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Non-existence of phase transition of oriented percolation on Sierpinski carpet lattices

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Abstract. A percolation problem on Sierpinski carpet lattices is considered. It is obtained that the critical probability $\overrightarrow{p_c}$ of oriented percolation is equal to 1. In contrast it was already shown that the critical probability p_c of percolation is strictly less than 1 in Kumagai [9]. This result shows a difference between fractal-like lattice and $\mathbb{Z}^d$ lattice.

1. Introduction

Percolation is studied as an important subject in statistical mechanics because this is one of the simplest models which contains phase transitions of disordered media. Percolation has close relations to disordered electrical networks, ferromagnetism, epidemic models and so on. Percolation models were proposed by Broadbent and Hammersley [1], and have been well studied in the last thirty years. See Grimmett [7] to view the whole of this field.

Percolation problems had been studied mostly on $\mathbb{Z}^d$ lattice until recent years. We note that $\mathbb{Z}^d$ lattice has translation-invariances. In this paper we consider percolation on *fractal-like lattices*. Fractal-like lattices are graphs which correspond to fractals. All of them have a kind of self-similarity, but most of them have no translation invariances. *The Sierpinski gasket* and *the Sierpinski carpet* are well-known examples of fractals. The former is a *finite* ramified fractal (that is, it can be disconnected by removing a finite number of points) and the latter is an *infinite* ramified fractal. See Mandelbrot [12] for details of fractals. In a previous paper [14] we have analysed percolation on the Sierpinski gasket lattice, which has no phase transition. The non-existence of phase transition is induced by the character of finite ramified fractals. Now we focus on *the Sierpinski carpet lattice*. The Sierpinski carpet lattice is a graph which corresponds to the Sierpinski carpet.

Let us define the Sierpinski carpet on $\mathbb{R}^2$ as follows. For $(i, j) \in \{0, 1, 2\}^2$ we set an affine map $\Psi_{(i,j)}$ from $[0, 1]^2$ to $[i/3, (i + 1)/3] \times [j/3, (j + 1)/3]$ which preserves the directions. Set $T = \{(i, j) \in \{0, 1, 2\}^2 \mid (i, j) \neq (1, 1)\}$. It is well-known (see Falconer [6] for example) that there exists a unique nonempty compact set $K \subset [0, 1]^2$ which satisfies the equation that $K = \bigcup_{t\in T} \Psi_t(K)$. We call this K the Sierpinski carpet. Let us define the graph corresponding to K. Set

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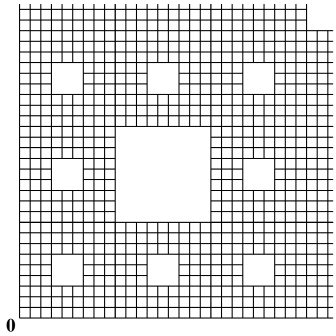

Fig. 1.1. The Sierpinski carpet lattice

$F^n = \bigcup_{t_1,t_2,\cdots,t_n \in T} \Psi_{t_1} \circ \Psi_{t_2} \circ \cdots \circ \Psi_{t_n}([0,1]^2)$. We note that K can be constructed as the limit of F^n. We write $kA = \{ka|\ a \in A\}$. Set $V^n = \mathbb{Z}^2 \cap 3^n F^n$. We denote by $||x||$ the Euclidean norm of x. For a vertex set W we define a bond set $E(W) = \{\langle u, v\rangle|\ u, v \in W, ||u - v|| = 1\}$. Here we wrote $\langle u, v\rangle$ as a bond with endvertices u and v. Set a graph $G^n = (V^n, E(V^n))$. Note that V^n and $E(V^n)$ are increasing sequences with respect to n. Set $G = \bigcup_{n=1}^{\infty} G^n$, that is $G = (V, E)$ where $V = \bigcup_{n=1}^{\infty} V^n$ and $E = \bigcup_{n=1}^{\infty} E(V^n)$. We call this G the Sierpinski carpet lattice. We will define a family of Sierpinski carpet lattices in Section 3.

We consider bond percolation and oriented bond percolation on G. Let $0 \le p \le 1$. Each $e \in E$ is declared to be *open* with probability p and *closed* with probability $1 - p$ independently. We denote by P_p the product measure. Next let us consider a sequence of vertices $\pi = (v_0, v_1, \cdots, v_m)$ where $v_i \in V$ for $0 \le i \le m$. We say π is a *path* when $\langle v_{i-1}, v_i\rangle \in E$ for $1 \le i \le m$ and $v_i \ne v_j$ for $i \ne j$. We give a partial order on $\mathbb{Z}^2$ such that $(x_1, x_2) \le (y_1, y_2)$ if and only if $x_1 \le y_1$ and $x_2 \le y_2$. We say π is an *oriented path* when π is a path and $v_{i-1} \le v_i$ for $1 \le i \le m$. We write $u \leftrightarrow v$ if and only if there exists a path π with $v_0 = u$, $v_m = v$ and $\langle v_{i-1}, v_i\rangle$ are open for $1 \le i \le m$. We denote $C(v) = \{u \in V|\ v \leftrightarrow u\}$. We call $C(v)$ the *open cluster containing* v, and we denote by C the open cluster containing the origin. We define $\theta(p) = P_p(|C| = \infty)$ where $|C|$ means the number of vertices in C. Set $p_c = \inf\{p|\ \theta(p) > 0\}$. We write $u \to v$ if and only if there exists an oriented path π with $v_0 = u$, $v_m = v$ and $\langle v_{i-1}, v_i\rangle$ are open for $1 \le i \le m$. We define $\overrightarrow{C}(v) = \{u \in V|\ v \to u\}$, $\overrightarrow{C}$, $\overrightarrow{\theta}(p)$ and $\overrightarrow{p_c}$ in the same way as $C(v)$, $C, \theta(p)$ and p_c. We write $p_c(S.C.)$ and $\overrightarrow{p_c}(S.C.)$ for p_c and $\overrightarrow{p_c}$ respectively when we want to emphasize its dependence on the graph (the Sierpinski carpet lattice in this case).

We explain studies of percolation on Sierpinski carpet lattices. Kumagai [9] showed that $p_c < 1$ for a family of Sierpinski carpet lattices (which includes the Sierpinski carpet lattice) and studied under an assumption its critical phenomena and uniqueness of infinite cluster for $p > p_c$. Lü [11] gave an alternative proof of $p_c < 1$ using a Peierls argument. Shinoda [15] gave sufficient conditions and necessary conditions to have $p_c < 1$ for generalized Sierpinski carpet lattices. Murai [13] studied an asymptotic behavior as $d \to \infty$ of the critical probability of d-dimensional Sierpinski carpet lattices. Dekking and Meester [5] studied the fractal percolation process (Mandelbrot percolation) on the Sierpinski carpet.

In this paper we study oriented percolation on Sierpinski carpet lattices. Oriented percolation is significant as a model of disordered media because it has close relations to media of semiconductors, contact processes and so on. On $\mathbb{Z}^2$ we may regard this model as a one-dimensional contact process in discrete time. See Durrett [4] and [7] for details. On $\mathbb{Z}^d$ $(d \geq 2)$, it is well-known that the critical probability $p_c(\mathbb{Z}^d)$ of percolation and that $\overrightarrow{p_c}(\mathbb{Z}^d)$ of oriented percolation are strictly less than 1. In particular, $p_c(\mathbb{Z}^2) = 1/2$ has been shown by Kesten [9] and $\overrightarrow{p_c}(\mathbb{Z}^2) \leq 2/3$ has been shown by Liggett [10]. We shall determine the critical probability $\overrightarrow{p_c}(S.C.)$ of oriented percolation on the Sierpinski carpet lattice. By definition $p_c(S.C.) \leq \overrightarrow{p_c}(S.C.)$ is clear. We obtain the following result.

Theorem 1.1. *The critical probability $\overrightarrow{p_c}(S.C.)$ of oriented percolation on the Sierpinski carpet lattice is equal to* 1.

This result is interesting because it shows a difference between the Sierpinski carpet lattice and $\mathbb{Z}^2$ lattice. Theorem 1.1 says that there exists no phase transition of oriented percolation on the Sierpinski carpet lattice, in spite of the existence of phase transition of percolation on it. This kind of extinction of phase transition had been shown by Chayes [2] and Chayes, Pemantle and Peres [3] in the case of the fractal percolation process on the unit square. Theorem 1.1 says also that the contact process will die out if $p < 1$ on the Sierpinski carpet lattice.

We give a proof of Theorem 1.1 in Section 2. In Section 3 we consider this problem on a family of Sierpinski carpet lattices, and give sufficient conditions for non-existence of phase transition.

2. Proof of main theorem

In this section we shall prove the main theorem. In this proof, events of a crossing in a rectangle play important roles. For a rectangle $R \subset \mathbb{R}^2$, we say *left-right crossing* (respectively *bottom-top crossing*) of R exists if $u \to v$ for some u on the left (respectively lower) side of R and some v on the right (respectively upper) side of R. We write $LR(R)$ (respectively $BT(R)$) for the event. This event depends on the configuration of $\{\langle u, v\rangle \mid u, v \in R\}$. For a positive integer k, we write $x_k^n(p) = P_p(LR([0, k \cdot 3^n] \times [0, 3^n]))$. Note that $x_k^n(p)$ is non-increasing with respect to k. In order to show Theorem 1.1 it is enough to prove

$$\lim_{n\to\infty} x_2^n(p) = 0, \tag{2.1}$$

because for any n

$$\{|\overrightarrow{C}| = \infty\} \subset LR([0, 2 \cdot 3^n] \times [0, 3^n]) \cup BT([0, 3^n] \times [0, 2 \cdot 3^n])$$

which implies $\overrightarrow{\theta}(p) \leq 2x_2^n(p)$ by symmetry. We will use the following lemmas.

Lemma 2.1. *Let $p < 1$. There exist $k_0 \geq 1$ and $\varphi > 0$ such that*

$$x_{k_0}^n(p) \leq \mathrm{e}^{-3^n\varphi} \tag{2.2}$$

for any n.

Lemma 2.2. *Let $k \geq 3$. For any n and p,*

$$x_k^{n+1}(p) \leq 2x_{k+1}^n(p). \tag{2.3}$$

Lemma 2.3. *For any n and p,*

$$x_2^{n+1}(p) \leq x_2^n(p)^2 + 2x_5^n(p). \tag{2.4}$$

Lemma 2.4. *For any n and p,*

$$x_2^n(p) \leq \left\{1 - (1-p)^{2^{n+1}}\right\}^2. \tag{2.5}$$

We give a proof of these lemmas one by one.

Proof of Lemma 2.1. For $m \geq 1$ we define a random variable

$$X_m^n = \inf\{j \mid \text{there exists } w \text{ such that } 0 \leq w \leq 3^n \text{ and } (0, w) \to (m, j)\}. \tag{2.6}$$

For convenience we set $X_0^n = 0$, and we set $X_m^n = \infty$ if the right-hand of (2.6) is empty. X_m^n is non-decreasing with respect to m. Set $V_m = ([0, m] \times [0, \infty)) \cap V$ and $E_m = E(V_m)$. Note that X_m^n is determined by the configuration of E_m. For any configuration ω_m of E_m we have

$$P_p(X_{m+1}^n = X_m^n \mid \omega_m) \leq p, \tag{2.7}$$

$$P_p(X_{m+1}^n \geq X_m^n + 1 \mid \omega_m) \geq 1 - p. \tag{2.8}$$

It is clear that

$$x_k^n(p) = P_p(X_{k\cdot 3^n}^n \leq 3^n)$$

by definition. Let $\{Y_i\}_{i=1,2,\ldots}$ be the sequence of independent random variables with $P(Y_i = 1) = 1 - P(Y_i = 0) = 1 - p$ for any i. Then we have

$$P_p(X_{k\cdot 3^n}^n \leq 3^n) \leq P\left(\sum_{i=1}^{k\cdot 3^n} Y_i \leq 3^n\right)$$

by (2.7) and (2.8). The event of right-hand side has been studied well as a sum of independent random variables, such as random walks (see Spitzer [16] for example). If $1/k < 1 - p$ then the probability decays exponentially with respect to 3^n. □

Remark. Lemma 2.1 is true also on $\mathbb{Z}^2$ lattice. In case of $\mathbb{Z}^2$ lattice the conditional probabilities in (2.7) and (2.8) are equal to p and $1 - p$ respectively.

Proof of Lemma 2.2. We set $s = \lfloor (k-1)/2 \rfloor$ where $\lfloor x \rfloor$ means the greatest integer not greater than x. Note that $2s + 1 \leq k$. We observe that

$$LR([0, k \cdot 3^{n+1}] \times [0, 3^{n+1}]) \subset A_1^n \cup A_2^n$$

where

$$A_1^n = LR([0, (3s+2)3^n] \times [0, 3^n]),$$
$$A_2^n = LR([(3s+1)3^n, (2s+1)3^{n+1}] \times [2 \cdot 3^n, 3^{n+1}]).$$

Here we used the property of G that there exists a hole with size $3^n \times 3^n$ centered at $[(2s+1)3^{n+1}/2, 3^{n+1}/2]$. Thus $x_k^{n+1}(p) \leq 2x_{3s+2}^n(p)$ follows. We note that $k + 1 \leq 3s + 2$ when $k \geq 3$, and we have completed the proof. □

Proof of Lemma 2.3. We observe that

$$LR([0, 2 \cdot 3^{n+1}] \times [0, 3^{n+1}]) \subset (A_3^n \cap A_4^n) \cup A_5^n \cup A_6^n$$

where

$$\begin{aligned} A_3^n &= LR([0, 2 \cdot 3^n] \times [0, 3^n]), \\ A_4^n &= LR([4 \cdot 3^n, 2 \cdot 3^{n+1}] \times [2 \cdot 3^n, 3^{n+1}]), \\ A_5^n &= LR([0, 5 \cdot 3^n] \times [0, 3^n]), \\ A_6^n &= LR([3^n, 2 \cdot 3^{n+1}] \times [2 \cdot 3^n, 3^{n+1}]). \end{aligned}$$

We have (2.4) immediately from this relation. □

Proof of Lemma 2.4. Set $E_m^n = \{\langle (m, w), (m+1, w) \rangle \mid 0 \le w \le 3^n\} \cap E$. If $LR([0, 2 \cdot 3^n] \times [0, 3^n])$ occurs, then at least one bond in $E_{(3^n-1)/2}^n$ must be open and so as in $E_{(3^{n+1}-1)/2}^n$. We obtain (2.5) immediately because $\mid E_{(3^n-1)/2}^n \mid = \mid E_{(3^{n+1}-1)/2}^n \mid = 2^{n+1}$. □

We give a proof of Theorem 1.1 by using of these lemmas.

Proof of Theorem 1.1. For $p < 1$ we pick k_0 and $\varphi > 0$ which satisfy (2.2). By (2.3) we obtain

$$x_5^n(p) \le 2x_6^{n-1}(p) \le \cdots \le 2^{k_0-5} x_{k_0}^{n-k_0+5}(p) \le 2^{k_0-5} \mathrm{e}^{-3^{n-k_0+5}\varphi}$$

for $n \ge k_0 - 5$. By this inequality and (2.4) we have

$$x_2^{n+1}(p) \le x_2^n(p)^2 + c\mathrm{e}^{-3^n \psi} \tag{2.9}$$

for some $c < \infty$ and $\psi > 0$. If $\liminf_{n \to \infty} x_2^n(p) < 1$ then (2.1) follows because $\lim_{n \to \infty} c\mathrm{e}^{-3^n \psi} = 0$. Suppose that $\lim_{n \to \infty} x_2^n(p) = 1$. Pick N such that $x_2^n(p) \ge 1/2$ for any $n > N$. By (2.9) and (2.5) we have

$$\begin{aligned} x_2^{n+1}(p) &\le x_2^n(p)^{3/2} \{x_2^n(p)^{1/2} + 2^{3/2} c\mathrm{e}^{-3^n \psi}\} \\ &\le x_2^n(p)^{3/2} \{1 - (1-p)^{2^{n+1}} + 2^{3/2} c\mathrm{e}^{-3^n \psi}\} \end{aligned}$$

for $n > N$. So we can pick N' such that $x_2^{n+1}(p) < x_2^n(p)^{3/2}$ for any $n > N'$. This contradicts to $\lim_{n \to \infty} x_2^n(p) = 1$. □

3. On generalized Sierpinski carpet lattices

In this section we consider oriented percolation on a family of Sierpinski carpet lattices in $\mathbb{Z}^d$, $d \ge 2$. Let a and b be positive integers. We write $L = 2a + b$. For $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \{0, 1, \ldots, L-1\}^d$ we set an affine map $\Psi_{\mathbf{i}}$ from $[0, 1]^d$ to $[i_1/L, (i_1+1)/L] \times [i_2/L, (i_2+1)/L] \times \cdots \times [i_d/L, (i_d+1)/L]$ which preserves the directions. Set

$$T_{a,b}^d = \left\{ (i_1, i_2, \ldots, i_d) \in \{0, 1, \ldots, L-1\}^d \;\middle|\; \left| \{j \mid a \le i_j \le a+b-1\} \right| \le 1 \right\}.$$

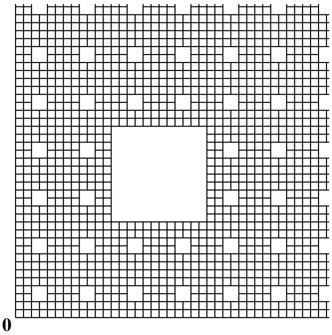

Fig. 3.1. The graph of $G^2_{2,2}$

We take the unique nonempty compact set $K^d_{a,b} \subset [0,1]^d$ which satisfies the equation that $K^d_{a,b} = \bigcup_{\mathbf{i}\in T^d_{a,b}} \Psi_{\mathbf{i}}(K^d_{a,b})$. We note that $K^d_{1,1}$ is called d-dimensional *Menger sponge* (see [12] for example). Set $F^{d,n}_{a,b} = \bigcup_{\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n\in T^d_{a,b}} \Psi_{\mathbf{i}_1}\circ\Psi_{\mathbf{i}_2}\circ\cdots\circ\Psi_{\mathbf{i}_n}([0,1]^d)$. Set $V^{d,n}_{a,b} = \mathbb{Z}^d \cap L^n F^{d,n}_{a,b}$ and $G^{d,n}_{a,b} = (V^{d,n}_{a,b}, E(V^{d,n}_{a,b}))$. We define a graph $G^d_{a,b} = \bigcup_{n=1}^{\infty} G^{d,n}_{a,b}$, that is $G^d_{a,b} = (V^d_{a,b}, E^d_{a,b})$ where $V^d_{a,b} = \bigcup_{n=1}^{\infty} V^{d,n}_{a,b}$ and $E^d_{a,b} = \bigcup_{n=1}^{\infty} E(V^{d,n}_{a,b})$. As an example, the graph of $G^2_{2,2}$ is illustrated in Figure 3.1.

We consider bond percolation and oriented bond percolation on $G^d_{a,b}$. We give a partial order on $\mathbb{Z}^d$ such that $(x_1,x_2,\ldots,x_d) \le (y_1,y_2,\ldots,y_d)$ if and only if $x_i \le y_i$ for $1 \le i \le d$. We define $\theta^d_{a,b}(p)$, $p_c(G^d_{a,b})$, $\overrightarrow{\theta}^d_{a,b}(p)$ and $\overrightarrow{p_c}(G^d_{a,b})$ in a similar fashion as in Section 1. In case of percolation, $p_c(G^d_{a,b}) < 1$ has been shown for all a and b in [9]. In contrast we obtain two theorems in case of oriented percolation.

Theorem 3.1. *Let $d = 2$ and $a \le b$. Then $\overrightarrow{p_c}(G^2_{a,b}) = 1$.*

Theorem 3.2. *Let $2 \le d \le b$. Then $\overrightarrow{p_c}(G^d_{1,b}) = 1$.*

Theorem 3.1 says that on two-dimensional Sierpinski carpet lattices if the ratio of its hole in $T^2_{a,b}$ is not smaller than $1/3^2$ then there is no phase transition. Theorem 3.2 says that for any $d \ge 2$ there exist d-dimensional Sierpinski carpet lattices on which there is no phase transition. We do not know whether $\overrightarrow{p_c}(G^d_{a,b}) = 1$ for *all* d,a and b or not.

Remark. We may define generalized Sierpinski carpet lattices in a different manner. Set $L = 3$ and $T^d_{sc} = \{0,1,2\}^d \setminus \{(1,1,\ldots,1)\}$. Let K^d_{sc} be the unique nonempty compact set which satisfies the equation that $K^d_{sc} = \bigcup_{\mathbf{i}\in T^d_{sc}} \Psi_{\mathbf{i}}(K^d_{sc})$. K^d_{sc} is called *d-dimensional Sierpinski carpet*. Both $K^d_{1,1}$ and K^d_{sc} are a generalization of the Sierpinski carpet in d dimensions. Let G^d_{sc} be the graph corresponding to K^d_{sc}. We note that G^d_{sc} contains $\mathbb{Z}^{d-1}$ lattice as a subgraph, and we observe that $\overrightarrow{p_c}(G^d_{sc}) \le \overrightarrow{p_c}(\mathbb{Z}^{d-1}) < 1$ when $d \ge 3$.

For a rectangle $R = [s_1, t_1] \times [s_2, t_2] \times \cdots \times [s_n, t_n] \subset \mathbb{R}^d$ we denote by $LR(R)$ the event $\{u \to v$ for some $u, v \in R$ with $u_1 = s_1, v_1 = t_1\}$ where u_1 and v_1 mean the first coordinate of u and v respectively. Set $x^n_{k,l}(p) = P_p(LR([0, kL^n] \times [0, lL^n]^{d-1}))$. We notice that $x^n_{k,l}(p)$ depends on d, a and b but we omit to write them. Note that $x^n_{k,l}(p)$ is non-increasing with respect to k and non-decreasing with respect to l.

First we shall prove Theorem 3.1. Recall that $d = 2$ and $a \le b$ in this case. It is enough to show that

$$\lim_{n\to\infty} x^n_{a+b,a}(p) = 0. \tag{3.1}$$

We have already shown this theorem in case of $a = b = 1$ in Section 2. Also in case of $a = 1$ and $b \ge 2$ we can prove (3.1) in exactly the same way. Hereafter we assume that $2 \le a \le b$. We will use the following lemmas.

Lemma 3.3. *Let $p < 1$. There exist $k_0 \ge 1$ and $\varphi > 0$ such that*

$$x^n_{k_0,2a}(p) \le \mathrm{e}^{-L^n\varphi} \tag{3.2}$$

for any n.

Lemma 3.4. *(i) Let $k \ge 2a + 3$. For any n and p,*

$$x^{n+1}_{k,a}(p) \le 2a x^n_{k+1,a}(p). \tag{3.3}$$

(ii) Let $k \ge 4a + 2$. For any n and p,

$$x^{n+1}_{k,2a}(p) \le 2x^n_{3a+2b,a}(p) + (2a-1)x^n_{k+1,2a}(p). \tag{3.4}$$

Lemma 3.5. *For any n and p,*

$$x^{n+1}_{a+b,a}(p) \le x^n_{a+b,a}(p)^2 + 2\left\{x^n_{3a+2b,a}(p) + (a-1)x^n_{4a+3b,2a}(p)\right\}. \tag{3.5}$$

Lemma 3.6. *For any n and p,*

$$x^n_{a+b,a}(p) \le \left\{1 - (1-p)^{(2a)^{n+1}}\right\}^{a+b}. \tag{3.6}$$

Lemma 3.3 and Lemma 3.6 are obtained in exactly the same way as Lemma 2.1 and Lemma 2.4 respectively. We give a proof of Lemma 3.4 and Lemma 3.5 briefly.

Proof of Lemma 3.4. Let $\alpha_1 = \lfloor \lfloor (k-1)/a \rfloor /2 \rfloor$ and $\alpha_2 = \lfloor (k-1)/a \rfloor$. Note that $2\alpha_1 + (a-1)\alpha_2 + 1 \le k$. We have the following relation:

$$LR([0, kL^{n+1}] \times [0, aL^{n+1}]) \subset \bigcup_{j=0}^{a} B^n_j$$

where

$$\begin{aligned} B^n_0 &= LR([0, (\alpha_1 L + a + b)L^n] \times [0, aL^n]), \\ B^n_j &= LR([(\alpha_1 L + (j-1)\alpha_2 L + a)L^n, (\alpha_1 L + j\alpha_2 L + a + b)L^n] \\ &\qquad \times [(jL - a)L^n, (jL + a)L^n]) \quad \text{for } 1 \le j \le a-1, \\ B^n_a &= LR([(\alpha_1 L + (a-1)\alpha_2 L + a)L^n, (2\alpha_1 + (a-1)\alpha_2 + 1)L^{n+1}] \\ &\qquad \times [(aL - a)L^n, aL^{n+1}]). \end{aligned}$$

Thus we obtain

$$x^{n+1}_{k,a}(p) \le 2x^n_{\alpha_1 L+a+b,a}(p) + (a-1)x^n_{\alpha_2 L+b,2a}(p).$$

We note that $x^n_{k,2l}(p) \le 2x^n_{\lfloor k/2\rfloor,l}(p)$ holds for any k and l. So we have

$$x^{n+1}_{k,a}(p) \le 2x^n_{\alpha_1 L+a+b,a}(p) + 2(a-1)x^n_{\lfloor(\alpha_2 L+b)/2\rfloor,a}(p).$$

We note that $\alpha_1 L+a+b \ge \lfloor(\alpha_2 L+b)/2\rfloor$ holds for any k, a and b by the definition of α_1 and α_2. If $k \ge 2a+3$ then $\lfloor(\alpha_2 L+b)/2\rfloor \ge k+1$ because

$$\begin{aligned}\left\lfloor \frac{\alpha_2 L+b}{2}\right\rfloor - (k+1) &\ge \frac{1}{2}\left\{\left\lfloor\frac{k-1}{a}\right\rfloor L + b - 1\right\} - (k+1)\\ &\ge \frac{k-a}{2a}L + \frac{b-1}{2} - (k+1)\\ &= \frac{bk-2a^2-3a}{2a}\end{aligned}$$

and $a \le b$. Thus we have proved (3.3). Let us prove (3.4) in a similar fashion. Let $\beta_1 = \lfloor\lfloor(k-1)/(2a)\rfloor/2\rfloor$ and $\beta_2 = \lfloor(k-1)/(2a)\rfloor$. Note that $2\beta_1 + (2a-1)\beta_2 + 1 \le k$. We obtain

$$x^{n+1}_{k,2a}(p) \le 2x^n_{\beta_1 L+a+b,a}(p) + (2a-1)x^n_{\beta_2 L+b,2a}(p).$$

We observe that $k \ge 4a+1$ implies $\beta_1 \ge 1$ and $bk \ge 4a^2+2a$ implies $\beta_2 L + b \ge k+1$. Thus we have obtained (3.4). □

Proof of Lemma 3.5. Let $\gamma = \lfloor(a+b-2)/(a-1)\rfloor$. We have

$$x^{n+1}_{a+b,a}(p) \le x^n_{a+b,a}(p)^2 + 2\left\{x^n_{3a+2b,a}(p) + (a-1)x^n_{\gamma L+b,2a}(p)\right\}.$$

We observe that $a \le b$ implies $\gamma \ge 2$, and we have obtained (3.5) similarly to the proof of Lemma 2.3. □

Proof of Theorem 3.1. Let $p < 1$. If we prove that there exist $c < \infty$ and $\psi > 0$ such that

$$x^n_{3a+2b,a}(p) + (a-1)x^n_{4a+3b,2a}(p) \le c\mathrm{e}^{-L^n\psi}, \tag{3.7}$$

then by (3.5) and (3.6) we can prove (3.1) in the same way as the proof of Theorem 1.1 in Section 2. Let us prove (3.7). Let k_0 and $\varphi > 0$ satisfy (3.2). By using (3.3) repeatedly we have

$$x^n_{3a+2b,a}(p) \le (2a)^q x^{n-q}_{k_0,a}(p) \le (2a)^q x^{n-q}_{k_0,2a}(p) \le (2a)^q \mathrm{e}^{-L^{n-q}\varphi} \tag{3.8}$$

where $q = k_0 - 3a - 2b$. We have also $x^n_{4a+3b,2a}(p) \le c' \mathrm{e}^{-L^n\varphi'}$ for some $c' < \infty$ and $\varphi' > 0$ in the same way by (3.4),(3.2) and (3.8). □

We turn to prove Theorem 3.2. Recall that $d \geq 2, a = 1$ and $L = 2+b \geq d+2$ in this case. It is enough to show that $\lim_{n\to\infty} x^n_{1+b,1}(p) = 0$. Theorem 3.2 follows immediately from the following two lemmas.

Lemma 3.7. *Let $p < 1$. There exist $k_0 \geq 1$ and $\varphi > 0$ such that*

$$x^n_{k_0,1}(p) \leq \mathrm{e}^{-L^n\varphi} \tag{3.9}$$

for any n.

Lemma 3.8. *Let $k \geq d + 1$. For any n and p,*

$$x^{n+1}_{k,1}(p) \leq d!\, x^n_{k+1,1}(p).$$

Proof of Lemma 3.7. Recall that $G^{d,n}_{1,b} = G^d_{1,b} \cap [0, L^n]^d$, and we regard $G^d_{1,b} \cap ([0, kL^n] \times [0, L^n]^{d-1})$ as a subset of $[0, kL^n] \times G^{d-1,n}_{1,b}$. We denote by Π the set of the oriented paths on $G^{d-1,n}_{1,b}$ starting at the origin. For $\pi \in \Pi$ we define $H(\pi) = \{v \in V^d_{a,b} | \, 0 \leq v_1 \leq kL^n$ and $(v_2, v_3, \ldots , v_d)$ is a vertex of $\pi\}$. We have

$$\begin{aligned} &LR([0, kL^n] \times [0, L^n]^{d-1}) \\ &\quad = \bigcup_{\pi\in\Pi} \{u \to v \text{ in } H(\pi) \text{ for some } u, v \text{ with } u_1 = 0, v_1 = kL^n\}. \end{aligned}$$

Note that the length of $\pi \in \Pi$ is not more than $(d - 1)L^n$. The number of the paths in Π is not more than $d^{(d-1)L^n}$. We have

$$x^n_{k,1}(p) \leq d^{(d-1)L^n} P\Big(\sum_{i=1}^{kL^n} Y_i \leq (d-1)L^n\Big)$$

where Y_i is the random variable defined in the proof of Lemma 2.1. We can pick k_0 sufficiently large to satisfy $P(\sum_{i=1}^{k_0L^n} Y_i \leq (d-1)L^n) \leq \mathrm{e}^{-L^n\varphi}$ and $\mathrm{e}^{-\varphi} < d^{-(d-1)}$. Then (3.9) follows. □

Proof of Lemma 3.8. Let Ξ be the set of the oriented paths from $(0, 0, \ldots , 0)$ to $(1, 1, \ldots , 1)$ on $\mathbb{Z}^{d-1}$: that is, $\xi = (\xi^1, \xi^2, \ldots , \xi^d) \in \Xi$ if and only if $\xi^1 = (0, 0, \ldots , 0)$, $\xi^d = (1, 1, \ldots , 1)$ and $\xi^i \leq \xi^{i+1}$ for $1 \leq i \leq d - 1$ with respect to the partial order on $\mathbb{Z}^{d-1}$. We write $A + x = \{a + x |\, a \in A\}$. For $\xi \in \Xi$ we set $R_{\xi,i} = [0, L^n]^{d-1} + (L - 1)L^n\xi^i$. Let $s = \lfloor (k - 1)/d \rfloor$. Note that $s \geq 1$ and $ds + 1 \leq k$. We observe that

$$LR([0, kL^{n+1}] \times [0, L^{n+1}]^{d-1}) \subset \bigcup_{\xi\in\Xi}\Big(\bigcup_{i=1}^{d} A_{\xi,i}\Big)$$

where $A_{\xi,i} = LR([((i - 1)sL + 1)L^n, (isL + 1 + b)L^n] \times R_{\xi,i})$. We have

$$x^{n+1}_{k,1}(p) \leq (d - 1)! \cdot d \cdot x^n_{sL+b,1}(p) = d!x^n_{sL+b,1}(p)$$

because the number of the paths in Ξ equals $(d-1)!$. Let us prove that $sL+b \geq k+1$ to complete this proof. If $k \geq 3d/2$ then

$$\begin{aligned} sL+b-(k+1) &= \left\lfloor \frac{k-1}{d} \right\rfloor L+b-(k+1) \\ &\geq \frac{k-d}{d} L+b-(k+1) \\ &= \frac{(b+2-d)k-3d}{d} \\ &\geq \frac{2k-3d}{d} \\ &\geq 0. \end{aligned}$$

Suppose that $d+1 \leq k < 3d/2$. Then $s=1$, and $sL+b = 2b+2 \geq 2d+2 \geq k+1$. □

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