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Weak interaction limits for one-dimensional random polymers

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Abstract. In this paper we present a new and flexible method to show that, in one dimension, various self-repellent random walks converge to self-repellent Brownian motion in the limit of weak interaction after appropriate space-time scaling. Our method is based on cutting the path into pieces of an appropriately scaled length, controlling the interaction between the different pieces, and applying an invariance principle to the single pieces. In this way, we show that the self-repellent random walk large deviation rate function for the empirical drift of the path converges to the self-repellent Brownian motion large deviation rate function after appropriate scaling with the interaction parameters. The method is considerably simpler than the approach followed in our earlier work, which was based on functional analytic arguments applied to variational representations and only worked in a very limited number of situations.

We consider two examples of a weak interaction limit: (1) vanishing self-repellence, (2) diverging step variance. In example (1), we recover our earlier scaling results for simple random walk with vanishing self-repellence and show how these can be extended to random walk with steps that have zero mean and a finite exponential moment. Moreover, we show that these scaling results are stable against adding self-attraction, provided the self-repellence dominates. In example (2), we prove a conjecture by Aldous for the scaling of self-avoiding walk with diverging step variance. Moreover, we consider self-avoiding walk on a two-dimensional horizontal strip such that the steps in the vertical direction are uniform over the width of the strip and find the scaling as the width tends to infinity.

1. Polymer measures

A linear polymer is a long chain of atoms or molecules, often referred to as monomers, which have a tendency to repel each other. This self-repellence is due to the excluded-volume-effect: two monomers cannot occupy the same space. The self-repellence causes the polymer to spread itself out more than it would do in the

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absence of self-repellence. The most widely used ways to describe a polymer are the *Domb-Joyce model*, respectively, the *Edwards model*, which start from random walk, respectively, Brownian motion and build in an appropriate penalty for self-intersections. In Sections 1.1 and 1.2 we introduce these two models (in dimension one) and list some known results about their space-time scaling. In previous work, the random walk was restricted to be symmetric with finite support, or even to be simple. One of the goals of this paper is to prove results for general random walks, and thereby to prove universality.

In Section 2 we consider a number of variations on the Domb-Joyce model and formulate our main results, which are weak interaction limits showing that all these models converge to the Edwards model, after appropriate space-time scaling, in the limit of vanishing self-repellence or diverging step variance. This convergence may be viewed as a stability property against perturbations in the interaction. In Section 3 we present a brief discussion of the method of proof and of some open ends. Section 4 reviews some large deviation results for the Domb-Joyce model and the Edwards model, while Sections 5–7 contain the proofs of the theorems in Section 2.

A general background on polymers from a physics and chemistry point of view may be found in [vdZ98], a survey of mathematical results for one-dimensional polymers appears in [vdHK01].

1.1. The Domb-Joyce model

Let $(S_n)_{n\in\mathbb{N}_0}$ be a random walk on $\mathbb{Z}$ starting at the origin ($S_0 = 0$). Let P be the law of this random walk and let E be expectation with respect to P. Assume that the random walk is irreducible (i.e., can travel between any pair of points with positive probability), and that

$$E(S_1) = 0, \qquad E(e^{\varepsilon|S_1|}) < \infty \quad \text{for some } \varepsilon > 0. \tag{1.1}$$

Throughout the paper,

$$\sigma^2 = E|S_1|^2 \in (0, \infty) \tag{1.2}$$

denotes the step variance.

Fix $n \in \mathbb{N}$, introduce a parameter $\beta \in [0, \infty]$, and define a probability law Q_n^β on n-step paths by setting

$$\frac{\mathrm{d}Q_n^\beta}{\mathrm{d}P}[\cdot] = \frac{1}{Z_n^\beta} e^{-\beta H_n[\cdot]}, \qquad Z_n^\beta = E(e^{-\beta H_n}), \tag{1.3}$$

with

$$H_n\left[(S_i)_{i=0}^n\right] = \sum_{\substack{i,j=0\\ i\neq j}}^{n} \mathbb{1}_{\{S_i=S_j\}} = \sum_{x\in\mathbb{Z}} \ell_n(x)^2 - (n+1) \tag{1.4}$$

the intersection local time up to time n, where

$$\ell_n(x) = \#\{0 \le i \le n \colon S_i = x\}, \qquad x \in \mathbb{Z}, \tag{1.5}$$

is the local time at site x up to time n. The law Q_n^β is called the *n-polymer measure with strength of self-repellence* β. The path receives a penalty $e^{-2\beta}$ for every self-intersection. The term $n+1$ in (1.4) can be trivially absorbed into the normalization. Note that $\{Q_n^\beta\}_{n\in\mathbb{N}}$ is not a *consistent* family of path measures, meaning that Q_n^β cannot be viewed as the projection of Q_{n+1}^β on the first n steps.

In the case $\beta = \infty$, with the convention $e^{-\infty H_n} = \mathbb{1}_{\{H_n=0\}}$, the path measure Q_n^∞ is the conditional probability law given that there are no self-intersections up to time n, i.e., $Q_n^\infty = P(\cdot \mid H_n = 0)$. If single steps are equally probable under P, then Q_n^∞ is the uniform distribution on all n-step self-avoiding paths having a strictly positive probability under P. The law Q_n^∞ is known as the *self-avoiding walk*, and is trivial for simple random walk but non-trivial when the random walk can make larger steps.[1]

For the special case where

$$S_1 \text{ is symmetric with support } \{-L, \ldots, -1, 1, \ldots, L\} \text{ for some } L \in \mathbb{N}, \tag{1.6}$$

the following is known.

Theorem 1.1 (CLT and partition function). *Fix* $\beta \in [0, \infty]$, *assume* (1.6), *and exclude the trivial case* $(\beta, L) = (\infty, 1)$. *Then there are functions* $r^* = r^*(\beta)$, $\theta^* = \theta^*(\beta)$, $\sigma^* = \sigma^*(\beta)$ *with values in* $(0, \infty)$ *(depending on the distribution of* S_1*) such that:*

(i) *Under the law* Q_n^β, *the distribution of the scaled and normalized endpoint* $(|S_n| - \theta^* n)/\sigma^* \sqrt{n}$ *converges weakly to the standard normal distribution.*
(ii) $\lim_{n\to\infty} \frac{1}{n} \log Z_n^\beta = -r^*$.

Theorem 1.1(i) is contained in [K96, Theorem 1.1], Theorem 1.1(ii) is proved in [K94] for $\beta < \infty$ and in [K93] for $\beta = \infty$. For $L = 1$, the law of large numbers contained in Theorem 1.1(i) first appeared in [GH93]. The proofs in the above papers use the theory of large deviations. In Section 4 we state the large deviation results known for the case (1.6) that are relevant for the present paper.

1.2. The Edwards model

Let $B = (B_t)_{t\ge 0}$ be a standard Brownian motion on $\mathbb{R}$ starting at the origin ($B_0 = 0$). Let $\widehat{P}$ be the Wiener measure and let $\widehat{E}$ be expectation with respect to $\widehat{P}$. For $T > 0$ and $\beta \in [0, \infty)$, define a probability law $\widehat{Q}_T^\beta$ on paths of length T by setting

$$\frac{\mathrm{d}\widehat{Q}_T^\beta}{\mathrm{d}\widehat{P}}[\cdot] = \frac{1}{\widehat{Z}_T^\beta} e^{-\beta \widehat{H}_T[\cdot]}, \qquad \widehat{Z}_T^\beta = \widehat{E}(e^{-\beta \widehat{H}_T}), \tag{1.7}$$

[1] For $\beta \in (0, \infty)$, Q_n^β is often referred to as the *weakly self-avoiding walk*.

with

$$\widehat{H}_T\left[(B_t)_{t\in[0,T]}\right] = \int_0^T \mathrm{d}u \int_0^T \mathrm{d}v\ \delta(B_u - B_v) = \int_{\mathbb{R}} L(T,x)^2\,\mathrm{d}x \tag{1.8}$$

the Brownian self-intersection local time up to time T. The middle expression in (1.8) is formal only. In the last expression the Brownian local times $L(T,x), x \in \mathbb{R}$, appear. The law $\widehat{Q}_T^\beta$ is called the *T-polymer measure with strength of self-repellence* β. The Brownian scaling property implies that

$$\left(L(t,x)\right)_{t\in[0,T],x\in\mathbb{R}} \stackrel{\mathcal{D}}{=} \left(\beta^{-\frac{1}{3}} L(\beta^{\frac{2}{3}}t, \beta^{\frac{1}{3}}x)\right)_{t\in[0,T],x\in\mathbb{R}}, \qquad \beta, T > 0 \tag{1.9}$$

(here $\stackrel{\mathcal{D}}{=}$ means equal in distribution under $\widehat{P}$), and hence that

$$\widehat{Q}_T^\beta\left((B_t)_{t\in[0,T]} \in \cdot\,\right) = \widehat{Q}^1_{\beta^{\frac{2}{3}}T}\left((\beta^{-\frac{1}{3}} B_{\beta^{\frac{2}{3}}t})_{t\in[0,T]} \in \cdot\,\right), \qquad \beta, T > 0. \tag{1.10}$$

Theorem 1.2 (CLT and partition function). *There are numbers a^*, b^*, c^* with values in $(0,\infty)$ such that, for any $\beta \in (0,\infty)$:*

(i) *Under the law $\widehat{Q}_T^\beta$, the distribution of the scaled and normalized endpoint $(|B_T| - b^*\beta^{\frac{1}{3}}T)/c^*\sqrt{T}$ converges weakly to the standard normal distribution.*
(ii) $\lim_{T\to\infty} \frac{1}{T} \log \widehat{Z}_T^\beta = -a^*\beta^{\frac{2}{3}}$.

Theorem 1.2 is proved in [vdHdHK97a]. Rigorous bounds on a^*, b^*, c^* appear in [vdH98, Theorem 3]. The numerical values are: $a^* \approx 2.19$, $b^* \approx 1.11$, $c^* \approx 0.63$. Note that $c^* < 1$ implies that there is a discontinuity of the variance in the CLT at $\beta = 0$, where the variance equals 1. The law of large numbers contained in Theorem 1.2(i) first appeared in [W84].

2. Main results

In this section we formulate and explain our main results, all of which are weak interaction limits for the large space-time scaling of the one-dimensional Domb-Joyce model introduced in Section 1.1 and various related models. In all cases the scaling is the same as that of the Edwards model introduced in Section 1.2, showing that universality holds. Two examples of a weak interaction limit are considered: $\beta \downarrow 0$ and $\sigma \to \infty$.

Section 2.1 considers the Domb-Joyce model, Section 2.2 the Domb-Joyce model with added self-attraction, and Section 2.3 self-avoiding walk on a two-dimensional strip. In Section 2.4 we describe some invariance principles that are needed in the proofs appearing in Sections 5–7. A brief discussion of our results and our method of proof can be found in Section 3, as well as some open ends.

2.1. Two weak interaction limits for self-repellent polymers

Consider an arbitrary random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ satisfying (1.1), respectively, the two technical conditions (2.23)–(2.26) introduced in Section 2.4.

Theorem 2.1 (LLN).

(i) *Fix* $\sigma \in (0, \infty)$. *Then, under* (1.1),

$$\lim_{\beta\downarrow 0} \limsup_{n\to\infty} Q_n^\beta\Big(\Big|\frac{|S_n|}{\beta^{\frac{1}{3}} n} - b^*\sigma^{\frac{2}{3}}\Big| \geq \varepsilon\Big) = 0 \qquad \forall \varepsilon > 0. \tag{2.1}$$

(ii) *Fix* $\beta = \infty$. *Then, under* (2.23)–(2.26),

$$\lim_{\sigma\to\infty} \limsup_{n\to\infty} Q_n^\infty\Big(\Big|\frac{|S_n|}{\sigma^{\frac{2}{3}} n} - b^*\Big| \geq \varepsilon\Big) = 0 \qquad \forall \varepsilon > 0. \tag{2.2}$$

Theorem 2.1 is proved in Sections 5–6. It is to be viewed as an *approximative law of large numbers* for the endpoint S_n of the polymer, since it shows that $|S_n| \approx b^*\sigma^{\frac{2}{3}}\beta^{\frac{1}{3}}n$, respectively, $|S_n| \approx b^*\sigma^{\frac{2}{3}}n$ when first n is taken large, and then β small, respectively, σ large. Note that in Theorem 2.1(i) the asymptotics does not depend on the details of the random walk other than its step variance, which shows that universality indeed holds.

In the special case of (1.6), where the central limit theorem is known (recall Theorem 1.1(i)), we obtain the following two corollaries for the scaling of the parameters $r^*(\beta)$ and $\theta^*(\beta)$ as $\beta \downarrow 0$, respectively, $r^*(\infty)$ and $\theta^*(\infty)$ as $\sigma \to \infty$. Both these corollaries are proved in Sections 5–6.

Corollary 2.2 (Scaling free energy and drift). *Fix* $\sigma \in (0, \infty)$. *Then, under* (1.6),

$$r^*(\beta) \sim a^*\sigma^{-\frac{2}{3}}\beta^{\frac{2}{3}}, \qquad \theta^*(\beta) \sim b^*\sigma^{\frac{2}{3}}\beta^{\frac{1}{3}}, \qquad \beta \downarrow 0. \tag{2.3}$$

For simple random walk (for which $\sigma^2 = 1$), the assertions in Corollary 2.2 were already proved in [vdHdH95, Theorems 4–6]. However, the proof used heavy functional analytic tools and gave no probabilistic insight. For other random walks satisfying (1.6) this route seems inaccessible, so it is nice that here the scaling comes out more generally.

Corollary 2.3 (Scaling free energy and drift). *Fix* $\beta = \infty$. *Then, under* (1.6) *and* (2.23)–(2.26),

$$r^*(\infty) \sim a^*\sigma^{-\frac{2}{3}}, \qquad \theta^*(\infty) \sim b^*\sigma^{\frac{2}{3}}, \qquad \sigma \to \infty. \tag{2.4}$$

The second assertion in Corollary 2.3 settles a conjecture due to Aldous [A86, Section 7(B)], although Aldous misses the factor b^*. The power $\sigma^{\frac{2}{3}}$ can be understood as follows. The steps of S_n are of size σ, so that S_n/σ has steps of variance 1. The effective strength of self-repellence, which comes from the fact that the walk needs to be self-avoiding, is for large σ proportional to $1/\sigma$. Therefore, a comparison with Corollary 2.2 (for $\sigma^2 = 1$) suggests that $S_n/\sigma n$ is proportional to $(1/\sigma)^{\frac{1}{3}}$.

We believe that also

$$\sigma^*(\beta) \to c^*, \quad \beta \downarrow 0, \qquad \text{respectively} \qquad \sigma^*(\infty) \to c^*, \quad \sigma \to \infty, \tag{2.5}$$

but we are unable to prove this. The reason why will become clear in Section 5.2. For nearest-neighbor random walk, the first assertion in (2.5) was proved in [vdHdHK97b].

Our approach is flexible enough to allow for a coupled limit $n \to \infty$ and $\beta \downarrow 0$, respectively, $\sigma \to \infty$.

Theorem 2.4 (Coupled LLN).

(i) *Fix* $\sigma \in (0, \infty)$*, and assume* (1.1)*. If* β *is replaced by* β_n *satisfying* $\beta_n \to 0$ *and* $\beta_n n^{\frac{3}{2}} \to \infty$ *as* $n \to \infty$*, then*

$$\lim_{n\to\infty} Q_n^{\beta_n}\Big(\Big|\frac{|S_n|}{\beta_n^{\frac{1}{3}} n} - b^* \sigma^{\frac{2}{3}}\Big| \geq \varepsilon\Big) = 0 \qquad \forall \varepsilon > 0. \tag{2.6}$$

(ii) *Fix* $\beta = \infty$*, and assume* (1.6) *and* (2.23)–(2.26)*. If* σ *is replaced by* σ_n *satisfying* $\sigma_n \to \infty$ *and* $\sigma_n n^{-\frac{3}{2}} \to 0$ *as* $n \to \infty$*, then*

$$\lim_{n\to\infty} Q_n^{\infty}\Big(\Big|\frac{|S_n|}{\sigma_n^{\frac{2}{3}} n} - b^* \beta^{\frac{1}{3}}\Big| \geq \varepsilon\Big) = 0 \qquad \forall \varepsilon > 0. \tag{2.7}$$

Theorem 2.4 is proved in Section 7.1. For simple random walk (for which $\sigma^2 = 1$), the assertion in Theorem 2.4(i) was already proved in [vdHdHK97b, Theorem 1.5]. The conditions on β_n, respectively, σ_n keep the scaling away from the central limit regime (i.e., $\beta_n^{\frac{1}{3}} n \gg n^{\frac{1}{2}}$, respectively $\sigma_n^{-\frac{1}{3}} n \gg n^{\frac{1}{2}}$).

2.2. *Weak interaction limit for self-repellent and self-attractive polymers*

The method introduced in this paper extends to the situation where self-attraction is added to the polymer. In (1.3), we replace βH_n by

$$\begin{aligned} H_n^{\beta,\gamma} &= \beta \sum_{\substack{i,j=0\\ i\neq j}}^{n} \mathbb{1}_{\{S_i = S_j\}} - \frac{\gamma}{2} \sum_{\substack{i,j=0\\ i\neq j}}^{n} \mathbb{1}_{\{|S_i - S_j| = 1\}} \\ &= (\beta - \gamma) \sum_{x\in\mathbb{Z}} \ell_n^2(x) + \frac{\gamma}{2} \sum_{x\in\mathbb{Z}} [\ell_n(x) - \ell_n(x+1)]^2 - \beta(n+1), \end{aligned} \tag{2.8}$$

where $\beta, \gamma \in (0, \infty)$ are parameters, and $(S_n)_{n\in\mathbb{N}_0}$ is an arbitrary random walk on $\mathbb{Z}$ satisfying (1.1). In words, $H_n^{\beta,\gamma}$ is equal to β times twice the number of self-intersections up to time n minus γ times twice the number of self-contacts up to time n. The law $Q_n^{\beta,\gamma}$ gives a penalty $e^{-2\beta}$ to every pair of monomers at the same site and a reward e^{γ} to every pair of monomers at neighboring sites. The term $\beta(n+1)$ in

(2.8) can again be trivially absorbed into the normalization. The term with γ models the presence of a repellent solution: the polymer tries to minimize the number of contacts with this solution by maximizing the number of self-contacts.

The scaling behavior under $Q_n^{\beta,\gamma}$ was studied (in arbitary dimension) in [vdHK00]. It was shown that there is a phase transition at $\beta = \gamma$, namely, the polymer collapses on a finite (random) number of sites when $\gamma > \beta$, while it visits order n sites when $\gamma < \beta$. Furthermore, in dimension one, a law of large numbers and a central limit theorem for the endpoint S_n under $Q_n^{\beta,\gamma}$, analogous to Theorem 1.1(i), were derived under the restriction $0 < \gamma < \beta - \frac{1}{2}\log 2$.

We want to obtain the analog of Theorem 2.1(i). In Theorem 2.5 below we abbreviate

$$\lim_{\beta,\gamma} \qquad \text{for} \qquad \beta,\gamma \downarrow 0 \text{ such that } 0 < \gamma < \beta \text{ and } \gamma(\beta-\gamma)^{-\frac{2}{3}} \to 0, \tag{2.9}$$

and likewise for lim inf and lim sup.

Theorem 2.5 (LLN). *Fix* $\sigma \in (0,\infty)$. *Then, under* (1.1)*,*

$$\lim_{\beta,\gamma} \limsup_{n\to\infty} Q_n^{\beta,\gamma}\Big(\Big|\frac{|S_n|}{(\beta-\gamma)^{\frac{1}{3}} n} - b^*\sigma^{\frac{2}{3}}\Big| \geq \varepsilon\Big) = 0 \qquad \forall \varepsilon > 0. \tag{2.10}$$

Theorem 2.5 is proved in Section 7.2. Note that no law of large numbers is known for small β, γ. If

$$\theta^*(\beta,\gamma) = \lim_{n\to\infty} E_{Q_n^{\beta,\gamma}}\Big(\frac{|S_n|}{n}\Big) \in (0,\infty) \tag{2.11}$$

would exist for every $\gamma < \beta$, then we could deduce from Theorem 2.5 that $\lim_{\beta,\gamma}(\beta-\gamma)^{-\frac{1}{3}}\theta^*(\beta,\gamma) = b^*\sigma^{\frac{2}{3}}$.

We believe that Theorem 2.5 fails without the restrictions on β, γ in (2.9). There is also a coupled limit version of Theorem 2.5 analogous to Theorem 2.4, but we refrain from writing this down.

2.3. Weak interaction limit for self-avoiding polymers on a two-dimensional strip

Let $(X_n)_{n\in\mathbb{N}_0} = (S_n, U_n^L)_{n\in\mathbb{N}_0}$ be a random walk on the strip $\mathbb{Z} \times \{-L,\dots,L\}$, where $(S_n)_{n\in\mathbb{N}_0}$ is a random walk on $\mathbb{Z}$ satisfying (1.1), and $(U_n^L)_{n\in\mathbb{N}_0}$ is an i.i.d. sequence, independent of $(S_n)_{n\in\mathbb{N}_0}$, such that U_0^L is uniformly distributed on $\{-L,\dots,L\}$. For this two-dimensional random walk, define its self-avoiding version by putting $Q_n^{\infty,L}(\cdot) = P^L(\cdot \mid H_n = 0)$, where P^L is the law of $(X_n)_{n\in\mathbb{N}_0}$ and

$$H_n = \sum_{\substack{i,j=0\\ i\neq j}}^{n} \mathbb{1}_{\{X_i = X_j\}} \tag{2.12}$$

is the intersection local time up to time n.

Theorem 2.6 below identifies the asymptotics of the endpoint of the *first* coordinate, S_n, under the law $Q_n^{\infty,L}$ in the limit as $n\to\infty$ followed by $L\to\infty$, and also when the two limits are coupled.

Theorem 2.6 (LLN and coupled LLN). *Fix $\sigma \in (0,\infty)$ and assume* (1.1).

(i) *Then*

$$\lim_{L\to\infty}\limsup_{n\to\infty} Q_n^{\infty,L}\Big(\Big|\frac{|S_n|}{(4L)^{-\frac{1}{3}}n} - b^*\sigma^{\frac{2}{3}}\Big| \geq \varepsilon\Big) = 0 \qquad \forall \varepsilon > 0. \tag{2.13}$$

(ii) *If L is replaced by L_n satisfying $L_n \to \infty$ and $L_n n^{-\frac{3}{2}} \to 0$ as $n \to \infty$, then*

$$\lim_{n\to\infty} Q_n^{\infty,L_n}\Big(\Big|\frac{|S_n|}{(4L_n)^{-\frac{1}{3}}n} - b^*\sigma^{\frac{2}{3}}\Big| \geq \varepsilon\Big) = 0 \qquad \forall \varepsilon > 0. \tag{2.14}$$

Theorem 2.6 is proved in Section 7.3. In [AJ90], it is shown that

$$\theta^*(L) = \lim_{n\to\infty} E_{Q_n^{\infty,L}}\Big(\frac{|S_n|}{n}\Big) \in (0,\infty) \tag{2.15}$$

exists for fixed L. Therefore we deduce from Theorem 2.6(i) that $\lim_{L\to\infty}(4L)^{\frac{1}{3}}\theta^*(L) = b^*\sigma^{\frac{2}{3}}$.

We close this section by making a comparison with self-avoiding walk on $\mathbb{Z}^2$. One of the prominent open problems for this process is the asymptotic analysis of its endpoint. The conjecture is that the endpoint runs on scale $n^{\frac{3}{4}}$. Now, interestingly, in Theorem 2.6(ii) it is precisely the choice $L_n = n^{\frac{3}{4}}$ that makes the two coordinates S_n and $U_n^{L_n}$ run on the *same* scale $n^{\frac{3}{4}}$. This suggests that for $L_n = n^{\frac{3}{4}}$ the behavior on the strip is a crude but reasonable approximation to the behavior on $\mathbb{Z}^2$.

Let us try to make this argument a bit more precise by appealing to an adaptation of the well-known Flory argument (see [MS93, Section 2.2]). Let $S = (S_n)_{n\in\mathbb{N}_0} = (S_n^{(1)}, S_n^{(2)})_{n\in\mathbb{N}_0}$ be two-dimensional simple random walk. We may assume that $S^{(1)} = (S_n^{(1)})_{n\in\mathbb{N}_0}$ and $S^{(2)} = (S_n^{(2)})_{n\in\mathbb{N}_0}$ are two independent one-dimensional simple random walks.[2] We want to investigate the quantity

$$\begin{aligned} Z_n^{\infty}(\nu) &= P\Big(\bigcap_{\substack{i,j=0\\ i\neq j}}^{n}\{S_i \neq S_j\} \cap \{|S_n| \asymp n^{\nu}\}\Big) \\ &= E^{(1)}\Big(\mathbb{1}\{|S_n^{(1)}| \asymp n^{\nu}\} P\Big(\bigcap_{\substack{i,j=0\\ i\neq j}}^{n}\{S_i \neq S_j\} \cap \{|S_n^{(2)}| \asymp n^{\nu}\}\Big| S^{(1)}\Big)\Big), \end{aligned} \tag{2.16}$$

where P is the law of S, $E^{(1)}$ is expectation with respect to $S^{(1)}$, and $\nu > 0$ is an exponent to be determined later. Denote the local times of $S^{(1)}$ by $\ell_n^{(1)}(x), x \in \mathbb{Z}$. Note that $S^{(1)}$ has $\ell_n^{(1)}(x)[\ell_n^{(1)}(x)-1]$ self-intersections at $x \in \mathbb{Z}$. In order that S has no self-intersections, $S^{(2)}$ must avoid a self-intersection at the $\sum_{x\in\mathbb{Z}} \ell_n^{(1)}(x)[\ell_n^{(1)}(x)-1]$ time pairs at which $S^{(1)}$ has self-intersections. Now, let us make the *crude approximation*

[2] Indeed, the projections of $S^{(1)}$ and $S^{(2)}$ onto the lines with slope 1 and -1 in $\mathbb{R}^2$, respectively, are two independent copies of one-dimensional simple random walk on $\sqrt{2}\,\mathbb{Z}$.

that $S_i^{(2)}, i = 0, \dots, n$, are i.i.d. uniformly distributed on $\{-|S_n^{(2)}|, \dots, |S_n^{(2)}|\}$. Then, on the event $\{|S_n^{(2)}| \asymp n^\nu\}$, the probability that a self-intersection of $S^{(2)}$ occurs at a given time pair $i \neq j$ at which $S_i^{(1)} = S_j^{(1)}$ is $\asymp n^{-\nu}$. (The idea behind the approximation is that for large n most self-intersections occur when $|i - j|$ is large.) The resulting model is precisely the one investigated in Theorem 2.6(ii) with $L_n \asymp n^\nu$. For this choice, (2.14) yields that $\{S_n^{(2)} \asymp n^{1-\frac{\nu}{3}}\}$ is typical. Putting $\nu = 1 - \frac{\nu}{3}$, we find $\nu = \frac{3}{4}$.

2.4. *Invariance principles and assumptions on variance scaling*

The proofs of our weak interaction limits in Sections 2.1–2.3 will be based on a number of invariance principles, which we describe now. Let $(B_t^\sigma)_{t\geq 0}$ be a Brownian motion with generator $\frac{1}{2}\sigma^2\Delta$, and write $\widehat{H}_T^\sigma$ for its intersection local time and $L^\sigma(T, x)$, $x \in \mathbb{R}$, for its local times up to time T.

1. The first invariance principle we will rely on was put forward in [BS95, Theorem 1.3]:[3]

$$\left(n^{-\frac{1}{2}}(S_{\lfloor nt\rfloor})_{t\in[0,T]}, n^{-\frac{3}{2}} H_{\lfloor nT\rfloor}\right) \overset{n\to\infty}{\Longrightarrow} \left((B_t^\sigma)_{t\in[0,T]}, \widehat{H}_T^\sigma\right), \qquad \sigma, T > 0. \quad (2.17)$$

This says that the Domb-Joyce model (for the random walk with variance σ^2) at time nT with strength of self-repellence $\beta n^{-\frac{3}{2}}$ converges, after appropriate space-time scaling, to the Edwards model (for the Brownian motion with generator $\frac{1}{2}\sigma^2\Delta$) at time T with strength of self-repellence β. Another version of the same invariance principle is the assertion

$$\left(\beta^{\frac{1}{3}}(S_{\lfloor \beta^{-\frac{2}{3}} t\rfloor})_{t\in[0,T]}, \beta H_{\lfloor \beta^{-\frac{2}{3}} T\rfloor}\right) \overset{\beta\downarrow 0}{\Longrightarrow} \left((B_t^\sigma)_{t\in[0,T]}, \widehat{H}_T^\sigma\right), \qquad \sigma, T > 0. \quad (2.18)$$

As was shown in [CR83], under weaker conditions than (1.1), the discrete local times process converges weakly to the continuous local times process:

$$\left(\beta^{\frac{1}{3}}\ell_{\lfloor \beta^{-\frac{2}{3}} T\rfloor}(\lfloor x\beta^{-\frac{1}{3}}\rfloor)\right)_{x\in\mathbb{R}} \overset{\beta\downarrow 0}{\Longrightarrow} \left(L^\sigma(T, x)\right)_{x\in\mathbb{R}}, \qquad \sigma, T > 0. \quad (2.19)$$

This explains the scaling of the second component in (2.17)–(2.18). Since $(B_t^\sigma)_{t\geq 0} \overset{\mathcal{D}}{=} (\sigma B_t)_{t\geq 0}$, we have that

$$\left(L^\sigma(T, x)\right)_{x\in\mathbb{R}} \overset{\mathcal{D}}{=} \left(\tfrac{1}{\sigma} L\big(T, \tfrac{x}{\sigma}\big)\right)_{x\in\mathbb{R}}, \qquad \widehat{H}_T^\sigma \overset{\mathcal{D}}{=} \tfrac{1}{\sigma}\widehat{H}_T, \qquad \sigma, T > 0. \quad (2.20)$$

[3] In fact, [BS95, Theorem 1.3] applies only to simple random walk, but an inspection of its proof reveals that it holds in the generality of our setting. Alternatively, one can use the weak convergence of the local times in (2.19) to obtain the weak convergence of $n^{-\frac{3}{2}} H_{\lfloor nT\rfloor}$ to $\widehat{H}_T^\sigma$.

2. The second invariance principle we will rely on was shown in [A86, Theorem 1.8], and states that

$$\Big(\sigma^{-\frac{4}{3}}(S_{\lfloor \sigma^{\frac{2}{3}} t\rfloor})_{t\in[0,T]}, \mathbb{1}_{\{H_{\lfloor \sigma^{\frac{2}{3}} T\rfloor}=0\}}\Big) \overset{\sigma\to\infty}{\Longrightarrow} \big((B_t)_{t\in[0,T]}, \mathbb{1}_{\{U>T\}}\big), \qquad T>0, \tag{2.21}$$

where the law of the random variable U is given by its conditional distribution given the underlying Brownian motion as

$$\widehat{P}\big(U>T\big|(B_t)_{t\in[0,T]}\big) = e^{-\widehat{H}_T}, \tag{2.22}$$

and the limit $\sigma\to\infty$ is to be taken subject to the following three technical restrictions:

$$\begin{aligned} &\text{(a)}\ \lim_{N\to\infty}\limsup_{\sigma\to\infty} E\big((S_1/\sigma)^2\mathbb{1}_{\{|S_1/\sigma|>N\}}\big)=0;\\ &\text{(b)}\ \lim_{\sigma\to\infty}\sigma^{\frac{2}{3}}\max_{x\in\mathbb{Z}} P(S_1=x)=0;\\ &\text{(c)}\ \min_{\sigma\geq 1}\min_{0<|x|\leq c_1\sigma}\sigma P(S_1=x)\geq c_2 \text{ for some } c_1,c_2>0. \end{aligned} \tag{2.23}$$

We continue by giving the heuristic explanation of the invariance principle in (2.21) provided by Aldous in [A86]. The convergence of the first coordinate comes from the fact that the random variable $S_{\lfloor \sigma^{\frac{2}{3}} t\rfloor}/\sigma$ is the sum of $\lfloor \sigma^{\frac{2}{3}} t\rfloor$ steps of variance 1. Therefore, after $\lfloor \sigma^{\frac{2}{3}} t\rfloor$ steps the displacement is close to $\lfloor \sigma^{\frac{2}{3}} t\rfloor^{\frac{1}{2}}$ times a standard normal distribution. The technical restrictions in (2.23) justify the extension of this argument to the process level.

To understand the weak convergence of the second coordinate, we estimate the expected number of self-intersections. First, note that since $P(S_k=0)\sim Ck^{-\frac{1}{2}}\sigma^{-1}$ as $k\to\infty$, we have from (1.4) that

$$E(H_n)=\sum_{k=1}^{n}(n+1-k)P(S_k=0)\sim C\sigma^{-1}n^{\frac{3}{2}} \qquad \text{as } n\to\infty, \tag{2.24}$$

which suggests that n must be of order $\sigma^{\frac{2}{3}}$ to have H_n exceed 1. Thus, the first self-intersection typically occurs after $\sigma^{\frac{2}{3}}$ steps. Furthermore, conditioned on $S=(S_0,\dots,S_{\lfloor \sigma^{\frac{2}{3}} t\rfloor})$, the probability of a self-intersection in a time interval of length $\sigma^{\frac{2}{3}}\delta$ right after time $\lfloor \sigma^{\frac{2}{3}} t\rfloor$ is the length of the time interval (i.e., $\sigma^{\frac{2}{3}}\delta$) times the density of S around its endpoint $S_{\lfloor \sigma^{\frac{2}{3}} t\rfloor}$. Under the assumption that the local times of S scale to the local times of Brownian motion, the latter density converges to $\sigma^{-\frac{2}{3}}L(t,B_t)$. Hence, the probability of a self-intersection in a time interval of length $\sigma^{\frac{2}{3}}\delta$ right after time $\lfloor \sigma^{\frac{2}{3}} t\rfloor$ is about $\delta L(t,B_t)$. Thus, the heuristic suggests that the probability of not having a self-intersection up to time T conditionally on the limiting Brownian motion $(B_s)_{s\in[0,T]}$ equals

$$\lim_{\delta\downarrow 0}\prod_{i=1}^{T/\delta}(1-\delta L(i\delta, B_{i\delta})) = e^{-\int_0^T L(s,B_s)\,\mathrm{d}s} = e^{-\widehat{H}_T} \tag{2.25}$$

(recall (1.8)). This explains (2.21).

The analog of (2.19) for $\sigma \to \infty$ under (2.23) is not known. Therefore, on top of (2.23), we will require a uniform exponential moment for S_1/σ, i.e.,

$$\sup_{\sigma\geq 1} E(e^{\varepsilon|S_1|/\sigma}) < \infty \quad \text{for some } \varepsilon > 0, \tag{2.26}$$

which is obviously stronger than (2.23)(a) and replaces the second condition in (1.1). Note that the random walk with $P(S_1 = x) = \frac{1}{2L}$ for $x \in \{-L, \dots, -1, 1, \dots, L\}$ satisfies (2.23)–(2.26) (for which $\sigma^2 \sim L^2/3$), and so does the random walk with $P(S_1 = x) = \frac{1}{2L}(\frac{L-1}{L})^{|x|-1}$ for $x \in \mathbb{Z} \setminus \{0\}$ (for which $\sigma^2 \sim L^2$).

3. Discussion

The weak interaction limit results in Section 2.1–2.3 will be proved in Sections 5–7 with the help of a new and flexible method. The idea is to cut the path into pieces of an appropriately scaled length, to control the interaction between the different pieces, and to apply the respective invariance principle to the single pieces. This method will allow us to prove scaling of the large deviation rate function for the empirical drift of the path, which in turn will imply the weak interaction limit results in Section 2.1–2.3.

The proof of Theorem 2.1(i) in Section 5 will be the guideline for the proofs of our other main results, and we will frequently refer back to it.

Large deviation arguments play an important role in the proof. In Section 4, we state some known results concerning the large deviation properties of the Domb-Joyce and the Edwards model that will be needed along the way. The scaling of the various large deviation rate functions formulated in Sections 5–7 is much stronger than the scaling results formulated in Sections 2.1–2.3. For one, they show that the probabilities under the respective polymer measures of the complements of the events in Theorems 2.1–2.6 are in fact *exponentially* small. Each of the proofs in Sections 5–7 starts with a proposition giving the precise large deviation statement we are after. Even though these statements are interesting in their own right, we have chosen not to present them in Sections 2.1–2.3, since they are technically more involved and only partially complete.

Our method has a number of advantages over the approach that was followed in our earlier work, which relied on a variational representation for the quantities in the central limit theorem and a functional analytic proof that this variational representation scales to a limit. Our new method is simple, works for a very large class of random walks in a variety of self-repelling and self-attracting situations, and allows for a coupled limit in which $n \to \infty$ and $\beta \downarrow 0$, respectively, $\sigma \to \infty$ together. We expect that it can be applied to other polymer models as well, such as branched polymers and heteropolymers, which we hope to investigate in the future.

The results in Section 2.1–2.3 show universality, in the sense that the scaling limits do not depend on the details of the underlying random walk other than its step variance and are all given in terms of the Edwards model. In order for the large deviation results to be true, we need the second condition in (1.1). We believe that the results in Section 2.1–2.3 remain true under weaker conditions, such as a finite third moment, but the large deviation results in Sections 5–7 will certainly fail without the second condition in (1.1).

Two items remain open. First, we cannot prove the scaling of the variance in the central limit theorem (recall (2.5)). This would require control of the second derivative of the rate function in its minimum (compare Theorem 4.1(iii) with Theorem 4.2(iii)). We only have good control over the first derivative of the rate function. The LDP does not imply the CLT, so even if we had obtained the scaling of the variance from the scaling of the second derivative, we would not be able to deduce the CLT anyway. Second, we cannot prove the scaling of the rate function in the linear regime (see Figs. 1 and 2 in Sections 4.1–4.2 and also the remark at the end of Sections 5.1 and 6.1). In this linear regime, we only derive the upper bound in the weak interaction limit. We have no doubt that the lower bound can be derived too, but this would require some further refinements. In particular, in the linear regime the path makes an overshoot, and we would need to control the interaction between overlapping pieces in this overshoot.

4. Large deviations

As already alluded to in Section 3, to prove the results in Sections 2.1–2.3, we will actually prove something much stronger, namely, *scaling of the large deviation rate function for the empirical drift of the path*. We will show that the rate function for the Domb-Joyce model and its variants scales to the rate function for the Edwards model. Now, the existence of the rate function for the Domb-Joyce model has been established only in a rather limited number of cases, namely, under the assumption in (1.6). In Section 4.1 we summarize what is known for this special case. For the variants of the Domb-Joyce model the existence is still open. Therefore we will have to work with liminf's and limsup's. The existence of the rate function for the Edwards model has been proved in our recent paper [vdHdHK02] and its properties will be described in Section 4.2. Another important object is the cumulant generating function for the Edwards model, which will be introduced in Section 4.3. More refined large deviation properties for the Edwards model also proved in [vdHdHK02], which will be needed in our proofs, are presented in Section 4.4.

4.1. Large deviations for the Domb-Joyce model

Throughout this section we assume (1.6). The main object of interest in this section is the rate function J_β defined by

$$-J_\beta(\theta) = \lim_{n\to\infty} \frac{1}{n} \log Q_n^\beta(S_n \approx \theta n), \qquad \theta \in \mathbb{R}, \tag{4.1}$$

where $S_n \approx \theta n$ means that either $S_n = \lfloor \theta n \rfloor$ or $S_n = \lfloor \theta n \rfloor + 1$ (possibly depending on the parity of these numbers). Actually, we prefer to work with the function I_β

defined by

$$-I_\beta(\theta) = \lim_{n\to\infty} \frac{1}{n} \log E\big(e^{-\beta H_n} \mathbb{1}_{\{S_n \approx \theta n\}}\big)$$
$$= \lim_{n\to\infty} \frac{1}{n} \log\big\{Z_n^\beta Q_n^\beta(S_n \approx \theta n)\big\}, \qquad \theta \in \mathbb{R}, \tag{4.2}$$

which according to Theorem 1.1(ii) differs from J_β by a constant, namely, $I_\beta = J_\beta + r^*$. For $\beta = \infty$ we adopt the convention $e^{-\infty H_n} = \mathbb{1}_{\{H_n=0\}}$. Obviously, $I_\beta(\theta) = I_\beta(-\theta)$, and $I_\beta(\theta) = \infty$ when $\theta > L$. Therefore we may restrict ourselves to $\theta \in [0, L]$.

Recall the three quantities r^*, θ^*, σ^* in Theorem 1.1. In the next theorem a fourth quantity θ^{**} appears, which, like the others, depends on β and on the distribution of S_1.

Theorem 4.1 (LDP). *Fix* $\beta \in [0, \infty]$*, assume* (1.6)*, and exclude the trivial case* $(\beta, L) = (\infty, 1)$.

(i) *For any* $\theta \in [0, L]$*, the limit* $I_\beta(\theta)$ *in* (4.2) *exists and is finite.*
(ii) I_β *is continuous and convex on* $[0, L]$*, and continuously differentiable on* $(0, L)$.
(iii) *There is a function* $\theta^{**} = \theta^{**}(\beta)$ *with values in* $(0, \theta^*)$ *such that* I_β *is linearly decreasing on* $[0, \theta^{**}]$*, real-analytic and strictly convex on* (θ^{**}, L)*, and attains its unique minimum at* θ^* *with height* $I_\beta(\theta^*) = r^*$ *and curvature* $I''_\beta(\theta^*) = 1/\sigma^{*2}$.

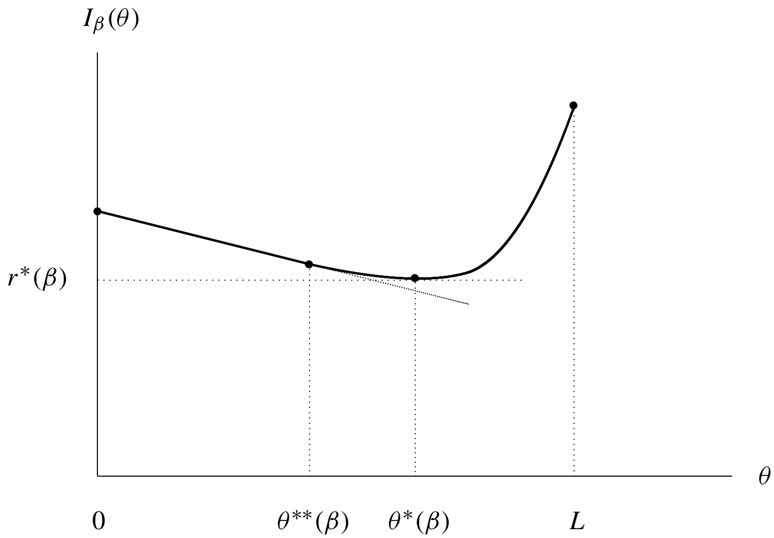

Fig. 1. Qualitative picture of $\theta \mapsto I_\beta(\theta)$ for $\theta \in [0, L]$.

Theorem 4.1 is proved for simple random walk ($L = 1$) in [dH00, Theorem IX.32], relying on the methods and results of [GH93]. We have checked that this proof can be extended to general $L \in \mathbb{N}$ with the help of the methods and results of [K94].

The main ingredients of the proof of Theorem 4.1 are reflection arguments and precise analytic knowledge of the contribution to the intersection local time coming from paths that satisfy the so-called "bridge condition", i.e., lie between their starting and ending locations S_0 and S_n. The linear piece of the rate function has the following intuitive explanation. If $\theta \geq \theta^{**}$, then the optimal strategy for the path to realize $S_n \approx \theta n$ is to assume local drift θ during n steps. In particular, the path then satisfies the bridge condition, and apparently this leads to the strict convexity and real-analyticity of the rate function on (θ^{**}, L). If, on the other hand, $0 \leq \theta < \theta^{**}$, then this strategy is too expensive, since too small a drift leads to too many self-intersections. Therefore the optimal strategy now is to move with local drift θ^{**} during $\frac{\theta^{**}+\theta}{2\theta^{**}}n$ steps and with local drift $-\theta^{**}$ during the remaining $\frac{\theta^{**}-\theta}{2\theta^{**}}n$ steps, thus making an overshoot of size $\frac{\theta^{**}-\theta}{2}n$, and this leads to the linearity of the rate function on $[0, \theta^{**}]$.

4.2. Large deviations for the Edwards model

The analogue of (4.1) for the Edwards model is the rate function $\widehat{J}_\beta$ defined by

$$-\widehat{J}_\beta(\theta) = \lim_{T\to\infty} \frac{1}{T} \log \widehat{Q}_T^\beta\left(B_T \approx bT\right), \qquad b \in \mathbb{R}, \tag{4.3}$$

where $B_T \approx bT$ means that

$$\begin{gathered} |B_T - bT| \leq \gamma_T \text{ for some } \gamma_T > 0 \text{ such that } \gamma_T/T \to 0 \\ \text{and } \gamma_T/\sqrt{T} \to \infty \text{ as } T \to \infty. \end{gathered} \tag{4.4}$$

Again, we prefer to work with the function $\widehat{I}_\beta$ defined by

$$\begin{aligned} -\widehat{I}_\beta(b) &= \lim_{T\to\infty} \frac{1}{T} \log \widehat{E}\left(e^{-\beta \widehat{H}_T} \mathbb{1}_{\{B_T \approx bT\}}\right) \\ &= \lim_{T\to\infty} \frac{1}{T} \log \left\{\widehat{Z}_T^\beta \widehat{Q}_T^\beta\left(B_T \approx bT\right)\right\}, \qquad b \in \mathbb{R}, \end{aligned} \tag{4.5}$$

which according to Theorem 1.2(ii) differs from $\widehat{J}_\beta$ by a constant, namely, $\widehat{I}_\beta(\theta) = \widehat{J}_\beta(\theta) + a^*\beta^{\frac{2}{3}}$. In [vdHdHK02] we proved that the limit in (4.5) exists and is independent of the choice of γ_T. From (1.10) it is clear that $\widehat{I}_\beta$ satisfies the scaling relation

$$\beta^{-\frac{2}{3}} \widehat{I}_\beta(\beta^{\frac{1}{3}} \cdot) = \widehat{I}_1(\cdot), \tag{4.6}$$

provided the limit in (4.5) exists for $\beta = 1$.

Recall the three quantities a^*, b^*, c^* in Theorem 1.2. In the next theorem a fourth quantity b^{**} appears.

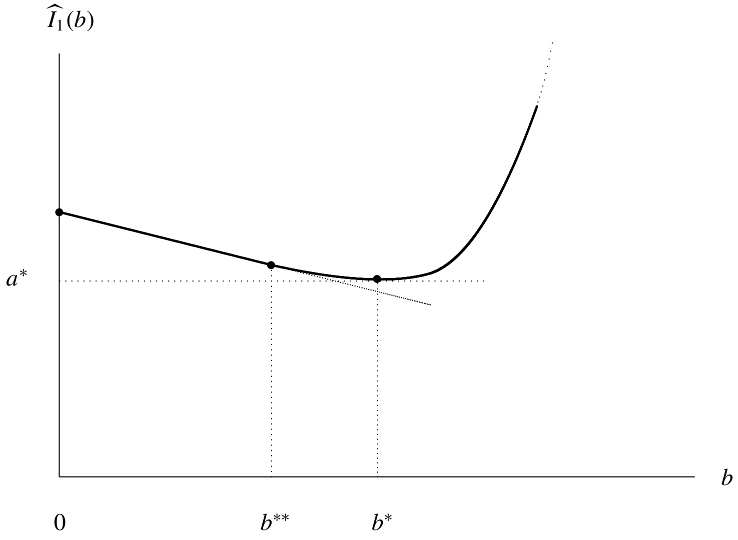

Fig. 2. Qualitative picture of $b \mapsto \widehat{I}_1(b)$.

Theorem 4.2 (LDP).

(i) *For any* $b \in [0, \infty)$*, the limit* $\widehat{I}_1(b)$ *in* (4.5) *exists and is finite (and is independent of the choice of* γ_T*).*

(ii) $\widehat{I}_1$ *is continuous and convex on* $[0, \infty)$*, and continuously differentiable on* $(0, \infty)$.

(iii) *There is a number* $b^{**} \in (0, b^*)$ *such that* $\widehat{I}_1$ *is linearly decreasing on* $[0, b^{**}]$*, real-analytic and strictly convex on* (b^{**}, ∞)*, and attains its unique minimum at* b^* *with height* $\widehat{I}_1(b^*) = a^*$ *and curvature* $\widehat{I}_1''(b^*) = 1/c^{*2}$.

Theorem 4.2 is proved in [vdHdHK02]. The numerical value of b^{**} is $b^{**} \approx 0.85$. Note the close analogy with Theorem 4.1. The linear piece has the same intuitive explanation in terms of overshoot and the bridge condition as given below Theorem 4.1.

Denote by $\widehat{I}^\sigma_\beta$ the rate function in (4.5) for the Brownian motion with generator $\frac{1}{2}\sigma^2\Delta$. Like $\widehat{I}_\beta$, it satisfies the scaling relation $\beta^{-\frac{2}{3}}\widehat{I}^\sigma_\beta(\beta^{\frac{1}{3}}\cdot) = \widehat{I}^\sigma_1(\cdot)$ in (4.6). Furthermore, from (2.20) we obtain the scaling relation

$$\widehat{I}^\sigma_1(\cdot) = \sigma^{-\frac{2}{3}}\widehat{I}_1(\sigma^{-\frac{2}{3}}\cdot). \tag{4.7}$$

4.3. Cumulant generating function for the Edwards model

There is an intimate connection between the rate function in (4.5) and the cumulant generating function $\Lambda^+\colon \mathbb{R} \to \mathbb{R}$ given by

$$\Lambda^+(\mu) = \lim_{T\to\infty} \frac{1}{T} \log \widehat{E}\big(e^{-\widehat{H}_T} e^{\mu B_T} 1\!\!1_{\{B_T \geq 0\}}\big), \qquad \mu \in \mathbb{R}. \tag{4.8}$$

Proposition 4.3 (Exponential moments).

(i) *For any* $\mu \in \mathbb{R}$*, the limit* $\Lambda^+(\mu)$ *in* (4.8) *exists and is finite.*
(ii) *There is a number* $\rho(a^{**}) > 0$ *such that* Λ^+ *is constant on* $(-\infty, -\rho(a^{**})]$*, and strictly increasing, strictly convex and real-analytic on* $(-\rho(a^{**}), \infty)$*. In* $-\rho(a^{**})$*,* Λ^+ *is continuous, but not differentiable.*
(iii) $\lim_{\mu\downarrow-\rho(a^{**})}(\Lambda^+)'(\mu) = b^{**}$, $(\Lambda^+)'(0) = b^*$, *and* $\lim_{\mu\to\infty}(\Lambda^+)'(\mu) = \infty$.
(iv) *The restriction of* $\widehat{I}_1$ *to* $[0, \infty)$ *is the Legendre transform of* Λ^+*, i.e.,*

$$\widehat{I}_1(b) = \max_{\mu\in\mathbb{R}}\big[\mu b - \Lambda^+(\mu)\big], \qquad b \geq 0. \tag{4.9}$$

Proposition 4.3 is proved in [vdHdHK02]. As is shown there, the statements in Proposition 4.3(ii-iii) are equivalent to the statement that there is a linear piece for the rate function in Theorem 4.2. The fact that the rate function is the Legendre transform of the cumulant generating function in Proposition 4.3(iv) is a common property in large deviation theory. In our case, since I_β is convex on $\mathbb{R}^+$ only, this property is restricted to $b \geq 0$. This explains that (4.9) is true only with an indicator on $B_T \geq 0$ in the definition of Λ^+ in (4.8). The numerical value of $\rho(a^{**})$ is $\rho(a^{**}) \approx 0.78$. By (4.9), $-\rho(a^{**})$ is the slope of the linear piece in Fig. 2. Note that $\Lambda^+(0) = -a^*$ by Theorem 1.2(ii) and (4.6).

As a consequence of Proposition 4.3(ii), the maximum on the right-hand side of (4.9) is attained in some $\mu > -\rho(a^{**})$ if $b > b^{**}$ and in $\mu = -\rho(a^{**})$ if $0 \leq b \leq b^{**}$.

Let Λ^- denote the cumulant generating function with $1\!\!1_{\{B_T\leq 0\}}$ instead of $1\!\!1_{\{B_T\geq 0\}}$. Then analogous assertions for Λ^- hold as well. In particular, the restriction of $\widehat{I}_1$ to $(-\infty, 0]$ is the Legendre transform of Λ^-. By symmetry, $\Lambda^+(-\mu) = \Lambda^-(\mu)$ for any $\mu \in \mathbb{R}$. Consequently, the cumulant generating function

$$\Lambda(\mu) = \lim_{T\to\infty}\frac{1}{T}\log\widehat{E}\big(e^{-\widehat{H}_T}e^{\mu B_T}\big) = \Lambda^+(\mu) \vee \Lambda^-(\mu) = \Lambda^+(|\mu|) \tag{4.10}$$

exists for any $\mu \in \mathbb{R}$ and is not differentiable at 0.

Let Λ^+_σ and Λ^-_σ denote the corresponding cumulant generating functions for the Edwards model with variance σ^2 (i.e., where the generator of the underlying Brownian motion is $\frac{1}{2}\sigma^2\Delta$). Then we have the scaling relation $\sigma^{\frac{2}{3}}\Lambda^+_\sigma(\sigma^{-\frac{4}{3}}\,\cdot) = \Lambda^+(\cdot)$. Moreover, we have

$$\widehat{I}_1^\sigma(b) = \max_{\mu\in\mathbb{R}}\big[\mu b - \Lambda^+_\sigma(\mu)\big] = \begin{cases} \max\limits_{\mu\geq 0}\big[\mu b - \Lambda^+_\sigma(\mu)\big] & \text{if } b \geq b^*\sigma^{\frac{2}{3}}, \\ \max\limits_{\mu\leq 0}\big[\mu b - \Lambda^+_\sigma(\mu)\big] & \text{if } 0 \leq b \leq b^*\sigma^{\frac{2}{3}}. \end{cases} \tag{4.11}$$

Analogous assertions hold for Λ^-_σ.

4.4. More refined large deviation properties for the Edwards model

In the proofs we will need some further refinements of Proposition 4.3, which are of a more technical nature. Abbreviate $B_{[0,T]} = (B_t)_{t\in[0,T]}$. For $T > 0, \delta, C \in (0,\infty]$ and $\alpha \in [0,\infty)$, define events

$$\widehat{\mathcal{E}}(\delta, T) = \big\{B_{[0,T]} \subset [-\delta, B_T + \delta]\big\}, \tag{4.12}$$

$$\widehat{\mathcal{E}}^{\leq}(\delta, C; T) = \Big\{ \max_{x\in[-\delta,\delta]} L(T,x) \leq C, \max_{x\in[B_T-\delta,B_T+\delta]} L(T,x) \leq C \Big\}, \tag{4.13}$$

$$\widehat{\mathcal{E}}^{\geq}(\delta, \alpha; T) = \Big\{ \max_{x\in[B_T-\delta,B_T+\delta]} L(T,x) \geq \alpha\delta^{-\frac{1}{2}} \Big\}. \tag{4.14}$$

In words, on $\widehat{\mathcal{E}}(\delta, T)$, the path does not leave the δ-neighborhood of the interval between its starting and ending location, and on $\widehat{\mathcal{E}}^{\leq}(\delta, C; T)$ and $\widehat{\mathcal{E}}^{\geq}(\delta, \alpha; T)$ its local times in the δ-neighborhood of the starting resp. the ending location do not exceed C, respectively, exceed $\alpha\delta^{-\frac{1}{2}}$. Note that $\widehat{\mathcal{E}}^{\leq}(\delta, \infty; T)$ and $\widehat{\mathcal{E}}^{\geq}(\delta, 0; T)$ are the full space.

Proposition 4.4 (Overshoots). *Fix $\mu > -\rho(a^{**})$. Then:*

(i) *For any $\delta, C \in (0,\infty]$ there exists a $K_1(\delta, C) \in (0,\infty)$ such that*

$$\begin{aligned} &e^{-\Lambda^+(\mu)T}\,\widehat{E}\Big(e^{-\widehat{H}_T}e^{\mu B_T}\mathbb{1}_{\widehat{\mathcal{E}}(\delta,T)}\mathbb{1}_{\widehat{\mathcal{E}}^{\leq}(\delta,C;T)}\mathbb{1}_{\{B_T\geq 0\}}\Big)\\ &\quad = K_1(\delta, C) + o(1), \qquad T\to\infty. \end{aligned} \tag{4.15}$$

Moreover, if $\mu = \mu_b$ solves $\widehat{I}_1(b) = \mu b - \Lambda^+(\mu)$, then the same is true when $\mathbb{1}_{\{B_T\geq 0\}}$ is replaced by $\mathbb{1}_{\{B_T\approx bT\}}$.

(ii) *For any $\delta, \alpha \in (0,\infty)$ there exists a $K_2(\delta, \alpha) \in (0,\infty)$ such that*

$$\begin{aligned} &e^{-\Lambda^+(\mu)T}\,\widehat{E}\Big(e^{-\widehat{H}_T}e^{\mu B_T}\mathbb{1}_{\widehat{\mathcal{E}}(\delta,T)}\mathbb{1}_{\widehat{\mathcal{E}}^{\geq}(\delta,\alpha;T)}\mathbb{1}_{\{B_T\geq 0\}}\Big)\\ &\quad = K_2(\delta, \alpha) + o(1), \qquad T\to\infty. \end{aligned} \tag{4.16}$$

(iii) *For any $\alpha \in (0,\infty)$,*

$$\lim_{\delta\downarrow 0}\frac{K_2(\delta,\alpha)}{K_1(\delta,\infty)} = 0. \tag{4.17}$$

Proposition 4.4 is proved in [vdHdHK02]. Proposition 4.4 plays an essential role in the proofs, since it allows us to estimate the interaction between the different pieces of the path. Indeed, Proposition 4.4(i-ii) shows that the cumulant generating function is unchanged by adding the indicators of the events $\widehat{\mathcal{E}}, \widehat{\mathcal{E}}^{\leq}$ and $\widehat{\mathcal{E}}^{\geq}$, while Proposition 4.4(iii) shows that $\widehat{\mathcal{E}}^{\geq}$ is asymptotically less likely than $\widehat{\mathcal{E}}^{\leq}$. Note that these assertions are statements up to and including order 1.

5. Proof of Theorem 2.1(i)

In this section we consider the limit $\beta \downarrow 0$. Let $(S_n)_{n\in\mathbb{N}_0}$ be a random walk satisfying (1.1). As announced at the beginning of Section 4, we will identify the scaling limit of the entire large deviation rate function (for the linear asymptotics of the endpoint) for the Domb-Joyce model in terms of that for the Edwards model, and we will deduce Theorem 2.1(i) from this scaling limit. However, as pointed out at the beginning of Section 4, the existence of the rate function has not been established in full generality for the Domb-Joyce model, and we will make no attempt to do so. Instead, we will be working with *approximative rate functions*, which are defined as a limsup or a liminf instead of a lim.

5.1. Approximative large deviations

It will be sufficient to deal with the event $\{S_n \geq \theta n\}$ for θ to the right of the scaled minimum point of the limiting rate function, and with $\{S_n \leq \theta n\}$ for θ to the left of it. To this end, define

$$I_\beta^+(\theta;\widetilde{\theta}) = \begin{cases} -\liminf\limits_{n\to\infty} \frac{1}{n}\log E\big(e^{-\beta H_n}\mathbb{1}_{\{S_n\geq\theta n\}}\big) & \text{if } \theta \geq \widetilde{\theta}, \\ -\liminf\limits_{n\to\infty} \frac{1}{n}\log E\big(e^{-\beta H_n}\mathbb{1}_{\{0\leq S_n\leq\theta n\}}\big) & \text{if } \theta \leq \widetilde{\theta}, \end{cases} \tag{5.1}$$

and define $I_\beta^-(\theta;\widetilde{\theta})$ in the same way with lim sup instead of lim inf. For $\beta = \infty$, recall the convention $e^{-\infty H_n} = \mathbb{1}_{\{H_n=0\}}$.

In the special case of (1.6), we know from Theorem 4.1 that the limit $I_\beta(\theta)$ in (4.2) exists. Since I_β is unimodal with unique minimiser θ^*, it follows that both limits in (5.1) exist and that

$$I_\beta^+(\theta;\theta^*) = I_\beta^-(\theta;\theta^*) = I_\beta(\theta), \qquad 0 \leq \theta \leq L. \tag{5.2}$$

Our main result in this section shows that the approximative rate function in (5.1) scales, as $\beta \downarrow 0$, to the rate function for the Edwards model with parameter σ.

Proposition 5.1. *Fix $\sigma \in (0,\infty)$. Then, under* (1.1)*,*

$$\liminf_{\beta\downarrow 0} \beta^{-\frac{2}{3}} I_\beta^-\big(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}}\big) \geq \widehat{I}_1^\sigma(b), \qquad b \geq 0, \tag{5.3}$$

$$\limsup_{\beta\downarrow 0} \beta^{-\frac{2}{3}} I_\beta^+\big(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}}\big) \leq \widehat{I}_1^\sigma(b), \qquad b > b^{**}\sigma^{\frac{2}{3}}. \tag{5.4}$$

In Section 5.2 we prove that Proposition 5.1 implies Theorem 2.1(i) and Corollary 2.2. Proposition 5.1 is proved in Section 5.3. In the special case of (1.6), we infer from Theorem 4.1 and Proposition 5.1 that

$$\lim_{\beta\downarrow 0} \beta^{-\frac{2}{3}} I_\beta\big(b\beta^{\frac{1}{3}}\big) = \widehat{I}_1^\sigma(b), \qquad b > b^{**}\sigma^{\frac{2}{3}}. \tag{5.5}$$

The proof of (5.4) for $0 \le b \le b^{**}\sigma^{\frac{2}{3}}$ remains open. To extend (5.4) to this regime would require some further refinements of our method. For instance, we would need to prove that the rate function for the Domb-Joyce model at $b = 0$ scales, in the weak interaction limit, to the rate function for the Edwards model at $b = 0$. In fact, by the linearity of $\widehat{I}_1^\sigma(b)$ below b^{**} together with the presumed convexity of I_β, respectively, $\widehat{I}_1^\sigma(b)$, this would imply the scaling of the rate function for every b in the linear piece.

5.2. Proof of Theorem 2.1(i) and Corollary 2.2 subject to Proposition 5.1

1. Fix $\varepsilon > 0$. We will show that, for $\beta > 0$ sufficiently small,

$$\lim_{n\to\infty} \frac{1}{n} \log Q_n^\beta\Big(\frac{|S_n|}{\beta^{\frac{1}{3}} n} - b^*\sigma^{\frac{2}{3}} > \varepsilon\Big) < 0. \tag{5.6}$$

This obviously implies the upper half of the statement in (2.1). The lower half can be derived in the same manner.

2. To prove (5.6), put $b' = b^*\sigma^{\frac{2}{3}} + \frac{\varepsilon}{2}$ and $b = b^*\sigma^{\frac{2}{3}} + \varepsilon$. Since $\widehat{I}_1^\sigma$ is strictly increasing on $[b^*\sigma^{\frac{2}{3}}, \infty)$, it is possible to pick $\gamma > 0$ so small (depending on ε) that

$$\widehat{I}_1^\sigma(b) - \widehat{I}_1^\sigma(b') - 2\gamma > 0. \tag{5.7}$$

According to Proposition 5.1, we may pick $\beta > 0$ so small (depending on γ) that

$$I_\beta^-\big(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}}\big) \ge \big[\widehat{I}_1^\sigma(b) - \gamma\big]\beta^{\frac{2}{3}}, \quad I_\beta^+\big(b'\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}}\big) \le \big[\widehat{I}_1^\sigma(b') + \gamma\big]\beta^{\frac{2}{3}}. \tag{5.8}$$

Now we can bound (recall (1.3))

$$\begin{aligned} &Q_n^\beta\Big(\frac{S_n}{\beta^{\frac{1}{3}} n} - b^*\sigma^{\frac{2}{3}} > \varepsilon\Big) \\ &= \frac{E\big(e^{-\beta H_n} 1\!\!1_{\{S_n > b\beta^{\frac{1}{3}} n\}}\big)}{E\big(e^{-\beta H_n}\big)} \le \frac{E\big(e^{-\beta H_n} 1\!\!1_{\{S_n > b\beta^{\frac{1}{3}} n\}}\big)}{E\big(e^{-\beta H_n} 1\!\!1_{\{S_n > b'\beta^{\frac{1}{3}} n\}}\big)} \\ &\le \exp\Big\{-n\big[I_\beta^-\big(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}}\big) - I_\beta^+\big(b'\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}}\big)\big] + o(n)\Big\}, \end{aligned} \tag{5.9}$$

where we use the definitions of I_β^- and I_β^+. Insert (5.7)–(5.8), to see that the term between square brackets in the exponent of (5.9) is strictly positive. This implies (5.6). In fact, this even shows that the complement of the event in (5.6) has an exponentially small probability.

3. The proof of Corollary 2.2 is as follows. Assume (1.6). First, by (5.5), the function f_β defined by $f_\beta(\cdot) = \beta^{-\frac{2}{3}} I_\beta\big(\beta^{\frac{1}{3}}\cdot\big)$ converges as $\beta \downarrow 0$ to $\widehat{I}_1^\sigma$ on $(b^{**}\sigma^{\frac{2}{3}}, \infty)$. In particular, the unique minimal value of f_β, which is $r^*(\beta)\beta^{-\frac{2}{3}}$ by Theorem 4.1, converges to the unique minimal value of $\widehat{I}_1^\sigma$, which is $a^*\sigma^{-\frac{2}{3}}$ by Theorem 4.2.

This proves the first assertion in (2.3). Next, by (5.5), f_β converges to $\widehat{I_1^\sigma}$ in the three points $b^*\sigma^{\frac{2}{3}}-\varepsilon$, $b^*\sigma^{\frac{2}{3}}$ and $b^*\sigma^{\frac{2}{3}}+\varepsilon$ for $\varepsilon>0$ small enough. For β small enough, both $f_\beta(b^*\sigma^{\frac{2}{3}}-\varepsilon)$ and $f_\beta(b^*\sigma^{\frac{2}{3}}+\varepsilon)$ are strictly larger than $f_\beta(b^*\sigma^{\frac{2}{3}})$. By unimodality, this implies that the unique minimiser of f_β, which is $\theta^*(\beta)\beta^{-\frac{1}{3}}$ by Theorem 4.1, lies in $(b^*\sigma^{\frac{2}{3}}-\varepsilon, b^*\sigma^{\frac{2}{3}}+\varepsilon)$. Let $\varepsilon\downarrow 0$ to obtain the second assertion in (2.3). □

Note that convexity of f_β, together with its pointwise convergence to $\widehat{I_1^\sigma}$, yields that even $(f_\beta)'$ converges to $(\widehat{I_1^\sigma})'$. However, we have no control over $(f_\beta)''$, which is why we are unable to prove (2.5).

5.3. *Proof of Proposition 5.1*

In Section 5.3.1 we prove (5.3), in Section 5.3.2 we prove (5.4). The main idea is to cut the path into smaller pieces to which the weak convergence assertion in (2.18) can be applied. The mutual interactions between the pieces has to be controlled appropriately. This is done by providing estimates in which either the pieces are independent or there is an interaction only between neighboring pieces. We define

$$H_n' = \sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbb{1}_{\{S_i=S_j\}} = H_n - 2(\ell_n(0)-1). \tag{5.10}$$

The proof runs via the moment generating function

$$Z_n^\beta(\mu) = E\big(e^{-\beta H_n'}e^{\mu\beta^{\frac{1}{3}}S_n}\big), \qquad n\in\mathbb{N}, \mu\in\mathbb{R} \tag{5.11}$$

which is the discrete analogue of the expectation in (4.8).

5.3.1. Proof of (5.3)

1. Fix $b\geq b^*\sigma^{\frac{2}{3}}$. Use the exponential Chebyshev inequality to get the following upper bound for $\mu\geq 0$:

$$E\big(e^{-\beta H_n}\mathbb{1}_{\{S_n\geq b\beta^{\frac{1}{3}}n\}}\big) \leq e^{-\mu b\beta^{\frac{2}{3}}n}Z_n^\beta(\mu). \tag{5.12}$$

Fix a large auxiliary parameter $T>0$ and abbreviate $T_\beta=\beta^{-\frac{2}{3}}T$. Split the path of length n into n/T_β pieces of length T_β. (To simplify the notation, assume that both n/T_β and T_β are integers.) Drop the interaction between any two of the pieces, to obtain an upper bound on $Z_n^\beta(\mu)$. After the pieces are decoupled they are independent of each other. This reasoning yields

$$Z_n^\beta(\mu) \leq \big(Z_{T_\beta}^\beta(\mu)\big)^{n/T_\beta}. \tag{5.13}$$

Substitute this estimate into (5.12), take logs, divide by $\beta^{\frac{2}{3}}n$ and let $n \to \infty$, to obtain (recall (5.1))

$$\begin{aligned}\beta^{-\frac{2}{3}} I_\beta^-(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}}) &\geq -\beta^{-\frac{2}{3}} \liminf_{n\to\infty} \frac{1}{n} \log\big(\text{l.h.s. of (5.12)}\big)\\ &\geq -\beta^{-\frac{2}{3}} \liminf_{n\to\infty} \frac{1}{n} \log\Big[e^{-\mu b\beta^{\frac{2}{3}}n}\big(Z_{T_\beta}^\beta(\mu)\big)^{n\beta^{\frac{2}{3}}/T}\Big]\\ &= \mu b - \frac{1}{T}\log Z_{T_\beta}^\beta(\mu). \end{aligned} \tag{5.14}$$

2. We will next use that, under (1.1), the expectation in the right-hand side of (5.14) converges to the corresponding Brownian expectation, i.e., assuming (1.1), for any $\mu \in \mathbb{R}$,

$$\lim_{\beta\downarrow 0} Z_{T_\beta}^\beta(\mu) = \widehat{E}(e^{-\widehat{H}_T^\sigma} e^{\mu B_T^\sigma}). \tag{5.15}$$

The proof of (5.15) is deferred to part **4**. We first complete the proof of (5.3) using (5.15). Indeed, using (5.15) applied to (5.14) gives

$$\liminf_{\beta\downarrow 0}[\beta^{-\frac{2}{3}} I_\beta^-(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}})] \geq \mu b - \frac{1}{T}\log \widehat{E}(e^{-\widehat{H}_T^\sigma} e^{\mu B_T^\sigma}), \qquad \mu \geq 0. \tag{5.16}$$

Now let $T \to \infty$ and use (4.8), to obtain

$$\liminf_{\beta\downarrow 0}[\beta^{-\frac{2}{3}} I_\beta^-(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}})] \geq \mu b - \Lambda_\sigma^+(\mu). \tag{5.17}$$

Maximize over $\mu \geq 0$ and use (4.11), to arrive at the assertion in (5.3).

3. The proof for $0 \leq b \leq b^*\sigma^{\frac{2}{3}}$ follows the same pattern. Estimate, for $\mu \leq 0$,

$$E\Big(e^{-\beta H_n} \mathbb{1}_{\{0\leq S_n \leq b\beta^{\frac{1}{3}}n\}}\Big) \leq e^{-\mu b\beta^{\frac{2}{3}}n} Z_n^\beta(\mu). \tag{5.18}$$

In the same way as above we obtain

$$\liminf_{\beta\downarrow 0}[\beta^{-\frac{2}{3}} I_\beta^-(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}})] \geq \mu b - \Lambda_\sigma^+(\mu). \tag{5.19}$$

Now maximize over $\mu \leq 0$ and again use (4.11). □

4. We finish by proving (5.15). Fix $\mu \in \mathbb{R}$. By the weak convergence assertion in (2.18), together with dominated convergence, we have for every $K > 0$,

$$\lim_{\beta\downarrow 0} E\Big(e^{-\beta H'_{T_\beta}} e^{\mu\beta^{\frac{1}{3}} S_{T_\beta}} \mathbb{1}_{\{\beta^{\frac{1}{3}}|S_{T_\beta}|<K\}}\Big) = \widehat{E}\left(e^{-\widehat{H}_T^\sigma} e^{\mu B_T^\sigma} \mathbb{1}_{\{|B_T^\sigma|<K\}}\right). \tag{5.20}$$

The right-hand side of (5.20) increases to $\widehat{E}(e^{-\widehat{H}_T^\sigma} e^{\mu B_T^\sigma})$ as $K \to \infty$. Therefore it suffices to show that

$$\lim_{K\to\infty} \limsup_{\beta\downarrow 0} E\Big(e^{-\beta H'_{T_\beta}} e^{\mu\beta^{\frac{1}{3}} S_{T_\beta}} \mathbb{1}_{\{\beta^{\frac{1}{3}}|S_{T_\beta}|\geq K\}}\Big) = 0. \tag{5.21}$$

To prove (5.21), use the Cauchy-Schwarz inequality:

$$E\Big(e^{-\beta H'_{T_\beta}} e^{\mu\beta^{\frac{1}{3}} S_{T_\beta}} \mathbb{1}_{\{\beta^{\frac{1}{3}}|S_{T_\beta}|\geq K\}}\Big)^2 \leq P\big(\beta^{\frac{1}{3}}|S_{T_\beta}| \geq K\big) E\big(e^{2\mu\beta^{\frac{1}{3}} S_{T_\beta}}\big). \tag{5.22}$$

The first term converges to $\widehat{P}(|B_T| \geq K)$ as $\beta \downarrow 0$, which vanishes as $K \to \infty$. Therefore it suffices to show that

$$\limsup_{\beta\downarrow 0} E\big(e^{2\mu\beta^{\frac{1}{3}} S_{T_\beta}}\big) < \infty. \tag{5.23}$$

To prove (5.23), denote the moment generating function of S_1 by $\varphi(t) = E(e^{tS_1})$, $t \in \mathbb{R}$. Then

$$E(e^{2\mu\beta^{\frac{1}{3}} S_{T_\beta}}) = \varphi(2\mu\beta^{\frac{1}{3}})^{T_\beta}. \tag{5.24}$$

By (1.1), the right-hand side is finite for β small enough (depending on μ). Now note that, by (1.1)–(1.2),

$$\varphi(t) = 1 + \frac{1}{2}\sigma^2 t^2 + \mathcal{O}(|t|^3), \qquad t \to 0. \tag{5.25}$$

Put $t = 2\mu\beta^{\frac{1}{3}}$ and combine (5.24)–(5.25), to get

$$E(e^{2\mu\beta^{\frac{1}{3}} S_{T_\beta}}) \leq e^{T_\beta[\frac{1}{2}\sigma^2 t^2 + \mathcal{O}(|t|^3)]} = e^{2\mu^2\sigma^2 T[1+\mathcal{O}(\beta^{\frac{1}{3}})]}, \qquad \beta \downarrow 0. \tag{5.26}$$

This proves (5.23) and completes the proof of (5.15). □

5.3.2. Proof of (5.4)

We again cut the path into pieces as in Section 5.3.1, but this time we keep control of the interaction between the pieces. Since we are looking for a lower bound on an expectation, we may freely require additional properties of the pieces in such a way that we can control their mutual interaction and still perform the limit $\beta \downarrow 0$.

1. Fix $b \geq b^*\sigma^{\frac{2}{3}}$. We require that in each piece the path has speed $\geq b\beta^{\frac{1}{3}}$, does not go too far beyond its starting and ending locations, and has local times in the overlapping areas that are uniformly bounded by a constant. To formulate this precisely, for $i = 1, \ldots, n/T_\beta$ denote by

$$S^{(i)} = (S_j^{(i)})_{j=0}^{T_\beta} \quad \text{with} \quad S_j^{(i)} = S_{j+(i-1)T_\beta} - S_{(i-1)T_\beta} \tag{5.27}$$

the i-th piece shifted such that it starts at the origin, and denote by

$$\ell^{(i)}(x) = \sum_{j=(i-1)T_\beta+1}^{iT_\beta} \mathbb{1}_{\{S_j - S_{(i-1)T_\beta} = x\}} = \sum_{j=1}^{T_\beta} \mathbb{1}_{\{S_j^{(i)} = x\}}, \qquad x \in \mathbb{Z}, \tag{5.28}$$

the local times of the i-th piece. Fix two parameters $\delta, C \in (0,\infty)$ and estimate

$$E\Big(e^{-\beta H_n}\mathbb{1}_{\{S_n\ge b\beta^{\frac13}n\}}\Big) \ge E\Big(e^{-\beta H_n}\prod_{i=1}^{n/T_\beta}\big[\mathbb{1}_{\mathcal{E}_i(\delta,T,\beta)}\mathbb{1}_{\mathcal{E}_i^{\le}(\delta,C,T,\beta)}\mathbb{1}_{\{S^{(i)}_{T_\beta}\ge b\beta^{\frac13}T_\beta\}}\big]\Big), \tag{5.29}$$

where the events $\mathcal{E}_i(\delta,T,\beta)$ and $\mathcal{E}_i^{\le}(\delta,T,C,\beta)$ are defined by

$$\mathcal{E}_i(\delta,T,\beta) = \Big\{S^{(i)} \subset [-\delta\beta^{-\frac13}, S^{(i)}_{T_\beta}+\delta\beta^{-\frac13}]\Big\}, \tag{5.30}$$

$$\mathcal{E}_i^{\le}(\delta,T,C,\beta) = \Big\{\max_{x\colon |x|\le\delta\beta^{-\frac13}}\ell^{(i)}(x)\le C\beta^{-\frac13}, \max_{x\colon |x-S^{(i)}_{T_\beta}|\le\delta\beta^{-\frac13}}\ell^{(i)}(x)\le C\beta^{-\frac13}\Big\}. \tag{5.31}$$

Observe that these events are discrete analogs of the events in (4.12) and (4.13).

2. Next, assume that $\delta < bT/2$ (i.e., $\delta\beta^{-\frac13} < b\beta^{\frac13}T_\beta/2$). Then, on the event $\bigcap_{i=1}^{n/T_\beta}[\mathcal{E}_i(\delta,T,\beta)\cap\mathcal{E}_i^{\le}(\delta,T,C,\beta)]$, the following hold: (a) there are no mutual intersections between the pieces unless they are neighbors of each other; (b) the i-th and the $(i+1)$-st piece have mutual intersections in an interval of length $2\delta\beta^{-\frac13}$ centered at S_{iT_β} only; (c) in this interval the local times of the i-th and the $(i+1)$-st piece are at most $C\beta^{-\frac13}$, so that the interaction between them satisfies

$$e^{-2\beta\sum_x \ell^{(i)}(x+S_{(i-1)T_\beta})\ell^{(i+1)}(x+S_{iT_\beta})} \ge e^{-4\delta C^2}. \tag{5.32}$$

Therefore, using (5.10) together with (5.32) and (5.31), we find that on the event $\bigcap_{i=1}^{n/T_\beta}[\mathcal{E}_i(\delta,T,\beta)\cap\mathcal{E}_i^{\le}(\delta,T,C,\beta)]$ we have

$$\begin{aligned} e^{-\beta H_n} &= e^{-2\beta(\ell_n(0)-1)-\beta H_n'} \ge e^{-2C\beta^{\frac23}}e^{-\beta H_n'} \\ &\ge e^{-2C\beta^{\frac23}}e^{-4\delta C^2 n/T_\beta}\prod_{i=1}^{n/T_\beta}e^{-\beta H'_{T_\beta}(i)}, \end{aligned} \tag{5.33}$$

where $H'_{T_\beta}(i)$ denotes H'_{T_β} computed for the i^{th} walk $S^{(i)}$. Substitute (5.33) into (5.29) and note that, after this is done, the pieces are independent. This reasoning yields

$$\begin{aligned} &E\Big(e^{-\beta H_n}\mathbb{1}_{\{S_n\ge b\beta^{\frac13}n\}}\Big) \\ &\quad\ge e^{-2C\beta^{\frac23}}e^{-4\delta C^2 n/T_\beta}E\Big(e^{-\beta H'_{T_\beta}}\mathbb{1}_{\mathcal{E}_1(\delta,T,\beta)}\mathbb{1}_{\mathcal{E}_1^{\le}(\delta,C,T,\beta)}\mathbb{1}_{\{S^{(1)}_{T_\beta}\ge b\beta^{\frac13}T_\beta\}}\Big)^{n/T_\beta}. \end{aligned} \tag{5.34}$$

3. Next, take logs, multiply by $\beta^{-\frac23}/n = T_\beta/Tn$ and let $n\to\infty$, to obtain

$$\begin{aligned} &\beta^{-\frac23}I_\beta^+(b\beta^{\frac13}; b^*\beta^{\frac13}\sigma^{\frac23}) \\ &\quad\le \frac{4\delta C^2}{T} - \frac1T\log E\Big(e^{-\beta H'_{T_\beta}}\mathbb{1}_{\mathcal{E}_1(\delta,T,\beta)}\mathbb{1}_{\mathcal{E}_1^{\le}(\delta,C,T,\beta)}\mathbb{1}_{\{S^{(1)}_{T_\beta}\ge b\beta^{\frac13}T_\beta\}}\Big). \end{aligned} \tag{5.35}$$

Let $\beta \downarrow 0$ and use the weak convergence assertions in (2.18)–(2.19), to obtain

$$\limsup_{\beta\downarrow 0}\big[\beta^{-\frac{2}{3}} I_\beta^+(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}})\big] \leq \frac{4\delta C^2}{T} - \frac{1}{T}\log \widehat{E}\big(e^{-\widehat{H}_T^\sigma}\mathbb{1}_{\widehat{\mathcal{E}}(\delta,T)}\mathbb{1}_{\widehat{\mathcal{E}}^{\leq}(\delta,C,T)}\mathbb{1}_{\{B_T^\sigma\geq bT\}}\big), \tag{5.36}$$

where the events $\widehat{\mathcal{E}}(\delta, T)$ and $\widehat{\mathcal{E}}^{\leq}(\delta, C, T)$ are defined in (4.12)–(4.13).

4. Finally, observe that $\mathbb{1}_{\{B_T^\sigma\geq bT\}} \geq \mathbb{1}_{\{B_T^\sigma\approx b'T\}}$ for any $b' > b$ and T sufficiently large (see (4.4)). Pick $\mu = \mu_{b'}$ with $\mu_{b'}$ the maximizer in (4.11), i.e., $\widehat{I}_1^\sigma(b') = \mu_{b'}b' - \Lambda_\sigma^+(\mu_{b'})$. Since $b \geq b^*\sigma^{\frac{2}{3}}$ and $b' > b$, we know that $\mu_{b'} > 0$ (recall (4.9)). Therefore we may bound

$$\widehat{E}\big(e^{-\widehat{H}_T^\sigma}\mathbb{1}_{\widehat{\mathcal{E}}(\delta,T)}\mathbb{1}_{\widehat{\mathcal{E}}^{\leq}(\delta,C,T)}\mathbb{1}_{\{B_T^\sigma\geq bT\}}\big) \geq e^{-\mu_{b'}b'T+o(T)}\widehat{E}\big(e^{-\widehat{H}_T^\sigma}e^{\mu_{b'}B_T^\sigma}\mathbb{1}_{\widehat{\mathcal{E}}(\delta,T)}\mathbb{1}_{\widehat{\mathcal{E}}^{\leq}(\delta,C,T)}\mathbb{1}_{\{B_T^\sigma\approx b'T\}}\big). \tag{5.37}$$

Insert (5.37) into (5.36), let $T \to \infty$ and use Proposition 4.4(i) (for the Brownian motion with variance σ^2 instead of 1), to arrive at

$$\limsup_{\beta\downarrow 0}[\beta^{-\frac{2}{3}} I_\beta^+(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}})] \leq \mu_{b'}b' - \Lambda_\sigma^+(\mu_{b'}) = \widehat{I}_1^\sigma(b'). \tag{5.38}$$

Let $b' \downarrow b$ and use the continuity of $\widehat{I}_1^\sigma$, to complete the proof of (5.4) for $b \geq b^*\sigma^{\frac{2}{3}}$.

5. The proof of (5.4) for $b^{**}\sigma^{\frac{2}{3}} < b \leq b^*\sigma^{\frac{2}{3}}$ is analogous. Indeed, (5.27)–(5.36) give that

$$\limsup_{\beta\downarrow 0}\big[\beta^{-\frac{2}{3}} I_\beta^+(b\beta^{\frac{1}{3}}; b^*\beta^{\frac{1}{3}}\sigma^{\frac{2}{3}})\big] \leq \frac{4\delta C^2}{T} - \frac{1}{T}\log \widehat{E}\big(e^{-\widehat{H}_T^\sigma}\mathbb{1}_{\widehat{\mathcal{E}}(\delta,T)}\mathbb{1}_{\widehat{\mathcal{E}}^{\leq}(\delta,C,T)}\mathbb{1}_{\{0\leq B_T^\sigma\leq bT\}}\big). \tag{5.39}$$

Complete the proof as in (5.37)–(5.38), via $\mathbb{1}_{\{0\leq B_T^\sigma\leq bT\}} \geq \mathbb{1}_{\{B_T^\sigma\approx b'T\}}$ for any $b' < b$ and T sufficiently large, and $\mu_{b'} < 0$ for any $b' < b$. □

6. Proof of Theorem 2.1(ii)

In this section we consider the limit $\sigma \to \infty$. Let $(S_n)_{n\in\mathbb{N}_0}$ be a random walk satisfying (2.23)–(2.26).

6.1. Approximative large deviations

Recall (4.2) and (5.1). Our main result in this section shows that the approximative rate function in (5.1) scales, as $\sigma \to \infty$, to the rate function for the Edwards model.

Proposition 6.1. *Fix $\beta = \infty$. Then, under* (2.23)–(2.26)*,*

$$\liminf_{\sigma\to\infty} \sigma^{\frac{2}{3}} I_\infty^-\big(b\sigma^{\frac{2}{3}}; b^*\sigma^{\frac{2}{3}}\big) \geq \widehat{I}_1(b), \qquad b \geq 0, \tag{6.1}$$

$$\limsup_{\sigma\to\infty} \sigma^{\frac{2}{3}} I_\infty^+\big(b\sigma^{\frac{2}{3}}; b^*\sigma^{\frac{2}{3}}\big) \leq \widehat{I}_1(b), \qquad b > b^{**}. \tag{6.2}$$

Proposition 6.1 implies Theorem 2.1(ii) and Corollary 2.3 in the same way as Proposition 5.1 implies Theorem 2.1(i) and Corollary 2.2 (see Section 5.1). We leave this for the reader to verify.

In the special case of (1.6), subject to (2.23)–(2.26), we know from Theorem 4.1 that the rate function I_β in (4.2) exists and so we can infer from Proposition 6.1 that

$$\lim_{\sigma\to\infty} \sigma^{\frac{2}{3}} I_\infty\big(b\sigma^{\frac{2}{3}}\big) = \widehat{I}_1(b), \qquad b > b^{**}. \tag{6.3}$$

Again, we leave open the convergence for $0 \leq b \leq b^{**}$.

6.2. Proof of Proposition 6.1

Like in Section 5.3, we decompose the path into pieces to which an appropriate weak convergence assertion can be applied, which is in this case (2.21). The arguments are similar and again revolve around controlling the interaction between neighboring pieces. In fact, the proof of (6.1) is almost identical to the proof of (5.3). However, in the proof of (6.2), it turns out to be more difficult to handle the mutual *avoidance* of neighboring pieces than to handle their mutual *intersection local times* as in Section 5.3. In order to overcome this problem, we use a technique that is reminiscent of the so-called "lace expansion", together with the refined large deviation properties in Proposition 4.4. The lace expansion has been used to prove diffusive behaviour for self-avoiding walk in dimensions greater than 4 (see e.g. [MS93]). Throughout the sequel we write "$(S_i)_{i=0}^n$ is SAW" if $S_i \neq S_j$ for all $0 \leq i < j \leq n$.

6.2.1. Proof of (6.1)

1. Fix $b \geq b^*$ and recall that

$$I_\infty^+(b\sigma^{\frac{2}{3}}; b^*\sigma^{\frac{2}{3}}) = -\liminf_{n\to\infty} \frac{1}{n} \log P\big((S_j)_{j=0}^n \text{ is SAW}, |S_n| \geq b\sigma^{\frac{2}{3}} n\big). \tag{6.4}$$

Instead of (5.11), now consider

$$Z_n^\infty(\mu) = E\Big(e^{\mu\sigma^{-\frac{4}{3}} S_n} \mathbb{1}_{\{(S_j)_{j=0}^n \text{is SAW}\}}\Big), \qquad n \in \mathbb{N},\ \mu \in \mathbb{R}. \tag{6.5}$$

Cut the path into n/T_σ pieces of length $T_\sigma = \sigma^{\frac{2}{3}} T$. (To simplify the notation, assume that both n/T_σ and T_σ are integers.) For $\mu > 0$, we estimate, like in (5.12)–(5.13),

$$P\big((S_j)_{j=0}^n \text{ is SAW}, S_n \geq b\sigma^{\frac{2}{3}} n\big) \leq e^{-\mu b\sigma^{-\frac{2}{3}} n}\, Z_n^\infty(\mu) \leq e^{-\mu b\sigma^{-\frac{2}{3}} n}\, [Z_{T_\sigma}^\infty(\mu)]^{n/T_\sigma}. \tag{6.6}$$

2. We claim that, under (2.23)–(2.26), the expectation in the right-hand side of (6.6) converges to the corresponding Brownian expectation, i.e., for any $\mu \in \mathbb{R}$,

$$\lim_{\sigma\to\infty} Z^{\infty}_{T_\sigma}(\mu) = \widehat{E}(e^{-\widehat{H}_T} e^{\mu B_T}). \tag{6.7}$$

We will first prove (6.7). As in the proof of (5.15), it suffices to show that

$$\limsup_{\sigma\to\infty} E(e^{2\mu\sigma^{-\frac{4}{3}} S_{T_\sigma}}) < \infty. \tag{6.8}$$

Denote the moment generating function of S_1/σ by $\varphi_\sigma(t) = E(e^{tS_1/\sigma})$. Then

$$E(e^{2\mu\sigma^{-\frac{4}{3}} S_{T_\sigma}}) = \varphi_\sigma(2\mu\sigma^{-\frac{1}{3}})^{T_\sigma}. \tag{6.9}$$

By (2.26), the right-hand side is finite for σ large enough. By (2.23)(a) we have, uniformly in $\sigma \geq 1$,

$$\varphi_\sigma(t) = 1 + \frac{1}{2}t^2 + \mathcal{O}(|t|^3), \qquad t \to 0. \tag{6.10}$$

Put $t = 2\mu\sigma^{-\frac{1}{3}}$ and combine (6.9)–(6.10), to get

$$E\Big(e^{2\mu\sigma^{-\frac{4}{3}} S_{T_\sigma}}\Big) \leq e^{T_\sigma[\frac{1}{2}t^2 + \mathcal{O}(1/\sigma)]} = e^{2T\mu^2[1+\mathcal{O}(\sigma^{-\frac{1}{3}})]}, \qquad \sigma \to \infty. \tag{6.11}$$

This completes the proof of (6.7).

3. The details of the remainder of the proof are the same as in Section 5.3.1, via (6.7) instead of (5.15). This completes the proof for $b \geq b^*$. The proof for $0 \leq b \leq b^*$ is analogous. □

6.2.2. Proof of (6.2)

The proof of (6.2) is quite a bit more involved than the proof of (5.4). The main reason is that we miss a version of the weak convergence of the local times as in (2.19). Therefore, we need to deal with the convergence issues in a different way.

1. Fix $b \geq b^*$. Pick any $b' > b$, fix $\sigma, T > 0$, and put $\gamma^{(n)} = \gamma_T \sigma^{\frac{2}{3}} n/T$. Then, for $\mu > 0$ and T large enough, we have

$$\mathbb{1}_{\{S_n \geq b\sigma^{\frac{2}{3}} n\}} \geq \mathbb{1}_{\{|S_n - b'\sigma^{\frac{2}{3}} n| \leq \gamma^{(n)}\}} e^{\mu\sigma^{-\frac{4}{3}}[S_n - b'\sigma^{\frac{2}{3}} n - \gamma^{(n)}]}. \tag{6.12}$$

This implies the lower bound

$$\begin{aligned} &\sigma^{\frac{2}{3}} \liminf_{n\to\infty} \frac{1}{n} \log P\left((S_j)_{j=0}^n \text{ is SAW}, S_n \geq b\sigma^{\frac{2}{3}} n\right) \\ &\quad \geq -\mu b' - \mu\frac{\gamma_T}{T} + \sigma^{\frac{2}{3}} \liminf_{n\to\infty} \frac{1}{n} \log E\Big(\mathbb{1}_{\{(S_j)_{j=0}^n \text{is SAW}\}} e^{\mu\sigma^{-\frac{4}{3}} S_n} \mathbb{1}_{\{|S_n - b'\sigma^{\frac{2}{3}} n| \leq \gamma^{(n)}\}}\Big). \end{aligned} \tag{6.13}$$

To handle the expectation in the right-hand side, we estimate

$$e^{\mu\sigma^{-\frac{4}{3}}S_n}\mathbb{1}_{\{|S_n-b'\sigma^{\frac{2}{3}}n|\le\gamma^{(n)}\}} \ge \prod_{i=1}^{n/T_\sigma}\Big[e^{\mu\sigma^{-\frac{4}{3}}S^{(i)}_{T_\sigma}}\mathbb{1}_{\{|\sigma^{-\frac{4}{3}}S^{(i)}_{T_\sigma}-b'T|\le\gamma_T\}}\mathbb{1}_{\mathcal{E}_i(\delta,T,\sigma)}\Big], \tag{6.14}$$

where we use the definition (5.27) of the shifted i-th piece with T_β replaced by T_σ, abbreviate $S^{(i)}=(S^{(i)}_j)_{j=0}^{T_\sigma}$, and introduce the event

$$\mathcal{E}_i(\delta,T,\sigma)=\Big\{S^{(i)}\subset\big[-\delta\sigma^{\frac{2}{3}},\,S^{(i)}_{T_\sigma}+\delta\sigma^{\frac{2}{3}}\big]\Big\}, \tag{6.15}$$

which is a discrete analog in the event in (4.12).

2. Assume that $\delta<bT/2$. On the event $\bigcap_{i=1}^{n/T_\sigma}\mathcal{E}_i(\delta,T,\sigma)$, the pieces $S^{(i)}$, $i=1,\dots,n/T_\sigma$, have no mutual intersection, unless they are neighbors of each other. Hence, we only need to estimate the interaction between the neighboring pieces. More precisely, $(S_j)_{j=0}^n$ is SAW as soon as all the pieces $S^{(i)}$ are SAW and neighboring pieces do not overlap in more than their connecting point. Introduce the indicator U_i of the event that the i-th and the $(i+1)$-st piece intersect each other in more than their connecting point:

$$U_i(T,\sigma)=\begin{cases}1 & \text{if } (S_j)_{j=(i-1)T_\sigma}^{iT_\sigma}\cap(S_j)_{j=iT_\sigma}^{(i+1)T_\sigma}\ne\{S_{iT_\sigma}\},\\ 0 & \text{otherwise.}\end{cases} \tag{6.16}$$

Then we have

$$\mathbb{1}_{\{(S_j)_{j=0}^n\text{ is SAW}\}}\prod_{i=1}^{n/T_\sigma}\mathbb{1}_{\mathcal{E}_i(\delta,T,\sigma)}=\prod_{i=1}^{n/T_\sigma}\Big[\mathbb{1}_{\{S^{(i)}\text{ is SAW}\}}\mathbb{1}_{\mathcal{E}_i(\delta,T,\sigma)}\Big]\prod_{i=1}^{n/T_\sigma-1}(1-U_i(T,\sigma)). \tag{6.17}$$

Using (6.14) and (6.17), we obtain the lower bound

$$E\Big(\mathbb{1}_{\{(S_j)_{j=0}^n\text{ is SAW}\}}e^{\mu\sigma^{-\frac{4}{3}}S_n}\mathbb{1}_{\{|S_n-b'\sigma^{\frac{2}{3}}n|\le\gamma^{(n)}\}}\Big)\ge c_{n/T_\sigma}(\delta,T,\sigma,b',\mu), \tag{6.18}$$

where

$$c_N=c_N(\delta,T,\sigma,b',\mu)=E\Big(\prod_{i=1}^{N-1}(1-U_i(T,\sigma))\prod_{i=1}^{N}X_i\Big),\qquad N\in\mathbb{N}, \tag{6.19}$$

with

$$X_i=e^{\mu\sigma^{-\frac{4}{3}}S^{(i)}_{T_\sigma}}\mathbb{1}_{\mathcal{E}_i(\delta,T,\sigma)}\mathbb{1}_{\{|\sigma^{-\frac{4}{3}}S^{(i)}_{T_\sigma}-b'T|\le\gamma_T\}}\mathbb{1}_{\{S^{(i)}\text{ is SAW}\}}. \tag{6.20}$$

3. Next use an expansion argument that is reminiscent of the "lace expansion technique", namely, expand the product $\prod_{i=1}^{N-1}(1-U_i)$ in (6.19) as

$$\prod_{i=1}^{N-1}(1-U_i)=\sum_{m=1}^{N}\prod_{i=1}^{m-1}(-U_i)\prod_{i=m+1}^{N-1}(1-U_i), \tag{6.21}$$

where the empty product is defined to be equal to 1. This expansion has the advantage that every summand splits into a product of two separated products. Substitute (6.21) into (6.19), to find that

$$c_N = \sum_{m=1}^{N} (-1)^{m-1} E\Big(\Big[\prod_{i=1}^{m-1} U_i X_i\Big] \times X_m \times \Big[\prod_{i=m+1}^{N-1} (1-U_i)X_i\Big] \times X_N\Big). \tag{6.22}$$

Since in the m-th summand the term U_m is absent, the two factors between the two pairs of large square brackets are independent: they depend on the path $(S_j)_{j=0}^n$ up time mT_σ, respectively, from time mT_σ onwards. Hence, the c_N satisfy the following renewal relation:

$$c_N = c_1 c_{N-1} + \sum_{m=2}^{N} (-1)^{m-1} \pi_m c_{N-m}, \qquad N \in \mathbb{N}, \tag{6.23}$$

where

$$\pi_m = \pi_m(\delta, T, \sigma, b', \mu) = E\Big(\prod_{i=1}^{m-1} U_i \prod_{i=1}^{m} X_i\Big). \tag{6.24}$$

4. Use the Cauchy-Schwarz inequality, to estimate

$$\begin{aligned} \pi_m &\le E\Big(\prod_{\substack{i=1\\ i \text{ odd}}}^{m-1} U_i \prod_{i=1}^{m} X_i\Big)^{1/2} E\Big(\prod_{\substack{i=1\\ i \text{ even}}}^{m-1} U_i \prod_{i=1}^{m} X_i\Big)^{1/2} \\ &= (\pi_2^{m/2})^{1/2} c_1 (\pi_2^{(m-2)/2})^{1/2}, \qquad m \in \mathbb{N} \text{ even}, \end{aligned} \tag{6.25}$$

and similarly for $m \in \mathbb{N}$ odd. Hence

$$\pi_m \le \varepsilon^{m-1} c_1^m, \qquad m \in \mathbb{N}, \tag{6.26}$$

where

$$\varepsilon = \frac{\sqrt{\pi_2}}{c_1}. \tag{6.27}$$

5. The following two auxiliary lemmas give us control over ε and c_N. From now on, we choose $\mu = \mu_{b'}$ with $\mu_{b'}$ the maximizer in (4.9), i.e., $\widehat{I}(b') = \mu_{b'} b' - \Lambda^+(\mu_{b'})$, which is possible when $b' > b^{**}$ (recall (4.9)).

Lemma 6.2. *Fix $b' > b^{**}$. Then*

$$\lim_{\delta \downarrow 0} \limsup_{T \to \infty} \limsup_{\sigma \to \infty} \varepsilon(\delta, T, \sigma, b', \mu_{b'}) = 0. \tag{6.28}$$

Lemma 6.3. *For $\eta > 0$ sufficiently small the following is true: If $\delta, T, \sigma > 0$ are chosen such that $\varepsilon = \varepsilon(\delta, T, \sigma, b', \mu_{b'}) < \eta$, then there are numbers $C, N_0 > 0$ (depending on ε and η only) such that*

$$c_N \ge C(1-3\eta)^N c_1^N, \qquad N > N_0. \tag{6.29}$$

The proofs of Lemmas 6.2–6.3 are deferred to Sections 6.3.1–6.3.2 below.

6. We complete the argument subject to Lemmas 6.2–6.3. Pick $\eta \in (0, \frac{1}{4})$ so small that Lemma 6.3 is satisfied for this η. According to Lemma 6.2, we may pick $\delta > 0$ so small that, when T is picked sufficiently large, we have $\varepsilon < \eta$ for any sufficiently large σ. Hence we may make use of the estimate in (6.29) for these T and σ.

We use (6.18) and Lemma 6.3 in (6.13), to obtain

$$\begin{aligned}\sigma^{\frac{2}{3}} I_\infty^+(b\sigma^{\frac{2}{3}}; b^*\sigma^{\frac{2}{3}}) &= -\sigma^{\frac{2}{3}} \liminf_{n\to\infty} \frac{1}{n} \log P\big((S_j)_{j=0}^n \text{ is SAW}, S_n \geq b\sigma^{\frac{2}{3}} n\big)\\ &\leq \mu_{b'} b' + \mu_{b'} \frac{\gamma_T}{T} - \liminf_{n\to\infty} \frac{\sigma^{\frac{2}{3}}}{n} \log c_{n/T_\sigma}\\ &\leq \mu_{b'} b' + \mu_{b'} \frac{\gamma_T}{T} - \liminf_{n\to\infty} \frac{\sigma^{\frac{2}{3}}}{n} \log\big[C(1-3\eta)^{n/T_\sigma} c_1^{n/T_\sigma}\big]\\ &= \mu_{b'} b' + \mu_{b'} \frac{\gamma_T}{T} - \frac{1}{T}\log(1-3\eta) - \frac{1}{T}\log c_1. \end{aligned} \tag{6.30}$$

Return to (2.21) and recall that $\{H_{T_\sigma} = 0\} = \{S^{(1)} \text{ is SAW}\}$. From the weak convergence assertion in (2.21) applied to (6.19) for $N = 1$, in combination with a statement like in (6.7), it follows that

$$\lim_{\sigma\to\infty} c_1(\delta, T, \sigma, b', \mu_{b'}) = \widehat{E}\big(e^{-\widehat{H}_T} e^{\mu_{b'} B_T} \mathbb{1}_{\widehat{\mathcal{E}}(\delta,T)} \mathbb{1}_{\{B_T \approx b'T\}}\big), \tag{6.31}$$

where $\widehat{\mathcal{E}}(\delta, T)$ is the event defined in (4.12). Combining (6.30)–(6.31), we obtain

$$\begin{aligned}&\limsup_{\sigma\to\infty} [\sigma^{\frac{2}{3}} I_\infty^+(b\sigma^{\frac{2}{3}}; b^*\sigma^{\frac{2}{3}})]\\ &\quad\leq \mu_{b'} b' + \mu_{b'} \frac{\gamma_T}{T} - \frac{1}{T}\log(1-3\eta) - \frac{1}{T}\log \widehat{E}\big(e^{-\widehat{H}_T} e^{\mu_{b'} B_T} \mathbb{1}_{\widehat{\mathcal{E}}(\delta,T)} \mathbb{1}_{\{B_T \approx b'T\}}\big).\end{aligned} \tag{6.32}$$

Now let $T \to \infty$ and use (4.15) for $C = \infty$, to see that the right-hand side of (6.32) tends to $\mu_{b'} b' - \Lambda^+(\mu_{b'})$, which is equal to $\widehat{I}_1(b')$. Finally, let $b' \downarrow b$ and use the continuity of $\widehat{I}_1$ to finish the proof of (6.2). □

6.3. *Proof of Lemmas 6.2–6.3*

We complete the proof of Proposition 6.1 by proving Lemmas 6.2–6.3. This will be done in Sections 6.3.1 and 6.3.2, respectively.

6.3.1. Proof of Lemma 6.2

Lemma 6.2 will follow from Proposition 4.4(iii). We will first need to reformulate the statement in Lemma 6.2 in terms of the statements in Proposition 4.4.

1. Fix δ, T. Introduce the Brownian event

$$\widehat{\mathcal{E}}_i(\delta, T) = \big\{B_{[(i-1)T, iT]} \subset [-\delta + B_{(i-1)T}, B_{iT} + \delta]\big\}, \qquad i = 1, 2, \tag{6.33}$$

and note that $\widehat{\mathcal{E}}_i(\delta, T)$ is identical to $\widehat{\mathcal{E}}(\delta, T)$ in (4.12) for the i-th piece. Write U_1 as $1-(1-U_1)$ in the definition of π_2 in (6.24), to obtain from (6.27) that

$$\varepsilon^2 = \frac{1}{c_1^2}\Big[E(X_1X_2) - E\Big(\mathbb{1}_{\{S^{(1)},S^{(2)}\text{ avoid each other}\}}X_1X_2\Big)\Big]$$
$$= 1 - \frac{1}{c_1^2}E\Big(\mathbb{1}_{\{(S_j)_{j=0}^{2T_\sigma}\text{ is SAW}\}}X_1X_2\Big)\Big]. \tag{6.34}$$

Now apply the weak convergence statement in (2.21) and recall (6.31), to obtain, analogously to (6.31), that

$$\lim_{\sigma\to\infty}\varepsilon^2 = 1 - \frac{\widehat{E}\Big(e^{-\widehat{H}_{2T}}e^{\mu B_{2T}}\mathbb{1}_{\widehat{\mathcal{E}}_1(\delta,T)\cap\widehat{\mathcal{E}}_2(\delta,T)}\mathbb{1}_{\{B_T\approx b'T\}}\mathbb{1}_{\{B_{2T}-B_T\approx b'T\}}\Big)}{\widehat{E}\Big(e^{-\widehat{H}_T}e^{\mu B_T}\mathbb{1}_{\widehat{\mathcal{E}}_1(\delta,T)}\mathbb{1}_{\{B_T\geq 0\}}\Big)^2}(1+o(1)), \tag{6.35}$$

where $o(1)$ refers to $T\to\infty$.

2. Denote the intersection local time of the i-th piece by $\widehat{H}_T^{(i)}$. Then (6.35) reads

$$\lim_{\sigma\to\infty}\varepsilon^2$$
$$= \frac{\widehat{E}\Big(\Big[e^{-\widehat{H}_T^{(1)}-\widehat{H}_T^{(2)}}-e^{-\widehat{H}_{2T}}\Big]e^{\mu B_{2T}}\mathbb{1}_{\widehat{\mathcal{E}}_1(\delta,T)\cap\widehat{\mathcal{E}}_2(\delta,T)}\mathbb{1}_{\{B_T\approx b'T\}}\mathbb{1}_{\{B_{2T}-B_T\approx b'T\}}\Big)}{\widehat{E}\Big(e^{-\widehat{H}_T}e^{\mu B_T}\mathbb{1}_{\widehat{\mathcal{E}}_1(\delta,T)}\mathbb{1}_{\{B_T\geq 0\}}\Big)^2}(1+o(1)). \tag{6.36}$$

Denote the local time of the i-th piece by $L^{(i)}(T,\cdot)$. Then, on the event $\widehat{\mathcal{E}}_1(\delta,T)\cap\widehat{\mathcal{E}}_2(\delta,T)$, we have

$$\widehat{H}_{2T} = \widehat{H}_T^{(1)} + \widehat{H}_T^{(2)} + 2\int_{B_T-\delta}^{B_T+\delta} L^{(1)}(T,x)L^{(2)}(T,x)\,\mathrm{d}x. \tag{6.37}$$

Now fix a small $\alpha>0$ and introduce the events

$$\widehat{\mathcal{E}}_1^{\geq,+}(\delta,\alpha,T) = \Big\{\max_{x\in[B_T-\delta,B_T+\delta]} L^{(1)}(T,x)\geq\alpha\delta^{-1/2}\Big\}, \tag{6.38}$$
$$\widehat{\mathcal{E}}_2^{\geq,-}(\delta,\alpha,T) = \Big\{\max_{x\in[B_T-\delta,B_T+\delta]} L^{(2)}(T,x)\geq\alpha\delta^{-1/2}\Big\}. \tag{6.39}$$

Observe that $\widehat{\mathcal{E}}_1^{\geq,+}(\delta,\alpha,T)$ and $\widehat{\mathcal{E}}_2^{\geq,+}(\delta,\alpha,T)$ are the events $\widehat{\mathcal{E}}^{\geq}(\delta,\alpha;T)$ for the first and second piece of the path, respectively, backwards in time.

We estimate the right-hand side of (6.35) differently on the event $\widehat{\mathcal{E}}_1^{\geq,+}\cup\widehat{\mathcal{E}}_2^{\geq,-}$ and on its complement. Namely, on the complement of $\widehat{\mathcal{E}}_1^{\geq,+}\cup\widehat{\mathcal{E}}_2^{\geq,-}$ we estimate

$$\widehat{H}_{2T}\leq\widehat{H}_T^{(1)}+\widehat{H}_T^{(2)}+4\alpha^2, \tag{6.40}$$

which implies

$$e^{-\widehat{H}_T^{(1)}-\widehat{H}_T^{(2)}}-e^{-\widehat{H}_{2T}}\leq\big[1-e^{-4\alpha^2}\big]e^{-\widehat{H}_T^{(1)}-\widehat{H}_T^{(2)}}, \tag{6.41}$$

while on the event $\widehat{\mathcal{E}}_1^{\geq,+} \cup \widehat{\mathcal{E}}_2^{\geq,-}$ we estimate $-e^{-\widehat{H}_{2T}} \leq 0$. By symmetry, $\widehat{\mathcal{E}}_1^{\geq,+}$ and $\widehat{\mathcal{E}}_2^{\geq,-}$ have the same probability. Summarizing, we obtain

$$\begin{aligned}\lim_{\sigma\to\infty} \varepsilon^2 \ &\leq 1 - e^{-4\alpha^2} \\ &+ 2\left(\frac{\widehat{E}\big(e^{-\widehat{H}_T} e^{\mu B_T} 1\!\!1_{\widehat{\mathcal{E}}(\delta,T)} 1\!\!1_{\widehat{\mathcal{E}}^{\geq}(\delta,\alpha,T)} 1\!\!1_{\{B_T\approx b'T\}}\big)}{\widehat{E}\big(e^{-\widehat{H}_T} e^{\mu B_T} 1\!\!1_{\widehat{\mathcal{E}}(\delta,T)} 1\!\!1_{\{B_T\geq 0\}}\big)}\right)^2 (1+o(1)),\end{aligned} \tag{6.42}$$

where we recall that the events $\widehat{\mathcal{E}}(\delta, T)$ and $\widehat{\mathcal{E}}^{\geq}(\delta, \alpha, T)$ are defined in (4.12), respectively, (4.14).

3. Let $T \to \infty$ in (6.42) and use Proposition 4.4(i–ii), to obtain

$$\limsup_{T\to\infty} \lim_{\sigma\to\infty} \varepsilon^2 \leq 1 - e^{-4\alpha^2} + 2\,\frac{K_2(\delta,\alpha)}{K_1(\delta,\infty)}. \tag{6.43}$$

Let $\delta \downarrow 0$ and use Proposition 4.4(iii), to obtain

$$\limsup_{\delta\downarrow 0} \limsup_{T\to\infty} \lim_{\sigma\to\infty} \varepsilon^2 \leq 1 - e^{-4\alpha^2}. \tag{6.44}$$

Let $\alpha \downarrow 0$, to arrive at the assertion in (6.28). □

6.3.2. Proof of Lemma 6.3

The proof of Lemma 6.3 uses induction on N, and Lemma 6.2 will play a crucial role.

1. Fix $\eta > 0$ and $\varepsilon \in (0, \eta)$. Define $z \in (0, \infty)$ by

$$1 - z = \sum_{m=2}^{\infty} (-1)^{m-1} \pi_m \left(\frac{z}{c_1}\right)^m. \tag{6.45}$$

Equation (6.26) implies that, for any $z \in (0, \frac{1}{\eta})$, the modulus of the right-hand side is bounded above by $\varepsilon z^2/(1-\varepsilon z) \leq \eta z^2/(1-\eta z)$. Since this function crosses $1-z$ in $z = \frac{1}{1+\eta}$ and since, for sufficiently small η, its negative value crosses $1 - z$ in $\frac{3}{2} + \mathcal{O}(\eta)$, there is indeed a solution z to (6.45) in $(0, \frac{3}{2} + \mathcal{O}(\eta)]$ as $\eta \downarrow 0$. Assume that η is so small that this solution exists and satisfies the estimate $z^{-1} \geq 1 - 3\eta$.

2. Abbreviate

$$A_N = c_N \left(\frac{z}{c_1}\right)^N, \qquad N \in \mathbb{N} \qquad (A_0 = 1). \tag{6.46}$$

We claim that, if η is small enough, then there are numbers $K > 0$ and $q \in (0, 1)$ (depending on η only) such that

$$|A_N - A_{N-1}| \leq K q^N, \qquad N \in \mathbb{N}. \tag{6.47}$$

The proof of this claim is given in part **3**. Because of (6.47), $A_\infty = \lim_{N\to\infty} A_N \in (0, \infty)$ exists and, for N sufficiently large, $A_N \geq \frac{1}{2}A_\infty$, which reads $c_N \geq \frac{1}{2}A_\infty(\frac{c_1}{z})^N$. Recall that $z^{-1} \geq 1 - 3\eta$, to finish the proof of the lemma.

3. The proof of (6.47) goes via induction on N. Pick $q = \sqrt{\eta z}$ and assume that η is so small that $q < 1$ and

$$\frac{\eta z^2}{(1-q)(q-\eta z)} \leq \frac{1}{2}. \tag{6.48}$$

Furthermore, pick $K \geq 1$ so large that $1 + Kq/(1-q) \leq K(1-\eta z)/2z$. Then the claim holds for $N = 1$, since $|A_1 - A_0| = |z - 1| \leq \eta z^2/(1-\eta z) \leq \frac{1}{2}(q - \eta z) \leq Kq$. Assume now that $N > 1$ and that the claim holds for all positive integers $< N$. From this induction hypothesis it follows that, for every $m = 2, \dots, N$,

$$|A_{N-1} - A_{N-m}| \leq \sum_{k=2}^{m} |A_{N-k+1} - A_{N-k}| \leq Kq^N \sum_{k=2}^{m} q^{-k+1} = \frac{Kq}{1-q} q^{N-m}. \tag{6.49}$$

Estimate, with the help of (6.23), (6.26), (6.45)–(6.46), the triangle inequality and (6.49),

$$\begin{aligned}
&|A_N - A_{N-1}| \\
&= \left| c_{N-1}\left(\frac{z}{c_1}\right)^{N-1}(z-1) + \left(\frac{z}{c_1}\right)^N \sum_{m=2}^{N} (-1)^{m-1}\pi_m c_{N-m} \right| \\
&= \left| -A_{N-1}\sum_{m=2}^{\infty}(-1)^{m-1}\pi_m\left(\frac{z}{c_1}\right)^m + \sum_{m=2}^{N}(-1)^{m-1}\pi_m\left(\frac{z}{c_1}\right)^m A_{N-m} \right| \\
&\leq A_{N-1}\left| \sum_{m=N+1}^{\infty} (-1)^{m-1}\pi_m\left(\frac{z}{c_1}\right)^m \right| + \left| \sum_{m=2}^{N}(-1)^{m-1}\pi_m\left(\frac{z}{c_1}\right)^m \big(A_{N-m} - A_{N-1}\big) \right| \\
&\leq \big(1 + |A_{N-1} - A_0|\big) \sum_{m=N+1}^{\infty} \varepsilon^{m-1} z^m + \sum_{m=2}^{N} \varepsilon^{m-1} z^m |A_{N-m} - A_{N-1}| \\
&\leq \left(1 + \frac{Kq}{1-q}\right)\frac{\eta^N z^{N+1}}{1-\eta z} + \frac{Kq}{1-q}\frac{q^N}{\eta}\sum_{m=2}^{\infty}\left(\frac{\eta z}{q}\right)^m \\
&= \left(1 + \frac{Kq}{1-q}\right)\frac{z(\eta z)^N}{1-\eta z} + Kq^N \frac{\eta z^2}{(1-q)(q-\eta z)}.
\end{aligned} \tag{6.50}$$

Now recall that $1 + Kq/(1-q) \leq K(1-\eta z)/2z$ and recall the estimate in (6.48). Furthermore, observe that $\eta z \leq \sqrt{\eta z} = q$. This implies that the right-hand side of (6.50) is at most Kq^N, which finishes the proof of the induction step. □

7. Remaining proofs

In this section we prove the remaining results in Section 2: Theorems 2.4–2.6. All the proofs are minor adaptations of the proof in Section 5, and we emphasize the differences only. Theorem 2.4 is proved in Section 7.1, Theorem 2.5 in Section 7.2, and Theorem 2.6 in Section 7.3.

7.1. Proof of Theorem 2.4

The main result proved in this section is the analog of Proposition 5.1 for the case where the strength of self-repellence β is coupled to the length of the polymer n:

Proposition 7.1. *Assume* (1.1). *If β is replaced by β_n satisfying $\beta_n \to 0$ and $\beta_n n^{\frac{3}{2}} \to \infty$ as $n \to \infty$, then*

$$-\lim_{n\to\infty} \frac{1}{\beta_n^{\frac{2}{3}} n} \log E\Big(e^{-\beta_n H_n} \mathbb{1}_{\{S_n \geq b\beta_n^{\frac{1}{3}} n\}}\Big) = \widehat{I_1^\sigma}(b), \quad b \geq b^* \sigma^{\frac{2}{3}}, \tag{7.1}$$

$$-\lim_{n\to\infty} \frac{1}{\beta_n^{\frac{2}{3}} n} \log E\Big(e^{-\beta_n H_n} \mathbb{1}_{\{0\leq S_n \leq b\beta_n^{\frac{1}{3}} n\}}\Big) = \widehat{I_1^\sigma}(b), \quad b^{**}\sigma^{\frac{2}{3}} < b \leq b^* \sigma^{\frac{2}{3}}, \tag{7.2}$$

$$-\liminf_{n\to\infty} \frac{1}{\beta_n^{\frac{2}{3}} n} \log E\Big(e^{-\beta_n H_n} \mathbb{1}_{\{0\leq S_n \leq b\beta_n^{\frac{1}{3}} n\}}\Big) \leq \widehat{I_1^\sigma}(b), \quad 0 \leq b \leq b^{**} \sigma^{\frac{2}{3}}. \tag{7.3}$$

The proof of Proposition 7.1 is identical to that of Proposition 5.1 after we replace the double limit $n \to \infty$, $\beta \downarrow 0$ (in this order) by the single limit $n \to \infty$ with the restrictions $\beta_n \to 0$, $\beta_n n^{\frac{3}{2}} \to \infty$. The latter implies that $T_{\beta_n} = \beta_n^{-\frac{2}{3}} T = o(n)$, and it is actually only this fact that is needed in the proof. Therefore we can simply copy the proofs in Sections 5.3.1–5.3.2 to derive Proposition 7.1. The reader is asked to check the details. Proposition 7.1 in turns implies Theorem 2.4(i).

A similar result holds when σ is coupled to n with the restrictions $\sigma_n \to \infty$, $\sigma_n n^{-\frac{3}{2}} \to 0$, and we refrain from writing this result down. The latter again implies that $T_{\sigma_n} = \sigma_n^{\frac{2}{3}} T = o(n)$. The approximate large deviation result in turn implies Theorem 2.4(ii). □

7.2. Proof of Theorem 2.5

Define rate functions $I^+_{\beta,\gamma}$ and $I^-_{\beta,\gamma}$ as in (5.1) with βH_n replaced by $H_n^{\beta,\gamma}$. Recall (2.9). The main result in this section is the following.

Proposition 7.2. *Fix $\sigma \in (0, \infty)$. Then, under* (1.1),

$$\liminf_{\beta,\gamma} (\beta-\gamma)^{-\frac{2}{3}} I^-_{\beta,\gamma}\big(b(\beta-\gamma)^{\frac{1}{3}}; b^*(\beta-\gamma)^{\frac{1}{3}}\sigma^{\frac{2}{3}}\big) \geq \widehat{I_1^\sigma}(b), \quad b \geq 0, \tag{7.4}$$

$$\limsup_{\beta,\gamma} (\beta-\gamma)^{-\frac{2}{3}} I^+_{\beta,\gamma}\big(b(\beta-\gamma)^{\frac{1}{3}}; b^*(\beta-\gamma)^{\frac{1}{3}}\sigma^{\frac{2}{3}}\big) \leq \widehat{I_1^\sigma}(b), \quad b > b^{**}\sigma^{\frac{2}{3}}. \tag{7.5}$$

Proposition 7.2 implies Theorem 2.5, analogously to the proof in Section 5.2. We believe that Proposition 7.2 and Theorem 2.5 fail without the restrictions on β, γ in (2.9).

We will next prove Proposition 7.2. The idea behind the proof is that, whenever (2.9) is satisfied, the model with self-repellence parameter β and self-attraction parameter γ is close to the Domb-Joyce model with self-repellence parameter $\beta-\gamma$ (see e.g. (7.6) below). We will need to obtain bounds on the difference between the approximate rate function in these two models. This is done in Sections 7.2.1–7.2.2.

7.2.1. Proof of (7.4)

Fix $b \geq b^*\sigma^{\frac{2}{3}}$. Fix $T > 0$ and put $T_{\beta,\gamma} = T_{\beta-\gamma} = T(\beta-\gamma)^{-\frac{2}{3}}$. (Again, assume for notational convenience that both $T_{\beta-\gamma}$ and $n/T_{\beta-\gamma}$ are integers.) First note that the interaction in (2.8) may be written as

$$H_n^{\beta,\gamma} = (\beta-\gamma)H_n + \frac{\gamma}{2}G_n, \tag{7.6}$$

where H_n is the interaction of the Domb-Joyce model in (1.4), and

$$G_n = \sum_{x\in\mathbb{Z}}[\ell_n(x) - \ell_n(x+1)]^2. \tag{7.7}$$

(Absorb the terms $n+1$ in (1.4) and $\beta(n+1)$ in (2.8) into the normalization.) Therefore, in the remaining proof, we will need to show that the extra term $\frac{\gamma}{2}G_n$ does not alter the approximate large deviation rate function.

Define

$$Y_n^{\beta,\gamma}(b) = E\big(e^{-H_n^{\beta,\gamma}}\mathbb{1}_{\{S_n \geq b(\beta-\gamma)^{\frac{1}{3}}n\}}\big). \tag{7.8}$$

To get the lower bound, simply estimate $H_n^{\beta,\gamma} \geq (\beta-\gamma)H_n$ in (7.6), which implies that $Y_n^{\beta,\gamma}(b) \leq Y_n^{\beta-\gamma,0}(b)$. Hence

$$\limsup_{n\to\infty}\frac{1}{n}\log Y_n^{\beta,\gamma}(b) \leq \limsup_{n\to\infty}\frac{1}{n}\log E\big(e^{-(\beta-\gamma)H_n}\mathbb{1}_{\{S_n \geq b(\beta-\gamma)^{\frac{1}{3}}n\}}\big). \tag{7.9}$$

The right-hand side is nothing but the approximative rate function $I^+_{\beta-\gamma}$ defined in (5.1). Hence, (7.4) follows from (5.3). □

7.2.2. Proof of (7.5)

1. Like in Section 5.3.2, we first show that (recall (5.35))

$$\begin{aligned}
&(\beta-\gamma)^{-\frac{2}{3}}I^+_{\beta,\gamma}\big(b(\beta-\gamma)^{-\frac{2}{3}}; b^*(\beta-\gamma)^{\frac{1}{3}}\sigma^{\frac{2}{3}}\big)\\
&\quad= -(\beta-\gamma)^{-\frac{2}{3}}\liminf_{n\to\infty}\frac{1}{n}\log Y_n^{\beta,\gamma}(b) \leq \frac{4\delta C^2}{T}\frac{\beta}{\beta-\gamma}\\
&\qquad-\frac{1}{T}\log E\Big(e^{-H^{\beta,\gamma}_{T_{\beta,\gamma}}}\mathbb{1}_{\mathcal{E}(\delta,T,\beta-\gamma)}\mathbb{1}_{\mathcal{E}^{\leq}(\delta,T,C,\beta-\gamma)}\mathbb{1}_{\{S_{T_{\beta-\gamma}} \geq b(\beta-\gamma)^{\frac{1}{3}}T_{\beta,\gamma}\}}\Big).
\end{aligned} \tag{7.10}$$

To see (7.10), we use that we can bound $G_{n+m} \geq G_n + G_m$, so that we only need to prove the bound for $(\beta - \gamma)H_n$ in (7.6). However, this is precisely what is proved in Section 5.3.2.

With the help of (7.6) for $n = T_{\beta-\gamma}$ and the inequality $e^{-x} \geq 1-x$, we estimate

$$\begin{aligned} e^{-H^{\beta,\gamma}_{T_{\beta,\gamma}}} &= e^{-(\beta-\gamma)H_{T_{\beta-\gamma}}} e^{-\frac{\gamma}{2}G_{T_{\beta-\gamma}}} \\ &\geq e^{-(\beta-\gamma)H_{T_{\beta-\gamma}}}\big[1 - \tfrac{\gamma}{2}G_{T_{\beta-\gamma}}\big] \geq e^{-(\beta-\gamma)H_{T_{\beta-\gamma}}} - \tfrac{\gamma}{2}G_{T_{\beta-\gamma}}. \end{aligned} \tag{7.11}$$

As to the second term on the right-hand side of (7.11), in part **2** we show that

$$\lim_{\beta,\gamma} \tfrac{\gamma}{2} E\big(G_{T_{\beta-\gamma}}\big) = 0, \tag{7.12}$$

where $\lim_{\beta,\gamma}$ is the limit in (2.9). Hence, applying $\lim_{\beta,\gamma}$ on the right-hand side of (7.10), we see that the remainder of the proof is now the same as in Section 5.3.1 after (5.35). Thus, the proof is finished as soon as (7.12) is proved.

2. In order to prove (7.12), we compute

$$\begin{aligned} &E\Big(\sum_{x\in\mathbb{Z}}[\ell_n(x) - \ell_n(x+1)]^2\Big) \\ &= \sum_{i,j=0}^{n} \big[2P(S_i = S_j) - P(S_i + 1 = S_j) - P(S_i - 1 = S_j)\big] \\ &= \sum_{i,j=0}^{n} \big[2P(S_{|i-j|} = 0) - P(S_{|i-j|} = 1) - P(S_{|i-j|} = -1)\big] \\ &= 2n + \sum_{k=1}^{n}\sum_{j=0}^{n-k} \big[2P(S_k = 0) - P(S_k = 1) - P(S_k = -1)\big] \\ &= 2n + \sum_{k=1}^{n}(n-k+1)\big[2P(S_k = 0) - P(S_k = 1) - P(S_k = -1)\big]. \end{aligned} \tag{7.13}$$

We must show that the right-hand side of (7.13) is $\mathcal{O}(n)$, because then (7.12) follows via our assumption that $\gamma(\beta-\gamma)^{-\frac{2}{3}} \to 0$. We thus complete the proof by showing that, as $n \to \infty$,

$$\sum_{k=1}^{n}(n-k+1)\big[2P(S_k = 0) - P(S_k = 1) - P(S_k = -1)\big] = \mathcal{O}(n). \tag{7.14}$$

In order to prove (7.14), let $\phi(t) = E(e^{\mathrm{i}tS_1})$ denote the characteristic function of S_1. We have

$$P(S_k = x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}tx}\phi(t)^k\,\mathrm{d}t, \qquad x\in\mathbb{Z}, k\in\mathbb{N}. \tag{7.15}$$

In particular,

$$2P(S_k = 0) - P(S_k = 1) - P(S_k = -1) = \frac{1}{\pi}\int_{-\pi}^{\pi} [1 - \cos t]\phi(t)^k \,\mathrm{d}t. \quad (7.16)$$

Abbreviate the left-hand side of (7.14) by B_n. Then (7.16) says that

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\Big[[1 - \cos t]\sum_{k=1}^{n}(n + 1 - k)\phi(t)^k\Big]\,\mathrm{d}t. \quad (7.17)$$

We next use that

$$\sum_{k=1}^{n}(n + 1 - k)\phi^k = \frac{n\phi}{1-\phi} - \phi\frac{1-\phi^n}{[1-\phi]^2}, \qquad \text{on } \{\phi \neq 1\}, \quad (7.18)$$

to arrive at

$$B_n = n\,\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(t)\frac{1-\cos t}{1-\phi(t)}\,\mathrm{d}t - \frac{1}{\pi}\int_{-\pi}^{\pi}\Big[\phi(t)\frac{1-\cos t}{1-\phi(t)}\,\frac{1-\phi^n(t)}{1-\phi(t)}\Big]\,\mathrm{d}t. \quad (7.19)$$

For the first term, we use that $|\phi(t)| \leq 1$, $t \in [-\pi, \pi]$, that the map $t \mapsto \frac{1-\cos t}{1-\phi(t)}$ is bounded on $[-\pi, \pi] \setminus \{0\}$, since the only value where $\phi(t) = 1$ is $t = 0$, and that

$$\lim_{t\to 0}\frac{1-\cos t}{1-\phi(t)} = \frac{1}{\sigma^2}.$$

This shows that the first term is of order $\mathcal{O}(n)$. For the second term, we use that

$$\Big|\frac{1-\phi^n(t)}{1-\phi(t)}\Big| = \Big|\sum_{k=0}^{n-1}\phi^k(t)\Big| \leq n, \qquad t \in [-\pi, \pi] \setminus \{0\}, \quad (7.20)$$

along with the reasoning for the first term, to conclude that also the second term in (7.19) is of order $\mathcal{O}(n)$. □

7.3. Proof of Theorem 2.6

Let I_L^+ and I_L^- denote the two approximative rate functions for the endpoint of the *first* coordinate, S_n, with the convention $e^{-\infty H_n} = \mathbb{1}_{\{H_n=0\}}$, i.e.,

$$I_L^+(\theta;\widetilde{\theta}) = \begin{cases} -\liminf\limits_{n\to\infty}\frac{1}{n}\log P^L\big(H_n = 0, S_n \geq \theta n\big) & \text{if } \theta \geq \widetilde{\theta}, \\ -\liminf\limits_{n\to\infty}\frac{1}{n}\log P^L\big(H_n = 0, 0 \leq S_n \leq \theta n\big) & \text{if } 0 \leq \theta \leq \widetilde{\theta}, \end{cases} \quad (7.21)$$

and similarly for I_L^- with lim sup. The result below identifies the asymptotics of these rate functions in the limit as $n \to \infty$ followed by $L \to \infty$, and also when the two limits are coupled in a certain way:

Proposition 7.3. *Fix* $\sigma \in (0, \infty)$ *and assume* (1.1).

(i) *Then*

$$\liminf_{L\to\infty} L^{\frac{2}{3}} I_L^-\big(b(4L)^{-\frac{1}{3}}; b^*(4L)^{-\frac{1}{3}}\sigma^{\frac{2}{3}}\big) \geq \widehat{I}_1^\sigma(b), \quad b \geq 0, \tag{7.22}$$

$$\limsup_{L\to\infty} L^{\frac{2}{3}} I_L^+\big(b(4L)^{-\frac{1}{3}}; b^*(4L)^{-\frac{1}{3}}\sigma^{\frac{2}{3}}\big) \leq \widehat{I}_1^\sigma(b), \quad b > b^{**}\sigma^{\frac{2}{3}} \tag{7.23}$$

(ii) *If L is replaced by L_n satisfying $L_n \to \infty$ and $L_n n^{-\frac{3}{2}} \to 0$ as $n \to \infty$, then*

$$-\lim_{n\to\infty} \frac{1}{(4L_n)^{-\frac{2}{3}} n} \log P^{L_n}\Big(H_n = 0, S_n \geq b(4L_n)^{-\frac{1}{3}} n\Big) = \widehat{I}_1^\sigma(b), \quad b \geq b^*\sigma^{\frac{2}{3}}, \tag{7.24}$$

$$-\lim_{n\to\infty} \frac{1}{(4L_n)^{-\frac{2}{3}} n} \log P^{L_n}\Big(H_n = 0, 0 \leq S_n \leq b(4L_n)^{-\frac{1}{3}} n\Big) = \widehat{I}_1^\sigma(b), \quad b^{**}\sigma^{\frac{2}{3}} < b \leq b^*\sigma^{\frac{2}{3}}, \tag{7.25}$$

$$-\liminf_{n\to\infty} \frac{1}{(4L_n)^{-\frac{2}{3}} n} \log P^{L_n}\Big(H_n = 0, 0 \leq S_n \leq b(4L_n)^{-\frac{1}{3}} n\Big) \leq \widehat{I}_1^\sigma(b), \quad 0 \leq b \leq b^{**}\sigma^{\frac{2}{3}}. \tag{7.26}$$

Analogously to the proof in Section 5.2, Theorem 2.6 is implied by Proposition 7.3.

Proof of Proposition 7.3. The main idea behind the proof is to compute the conditional probability of the event $\{H_n = 0\}$ given the path $S = (S_0, \dots, S_n)$ of the first coordinate. For L large, this turns out to be close to the Domb-Joyce model with parameter $\beta = 1/(4L)$. Then the proof follows the line of reasoning in Section 5.

1. Let us compute the conditional probability of the event $\{H_n = 0\}$, i.e., the path $(X_0, \dots, X_n)$ is self-avoiding, given the path $S = (S_0, \dots, S_n)$ of the first coordinate. Given S, the event $\{H_n = 0\}$ is equal to the event that $U_i^L \neq U_j^L$ for all time pairs $0 \leq i < j \leq n$ at which $S_i = S_j$. Let us denote by $\ell_n(x)$, $x \in \mathbb{Z}$, the local times of S as in (1.5), and by $i_1^x, \dots, i_{\ell_n(x)}^x$ the times at which S hits x. Then $\{H_n = 0\}$ is the event that, for all $x \in \mathbb{Z}$, the random variables $U_{i_1^x}^L, \dots, U_{i_{\ell_n(x)}^x}^L$ are distinct. Since $U_0^L, \dots, U_n^L$ are i.i.d. and uniform on $\{-L, \dots, L\}$, the conditional probability of this event is easily computed:

$$P^L\big(H_n = 0 \,\big|\, S\big) = \prod_{x\in\mathbb{Z}} \prod_{k=0}^{\ell_n(x)-1} \Big(1 - \frac{k}{2L+1}\Big) = \exp\Big\{\sum_{x\in\mathbb{Z}} \sum_{k=0}^{\ell_n(x)-1} \log\Big(1 - \frac{k}{2L+1}\Big)\Big\}. \tag{7.27}$$

2. Fix $b \geq b^*\sigma^{\frac{2}{3}}$. To prove (7.23), use the inequality $\log(1-x) \leq -x$ and the fact that $\sum_{k=0}^{l-1} k = \frac{1}{2} l(l-1)$, to estimate

$$\begin{aligned} &P^L\big(H_n = 0, S_n \geq b(4L)^{-\frac{1}{3}} n\big) \\ &\quad = E^L\Big(\mathbb{1}_{\{S_n \geq b(4L)^{-\frac{1}{3}} n\}} P^L\big(H_n = 0 \,\big|\, S\big)\Big) \\ &\quad \leq E^L\Big(\exp\Big\{-\frac{1}{4L+2}\sum_{x\in\mathbb{Z}} \ell_n(x)[\ell_n(x)-1]\Big\}\mathbb{1}_{\{S_n \geq b(4L)^{-\frac{1}{3}} n\}}\Big) \\ &\quad = E^L\Big(e^{-\frac{1}{4L+2}H_n}\mathbb{1}_{\{S_n \geq b(4L)^{-\frac{1}{3}} n\}}\Big), \end{aligned} \tag{7.28}$$

with H_n denoting the self-intersection local time of S as in (1.4). The right-hand side of (7.28) is nothing but the quantity appearing in (5.1) for the Domb-Joyce model with strength of self-repellence $\beta = \frac{1}{4L+2}$. For $0 \leq b \leq b^{**}\sigma^{\frac{2}{3}}$, the same argument works with $\leq$ replacing $\geq$. Hence, (7.22) directly follows from Proposition 5.1.

3. Fix $b > b^{**}\sigma^{\frac{2}{3}}$. To prove (7.23), we insert the condition that $\max_{x\in\mathbb{Z}} \ell_n(x) \leq \sqrt{L}$. We then have that, for all $0 \leq k (< \ell_n(x)) \leq \sqrt{L}$ and L sufficiently large,

$$\log\Big(1 - \frac{k}{2L+1}\Big) \geq -\frac{k}{2L+1}\Big(1 - \frac{k}{L}\Big) \geq -\frac{k}{2L+1}\Big(1 - \frac{1}{\sqrt{L}}\Big), \tag{7.29}$$

and substituting this into (7.27) we get that

$$\begin{aligned} &P^L\big(H_n = 0, S_n \geq b(4L)^{-\frac{1}{3}} n\big) \\ &\quad \geq E^L\Big(e^{-\frac{1}{4L+2}(1-\frac{1}{\sqrt{L}})H_n}\mathbb{1}_{\{S_n \geq b(4L)^{-\frac{1}{3}} n\}}\mathbb{1}_{\{\max_{x\in\mathbb{Z}} \ell_n(x) \leq \sqrt{L}\}}\Big). \end{aligned} \tag{7.30}$$

Now we can follow the same argument as in Section 5.3.2, noting that the condition $\max_{x\in\mathbb{Z}} \ell_n(x) \leq \sqrt{L}$ is asymptotically negligible as $L \to \infty$.

4. The proof for $L = L_n$ is identical to the above proof and relies on Proposition 7.1. □

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