with discrete, central, kernel. The same is then still true if we replace G' by its simply connected covering group $\tilde{H}$. In other words, G is the quotient of a group $G^+ := U(1)^k \times \tilde{H}$ by a discrete subgroup of its center. The corresponding projection $(\Lambda^{G^+})^* = T(G^+) \to T(G) = (\Lambda^G)^*$ thus induces an imbedding of Λ^G in Λ^{G^+}. Now, the roots of G are by definition characters of $T(G)$, so correspond to elements of Λ^G; similarly, the Weyl group W of G acts on $T(G)$, so by duality on Λ^G. Both structures are respected by the projection $G^+ \to G$; since $\Lambda^{G^+} = \mathbb{Z}^k \times \Lambda^{\tilde{H}}$, and $\Lambda^{\tilde{H}}$ is the weight lattice of W, we conclude that Λ^G is a subweight lattice of W.

Now, the density of the eigenvalue distribution (the distribution of the eigenvalue generators) of G (with respect to Haar measure) has the form $f(\lambda)dT$ where

$$f(\lambda) \propto \sum_{\rho' \in W} \rho' \cdot \left(\sum_{\rho \in W} \sigma(\rho) \lambda^{\delta - \rho\delta} \right), \tag{52}$$

where δ is the Weyl vector of W, i.e. half the sum of the positive roots. We thus find that

$$f^{(p)}(\lambda^{1/p}) \propto \sum_{\rho' \in W} \rho' \cdot \left(\sum_{\rho \in W^{\delta/p + \Lambda^G}} \sigma(\rho) \lambda^{(\delta - \rho\delta)/p} \right) \tag{53}$$

$$= \sum_{\rho' \in W} \rho' \cdot \left(\sum_{\rho \in W^{(p)}} \sigma(\rho) \lambda^{\widetilde{\delta/p} - \rho\widetilde{\delta/p}} \right). \tag{54}$$

$$\propto \sum_{\rho' \in W} \rho' \cdot \left(\sum_{\rho_1, \rho_2 \in W^{(p)}} \sigma(\rho_1 \rho_2^{-1}) \lambda^{\rho_1 \widetilde{\delta/p} - \rho_2 \widetilde{\delta/p}} \right), \tag{55}$$

where we define

$$W^{(p)} := W^{\widetilde{\delta/p} + \Lambda^G}. \tag{56}$$

The inner sum thus looks roughly like the eigenvalue density of some different Lie group. For it to actually *be* an eigenvalue density (or rather, for our methods to *prove* it an eigenvalue density[1]), we need three things to happen. First, we need Λ^G to be a subweight lattice of $W^{(p)}$; luckily, this is trivial, since every root of $W^{(p)}$ is a root of W. Second, we need

$$W^{(p)} = W^{\widetilde{\delta/p} + \Lambda_a}, \tag{57}$$

so that the stabilizer is actually a Weyl group. Finally, we need the projection of $\widetilde{\delta/p}$ to the root space of $W^{(p)}$ to be equal to the Weyl vector of $W^{(p)}$.

[1] In particular, we observe that $O^+(2n)^p$ and $Sp(2n)^p$ violate these conditions!

These last two conditions are, unfortunately, not always satisfied, although we can at least readily determine when they are; see for instance Section 5 below. One partial result is:

Theorem 3.1. *The center of G is connected if and only if $\Lambda_0 = \Lambda_a$. Thus if the center of G is connected (in particular if G is adjoint), then*

$$W^{(p)} = W^{\widetilde{\delta/p}+\Lambda_a}, \tag{58}$$

and $W^{(p)}$ is a Weyl group.

Proof. Writing $G = G^+/Z$ as above, we observe that G has connected center if and only if for every element $z \in Z(\tilde{H})$, there exists some element of Z that projects to z. Dualizing, this is precisely the requirement that $\Lambda_0 = \Lambda_a$. □

Thus, at least in the connected center case, the only obstacle is the third condition. That this is, indeed, a problem can be seen from the tables of Section 5; this is the main obstacle to a truly satisfying result in general.

Similar remarks hold if G is not connected. Indeed, much of the structure theory can be extended to this case (see [9] and [3][2]). A (pointed) connected component of a compact Lie group is specified by a quadruple (G_0, a, o, z), where G_0 is a connected compact Lie group and $a \in \mathrm{Aut}(G_0)$, $o \in \mathbb{Z}^+$, $z \in G_0$ are such that a^o is the inner automorphism of conjugation by z, with $z^a = z$; here z^a denotes the image of z under a. This corresponds to a pair $(C, x \in C)$ where C is a component of G, $[\langle C\rangle : G] = o$, x induces a on G_0, and $x^o = z$. Two quadruples specify isomorphic components if they are related by a combination of the following transformations:

$$(G_0, a, o, z) \to (G_0, t^{-1}at, o, z^t), \tag{59}$$

$$(G_0, a, o, z) \to (G_0, ga, o, zgg^a g^{a^2} \cdots g^{a^{o-1}}), \tag{60}$$

where t is an automorphism of G_0 and $g \in G_0$. In particular, we may always arrange for a to have finite order dividing o (making z central). Once we have chosen such an a, we find that the nature and density of the eigenvalue generators for the component are independent of o and z; thus for our purposes we need only consider the (split) case $z = 1$, a of order o, corresponding to a component of a semidirect product. Using the classification of simple Lie groups, one readily obtains the following irreducible local possibilities, each indexed by a positive integer n:

- ${}^nU$: $G_0 = U(1)^{\phi(n)}$, a satisfies the cyclotomic polynomial of order n (acting on the Lie algebra).
- ${}^nH$: $G_0 = H^n$, with H simple; a acts as a cyclic shift.
- ${}^nA_m^{(2)}$: $G_0 = (SU(m+1))^n$; a acts as a cyclic shift, twisted by the outer automorphism of $SU(m+1)$.

[2] The author was unaware of the results in [3] when writing [9], and thus most of the structural results of [9] had already appeared (with different proofs) in [3], with the notable exceptions of the density formula and the independence result (see below).

- ${}^{n}D_m^{(2)}$: $G_0 = (\widetilde{SO}(2m))^n$; a acts as a cyclic shift, twisted by the (classical) outer automorphism of $\widetilde{SO}(2m)$.
- ${}^{n}E_6^{(2)}$: $G_0 = (E_6)^n$; a acts as a cyclic shift, twisted by the outer automorphism of E_6.
- ${}^{n}D_4^{(3)}$: $G_0 = (\widetilde{SO}(8))^n$; a acts as a cyclic shift, twisted by the triality automorphism of $\widetilde{SO}(8)$.

The case ${}^{n}U$ is essentially trivial; either $n = 1$, in which case the (single) eigenvalue generator is uniformly distributed, or $n > 1$, in which case there are no eigenvalue generators (because the eigenvalues are constant over the component). In the other cases, if we let $f_X(\lambda)$ denote the density for ${}^{1}X$, then the density for ${}^{n}X$ is given by $f_X(\lambda^n)$. This implies:

Corollary 3.2. *Let X either denote a simply connected, compact, simple Lie group or one of $A_m^{(2)}$, $D_m^{(2)}$, $E_6^{(2)}$, or $D_4^{(3)}$. Then for all positive integers n and p, the eigenvalue density of $({}^{n}X)^p$ is given by*

$$f(\lambda^{n/\gcd(n,p)}), \tag{61}$$

where f is the density for $({}^{1}X)^{p/\gcd(n,p)}$.

Thus it suffices to consider the five "interesting" cases with $n = 1$. In each case, we find that the density has the expected form corresponding to an appropriate Weyl group (keeping the same Weyl vector), but using a different lattice in place of the root lattice. The effective Weyl group and Λ^G are given as follows:

- $A_{2m-1}^{(2)}$: $W = B_m$. $\Lambda^G = (1/2)D_m$ when G_0 has even fundamental group, and $(1/2)Z^m$ when G_0 has odd fundamental group.
- $A_{2m}^{(2)}$: $W = C_m$. $\Lambda^G = (1/2)Z^m$.
- $D_m^{(2)}$: $W = C_{m-1}$. $\Lambda^G = Z^{m-1}$ for the adjoint and orthogonal groups, and $\Lambda^G = C_{m-1}^{\perp}$ otherwise.
- $E_6^{(2)}$: $W = F_4$. $\Lambda^G = F_4^{\perp}$.
- $D_4^{(3)}$: $W = G_2$. $\Lambda^G = G_2^{\perp}$.

In each case, if we define Λ_a to be the lattice corresponding to the adjoint group, we find that $W \ltimes \Lambda_a$ is an affine Weyl group, and the above results carry over. We observe the following relations in the adjoint case:

$${}^{n}A_{2m}^{(2)} \sim {}^{2n}D_{m+1}^{(2)} \sim {}^{2n}C_m. \tag{62}$$

4. Independence results

In [9], it was shown that for any connected component C of any compact Lie group G, there exists a threshold P such that for any $p > P$ and any representation of G, we have the relation

$$C^p \sim U(1)^r \tag{63}$$

on the eigenvalue generators; that is, the eigenvalue generators of C^p are independent and uniform. Using the above considerations, we can now give an explicit value to the threshold:

Theorem 4.1. *Let G be a connected compact Lie group, let h be the maximum Coxeter number of the simple factors of $W(G)$, and let r be the rank of G. Then*

$$p > h \implies G^p \sim U(1)^r \tag{64}$$

and conversely

$$G^p \sim U(1)^r \implies p \geq h. \tag{65}$$

If the center of G is connected, then $G^h \sim U(1)^r$.

Proof. The key observation is that δ is not stabilized by any element of W. Consequently, the sum

$$\sum_{\rho \in W^{(p)}} \sigma(\rho)\lambda^{\delta-\rho(\delta)} \tag{66}$$

is equal to 1 precisely when the group $W^{(p)}$ is trivial. If the sum is 1, then clearly $G^p \sim U(1)^r$; conversely, if $G^p \sim U(1)^r$, then the sum must equal 1.

It thus remains to consider the group $W^{(p)}$. Aside from diagram automorphisms, $W^{(p)}$ is the product of the groups corresponding to the simple factors of G; we may thus assume that G is simple. In this case, we can explicitly verify that when $p < h$, $W^{(p)}$ is nonempty (it is straightforward for the classical groups, and a short computation for the exceptional groups). For $p \geq h$, we have the following lemma (from Prop. 7.3 of [8]):

Lemma 4.2. *Let W be a simple (finite) Weyl group, and let $W^+ = W \ltimes \Lambda_a$. If h is the Coxeter number and δ the Weyl vector of W, then δ/h is the centroid of the fundamental chamber of W^+, and is thus invariant under all diagram automorphisms of W^+.*

Thus when $p \geq h$, δ/p is strictly in the interior of W^+, so any nontrivial element of $W^{(p)}$ must come from a diagram automorphism; since this is impossible when the center of G is connected, we obtain the desired result for $p = h$. When $p > h$, δ/p treats the highest root of W^+ differently from the other roots, so any diagram automorphism preserving δ/p must preserve the highest root. But this precludes the diagram automorphisms corresponding to the cosets of the root lattice in the weight lattice, giving the desired result. □

A similar result holds in the disconnected case (with an appropriate definition of Coxeter number), with p replaced by $p/\gcd(p,n)$, except for ${}^nA_m^{(2)}$ and ${}^nE_6^{(2)}$, when p must be replaced by $p/\gcd(p,2n)$, and for ${}^nD_4^{(3)}$, when p must be replaced by $p/\gcd(p,3n)$.

Thus for instance, we find that

$$E_8(n)^p \sim U(1)^8 \tag{67}$$

precisely when $p \geq 30$. Similarly,

$$SU(n)^p \sim U(1)^{n-1} \tag{68}$$

precisely when $p > n$, since $SU(n)$ is not adjoint (and we readily verify that $W^{(n)}$ has n elements in this case); this threshold was incorrectly given as $n-1$ in [9].

5. Tables

We conclude the paper by giving a table of $\widetilde{\delta/p}$ and $\overline{\delta/p}$ (for $1 \le p \le h$) for the exceptional groups; from this, it is straightforward to read off the appropriate power-relations.

We again caution the reader that the standard conventions for affine Weyl groups are based on the dual of the weight lattice, not the root lattice as occurs above. In particular, the extended root is the negative of the highest *short* root in the connected group cases.

The table for each group begins with the name of the group and the affine Dynkin diagram; here the node labelled i corresponds to root r_i. Each line then gives $\widetilde{\delta/p}$ and $\overline{\delta/p}$ for p ranging from 1 to the Coxeter number. An entry $\mathbf{k}$ in position i of line p indicates that

$$\frac{2\langle\widetilde{\delta/p}, r_i\rangle}{|r_i|^2} = k, \tag{69}$$

$$\frac{2\langle\overline{\delta/p}, r_i\rangle}{|r_i|^2} = -\delta_{i0}, \tag{70}$$

while an entry $\overline{k}$ indicates that

$$\frac{2\langle\widetilde{\delta/p}, r_i\rangle}{|r_i|^2} = 1/p - k, \tag{71}$$

$$\frac{2\langle\overline{\delta/p}, r_i\rangle}{|r_i|^2} = 1/p - \delta_{i0}. \tag{72}$$

The Dynkin diagram of $W^{(p)}$ is then read off as the subdiagram spanned by the indices without bars, and the projection of $\widetilde{\delta/p}$ to the root space of $W^{(p)}$ is a Weyl vector precisely when all of those indices are 1. Finally, in light of Theorem 2.5, we append an asterisk when $\overline{\delta/p}$ is symmetric under a diagram transformation of the affine Weyl group.

Thus, for instance, in the table for G_2, we have

$$0 - 1 <\equiv 2, \tag{73}$$

indicating that r_1 is the short simple root, r_2 is the long simple root, and $r_0 = -(2r_1 + r_2)$ is the negative of the highest short root. The entry $p = 2 : (\mathbf{2}, \overline{3}, \mathbf{1})$ indicates that

$$\frac{2\langle\widetilde{\delta/2}, r_i\rangle}{|r_i|^2} = (2, 1/2 - 3, 1) \tag{74}$$

$$\frac{2\langle\overline{\delta/2}, r_i\rangle}{|r_i|^2} = (-1, 1/2, 0), \tag{75}$$

and thus

$$\widetilde{\delta/2} = -2r_1 - r_2/2, \quad \overline{\delta/2} = r_1 + r_2/2 \tag{76}$$

(Note that the induced fundamental chamber for $W^{\overline{\delta/2}+\Lambda_a}$ consists of those vectors v such that $\langle r_0, v\rangle, \langle r_2, v\rangle \geq 0$.)

We remark that it is not at all clear why this encoding scheme should happen to work!

- G_2 0−1⇚2

$p = 1 : (\overline{6}\ \mathbf{1}\ \mathbf{1})$
$p = 2 : (\mathbf{2}\ \overline{3}\ \mathbf{1})$
$p = 3 : (\overline{0}\ \overline{2}\ \mathbf{1})$
$p = 4 : (\overline{0}\ \mathbf{1}\ \overline{1})$
$p = 5 : (\mathbf{1}\ \overline{1}\ \overline{0})$
$p = 6 : (\overline{1}\ \overline{0}\ \overline{0})$

- F_4 0−1−2⇐3−4

$p = 1 : (\overline{12}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 2 : (\mathbf{2}\ \overline{6}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 3 : (\mathbf{1}\ \mathbf{2}\ \overline{4}\ \mathbf{1}\ \mathbf{1})$
$p = 4 : (\overline{0}\ \mathbf{1}\ \overline{3}\ \mathbf{1}\ \mathbf{1})$
$p = 5 : (\mathbf{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \overline{1})$
$p = 6 : (\overline{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \overline{0})$
$p = 7 : (\overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1})$
$p = 8 : (\overline{0}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1})$
$p = 9 : (\overline{0}\ \overline{0}\ \mathbf{1}\ \overline{1}\ \overline{0})$
$p = 10 : (\overline{0}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0})$
$p = 11 : (\mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0})$
$p = 12 : (\overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0})$

- E_6

```
          0
          |
          1
          |
2− 3− 6 −5 −4
```

$p = 1 : (\overline{12}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 2 : (\mathbf{2}\ \overline{6}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 3 : (\mathbf{1}\ \mathbf{2}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{4})^*$
$p = 4 : (\overline{0}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{3})$
$p = 5 : (\mathbf{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{2})$
$p = 6 : (\overline{1}\ \mathbf{1}\ \overline{0}\ \mathbf{1}\ \overline{0}\ \mathbf{1}\ \overline{2})^*$
$p = 7 : (\overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1})$
$p = 8 : (\overline{0}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1})$
$p = 9 : (\overline{0}\ \overline{0}\ \overline{0}\ \overline{1}\ \overline{0}\ \overline{1}\ \mathbf{1})^*$
$p = 10 : (\overline{0}\ \mathbf{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{1})$
$p = 11 : (\mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0})$
$p = 12 : (\overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0})^*$

- E_7

```
          4
          |
0− 1− 2− 3 −5 −6 −7
```

$p = 1 : (\mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{18})$
$p = 2 : (\mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{9}\ \mathbf{1}\ \mathbf{1}\ \mathbf{2})^*$
$p = 3 : (\mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{6}\ \mathbf{2}\ \mathbf{1})$
$p = 4 : (\overline{3}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{4}\ \mathbf{1}\ \mathbf{1})$
$p = 5 : (\mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{3}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{2})$
$p = 6 : (\overline{1}\ \mathbf{1}\ \mathbf{1}\ \overline{3}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{0})^*$
$p = 7 : (\mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{2})$
$p = 8 : (\overline{1}\ \mathbf{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{1}\ \overline{0})$
$p = 9 : (\mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{0}\ \overline{0})$
$p = 10 : (\overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \overline{1}\ \mathbf{1}\ \overline{0})^*$
$p = 11 : (\overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \mathbf{1}\ \overline{1}\ \mathbf{1})$
$p = 12 : (\overline{0}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1})$
$p = 13 : (\overline{0}\ \overline{0}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \mathbf{1}\ \overline{1}\ \overline{0})$
$p = 14 : (\overline{0}\ \overline{0}\ \overline{0}\ \mathbf{1}\ \overline{1}\ \overline{1}\ \overline{0}\ \overline{0})^*$
$p = 15 : (\overline{0}\ \overline{0}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0})$
$p = 16 : (\overline{0}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0})$
$p = 17 : (\mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0})$
$p = 18 : (\overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0})^*$

- E_8

```
                    6
                    |
0− 1− 2− 3− 4− 5 −7 −8
```

$p = 1 : (\overline{30}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 2 : (\mathbf{2}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{15})$
$p = 3 : (\mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{10}\ \mathbf{1}\ \mathbf{2})$
$p = 4 : (\mathbf{2}\ \mathbf{1}\ \mathbf{1}\ \overline{7}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 5 : (\mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{2}\ \overline{6}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 6 : (\overline{0}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{5}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 7 : (\mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{4}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 8 : (\mathbf{2}\ \overline{3}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{3}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 9 : (\overline{0}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \overline{3}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 10 : (\overline{0}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \mathbf{1}\ \overline{3}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 11 : (\mathbf{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{1})$
$p = 12 : (\overline{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{0})$
$p = 13 : (\overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1})$
$p = 14 : (\overline{0}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{1})$
$p = 15 : (\overline{0}\ \overline{0}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{0})$
$p = 16 : (\overline{0}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \overline{1}\ \mathbf{1})$
$p = 17 : (\mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \mathbf{1}\ \overline{1})$
$p = 18 : (\overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{0}\ \overline{1}\ \overline{0})$
$p = 19 : (\overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{0}\ \overline{0})$
$p = 20 : (\overline{0}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0})$
$p = 21 : (\overline{0}\ \overline{0}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \mathbf{1}\ \overline{0})$
$p = 22 : (\overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \overline{0}\ \overline{1}\ \mathbf{1})$
$p = 23 : (\overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{1}\ \mathbf{1}\ \mathbf{1}\ \overline{1})$
$p = 24 : (\overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \mathbf{1}\ \overline{1}\ \overline{1}\ \overline{0})$
$p = 25 : (\overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0})$

$p = 26 : (\,\overline{0}\ \overline{0}\ \overline{0}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\,)$
$p = 27 : (\,\overline{0}\ \overline{0}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\,)$
$p = 28 : (\,\overline{0}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\,)$
$p = 29 : (\,\mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\,)$
$p = 30 : (\,\overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\,)$

• $(D_4^{(3)})^3$ 0−1≡>2

$p = 1 : (\,\mathbf{1}\ \mathbf{2}\ \overline{6}\,)$
$p = 2 : (\,\overline{0}\ \mathbf{1}\ \overline{3}\,)$
$p = 3 : (\,\mathbf{1}\ \overline{1}\ \overline{0}\,)$
$p = 4 : (\,\overline{1}\ \overline{0}\ \overline{0}\,)$

• $(E_6^{(2)})^2$ 0−1−2⇒3−4

$p = 1 : (\,\mathbf{2}\ \mathbf{1}\ \mathbf{1}\ \mathbf{2}\ \overline{12}\,)$
$p = 2 : (\,\mathbf{1}\ \mathbf{1}\ \mathbf{2}\ \overline{6}\ \mathbf{2}\,)$
$p = 3 : (\,\overline{0}\ \mathbf{1}\ \mathbf{1}\ \overline{4}\ \mathbf{2}\,)$
$p = 4 : (\,\overline{2}\ \mathbf{1}\ \mathbf{1}\ \overline{2}\ \overline{0}\,)$
$p = 5 : (\,\overline{0}\ \overline{1}\ \mathbf{1}\ \overline{2}\ \mathbf{2}\,)$
$p = 6 : (\,\overline{0}\ \overline{0}\ \mathbf{1}\ \overline{2}\ \overline{0}\,)$
$p = 7 : (\,\overline{0}\ \mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\,)$
$p = 8 : (\,\mathbf{1}\ \overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\,)$
$p = 9 : (\,\overline{1}\ \overline{0}\ \overline{0}\ \overline{0}\ \overline{0}\,)$

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