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Nathalie Eisenbaum

On the infinite divisibility of squared Gaussian processes

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Abstract. We show that fractional Brownian motions with index in (0, 1] satisfy a remarkable property: their squares are infinitely divisible. We also prove that a large class of Gaussian processes are sharing this property. This property then allows the construction of two-parameters families of processes having the additivity property of the squared Bessel processes.

1. Introduction

The question of the infinite divisibility of squared Gaussian processes was first raised by Lévy (1948) who conjectured that for any Gaussian vector of dimension 2 (X, Y), the distribution of (X^2, Y^2) was not infinitely divisible. Vere-Jones (1967) proved that this conjecture was false and that actually this distribution is always infinitely divisible. Later Griffiths (1970), Evans (1991) have shown examples of p-dimensional Gaussian vectors for $p \geq 3$, such that the corresponding vector of the squares is not infinitely divisible. Then Griffiths (1984) (see also Bapat (1989)) established a characterization of the p-dimensional Gaussian vectors such that the vector of the squares is infinitely divisible. Except for special examples, his criterion is difficult to use since it requires the computation of all the cofactors of the covariance matrix. In particular, till now the condition of Griffiths could not be checked for fractional Brownian motions and $p \geq 3$.

Here we solve this problem by showing that a squared fractional Brownian motion with an index β is infinitely divisible if and only if β is in (0, 1]. This is done in Section 2.

More generally, in Section 3, we show that the condition of Griffiths is satisfied by a large class of Gaussian processes. These Gaussian processes are characterized by the fact that their covariance is equal to the Green function of a transient Markov process. We actually show that if the Green function of a transient Markov process is symmetric, then it is the covariance of a Gaussian process with an infinitely divisible square.

N. Eisenbaum: Laboratoire de Probabilités, UMR 7599, CNRS - Université de Paris VI 4, Place Jussieu, 75252, Paris Cedex 05, France. e-mail: nae@ccr.jussieu.fr

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We are then able to answer a question which is deeply connected to the property of infinite divisibility. Indeed, Shiga and Watanabe (1973) characterized all the two-parameters families of real-valued Markov processes $(Y_{d,x})_{d,x\geq 0}$ indexed by the same set, satisfying

$$Y_{d,b} + Y_{d',b'} \stackrel{\text{(law)}}{=} Y_{d+d',b+b'} \tag{A}$$

for independent processes $Y_{d,b}$ and $Y_{d',b'}$.

A well-known example is given by the family of the squared Bessel processes. The property (A) is called the additivity property. Here, we ask the following question: is there a two-parameters family $(Y_{d,x})_{d,x\geq 0}$ of non-Markovian processes satisfying (A)?

We note that a squared Bessel process of dimension 1, is a squared Brownian motion. In Section 4, we show that similarly to Brownian motion, a Gaussian process with an infinitely divisible square can generate a two-parameter family of processes satisfying (A). The obtained processes are Markovian if and only if the corresponding Gaussian process is itself Markovian.

The connection, described in Section 2 and 3, between Gaussian processes and Markov processes via their symmetric Green function has often been used to obtain from the Gaussian process informations on the Markov process (see for example Marcus and Rosen (1992), (1996) or Bass et al. (2000)). Here we see that it can also be efficient as a tool to solve questions about Gaussian processes. In Section 5, we show how to connect a Gaussian process to a Markov process with a non-symmetric Green function. The arguments developed in the previous sections are not, a priori, available for this class of Gaussian processes. Nevertheless, we suspect that it also should be possible to deduce properties of these Gaussian processes from the associated Markov processes. To illustrate this idea, we show that Barlow's necessary and sufficient condition for the continuity of the local time process of a Lévy process (Barlow (1988)), can be translated as a condition for the continuity of a Gaussian process even when the Lévy process is not symmetric.

2. On the infinite divisibility of squared fractional Brownian motion

A fractional Brownian motion is a real-valued centered Gaussian process with a covariance given by:

$$g(x, y) = |x|^\beta + |y|^\beta - |x - y|^\beta$$

where the index β is in $(0, 2)$.

Theorem 2.1. *A squared fractional Brownian motion is infinitely divisible if and only if its index is in* $(0, 1]$.

We will prove the above theorem by using the following criterion of Griffiths. Indeed, in his paper, Griffiths (1984) considers a p-dimensional multivariate exponential distribution with a Laplace transform

$$\psi(t) = [\det(I + GT)]^{-1}$$

where $t = (t_i)_{1\le i\le p}$, G is a positive definite $p \times p$-matrix, T is the diagonal matrix $T_{ii} = t_i$, and I is the $p \times p$-identity matrix.

He obtains the following characterization of the matrices G for which the Laplace transform ψ is infinitely divisible, i.e. ψ^δ is a Laplace transform for any $\delta > 0$.

Theorem (Griffiths 1984). *The Laplace transform ψ is infinitely divisible if and only if for every product of k cofactors, we have*

$$(-1)^k G^{i_1 i_2} G^{i_2 i_3} \dots G^{i_{k-1} i_k} G^{i_k i_1} \ge 0$$

for $\{i_1, \dots, i_k\}$ any subset of $\{1, \dots, p\}$, $k = 3, \dots, p$.

The function $\sqrt{\psi}$ can be viewed as the Laplace transform of a squared centered Gaussian vector with a covariance matrix G. Hence Griffiths Theorem also provides a criterion for the infinite divisibility of squared Gaussian vectors.

Proof of Theorem 2.1. Consider a fractional Brownian motion with an index β in $(0, 1]$. Multiplied by an appropriate constant, its covariance function g is then equal to the Green function of a symmetric stable process X with an index $(\beta + 1)$ killed at its first hitting time of 0 (see for example Eisenbaum et al. (2000)).

Let $(x_i)_{1\le i\le n}$ be a sequence of distinct real numbers. Let G be the matrix $(g(x_i, x_j)_{1\le i,j\le n}$. To compute the cofactors of G we introduce the following stopping time

$$\sigma = \inf\{t \ge 0 : X_t \in \{x_1, x_2, \dots, x_n\} \setminus \{X_0\}\}.$$

The time σ may be infinite in that case the value of X_σ is a cemetery point. Let $(L^x_t, x \in \mathbb{R}, t \ge 0)$ be the local time process of X. We set: $b_{ij} = \mathbb{E}_{x_i}(L^{x_j}_\sigma)$, and $p_{ij} = \mathbb{P}_{x_i}(X_\sigma = x_j)$. Note that: $b_{ij} = 0$ for $i \ne j$, $p_{ii} = 0$ and $\sum_{j=1}^n p_{ij} = 1 - \mathbb{P}_{x_i}(\sigma = \infty)$. Thanks to the Markov property we have:

$$\begin{aligned} g(x_i, x_j) = \mathbb{E}_{x_i}(L^{x_j}_\infty) &= \mathbb{E}_{x_i}(L^{x_j}_\sigma) + \mathbb{E}_{x_i}(\sigma < +\infty; \mathbb{E}_{X_\sigma}(L^{x_j}_\infty)) \\ &= b_{ij} + \sum_{k=1}^n p_{ik}\, g(x_k, x_j). \end{aligned}$$

Let B and P be the matrices defined by: $B = (b_{ij})_{1\le i,j\le n}$ and $P = (p_{ij})_{1\le i,j\le n}$. The above equation can be written as: $G = B + PG$. Or equivalently as: $B = (I - P)G$. This shows that both $(I - P)$ and G are invertible and that:

$$G^{-1} = B^{-1}(I - P)$$

This implies that the cofactors of G are given by:

$$G^{ij} = \begin{cases} \det(G)\,(b_{ii})^{-1} & \text{if } i = j \\ -\det(G)\, p_{ji}\,(b_{jj})^{-1} & \text{if } i \ne j \end{cases}$$

Once we note that $\det(G)$ is positive, the criterion of Griffiths can immediately be checked and we see that it is satisfied. Hence, a fractional Brownian motion with index β in $(0, 1]$ has an infinitely divisible square.

Assume now that β is in $(1, 2)$. This time we use Griffiths criterion for $k = p = 3$. Indeed, let (η_1, η_2, η_3) be a centered Gaussian vector of dimension 3 with a covariance matrix equal to $(g(x_i, x_j)_{1\leq i,j\leq 3}$ where $x_1 = 1, x_2 = 2$ and $x_3 = 3$. The necessary and sufficient condition of Griffiths for the infinite divisibility of $(\eta_1^2, \eta_2^2, \eta_3^2)$ becomes:

$$-K_\beta(1, 2, 3)K_\beta(1, 3, 2)K_\beta(2, 3, 1) \leq 0$$

where $K_\beta(x, y, z) = g(x, y)g(z, z) - g(x, z)g(y, z)$.

We are going to prove successively that $K_\beta(1, 3, 2) < 0$, $K_\beta(1, 2, 3) > 0$ and $K_\beta(2, 3, 1) > 0$.

We have: $K_\beta(1, 3, 2) = 3.2^\beta(3^{\beta-1} + 1 - 2^\beta)$. It is not difficult to see that the function $(3^{\beta-1} + 1 - 2^\beta)$, defined on $[1, 2]$, realises its minimum at a value $\beta_0 \in (1, 2)$, is strictly decreasing on $[1, \beta_0)$ and strictly increasing on $(\beta_0, 2]$. Since at 1 and 2, this function takes the value 0, we obtain: $K_\beta(1, 3, 2) < 0$ on $(1, 2)$.

Then we have: $K_\beta(1, 2, 3) = 2.6^\beta - 9^\beta + (2^\beta - 1)^2$. We first note that we can write: $K_\beta(1, 2, 3) = (2^\beta)^2 + (2^\beta - 1)^2 - (3^\beta - 2^\beta)^2$. The function $(y^2 + (y - 1)^2 - (3^\beta - y)^2)$ is an increasing function of y on $\mathbb{R}_+$. Moreover the function $((x + 1)^2 + x^2 - (2x - 1)^2)$ is a strictly positive function of x on $(0, 3)$. Choosing $x = 3^{\beta-1}$, we obtain: $(3^{\beta-1} + 1)^2 + (3^{\beta-1})^2 - (3^{\beta-1} - (1 - 3^{\beta-1}))^2 > 0$. We use then the previous argument according which: $3^{\beta-1} + 1 < 2^\beta$, to finally obtain: $(2^\beta)^2 + (2^\beta - 1)^2 - (3^\beta - 2^\beta)^2 > 0$.

The third expression is: $K_\beta(2, 3, 1) = 2^\beta + 4^\beta - 6^\beta + 2.3^\beta - 2$. This can be rewritten as: $K_\beta(2, 3, 1) = 2[2.2^{2(\beta-1)} - (3^\beta - 2)2^{\beta-1} + 3^\beta - 1]$. We note then that the polynomial $(2x^2 - (3^\beta - 2)x + 3^\beta - 1)$ has a discreminant equal to: $((3^\beta - 1)^2 - 10(3^\beta - 1) + 1)$. This discreminant is always negative for $\beta \in (1, 2)$. and hence $K_\beta(2, 3, 1)$ is always strictly positive.

Finally, we have obtain:

$$-K_\beta(1, 2, 3)K_\beta(1, 3, 2)K_\beta(2, 3, 1) > 0.$$

Consequently, we have proved that squared fractional Brownian motions with index in $(1, 2)$ are not infinitely divisible. □

As an immediate consequence of Theorem 2.1, we know that for β in $(1, 2)$, the function g can not be interpreted as a Green function. If it was so then the arguments used for the case $\beta \in (0, 1]$ would hold and the corresponding squared fractional Brownian motion would be infinitely divisible.

3. A class of Gaussian processes with infinitely divisible squares

Here is a way to associate a Gaussian process with a not necessarily symmetric Markov process. Consider a transient Markov process X with a state space E admitting a local time process $(L_t^x, x \in E, t \geq 0)$. For any $(x, y) \in E^2$, we have:

$$\mathbb{E}_x(L_\infty^y) < +\infty$$

We set: $g(x, y) = \mathbb{E}_x(L_\infty^y)$. The function g is called the Green function of X.

Theorem 3.1. *If the Green function of a transient Markov process is symmetric then it is definite positive.*

Proof. We will make use of the following property. Let a, x, y be three elements of E, we have then:

$$\mathbb{E}_a(L^x_\infty L^y_\infty) = g(a,x)g(x,y) + g(a,y)g(y,x) \tag{1}$$

This identity can be easily established by a direct computation, but it can also be seen as a simple consequence of the Feynman-Kac Formula (see for example Rogers and Williams (1994) section III.19).

Let a be an element of the state space. Assume that a is such that $g(a,a) > 0$. We define the following h-transform of X.

$$\overline{\mathbb{P}}_a|_{\mathcal{F}_t} = \frac{g(X_t, a)}{g(a,a)} \mathbb{P}_a|_{\mathcal{F}_t}$$

where $\mathcal{F}_t$ denotes the field generated by $(X_s, 0 \le s \le t)$. Under $\overline{\mathbb{P}}_a$, the process X starts at a and is killed at its last visit to a. Since this h-transform of X is still a homogenous Markov process, we can use (1) to obtain:

$$\overline{\mathbb{E}}_a(L^x_\infty L^y_\infty) = \overline{g}(a,x)\overline{g}(x,y) + \overline{g}(a,y)\overline{g}(y,x)$$

where $\overline{g}$ is the Green function of X under $\overline{\mathbb{P}}_a$. Since we have:

$$\overline{g}(x,y) = g(x,y)\frac{g(y,a)}{g(x,a)}$$

we obtain:

$$\overline{\mathbb{E}}_a(L^x_\infty L^y_\infty) = \frac{g(a,x)g(y,a)}{g(a,a)} g(x,y) + \frac{g(a,y)g(x,a)}{g(a,a)} g(y,x) \tag{2}$$

Assume now that the function g is symmetric. Let $(a_i)_{1 \le i \le n}$ be a sequence of $\mathbb{R}$ and $(x_i)_{1 \le i \le n}$ a sequence of E. We set: $f(x_i) = \frac{a_i}{g(a,x_i)} 1_{\{g(a,x_i)\}>0}$. We have then thanks to (2):

$$\sum_{i,j} a_i a_j g(x_i, x_j) = \frac{g(a,a)}{2} \overline{\mathbb{E}}_a[(\sum_i f(x_i) L^{x_i}_\infty)^2]$$

Consequently, g is definite positive. □

Thanks to Theorem 3.1, we know that a symmetric Green function can be interpreted as a covariance function. We can then use the proof of Theorem 2.1 to obtain similarly the following theorem.

Theorem 3.2. *Let X be a transient Markov process with a symmetric Green function g. Let η be a centered Gaussian process with a covariance equal to g. Then η^2 is infinitely divisible.*

4. Two-parameters families with additivity properties

Let $(Y_{d,b})_{d,b\geq 0}$ be a family of random processes indexed by the same set E, and satisfying the following additivity property

$$Y_{d,b} \;+\; Y_{d',b'} \overset{\text{(law)}}{=} Y_{d+d',b+b'} \tag{A}$$

for independent processes $Y_{d,b}$ and $Y_{d',b'}$.

As an immediate consequence of the property (A), for any (d, b), the process $Y_{d,b}$ is infinitely divisible. At this level, Shiga and Watanabe (1973) made use of the Lévy-Khintchine formula for each marginal and assumed that each $Y_{d,b}$ was a Markov process indexed by $\mathbb{R}_+$. In order to show other families satisfying (A), we would like to obtain a family $(Y_{d,b})_{d,b\geq 0}$ such that $Y_{1,0}$ is a squared centered Gaussian process.

Let $(\eta_x)_{x\in E}$ be a centered Gaussian process indexed by a topological space E. We denote by o a fixed element of E. Assume we know that η^2 is infinitely divisible and that $\eta_o = 0$, we set then: $Y_{1,0} = \eta^2$. If we exclude the trivial solution: $Y_{1,b} = \eta^2 + b$, the right candidate for being $Y_{1,b}$ becomes $(\eta + \sqrt{b})^2$. Hence a natural question arises: Is the process $(\eta + r)^2$ also infinitely divisible for any real number r? There is no known criterion for this last property, but we have the following lemma.

Lemma 4.1. *Let $(\eta_x)_{x\in E}$ be a centered Gaussian process such that $\eta_o = 0$. We have then:*

(i) If $(\eta_x + r)^2_{x\in E}$ is infinitely divisible for any real number r then for any $b > 0$, there exists a process $(Z_b(x))_{x\in E}$ independent of $(\eta_x)_{x\in E}$ such that $Z_b(o) = b$ and for any finite subset F of E

$$(Z_b(x) + \eta_x^2)_{x\in F} \overset{\text{(law)}}{=} (\eta_x + \sqrt{b})^2_{x\in F} \tag{3}$$

(ii) If $(\eta_x)_{x\in E}$ is infinitely divisible and for any $b > 0$ there exists a process $(Z_b(x))_{x\in E}$ satisfying (3)*, then $(\eta_x + \sqrt{b})^2_{x\in E}$ is infinitely divisible.*

Proof. The proof relies on the well-known expression of the Laplace transform of $(\eta_x + r)^2_{x\in F}$. Indeed, let $x = (x_i)_{1\leq i\leq n}$ be a sequence of elements of E and $\alpha = (\alpha_i)_{1\leq i\leq n}$ a sequence of positive real numbers. There exist then two constants $c(\alpha, x)$ and $f(\alpha, x)$ such that for any real r:

$$\mathbb{E}(\exp\{-\sum_{i=1}^{n} \alpha_i \frac{(\eta_{x_i} + r)^2}{2}\}) \;=\; c(\alpha, x)\exp\{-\frac{r^2}{2} f(\alpha, x)\} \tag{4}$$

By the assumption of (i), we know that for any $n \in \mathbb{N}^*$ the expression $[c(\alpha, x)]^{1/n}\exp\{-\frac{r^2}{2} f(\alpha, x)\}$ is still a Laplace transform in α. On the other hand, $\exp\{-\frac{r^2}{2} f(\alpha, x)\}$ is a continuous function of α. Consequently, when n tends to ∞ in the above expression, the limit $\exp\{-\frac{r^2}{2} f(\alpha, x)\}$ appears as a Laplace transform in α. This proves (i).

The sum of two independent infinitely divisible variables being infinitely divisible, the proof of (ii) is immediate. □

Assume now that $(\eta_x)_{x\in E}$ is a centered Gaussian process with an infinitely divisible square and such that for any $b > 0$ there exists a process Z_b satisfying (3). Then we can set: $Y_{1,b} = (\eta + \sqrt{b})^2$ and $Y_{0,b} = Z_b$. Moreover since for any r $(\eta + r)^2$ is infinitely divisible, then for any $d > 0$ and any $b \geq 0$, there exists a process $Y_{d,b}$ such that for any sequence of positive numbers $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and any sequence of $E(x_1, x_2, \dots, x_n)$:

$$\mathbb{E}(\exp\{-\sum_{i=1}^{n} \alpha_i Y_{d,b}(x_i)\}) = [c(\alpha, x)]^d \exp\{-\frac{b}{2} f(\alpha, x)\}$$

Setting: $Y_{0,0} = 0$, we immediately check that the family $(Y_{d,b})_{d,b\geq 0}$ satisfies the additivity property (A).

The problem becomes now to find a Gaussian process η satisfying these assumptions. The following theorem provides a large class of such Gaussian processes.

Theorem 4.2. *Let X be a recurrent Markov process with a state space E. Let o be an element of E. Assume that the Green function of X killed at the first hitting time of the value o is symmetric. Let $(\eta_x)_{x\in E}$ be a centered Gaussian process with a covariance equal to this Green function. Then for any real r, the process $(\eta_x + r)^2_{x\in E}$ is infinitely divisible. Morover there exists a two-parameter family $(Y_{\delta,b})_{\delta,b\geq 0}$ of processes indexed by E such that $Y_{1,b} = (\eta + \sqrt{b})^2$ and satisfying the additivity property (A).*

Proof. Thanks to Section 3, we know that there exists a centered Gaussian process η with a covariance equal to the Green function of X killed at o and that its square is infinitely divisible. We use then a Ray-Knight Theorem established in Eisenbaum et al. (2000). Indeed, let $(L_t^x, x \in E, t \geq 0)$ be the local time process of X, then setting for any $r > 0$, $\tau_r = \inf\{t \geq 0 : L_t^o > r\}$, we have

$$(L_{\tau_r}^x + \frac{\eta_x^2}{2})_{x\in E} \overset{(\text{law})}{=} (\frac{(\eta_x + \sqrt{2r})^2}{2})_{x\in E}. \tag{5}$$

Actually, this last identity has been established in Eisenbaum et al. (2000) for strongly symmetric Markov processes. But assuming the symmetry of the Green function is clearly sufficient to establish (5) since once the existence of η is established, the proof relies on the Feynman Kac's Formula.

Thanks to Lemma 4.1 and the previous remark, Theorem 4.2 is immediately obtained. □

Example. Let η be a fractional Brownian motion with an index β in (0, 1]. Then we know, thanks to Theorem 4.2, that the process $(\eta + r)^2$ is infinitly divisible for any real r. Moreover there exists a family $(Y_{d,b})_{d,b\geq 0}$ satisfying (A) such that $Y_{1,b} = (\eta + \sqrt{b})^2$.

What is the link between Shiga and Watanabe's families and those provided by Theorem 4.3?

The intersection of the two types of families is not empty and corresponds to the case when the process η^2 is Markovian. This is exactly the case when X is a diffusion. If X is a Brownian motion, then η is also a Brownian motion and the obtained family represents the squared Bessel processes. For a general diffusion, the corresponding family is obtained by multiplying the squared Bessel processes by a deterministic function, and making a deterministic time change.

But when X does not have continuous paths, the process $(L^x_{\tau_r}, x \in E)$ is not Markovian, neither the process η^2. Consequently the corresponding family $(Y_{d,b})_{d,b\geq 0}$ is not a Shiga and Watanabe's family.

5. Gaussian processes associated to non-symmetric Green functions

To enlarge the class of Gaussian processes connected to Markov processes, we have the following theorem.

Theorem 5.1. *Let X be a Markov process. Let a be a value of its state space. Denote by g_{T_a} the Green function of X killed at T_a the first hitting time of a. Then the function $(g_{T_a}(x,y) + g_{T_a}(y,x))_{(x,y)\in E^2}$ is definite positive.*

Proof. We assume first that X is recurrent. For $t > 0$, let τ_t be the stopping time defined by: $\tau_t = \inf\{s \geq 0 : L^a_s > t\}$. Let T be an exponential time independent of X, with mean θ. Then X killed at τ_T is still a homogenous Markov process. Applying identity (1) of the proof of Theorem 3.1, to X killed at τ_T, we have:

$$\mathbb{E}_a(L^x_{\tau_T} L^y_{\tau_T}) = \mathbb{E}_a(L^x_{\tau_T})\mathbb{E}_x(L^y_{\tau_T}) + \mathbb{E}_a(L^y_{\tau_T})\mathbb{E}_y(L^x_{\tau_T})$$

Making use, for example, of the proof of Lemma 12 of Bertoin ((1996), p.145), we know that: $\mathbb{E}_a(L^y_{\tau_t}) = t$, for any y. Hence we obtain:

$$\mathbb{E}_a(L^y_{\tau_T}) = \mathbb{E}(T)$$

then:

$$\mathbb{E}_a(L^x_{\tau_T} L^y_{\tau_T}) = \mathbb{E}(T)\big(\mathbb{E}_x(L^y_{\tau_T}) + \mathbb{E}_y(L^x_{\tau_T})\big)$$

Consequently:

$$\sum_{i,j} a_i a_j \big(\mathbb{E}_{x_i}(L^{x_j}_{\tau_T}) + \mathbb{E}_{x_j}(L^{x_i}_{\tau_T})\big) = \frac{1}{\mathbb{E}(T)}\mathbb{E}_a\Big(\sum_i a_i L^{x_i}_{\tau_T}\Big)^2$$

which shows that the function $(\mathbb{E}_x(L^y_{\tau_T}) + \mathbb{E}_y(L^x_{\tau_T}))_{(x,y)\in E^2}$ is definite positive. Note that we have:

$$\mathbb{E}_x(L^y_{\tau_T}) = \mathbb{E}_x(L^y_{T_a}) + \mathbb{E}_a(L^y_{\tau_T})$$

hence for any $\theta > 0$, the function $(\mathbb{E}_x(L^y_{T_a}) + \mathbb{E}_y(L^x_{T_a}) + 2\theta)_{(x,y)\in E^2}$ is definite positive. By letting θ tend to 0 we finally obtain that $(\mathbb{E}_x(L^y_{T_a}) + \mathbb{E}_y(L^x_{T_a}))_{(x,y)\in E^2}$ is definite positive.

Assume now that X is transient. We can then associate to X a recurrent Markov process $\hat{X}$ as follows. Let $(e_s, s > 0)$ denote the excursion process of X around a. Let $R(e)$ be the length of the excursion e. We assume that the Poisson point process of the finite excursions of X has an infinite life time. Let n^f be the restriction to the finite excursions of n, the Ito measure of X. That is, for $0 < t_1 < t_2 < \ldots < t_k$ and $A_1, \ldots, A_k$ measurable sets of E

$$\begin{aligned} n^f(X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k) &= n(X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k, R < \infty) \\ &= n(X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k; \mathbb{P}_{X_{t_k}}(T_a < \infty)) \end{aligned}$$

Then $\hat{X}$ is a Markov process with a law constructed from the excursion law n^f (see Ito (1970)) and possibly the drift of (τ_t) if the time that X spends at a has positive Lebesgue measure.

Let X^{T_a} be the process X killed at T_a. Let $(\hat{L}_t^x, x \in E, t \geq 0)$ be the local time process of $\hat{X}$ and $\hat{T}_a$ its first hitting time of a. Note that $\hat{X}$ killed at its first hitting time of a coincides with X^{T_a} conditionned to die at a, i.e. the h-transform of X^{T_a}, with $h(x) = \mathbb{P}_x(T_a < \infty)$. Hence:

$$\mathbb{E}_x(\hat{L}^y_{\hat{T}_a}) = \frac{h(y)}{h(x)} g_{T_a}(x, y)$$

But $(\frac{1}{h^2(x)}\hat{L}_t^x, x \in Et \geq 0)$ is also a local time process for $\hat{X}$. Consequently thanks to the previous remark, we know that $(\frac{g_{T_a}(x,y)+g_{T_a}(y,x)}{h(x)h(y)}, (x, y) \in E^2)$ is definite positive, and so is $(g_{T_a}(x, y) + g_{T_a}(y, x), (x, y) \in E^2)$. □

Consequently there exists a centered Gaussian process with a covariance equal to $(g_{T_a}(x, y) + g_{T_a}(y, x), (x, y) \in E^2)$. The arguments of the previous sections are not available for this process but the following remark shows that it is possible to deduce properties of this Gaussian process from the associated Markov process.

Remark 5.2. Marcus and Rosen have established (1992) an equivalence between the continuity of the local time process of symmetric Markov processes and the continuity of associated Gaussian processes. In the particular case of a Lévy process, they gave a much simpler proof of this result in Marcus and Rosen (2001). A carefull reading of this proof reveals that if g_{T_0} is the Green function of a recurrent symmetric Lévy process X killed at the first hitting time of 0, then X admits continuous local times if and only if the centered Gaussian process with a covariance equal to g_{T_0} is continuous. This last remark can also be directly done by using, under the assumption of symmetry for X, the necessary and sufficient condition of Barlow (1988) as it has been stated by Bertoin ((1996), p.148).

The next proposition shows that even when X is not symmetric this remark remains true.

Proposition 5.2. *Let X be a recurrent Lévy process and g_{T_0} be the Green function of X killed at its first hitting time of* 0*. Let $(\eta_x)_{x\in\mathbb{R}}$ be a centered Gaussian process with a covariance given by:*

$$\mathbb{E}(\eta_x\eta_y) = \frac{1}{2}(g_{T_0}(x, y) + g_{T_0}(y, x)).$$

The process η is continuous if and only if the local time process of X admits a continuous version.

The proof of Proposition 5.2 consists in translating Barlow's condition in terms of a Gaussian process. This translation together with Theorem 5.1, allows to suspect that Proposition 5.2 might have an extension to recurrent non symmetric Markov processes.

Proposition 5.2 admits a version for the transient Lévy processes. In that case, the covariance is equal to: $\frac{1}{2}(g_{T_0}(x, y) + g_{T_0}(y, x) + f(x)f(-y) + f(-x)f(y))$ where $f(x) = \mathbb{P}_x(T_0 = \infty)$.

Proof of Proposition 5.2. Let ψ be the Fourier exponent of X_1. Define the function h on $\mathbb{R}$ by:

$$h(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} (1 - \cos(\zeta x))\mathcal{R}(\frac{1}{\psi(\zeta)})d\zeta$$

Denote by $\overline{h}$ the monotone rearrangement of h. Then the local time process of X admits a continuous version if and only if

$$\int_{0+} \frac{\overline{h}(u)du}{u(\log(1/u))^{1/2}} < +\infty. \tag{6}$$

This is Barlow's necessary and sufficient condition (Barlow (1988)) as it has been formulated by Bertoin ((1996), p. 148).

In the symmetric case ψ is a real function, and it is known that: $g_{T_0}(x, y) = \frac{1}{2}(h(x)+h(y)-h(x-y))$. Applying then the results of Dudley (1973) and Fernique (1974) one can claim that (6) is a necessary and sufficient condition for the continuity of the centered Gaussian process with a covariance equal to g_{T_0}.

Here we are going to prove that in the general case, we have:

$$g_{T_0}(x, y) + g_{T_0}(y, x) = h(x) + h(y) - h(x - y) \tag{7}$$

Once we have (7), Proposition 5.2 follows similarly from Dudley and Fernique's results.

We set: $u^\alpha(x, y) = \mathbb{E}_x(\int_0^\infty e^{-\alpha t} dL_t^y)$, for $(L_t^x, x \in \mathbb{R}, t \geq 0)$ the local time process of X. Let T_0 be the first hitting time of 0 by X. We have then thanks to the Markov property:

$$u^\alpha(x, y) = \mathbb{E}_x(\int_0^{T_0} e^{-\alpha t} dL_t^y) + \mathbb{E}_x(\mathrm{e}^{-\alpha T_0})\mathbb{E}_0(\int_0^\infty e^{-\alpha t} dL_t^y)$$

Setting: $u_{T_0}^\alpha(x, y) = \mathbb{E}_x(\int_0^{T_0} e^{-\alpha t} dL_t^y)$, we obtain:

$$u_{T_0}^\alpha(x, y) = u^\alpha(x, y) - \mathbb{E}_x(\mathrm{e}^{-\alpha T_0})u^\alpha(0, y)$$

Since X is a Lévy process, we have: $u^\alpha(x, y) = u^\alpha(y-x)$. Note also that we have: $\mathbb{E}_x(\mathrm{e}^{-\alpha T_0}) = \frac{u^\alpha(-x)}{u^\alpha(0)}$.

Hence , we obtain:

$$u_{T_0}^\alpha(x, y) + u_{T_0}^\alpha(y, x) = (u^\alpha(y - x) + u^\alpha(x - y)) - \frac{u^\alpha(-x)}{u^\alpha(0)}u^\alpha(y) - \frac{u^\alpha(-y)}{u^\alpha(0)}u^\alpha(x) \tag{8}$$

But we have:

$$u^\alpha(0) - \frac{u^\alpha(-x)}{u^\alpha(0)} u^\alpha(y) = 2u^\alpha(0) - u^\alpha(-x) - u^\alpha(y) - (1 - \frac{u^\alpha(-x)}{u^\alpha(0)})(1 - \frac{u^\alpha(y)}{u^\alpha(0)})$$

Consequently, (8) becomes:

$$\begin{aligned} u^\alpha_{T_0}(x,y) + u^\alpha_{T_0}(y,x) &= \big(u^\alpha(x-y) + u^\alpha(y-x) - 2u^\alpha(0)\big) \\ &+ (2u^\alpha(0) - u^\alpha(x) - u^\alpha(-x)) + (2u^\alpha(0) - u^\alpha(y) - u^\alpha(-y)) \\ &- (1 - \frac{u^\alpha(-x)}{u^\alpha(0)})(1 - \frac{u^\alpha(y)}{u^\alpha(0)}) - (1 - \frac{u^\alpha(-y)}{u^\alpha(0)})(1 - \frac{u^\alpha(x)}{u^\alpha(0)}) \end{aligned}$$

First we note that: $\lim_{\alpha\to 0} \frac{u^\alpha(-x)}{u^\alpha(0)} = \lim_{\alpha\to 0} \mathbb{E}_x(\mathrm{e}^{-\alpha T_0}) = 1$. Then in order to let α tend to 0 in the above equation, we use a consequence of an argument of Bertoin ((1996) p.144). Indeed, he established that as α tends to 0, the expression $(2u^\alpha(0) - u^\alpha(x) - u^\alpha(-x))$ converges to $h(x)$. This gives (7).

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