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Backward SDEs and Cauchy problem for semilinear equations in divergence form

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Abstract. We extend the definition of solutions of backward stochastic differential equations to the case where the driving process is a diffusion corresponding to symmetric uniformly elliptic divergence form operator. We show existence and uniqueness of solutions of such equations under natural assumptions on the data and show its connections with solutions of semilinear parabolic partial differential equations in Sobolev spaces.

1. Introduction

Connections between solutions of backward stochastic differential equations (BSDEs) driven by strong solutions of Itô SDEs and viscosity solutions of semilinear second order (nondivergence) PDEs are well examined. We refer the reader to [12] for a nice presentation of the theory and extensive bibliography of the subject.

The aim of the present paper is to extend the theory of BSDEs to the case where the driving forward process is a time-inhomogeneous diffusion corresponding to uniformly elliptic divergence form operator, and to show its connections with the theory of solutions of semilinear divergence form PDEs in Sobolev spaces. We impose no regularity assumption on the diffusion coefficient, so in general, our forward process is not a semimartingale. Therefore the classical definition of solutions of BSDEs is not applicable. Following [2, 9] we propose slightly more general definition which make use of Fukushima's decomposition of additive functionals of time-inhomogeneous diffusions with divergence form generators (see [11, 16]) and seems to be well adjusted to problems with nonsmooth data. Our main theorem says that under standard, in the $\mathbb{L}_2$-theory of PDEs, assumptions on the data of the Cauchy problem there exists a unique solution of the associated BSDE and it is represented in terms of the analytical solution in a similar way as in the case of usual BSDEs and viscosity solutions. This strenghtens the corresponding results from [2, 3, 9].

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Let us now present and justify our definition of solutions of BSDEs and describe more precisely the content of the paper.

Let $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ be a measurable, symmetric matrix valued function satisfying

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{d} a^{ij}(t,x)\xi_i\xi_j \le \Lambda|\xi|^2, \ \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \quad a^{ij} = a^{ji} \quad (1.1)$$

for some $0 < \lambda \le \Lambda$. Define the operator

$$A_t = (1/2) \sum_{i,j=1}^{d} D_j(a^{ij}(t,x)D_i)$$

and for given $\varphi : \mathbb{R}^d \to \mathbb{R}$, $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ consider the Cauchy problem

$$\begin{cases} (D_t + A_t)u(t,x) = -F_u(t,x), & (t,x) \in [0,T) \times \mathbb{R}^d, \\ u(T,x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

where

$$F_u(t,x) = f(t,x,u(t,x),(\sigma\nabla u)(t,x))$$

and σ is the symmetric square-root of a. Let $\Omega = C([0, T]; \mathbb{R}^d)$ denote the space of continuous $\mathbb{R}^d$-valued functions on $[0, T]$ and let X be the canonical process on it. Let us recall that given A_t with a satisfying (1.1) one can construct a weak fundamental solution $p(s, x, t, y)$ for A_t and a Markov diffusion $\mathbb{X} = \{(X, P_{s,x}); (s, x) \in [0, T) \times \mathbb{R}^d\}$ for which p is a transition density function, that is

$$P_{s,x}(X_t = x, 0 \le t \le s) = 1, \quad P_{s,x}(X_t \in \Gamma) = \int_\Gamma p(s,x,t,y)\,dy, \quad t \in (s, T]$$

for any Borel $\Gamma \subset \mathbb{R}^d$ (see [14, 19]). We will call $\mathbb{X}$ a diffusion corresponding to a.

Assume for the moment that a, f, φ are smooth and f, φ satisfy growth conditions ensuring existence of a classical solution u to (1.2). Then by Itô's formula, for each $(s, x) \in [0, T) \times \mathbb{R}^d$,

$$u(t,X_t) = \varphi(X_T) + \int_t^T F_u(\theta, X_\theta)\,d\theta - \int_t^T < \nabla u(\theta, X_\theta), dM_{s,\theta} >, \quad t \in [s, T]$$

$P_{s,x}$ - a.s., where

$$M_{s,t} = X_t - X_s - A_{s,t}\,, \quad 0 \le s \le t \le T \quad (1.3)$$

and $A_{s,t} = \int_s^t b(\theta, X_\theta)\,d\theta$, $b = (b^1, \dots, b^d)$, $b^i = (1/2)\sum_{j=1}^d D_j a^{ij}$, $i = 1, \dots, d$. Therefore, if we set

$$(Y_t^{s,x}, Z_t^{s,x}) = (u(t,X_t), (\sigma\nabla u)(t,X_t)), \quad t \in [s, T] \quad (1.4)$$

and

$$\Phi_{t,T}^{s,x} = \varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_\theta^{s,x}, Z_\theta^{s,x})\, d\theta, \quad t \in [s,T],$$

then

$$Y_t^{s,x} = \Phi_{t,T}^{s,x} - \int_t^T < Z_\theta^{s,x}, \sigma^{-1}(\theta, X_\theta)\, dM_{s,\theta} >, \quad t \in [s,T], \quad P_{s,x}\text{-}a.s. \tag{1.5}$$

If, for instance, b is bounded, then $\{M_{s,t}\}$ is a martingale additive functional (MAF) of $\mathbb{X}$ of finite energy and $\{A_{s,t}\}$ is a continuous additive functional (CAF) of $\mathbb{X}$ of zero energy, and so (1.3) is the Fukushima decomposition of the additive functional (AF) $\{X_{s,t} = X_t - X_s\}$. We will prove in Section 2 that similar decomposition holds for any measurable a satisfying (1.1). Moreover, the martingale part $\{M_{s,t}\}$ of it is uniquely determined, does not depend on the starting point x and for any $(s,x) \in [0,T) \times \mathbb{R}^d$ the process $M_{s,\cdot}$ is an $P_{s,x}$ - square-integrable martingale on $[s,T]$ with the co-variation process

$$\langle M_{s,\cdot}^i\ , M_{s,\cdot}^j \rangle_t = \int_s^t a^{ij}(\theta, X_\theta)\, d\theta,\ t \in [s,T], \quad i,j = 1,\dots,d. \tag{1.6}$$

It is therefore reasonable, if φ is regular, to define a solution to the BSDE associated with $(X, P_{s,x})$ as a pair of processes satisfying (1.5). If we assume only that φ is square-integrable, one cannot expect that (1.5) holds for $t = T$. A natural requirement in such a case is that $Y_t^{s,x} \to \varphi(X_T)$ in $\mathbb{L}_2(P_{s,x})$ as $t \uparrow T$. A precise definition of solutions is given in Section 2.

In our main theorems we prove that if a satisfies (1.1), $\varphi \in \mathbb{L}_2(\mathbb{R}^d)$, f is Lipschitz continuous in y, z, satisfies the linear growth condition in y, z and $f(t,x,0,0) \in \mathbb{L}_{q,p}((0,T) \times \mathbb{R}^d)$ for suitably chosen q, p, then there exist a unique solution to the problem (1.2) and a unique solution to the BSDE (1.5), and moreover, (1.4) holds for every $(s,x) \in [0,T) \times \mathbb{R}^d$. Let us note here that some results on solutions to BSDEs with driving process being a time-homogeneous diffusion corresponding to second order divergence form operator are given in [2, 3, 9]. In particular, in [3] it is proved that (1.4) holds under the assumption that a is regular, so that the driving process is in fact an Itô process. But let us remark, that the most interesting results of [3] concern degenerate diffusions, which are not considered here. In [2, 9] no regularity assumption on a is imposed. In [2], where as in [3] the Cauchy problem is considered and uniform ellipticity of a is not required, (1.4) is proved for quasi-every starting point. In [9] Cauchy-Dirichlet problem in bounded domain is considered. It is shown there that under (1.1) a relation similar to (1.4) holds for almost every starting point. Thus the main feature of our paper, compared with [2, 9], is that the forward driving process is a time-inhomogeneous diffusion and, what is more important, that we prove existence of solutions of BSDE (1.5) and validity of the representation (1.4) for every starting point.

In proofs we combine some methods from PDE's theory with methods of BSDEs. On the one hand we use some standard facts from $\mathbb{L}_2$ - theory of linear PDEs and two deep results: Aronson's upper estimate on the transition density of $\mathbb{X}$

and Nash's continuity theorem [1, 8]. On the other hand we use stochastic calculus and ideas from the theory of BSDEs to prove a priori estimates on solutions of (1.2) in a Sobolev space with weight, the weight being the transition density of $\mathbb{X}$. The advantage of using this space lies in the fact that if a solution u to (1.2) is in it, then the integrals on the right-hand side of (1.5) are integrable under $P_{s,x}$, which allows to consider every starting point.

In the paper we will use the following notation.

$D_t = \partial/\partial t,\ D_i = \partial/\partial x_i,\ \nabla = (D_1, \dots, D_d)$. $C_0^\infty(\mathbb{R}^d)$ is the set of infinitely differentiable functions on $\mathbb{R}^d$ with compact supports. $\mathbb{L}_p$ ($\mathbb{L}_p(s, T)$) is the Banach space of functions on $\mathbb{R}^d$ (on $(s, T) \times \mathbb{R}^d$) that are pth-power summable on $\mathbb{R}^d$ ($(s, T) \times \mathbb{R}^d$). $W_p(\mathbb{R}^d)$ is the Banach space consisting of all elements u of $\mathbb{L}_p(\mathbb{R}^d)$ having generalized derivatives $D_i u$ in $\mathbb{L}_p$. $W_p(0, T)$ is the Banach space consisting of all elements u of $\mathbb{L}_p(0, T)$ having generalized derivatives $D_i u$ from $\mathbb{L}_p(s, T)$. By $(\cdot, \cdot)_2$, $\|\cdot\|_2$ we denote the scalar product and the norm in $\mathbb{L}_2$, and by $\|\cdot\|_{2;s,T}$ the norm in $\mathbb{L}_2(s, T)$.

2. Decomposition of diffusions

Given the family $\{P_{s,x}; (s, x) \in [0, T] \times \mathbb{R}^d\}$ corresponding to some a and a probability measure μ on a Borel σ - field $\mathcal{B}$ of $\mathbb{R}^d$ for some fixed s define the measure $P_{s,\nu}$ by $P_{s,\nu}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot)\,\nu(dx)$ and set $\mathcal{P} = \{P_{s,\nu} : \nu$ is a probability measure on $\mathcal{B}\}$. Let us set now $\mathcal{F}_t^s = \sigma(X_u, u \in [s, t])$ and define $\mathcal{G}$ as the completion of $\mathcal{F}_T^s$ with respect to the family $\mathcal{P}$ and $\mathcal{G}_t^s$ as the completion of $\mathcal{F}_t^s$ in $\mathcal{G}$ with respect to $\mathcal{P}$.

We say that a family of random variables $B = \{B_{s,t},\ 0 \le s \le t \le T\}$ is an additive functional AF of $\mathbb{X}$ on $[0, T]$ if $B_{s,t}$ is $\mathcal{G}_t^s$ - measurable for $0 \le s \le t \le T$ and $P_{s,x}(B_{s,t} = B_{s,u} + B_{u,t}, s \le u \le t \le T) = 1$ for every $(s, x) \in [0, T) \times \mathbb{R}^d$. If, in addition, $P_{s,x}(\{\omega \in \Omega : [s, T] \ni t \mapsto B_{s,t}(\omega)$ is continuous$\})=1$ for every $(s, x) \in [0, T) \times \mathbb{R}^d$ then B is called a continuous AF (CAF).

An AF B is called a continuous martingale AF (MAF), if it is a CAF of $\mathbb{X}$ such that $E_{s,x} B_{s,t}^2 < \infty,\ E_{s,x} B_{s,t} = 0$ for every $0 \le s \le t \le T,\ x \in \mathbb{R}^d$ ($E_{s,x}$ stands for the expectation with respect to $P_{s,x}$). Let us recall that if B is a MAF of $\mathbb{X}$ then $B_{s,\cdot}$ is a $(\{\mathcal{G}_t^s\}, P_{s,x})$-martingale on $[s, T]$.

For an AF B of $\mathbb{X}$ on $[0, T]$ we define its energy $e(B)$ by

$$e(B) = \lim_{\alpha \to \infty} \alpha^2 \int_0^T \int_0^T \mathbf{1}_{[0,T]}(s + t) e^{-\alpha t} \left(\int_{\mathbb{R}^d} E_{s,x} B_{s,s+t}^2\, dx \right) ds\, dt,$$

whenever the limit exists. We say that B is of finite energy (zero energy) if $e(B)$ exists and $e(B) < \infty$ ($e(B) = 0$).

A family of random vectors $B = (B^1, \dots, B^d)$ is called an AF (CAF, MAF, etc.) of $\mathbb{X}$ if its each component B^i is an AF (CAF, MAF, etc.) of $\mathbb{X}$.

In [16] (see also [11]) it is shown that for any continuous $\varphi \in W_p$ with $p = 2$ if $d = 1$ and $p > d$ if $d > 1$ there is a unique continuous MAF of finite energy M^φ and a CAF of zero energy A^φ such that

$$X_{s,t}^\varphi \equiv \varphi(X_t) - \varphi(X_s) = M_{s,t}^\varphi + A_{s,t}^\varphi, \quad t \in [s, T] \tag{2.1}$$

and

$$\langle M^{\varphi}_{s,\cdot}\rangle_t = \int_s^t (a\nabla\varphi, \nabla\varphi)(\theta, X_\theta)\, d\theta, \quad t \in [s, T] \tag{2.2}$$

$P_{s,x}$ - a.s. The above decomposition can be extended to functions φ which are locally in W_p. In our paper we will need such extension only for coordinate functions.

Theorem 2.1. *Let $\mathbb{X}$ be a diffusion corresponding to some a satisfying (1.1) and for $i = 1, \ldots, d$ let $\{\varphi_{i,n}\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ be a sequence of functions such that $\varphi_{i,n}(x) = x^i$ if $|x| < n$. Then there exist CAFs M, A of $\mathbb{X}$ on $[0, T]$ such that*

$$X_t - X_s = M_{s,t} + A_{s,t}, \;\; t \in [s, T], \;\; P_{s,x}\text{-}a.s. \tag{2.3}$$

and

$$M^i_{s,t\wedge\tau_n(s)} = M^{i,n}_{s,t\wedge\tau_n(s)}, \;\; A^i_{s,t\wedge\tau_n(s)} = A^{i,n}_{s,t\wedge\tau_n(s)}, \;\; t \in [s, T], \;\; P_{s,x}\text{-}a.s. \tag{2.4}$$

for $i = 1, \ldots, d$, $(s, x) \in [0, T) \times \mathbb{R}^d$, where $\tau_n(s) = \inf\{t > s : |X_t| > n\}$ and $M^{i,n}$ ($A^{i,n}$) is the MAF (CAF) of the decomposition of $X^{\varphi_{i,n}}$. The decomposition (2.3) is unique in the sense that if $\{\psi_{i,n}\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$, $i = 1, \ldots, d$ are another sequences of functions with the property that $\psi_{i,n}(x) = x^i$ if $|x| < n$ and if N, B are CAFs such that

$$X_t - X_s = N_{s,t} + B_{s,t}, \;\; N^i_{s,t\wedge\tau_n(s)} = N^{i,n}_{s,t\wedge\tau_n(s)}, \;\; B^i_{s,t\wedge\tau_n(s)} = B^{i,n}_{s,t\wedge\tau_n(s)}, \;\; t \in [s, T]$$

$P_{s,x}$-a.s for $i = 1, \ldots, d$, $(s, x) \in [0, T) \times \mathbb{R}^d$, where $N^{i,n}$ ($B^{i,n}$) is the MAF (CAF) of the decomposition of $X^{\psi_{i,n}}$, then

$$M^i_{s,t} = N^i_{s,t}, \quad A^i_{s,t} = B^i_{s,t}, \;\; t \in [s, T], \;\; P_{s,x}\text{-}a.s. \tag{2.5}$$

for $i = 1, \ldots, d$, $(s, x) \in [0, T) \times \mathbb{R}^d$.

Proof. We first prove (2.5). Due to [16, Theorem 4.2] the AF $X^{\varphi_{i,n}-\psi_{i,n}}$ admits a unique decomposition of the form (2.1) into a continuous MAF of finite energy $M^{\varphi_{i,n}-\psi_{i,n}}$ and a CAF of zero energy $A^{\varphi_{i,n}-\psi_{i,n}}$. Since $X^{\varphi_{i,n}}_{s,t\wedge\tau_n(s)} = X^{\psi_{i,n}}_{s,t\wedge\tau_n(s)}$, it follows from uniqueness that

$$M^{\varphi_{i,n}-\psi_{i,n}}_{s,t\wedge\tau_n} = M^{i,n}_{s,t\wedge\tau_n} - N^{i,n}_{s,t\wedge\tau_n} = -(A^{i,n}_{s,t\wedge\tau_n} - B^{i,n}_{s,t\wedge\tau_n}) = -A^{\varphi_{i,n}-\psi_{i,n}}_{s,t\wedge\tau_n}, \quad t \in [s, T]$$

$P_{s,x}$ - a.s. Hence, again by uniqueness of the decomposition of $X^{\varphi_{i,n}-\psi_{i,n}}$,

$$M^{i,n}_{s,t\wedge\tau_n(s)} = N^{i,n}_{s,t\wedge\tau_n(s)}, \;\; A^{i,n}_{s,t\wedge\tau_n(s)} = B^{i,n}_{s,t\wedge\tau_n(s)}, \;\; t \in [s, T], \;\; P_{s,x}\text{-}a.s.,$$

which gives (2.5), because $\tau_n(s) \to \infty P_{s,x}$ - a.s. as $n \to \infty$. To prove existence, we observe that by the uniquenes of the decomposition (2.1), $M^{i,n}_{s,t\wedge\tau_n(s)} = M^{i,n+1}_{s,t\wedge\tau_n(s)}$, $t \in [s, T]$, $P_{s,x}$ -a.s. for $n \in \mathbb{N}$. Therefore putting $M^i_{s,t} = \lim_{n\to\infty} M^{i,n}_{s,t}$, $A^i_{s,t} = X^i_{s,t} - M^i_{s,t}$ for $0 \le s \le t \le T$, $i = 1, \ldots, d$ we get well-defined CAFs satisfying (2.3), (2.4). □

From (2.2), (2.4) and the fact that $E_{s,x}\int_s^T a^{ii}(t,X_t)\,dt<\infty$ for $i=1,\dots,d$ it follows that for any fixed $(s,x)\in[0,T)\times\mathbb{R}^d$ the process $M_{s,\cdot}$ is an $(\{\mathcal{G}_t^s\}_{t\in[s,T]}, P_{s,x})$-square-integrable martingale on $[s,T]$ with the co-variation process given by (1.6).

Let us note also that $e(M^{i,n})<\infty$, $e(A^{n,i})=0$ for $n\in\mathbb{N}$, $i=1,\dots,d$, so following [7, Section 5.5] we may call M a MAF locally of finite energy and A a CAF locally of zero energy.

We are now ready to give definition of the solution of the BSDE associated with the Markov process $(X,P_{s,x})$.

Definition. Let $\varphi:\mathbb{R}^d\to\mathbb{R}$, $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ be measurable functions and let $(s,x)\in[0,T)\times\mathbb{R}^d$. We say that a pair $\{(Y_t^{s,x},Z_t^{s,x});\,t\in[s,T]\}$ of $\{\mathcal{G}_t^s\}_{t\in[s,T]}$ - progressively measurable processes is a solution to the BSDE (φ,f) associated with $(X,P_{s,x})$ if

(a) $Y^{s,x}$ is continuous on $[s,T)$, $\int_s^T|Z_t^{s,x}|^2\,dt<\infty$, $P_{s,x}$ - a.s.;
(b) (1.5) holds with $[s,T)$ in place of $[s,T]$, where M is a (unique) CAF of the decomposition (2.3);
(c) $\lim_{t\uparrow T}E_{s,x}|Y_t^{s,x}-\varphi(X_T)|^2=0$.

We say that the BSDE (φ,f) associated with $(X,P_{s,x})$ has a unique solution if for any its solutions $(Y^{s,x,i},Z^{s,x,i})$, $i=1,2$ we have $Y_t^{s,x,1}=Y_t^{s,x,2}$, $t\in[s,T)$, $P_{s,x}$ - a.s. and $Z_\cdot^{s,x,1}=Z_\cdot^{s,x,2}$, $dt\otimes P_{s,x}$ - a.s. on $(s,T)\times\Omega$.

3. Some limit theorems

In this section we collect some auxiliary results that will be needed in Sections 5 and 6.

Suppose we are given measurable functions $a_n:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$ satisfying (1.1) and such that $a_n^{ij}\to a^{ij}$ for $i,j=1,\dots,d$ almost everywhere with respect to the Lebesgue measure. For $n,k\in\mathbb{N}$ let $\varphi_k^n\in W_2(0,T)$ denote a unique weak solution to the problem

$$(k-D_t-A_t^n)\varphi_k^n=\varphi \text{ in } (0,T)\times\mathbb{R}^d,\quad \varphi_k^n(T,\cdot)=0 \text{ on } \mathbb{R}^d,$$

where $A_t^n=(1/2)\sum_{i,j=1}^d D_j(a_n^{ij}(t,x)D_i)$, and let $\varphi_k\in W_2(0,T)$ denote a weak solution to the above problem with A_t in place of A_t^n.

Lemma 3.1. *Let $\varphi\in C_0^\infty(\mathbb{R}^d)$. Then $k\varphi_k\to\varphi$ in $W_2(0,T)$ as $k\to\infty$ and for each $k\in\mathbb{N}$, $\varphi_k^n\to\varphi_k$ in $W_2(0,T)$ as $n\to\infty$.*

Proof. For the first assertion see [18, Proposition 3.7]. The second one follows from [8, Theorem III. 4.5]. □

Let $(X,P_{s,x}^n)$ be a diffusion corresponding to a_n and let

$$X_t-X_s=M_{s,t}^n+A_{s,t}^n,\quad t\in[s,T]$$

be a decomposition of Theorem 2.1 given the family $\{P_{s,x}^n\,;\,(s,x)\in[0,T)\times\mathbb{R}^d\}$. It is known (see, e.g., [14, 19]) that $\mathcal{L}[X|P_{s,x}^n]\Rightarrow\mathcal{L}[X|P_{s,x}]$ as $n\to\infty$, that is the law of X under $P^n_{s,x}$ converges weakly to the law of X under $P_{s,x}$. In fact, we have the following stronger result.

Lemma 3.2. *$\mathcal{L}[(X, M^n_{s,\cdot})|P^n_{s,x}] \Rightarrow \mathcal{L}[(X, M_{s,\cdot})|P_{s,x}]$ in $C([s,T];\mathbb{R}^{2d})$ for each $(s,x) \in [0,T) \times \mathbb{R}^d$.*

Proof. It is sufficient to prove that for each $(s,x) \in [0,T) \times \mathbb{R}^d$

$$\mathcal{L}[(X^{\varphi}_{s,\cdot}, M^{n,\varphi}_{s,\cdot})|P^n_{s,x}] \Rightarrow \mathcal{L}[(X^{\varphi}_{s,\cdot}, M^{\varphi}_{s,\cdot})|P_{s,x}] \tag{3.1}$$

in $C([s,T];\mathbb{R}^2)$ for any $\varphi \in C_0^{\infty}(\mathbb{R}^d)$, where $M^{n,\varphi}$ denotes the martingale part of the decomposition of X^{φ} of the form (2.1) under $\{P^n_{s,x}\,;\,(s,x) \in [0,T) \times \mathbb{R}^d\}$. For this purpose we fix $\varphi \in C_0^{\infty}(\mathbb{R}^d)$ and define φ_k, φ^n_k as in Lemma 3.1. Let $\psi_k = k\varphi_k$, $\psi^n_k = k\varphi^n_k$ and let M^{φ_k} (M^{n,φ^n_k}) be the martingale part of the decomposition of X^{φ_k} $(X^{\varphi^n_k})$ under $P_{s,x}$ $(P^n_{s,x})$. By [14, Lemma 1.3],

$$M^{\psi_k}_{s,t} = \psi_k(t, X_t) - \psi_k(s, X_t) - \int_s^t k(\psi_k - \varphi)(u, X_u)\,du. \tag{3.2}$$

Similarly, by Itô's formula, M^{n,ψ^n_k} is given by the right-hand side of (3.2) with ψ^n_k in place of ψ_k. For given AF Y and $\delta \in (0, T-s)$ let us use ${}^{\delta}Y_{s,\cdot}$ to denote the process $Y_{s,t\vee(s+\delta)} - Y_{s,s+\delta}$, $t \in [s,T]$. By Lemma 3.1 and Nash's continuity theorem (see, e.g., [1]), $\psi^n_k \to \psi_k$ uniformly in compact sets in $(0,T] \times \mathbb{R}^d$ as $n \to \infty$. Hence

$$\mathcal{L}[(X^{\varphi}_{s,\cdot}, {}^{\delta}M^{n,\psi^n_k}_{s,\cdot})|P^n_{s,x}] \Rightarrow \mathcal{L}[(X^{\varphi}_{s,\cdot}, {}^{\delta}M^{\psi_k})|P_{s,x}] \tag{3.3}$$

in $C([s,T];\mathbb{R}^d)$. By Aronson's estimates (see [1, 19]) there exists a constant $C > 0$ depending only on λ, Λ, d, T such that

$$p(s,x,t,y) \le C(t-s)^{-d/2} \exp\left(-\frac{|y-x|^2}{C(t-s)}\right) \tag{3.4}$$

for all $0 \le s < t \le T$. Therefore, due to (2.2) and uniqueness of the decomposition (2.1) we have

$$\begin{aligned} E_{s,x}\langle {}^{\delta}M^{\psi_k}_{s,\cdot} - {}^{\delta}M^{\varphi}_{s,\cdot}\rangle_T &= E_{s,x}\langle {}^{\delta}M^{\psi_k-\varphi}_{s,\cdot}\rangle_T \\ &= E_{s,x}\int_{s+\delta}^T (a\nabla(\psi_k-\varphi), \nabla(\psi_k-\varphi))(t, X_t)\,dt \\ &\le \Lambda C\delta^{-d/2}\|\nabla(\psi_k-\varphi)\|^2_{2;s+\delta,T}. \end{aligned}$$

Hence, by Doob's inequality and Lemma 3.1,

$$\lim_{k\to\infty} E_{s,x} \sup_{s\le t\le T} |{}^{\delta}M^{\psi_k}_{s,t} - {}^{\delta}M^{\varphi}_{s,t}|^2 = 0. \tag{3.5}$$

Due to Lemma 3.1, $\psi_k^n \to \psi_k$ in $W_2(s, T)$ as $n \to \infty$, so in much the same way as above we obtain

$$\lim_{k\to\infty} \limsup_{n\to\infty} E_{s,x} \sup_{s\le t\le T} |{}^{\delta}M_{s,t}^{n,\psi_k^n} - {}^{\delta}M_{s,t}^{n,\varphi}|^2$$
$$\le 4\Lambda C\delta^{-d/2} \lim_{k\to\infty} \limsup_{n\to\infty} \|\nabla(\psi_k^n - \varphi)\|_{2;s+\delta,T}^2 = 0. \tag{3.6}$$

Putting (3.3), (3.5), (3.6) together and using [5, Theorem 4.2] gives

$$\mathcal{L}[(X_{s,\cdot}^{\varphi}, {}^{\delta}M_{s,\cdot}^{n,\varphi})|P_{s,x}^n] \Rightarrow \mathcal{L}[(X_{s,\cdot}^{\varphi}, {}^{\delta}M_{s,\cdot}^{\varphi})|P_{s,x}]. \tag{3.7}$$

Since

$$E_{s,x}\langle {}^{\delta}M_{s,\cdot}^{\varphi} - M_{s,\cdot}^{\varphi}\rangle_T = \int_s^{s+\delta} (a\nabla\varphi, \nabla\varphi)(u, X_u)\, du$$

and

$$E_{s,x}\langle {}^{\delta}M_{s,\cdot}^{n,\varphi} - M_{s,\cdot}^{n,\varphi}\rangle_T = \int_s^{s+\delta} (a_n\nabla\varphi, \nabla\varphi)(u, X_u)\, du,$$

we have also

$$\lim_{\delta\downarrow 0} E_{s,x} \sup_{s\le t\le T} |{}^{\delta}M_{s,t}^{\varphi} - M_{s,t}^{\varphi}|^2 = 0, \quad \lim_{\delta\downarrow 0} \limsup_{n\to\infty} E_{s,x} \sup_{s\le t\le T} |{}^{\delta}M_{s,t}^{n,\varphi} - M_{s,t}^{n,\varphi}|^2 = 0,$$

which yields (3.1) when combined with (3.7). □

As a corollary to the above lemma we get

Lemma 3.3. *If $F_n \to F$, $G_n^i \to G^i$, $i = 1, \ldots, d$ in $\mathbb{L}_2(s, T)$ then*

$$\mathcal{L}[(X, \int_\cdot^T F_n(\theta, X_\theta)\, d\theta, \int_\cdot^T < G_n(\theta, X_\theta),\ dM_{s,\theta}^n >)|P_{s,x}^n]$$
$$\Rightarrow \mathcal{L}[(X, \int_\cdot^T F(\theta, X_\theta)\, d\theta, \int_\cdot^T < G(\theta, X_\theta),\ dM_{s,\theta} >)|P_{s,x}]$$

in $C([s+\delta, T]; \mathbb{R}^{d+2})$ for each $\delta \in (0, T-s)$.

Proof. It suffices to repeat arguments from the proof of [15, Lemma 1.1] or [17, Lemma 3.3]. □

4. Uniqueness and a priori estimates for solutions of BSDEs

We will need the following assumptions.

(i) $\varphi \in \mathbb{L}_2(\mathbb{R}^d)$;

(ii) f is measurable and there is $L > 0$ such that $|f(t, x, y_1, z_1) - f(t, x, y_2, z_2)| \le L(|y_1 - y_2| + |z_1 - z_2|)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$;

(iii) there exist $K > 0$ and $g \in \mathbb{L}_2(0, T)$ such that $|f(t, x, y, z)| \le g(t, x) + K(|y| + |z|)$ for all $(t, x, y, z) \in (0, T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$;

(iv) condition (iii) holds with $g \in \mathbb{L}_2(0,T) \cap \mathbb{L}_{q,p}(0,T)$, where $\mathbb{L}_{q,p}(0,T)$ is the space of measurable functions on $(0,T)\times\mathbb{R}^d$ having a finite norm $\|u\|_{q,p} = (\int_0^T(\int_{\mathbb{R}^d}|u(t,x)|^p\,dx)^{q/p}dt)^{1/q}$ and q, p are such that $q, p \in (2,\infty]$ and $(2/q)+(d/p) < 1$.

Let $\mathbb{H}_T^{(d)}(P_{s,x})$ denote the space of $\{\mathcal{G}_t^s\}_{t\in[s,T]}$-progressively measurable d-dimensional processes η on $[s,T]$ such that $E_{s,x}\int_s^T|\eta_t|^2\,dt < \infty$.

Proposition 4.1. *Assume (1.1) and (i), (ii), (iv). Then for every $(s,x)\in[0,T)\times\mathbb{R}^d$ the BSDE (φ, f) associated with $(X, P_{s,x})$ has at most one solution in $\mathbb{H}_T^{(1+d)}(P_{s,x})$.*

Proof. It follows from Itô's formula and Gronwall's lemma in much the same way as for BSDEs driven by Itô's diffusions; see, e.g., the proof of [12, Theorem 2.1]. □

Let us remark that assumption (ii) in Proposition 4.1 can be relaxed. Namely, it suffices to assume that

(ii') f is measurable, Lipschitz continuous in z and satisfies the monotonicity condition: $(y_1-y_2)\cdot(f(t,x,y_1,z)-f(t,x,y_2,z)) \le L'|y_1-y_2|^2$ for some $L'>0$.

Proposition 4.2. *Under the assumptions of Proposition 4.1, if the pair $(Y^{s,x}, Z^{s,x}) \in \mathbb{H}_T^{(1+d)}(P_{s,x})$ is a solution to the BSDE (φ, f), then there exist a constant C_1 depending only on K, T such that*

$$E_{s,x}\left(\sup_{s\le t\le T}|Y_t^{s,x}|^2 + \int_s^T |Z_t^{s,x}|^2\,dt\right) \le C_1 E_{s,x}\left(|\varphi(X_T)|^2 + \int_s^T |g(t,X_t)|^2\,dt\right).$$

Proof. The proof in analogous to that of [12, Proposition 1.1], so we omit it. □

For fixed $(s,x)\in[0,T)\times\mathbb{R}^d$ let $\mathcal{L}_2(x,s,T)$ denote the Banach space of functions on $(s,T)\times\mathbb{R}^d$ having a finite norm $\|u\|^2_{2;x,s,T} = \int_s^T\int_{\mathbb{R}^d}|u(t,y)|^2p(s,x,t,y)\,dt\,dy$ and let $\mathcal{W}_2(x,s,T)$ denote the space of all elements u of $\mathcal{L}_2(x,s,T)$ having generalized derivatives D_iu in $\mathcal{L}_2(x,s,T)$ equipped with the norm $\|u\|^2_{\mathcal{W}_2(x,s,T)} = \|u\|^2_{2;x,s,T} + \|\nabla u\|^2_{2;x,s,T}$.

From Proposition 4.2 and (3.4) we conclude that

$$|Y_s^{s,x}|^2 + E_{s,x}\int_s^T(|Y_t^{s,x}|^2 + |Z_t^{s,x}|^2)\,dt \le C_2((T-s)^{-d/2}\|\varphi\|_2^2 + \|g\|^2_{2;x,s,T}) \tag{4.1}$$

for some C_2 depending on $\lambda, \Lambda, K, d, T$. In case there exists a solution u to (1.2) and (1.4) holds true, (4.1) gives estimates for $|u(s,x)|$ and $\|u\|_{\mathcal{W}_2(x,s,T)}$. These estimates will play a key role in the proof of existence of solutions to the BSDE (φ, f). Notice also that if q, p satisfy the conditions formulated in assumption (iv) then from Aronson's estimate (3.4) and Hölder's inequality it follows that

$$\|g\|_{2;x,s,T} \le C_3\|g\|_{q,p} \tag{4.2}$$

for some C_3 depending only on λ, Λ, d, T and q, p.

5. Linear problem

In this section we consider the problem

$$(D_t + A_t)u = -F \text{ in } (s,T)\times\mathbb{R}^d, \quad u(T,\cdot) = \varphi \text{ on } \mathbb{R}^d. \tag{5.1}$$

We will seek solutions in the Banach space $V(s,T)$ consisting of all elements of $W_2(s,T)$ that are continuous in t in the norm of $\mathbb{L}_2$ and have a finite norm defined by $\|u\|^2_{V(s,T)} = \sup_{s<t<T}\|u(t,\cdot)\|_2^2 + \|u\|^2_{W_2(s,T)}$.

Proposition 5.1. *Assume (1.1) and let $\varphi\in\mathbb{L}_2$, $F\in\mathbb{L}_2(s,T)$. Then there exists a unique weak solution $v\in V(s,T)$ to the problem (5.1). If moreover*

$$\forall\ H\subset[s,T)\times\mathbb{R}^d,\ H - \text{compact} \quad \sup_{(t,x)\in H}\|F\|_{2;x,t,T}<\infty \tag{5.2}$$

then there is a version u of v that is continuous on $[0,T)\times\mathbb{R}^d$ and the pair

$$(Y^{s,x}_t, Z^{s,x}_t) = (u(t,X_t), (\sigma\nabla u)(t,X_t)), \quad t\in[s,T) \tag{5.3}$$

is a solution, in $\mathbb{H}^{(1+d)}_T(P_{s,x})$, to the BSDE (φ,F) associated with $(X,P_{s,x})$.

Proof. The first assertion is a well known classical result (see, e.g., [10, Theorem 6.2]). To prove the second, we first assume that F is bounded. By using a mollification one can construct smooth functions $a_n:[s,T]\times\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$ satisfying (1.1) with a_n in place of a such that $a_n^{ij}\to a^{ij}$, $i,j=1,\dots,d$ almost everywhere and smooth bounded functions $\varphi_n:\mathbb{R}^d\to\mathbb{R}$, $f_n:[s,T]\times\mathbb{R}^d\to\mathbb{R}$ such that $\varphi_n\to\varphi$ in $\mathbb{L}_2$, $f_n\to F$ in $\mathbb{L}_2(s,T)$ and $\|f_n\|_\infty\le\|F\|_\infty$. Let u_n denote a classical solution to the problem

$$(D_t + A^n_t)u_n = -f_n \text{ in } [s,T)\times\mathbb{R}^d, \quad u_n(T,\cdot)=\varphi_n \text{ on } \mathbb{R}^d,$$

where A^n_t is defined as A but with a replaced by a_n. Then, by Itô's formula,

$$(Y^{s,x,n}_t, Z^{s,x,n}_t) = (u_n(t,X_t), \sigma_n\nabla u_n(t,X_t)), \quad t\in[s,T] \tag{5.4}$$

is a solution to the BSDE (φ_n, f_n) associated with $(X, P^n_{s,x})$, that is

$$Y^{s,x,n}_t = \Phi^{s,x,n}_{t,T} - \int_t^T < Z^{s,x,n}_\theta, \sigma_n^{-1}(\theta,X_\theta)\,dM^n_{s,\theta} >,\ t\in[s,T],\ P^n_{s,x}\text{-}a.s., \tag{5.5}$$

where

$$\Phi^{s,x,n}_{t,T} = \varphi_n(X_T) + \int_t^T f_n(\theta,X_\theta)\,d\theta$$

and $M^n=\{M^n_{s,t}, 0\le s\le t\le T\}$ is a MAF from the decomposition of X under $\{P^n_{s,x};(s,x)\in[0,T)\times\mathbb{R}^d\}$. Let $v\in V(0,T)$ be a weak solution to (5.1). Since $\varphi_n\to\varphi$ in $\mathbb{L}_2$ and $f_n\to F$ in $\mathbb{L}_2(s,T)$, it follows from the energy inequality (see, e.g., [10, Theorem 6.1]) that $u_n\to v$ in $V(s,T)$. On the other hand, (1.1), (4.1), (5.4) and (3.4) imply that

$$|u_n(t,x)|^2 + \|u_n\|^2_{\mathcal{W}_2(x,t,T)} \le C_2(1\vee\lambda^{-1})((T-t)^{-d/2}\|\varphi_n\|_2^2 + \|f_n\|^2_{2;x,t,T}) \tag{5.6}$$

for $(t,x)\in[s,T)\times\mathbb{R}^d$. Therefore $\{u_n\}$ is uniformly bounded on $[s,T-\delta]\times\mathbb{R}^d$ for each $\delta\in(0,T-s)$, and hence, by Nash's continuity theorem (see, e.g., [1]), $\{u_n\}$ is uniformly bounded and equicontinuous in any compact subset of $[s,T)\times\mathbb{R}^d$. Therefore there is a continuous version u of v such that $u_n\to u$ in any compact subset of $[s,T)\times\mathbb{R}^d$. From the above and Lemma 3.3 we conclude that

$$\mathcal{L}[(Y^{s,x,n},\Phi^{s,x,n}_{\cdot,T},\int_{\cdot}^{T}<Z^{s,x,n}_\theta,\sigma_n^{-1}(\theta,X_\theta)\,dM^n_{s,\theta}>)|P^n_{s,x}]$$

$$\Rightarrow\mathcal{L}[(u(\cdot,X_\cdot),\varphi(X_T)+\int_{\cdot}^{T}F(\theta,X_\theta)\,d\theta,\int_{\cdot}^{T}<\nabla u(\theta,X_\theta),\,dM_{s,\theta}>)|P_{s,x}]$$

in $C([s+\delta,T-\delta];\mathbb{R}^3)$ for every $\delta\in(0,(T-s)/2)$. Hence, by (5.5) and the continuous mapping theorem,

$$u(t,X_t)=\varphi(X_T)+\int_t^T F(\theta,X_\theta)\,d\theta-\int_t^T<\nabla u(\theta,X_\theta),\,dM_{s,\theta}>,\quad t\in(s,T) \tag{5.7}$$

$P_{s,x}$ -a.s. Furthermore, (5.6) implies that $\{u_n\}$ is uniformly bounded in $\mathcal{W}_2(x,s,T)$. Therefore, it is weakly relatively compact in $\mathcal{W}_2(x,s,T)$, and consequently, $\nabla u\in\mathcal{L}_2(x,s,T)$. From this and the fact that $F\in\mathcal{L}_2(x,s,T)$ we conclude that

$$\int_s^t F(\theta,X_\theta)\,d\theta\to 0,\qquad\int_s^t<\nabla u(\theta,X_\theta),\,dM_{s,\theta}>\to 0$$

$P_{s,x}$ - a.s. as $t\downarrow s$, which together with continuity of u shows that (5.7) holds for $t=s$, too. Thus, conditions (a), (b) of the definition of a solution of BSDE are satisfied. Now set $g(t,y)=E_{t,y}\{u^2(t,y)-2u(t,y)\varphi(X_T)+\varphi^2(X_T)\}$ and observe that by the Markov property and (3.4),

$$\begin{aligned}E_{s,x}|u(t,X_t)-\varphi(X_T)|^2&=E_{s,x}E_{s,x}(\{u^2(t,X_t)-2u(t,X_t)\varphi(X_T)\\&\quad+\varphi^2(X_T)\}|\mathcal{F}^X_{[s,t]})=E_{s,x}g(t,X_t)\\&\le\Lambda C(t-s)^{-d/2}\int_{\mathbb{R}^d}g(t,y)\,dy\,dy.\end{aligned}$$

Let $w\in V(s,T)$ denote a weak solution to (5.1) with $F\equiv 0$. Then

$$\begin{aligned}\int_{\mathbb{R}^d}g(t,y)\,dy&=\int_{\mathbb{R}^d}(u^2(t,y)-2u(t,y)w(t,y)+\varphi^2(y))\,dy\\&=(u(t,\cdot),u(t,\cdot)-\varphi)_2+(u(t,\cdot),\varphi-w(t,\cdot))_2\\&\quad-(u(t,\cdot)-\varphi,w(t,\cdot))_2-(\varphi,w(t,\cdot)-\varphi)_2,\end{aligned}$$

which converges to 0 as $t\uparrow T$, since $u,w\in V(s,T)$. This shows (c), and the proof of the theorem in the case of bounded F is complete. To prove the general case, we now set $f_n=(-n)\vee F\wedge n$ and denote by u_n the unique, in $V(s,T)$, weak solution to the problem

$$(D_t+A_t)u_n=-f_n\text{ in }[s,T)\times\mathbb{R}^d,\quad u_n(T,\cdot)=\varphi\text{ on }\mathbb{R}^d,$$

and by $(Y^{s,x,n}, Z^{s,x,n})$ a solution to the BSDE (φ, f_n) associated with $(X, P_{s,x})$. Since we know already that $(Y^{s,x,n}, Z^{s,x,n})$ is given by (5.4) with σ_n replaced by σ, from (4.1) we obtain the estimate (5.6) with φ in place of φ_n and F in place of f_n. Hence, by (5.2), $\{u_n\}$ is uniformly bounded on compact sets in $[s, T) \times \mathbb{R}^d$ and in $\mathcal{W}_2(x, s, T)$, and therefore the rest of the proof runs as before. □

6. Semilinear PDEs and BSDEs

Our next theorem concerns analytical solutions of (1.2). We find interesting that the main simple idea of its proof comes from the theory of BSDEs.

Theorem 6.1. *Assume (1.1) and (i)–(iii). Then there exists a unique weak solution $u \in V(0, T)$ to the Cauchy problem (1.2). If moreover (iv) holds, then $u \in \mathcal{W}_2(x, s, T)$ for every $(s, x) \in [0, T) \times \mathbb{R}^d$ and $F = F_u$ satisfies (5.2) with $s = 0$.*

Proof. For $\gamma > 0$ to be determined later define the norm $|||\cdot|||_\gamma$ in $V(0, T)$ by $|||u|||_\gamma^2 = \sup_{0\le s\le T} \|u_\gamma(s, \cdot)\|_2^2 + \|u_\gamma\|_{2;0,T}^2 + \lambda\|\nabla u_\gamma\|_{2;0,T}^2$, where $u_\gamma(s, x) = e^{\gamma s/2}u(s, x)$. Clearly the norm $|||\cdot|||_\gamma$ is equivalent to $\|\cdot\|_{V(0,T)}$, so $(V(0, T), |||\cdot|||_\gamma)$ is a Banach space, which we denote by V_γ. Define the mapping $\Phi : V_\gamma \to V_\gamma$ by putting $\Phi(u)$ to be the solution to (5.1) with F_u in place of F. Suppose that $v_1, v_2 \in V$ and set $v = \Phi(v_1) - \Phi(v_2)$. Multiplying v by the function $v(s, x)e^{\gamma s}$ and integrating by parts gives

$$
\begin{aligned}
e^{\gamma s}\|v(s, \cdot)\|_2^2 + \gamma \int_s^T e^{\gamma t}\|v(t, \cdot)\|_2^2\,dt + \int_s^T e^{\gamma t}(a(t, \cdot)\nabla v(t, \cdot), \nabla v(t, \cdot))_2\,dt \\
= 2\int_s^T e^{\gamma t}((F_{v_1} - F_{v_2})(t, \cdot), v(t, \cdot))_2\,dt. \qquad (6.1)
\end{aligned}
$$

Using (1.1), (ii) and the elementary inequality $2ab \le \varepsilon^{-1}a^2 + \varepsilon b^2$ with $\varepsilon = (8\Lambda)^{-1}\lambda$ we see that

$$
\begin{aligned}
2((F_{v_1} - F_{v_2})(t, \cdot), v(t, \cdot))_2 &\le 8\lambda^{-1}\Lambda L^2\|v(t, \cdot)\|_2^2 + (4\Lambda)^{-1}\lambda\|(v_1 - v_2)(t, \cdot)\|_2^2 \\
&\quad + 4^{-1}\lambda\|\nabla(v_1 - v_2)(t, \cdot)\|_2^2.
\end{aligned}
$$

Hence, if we put $\gamma = 1+8\lambda^{-1}\Lambda L^2$, then from (1.1), (6.1) we get $|||v|||_\gamma \le 2^{-1}|||v_1 - v_2|||_\gamma$. Thus, Φ is a contraction on V_γ, and therefore has a unique fixed point u, which is a unique solution to (1.2) in $V(0, T)$. Moreover, if we set $u_0 \equiv 0$ and $u_{n+1} = \Phi(u_n)$ for $n \in \mathbb{N}\cup\{0\}$, then $u_n \to u$ in $V(0, T)$. Let us now fix $(s, x) \in [0, T) \times \mathbb{R}^d$ and assume (iv) holds. We will show that $\{u_n\} \subset \mathcal{W}_2(x, s, T)$. For this purpose we will prove that

$$
|u_{n+1}(s, x)|^2 + \|u_{n+1}\|_{\mathcal{W}_2(x,s,T)}^2 \le C_2(1 \vee \lambda^{-1})((T - s)^{-d/2}\|\varphi\|_2^2 + \|F_{u_n}\|_{2;x,s,T}^2) \qquad (6.2)
$$

for $n \in \mathbb{N} \cup \{0\}$. Since $|F_{u_0}| \le g$, it follows from Proposition 5.1 that $u_1(\cdot, X_\cdot)$, $\sigma\nabla u_1(\cdot, X_\cdot)$ solves the BSDE (φ, F_{u_0}) associated with $(X, P_{s,x})$. From this and (4.1) we see that (6.2) holds for $n = 0$. Hence, in particular, (5.2) holds with F_{u_1} in place of F. We can therefore use once again Proposition 5.1 to get (6.2) for $n = 1$.

Working by induction we see that (6.2) holds for $n \in \mathbb{N}$, and so $\{u_n\} \subset \mathcal{W}_2(x, s, T)$. Let $\mathcal{V}_\gamma(x, s, T)$ denote the space $\mathcal{W}_2(x, s, T)$ equipped with the norm

$$|||u|||_{\gamma;x,s,T} = \left(\int_s^T \int_{\mathbb{R}^d} e^{\gamma t}(|u(t, y)|^2 + \lambda|\nabla u(t, y)|^2) p(s, x, t, y)\, dt\, dy\right)^{1/2}$$

and let $w_n = u_n - u_{n-1}$, $n \in \mathbb{N}$. Since the pair $(w_{n+1}(\cdot, X_\cdot), \sigma\nabla w_{n+1}(\cdot, X_\cdot))$ solves the BSDE $(0, F_{u_n} - F_{u_{n-1}})$, applying Itô's formula gives

$$\begin{aligned}
&e^{\gamma t} E_{s,x}|w_{n+1}(t, X_t)|^2 + \gamma E_{s,x} \int_t^T e^{\gamma\theta}|w_{n+1}(\theta, X_\theta)|^2 \\
&\qquad + E_{s,x} \int_t^T e^{\gamma\theta} < a\nabla w_{n+1}(\theta, X_\theta), \nabla w_{n+1}(\theta, X_\theta) > d\theta \\
&\quad = 2E_{s,x} \int_t^T e^{\gamma\theta} w_{n+1}(\theta, X_\theta) \cdot (F_{u_n} - F_{u_{n-1}})(\theta, X_\theta)\, d\theta \\
&\quad \le 2LE_{s,x} \int_t^T e^{\gamma\theta} w_{n+1}(\theta, X_\theta) \cdot (|w_n| + |\sigma\nabla w_n|)(\theta, X_\theta)\, d\theta \\
&\quad \le 8\lambda^{-1}\Lambda L^2 E_{s,x} \int_t^T e^{\gamma\theta}|w_{n+1}(\theta, X_\theta)|^2\, d\theta \\
&\qquad + 4^{-1} E_{s,x} \int_t^T e^{\gamma\theta}(|w_n(\theta, X_\theta)|^2 + \lambda|\nabla w_n(\theta, X_\theta)|^2)\, d\theta.
\end{aligned}$$

Therefore

$$\begin{aligned}
&E_{s,x} \int_s^T e^{\gamma t}(|w_{n+1}|^2 + \lambda|\nabla w_{n+1}|^2)(t, X_t)\, dt \\
&\qquad \le 4^{-1} \int_s^T e^{\gamma t}(|w_n|^2 + \lambda|\nabla w_n|^2)(t, X_t)\, dt,
\end{aligned}$$

that is $|||w_{n+1}|||_{\gamma;x,s,T} \le 2^{-1}|||w_n|||_{\gamma;x,s,T}$. Hence, by elementary calculations, $|||u_{n+m} - u_n|||_{\gamma;x,s,T} \le 2^{-n+1}|||u_1 - u_0|||_{\gamma;x,s,T}$ for $n, m \in \mathbb{N}$, which shows that $\{u_n\}$ is a Cauchy sequence in $\mathcal{V}_\gamma(x, s, T)$. In fact, since $u_n \to u$ in $V(0, T)$, it follows from (3.4) that $u_n \to u$ $\mathcal{V}_\gamma(x, s, T)$. Moreover, $|||u - u_1|||_{\gamma;x,s,T} \le |||u_1 - u_0|||_{\gamma;x,s,T} = |||u_1|||_{\gamma;x,s,T}$, and hence, by (4.1),

$$\begin{aligned}
(1 \wedge \lambda)^{1/2}\|u\|_{\mathcal{W}_2(s,x,T)} &\le |||u|||_{\gamma;x,s,T} \le 2|||u_1|||_{\gamma;x,s,T} \le C'\|u_1\|_{\mathcal{W}_2(x,s,T)} \\
&\le C_2^{1/2} C'(1 \vee \lambda^{-1})^{1/2}((T - s)^{-d/2}\|\varphi\|_2^2 + \|g\|_{2;x,s,T}^2)^{1/2}
\end{aligned}$$

with $C' = 2(1 \vee \lambda)^{1/2} e^{\gamma T/2}$. From the above, (iv) and (4.2) we see that F_u satisfies (5.2), and the proof is complete. □

Notice that from Theorem 4 in [6] one can deduce that the first assertion of Theorem 6.1 still holds if we replace (ii) by (ii').

The technique of equivalent norms used in the proof of Theorem 6.1 goes back to [4]. Applications of this technique to backward or forward-backward SDEs can be found in [13] and in references given in Remark 2.5 there.

Theorem 6.2. *If (1.1) and (i), (ii), (iv) are satisfied, then for each* $(s, x) \in [0, T) \times \mathbb{R}^d$ *there exists a unique solution* $(Y^{s,x}, Z^{s,x}) \in \mathbb{H}_T^{(1+d)}(P_{s,x})$ *to the BSDE* (φ, f) *associated with* $(X, P_{s,x})$*. Moreover, it is given by (5.3), where* $u \in V(0, T)$ *is a unique continuous solution to the problem (1.2).*

Proof. Uniqueness follows from Proposition 4.1. By Theorem 6.1, F_u satisfies (5.2) with $s = 0$, so to prove existence of solutions it suffices to put $F = F_u$ and repeat the proof of Proposition 5.1. □

From Theorem 6.2 and (4.1), (4.2) we have

Corollary 6.3. *Under the assumptions of Theorem 6.2 there is* C_4 *depending only on* $\lambda, \Lambda, K, d, T$ *and* q, p *such that*

$$|u(s, x)| + \|u\|_{\mathcal{W}_2(x,s,T)} \le C_4((T - s)^{-d/2}\|\varphi\|_2^2 + \|g\|_{q,p}^2)^{1/2}$$

for $(s, x) \in [0, T) \times \mathbb{R}^d$.

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