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On-line routing of random calls in networks

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Abstract. We consider a random sequence of calls between nodes in a complete network with link capacities. Each call first tries the direct link. If that link is saturated, then the 'first-fit dynamic alternative routing algorithm' sequentially selects up to *d* random two-link alternative routes, and assigns the call to the first route with spare capacity on each link, if there is such a route. The 'balanced dynamic alternative routing algorithm' simultaneously selects *d* random two-link alternative routes; and the call is accepted on a route minimising the maximum of the loads on its two links, provided neither of these two links is saturated.

We determine the capacities needed for these algorithms to route calls successfully, and find that the second 'balanced' algorithm requires a much smaller capacity. Our results strengthen and extend the discrete-time results presented in [10].

1. Introduction

Let us first consider briefly a version of the 'online load-balancing problem'. Here there is a set of servers each with the same capacity, and a random sequence of jobs. Each job may be allocated to any server, and takes up unit capacity. One allocation strategy is as follows: when a job arrives, sequentially select up to *d* random servers, and allocate the job to the first server with spare capacity, if there is one. Another strategy is simultaneously to select *d* random servers, and allocate the job to a server with most spare capacity, if there is one. It is well known that the latter 'balanced allocation' strategy requires much smaller capacity to allocate most jobs successfully, see [3, 5, 6, 12].

In this paper, we investigate a related problem in a more complicated setting, namely the online routing of a random sequence of calls in a network, and find a similar conclusion. Initial results on this subject were presented in [10]: here we give strengthened results with full proofs. We naturally build to some extent on ideas from [10], which itself builds on the work mentioned above on balanced allocations. An earlier version of what follows can be found in [9]. This last reference also investigates the technical difficulties (involving infinite dimensional spaces) in the methods used in [10] for analysing the continuous-time model proposed there:

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the continuous-time results stated in [10] should, at least in the meantime, be taken as merely indicating the way things may work.

The model we consider here is as follows. There are n nodes, each pair of which may wish to communicate. A *call* is an unordered pair $\{u, v\}$ of distinct nodes, that is an edge of the complete graph K_n on these nodes. For each of the $N = \binom{n}{2}$ unordered pairs $\{u, v\}$ of distinct nodes, there is a *direct link*, also denoted by $\{u, v\}$, with capacity $D_1 = D_1(n)$. This direct link is used as long as it has available capacity. There are also two *indirect links*, denoted by uv and vu, each with capacity $D_2 = D_2(n)$. The indirect link uv may be used when for some w a call $\{u, w\}$ finds its direct link saturated, and we seek an alternative route via node v. Similarly vu may be used for alternative routes for calls $\{v, w\}$ via u.

We are given a sequence of calls $x_1, x_2, \ldots, x_M$ one at a time. For each call in turn, we must choose a route (either a direct link or an alternative two-link route via an intermediate node) if this is possible, before seeing later calls. These routes cannot be changed later, and calls do not end. The aim is to minimise the number of calls that fail to be routed successfully. Thus we are considering a static online discrete time routing problem.

We assume that the calls are independent random variables $X_1, X_2, \ldots, X_M$, where each X_j is uniformly distributed over the edges of K_n. We consider the case when $M = N = \binom{n}{2}$, so that there is on average one call per edge (see also Section 7 below). Let d be a positive integer (think of d as 2). A *general dynamic alternative routing* algorithm GDAR operates as follows. For each call $\{u, v\}$ in turn, the call is routed on the direct link if possible; and otherwise d nodes $w_1, \ldots, w_d$ are selected uniformly at random with replacement from $V \setminus \{u, v\}$ and the call is routed via one of these nodes if possible, along the two corresponding indirect links. The *first-fit dynamic alternative routing* algorithm FDAR is the version when we always choose the first possible alternative route. The *balanced dynamic alternative routing* algorithm BDAR is the version when we choose an alternative route which minimises the maximum of the current loads on its two indirect links, if possible. Implementations of such techniques include the Dynamic Alternative Routing (DAR) algorithm used by British Telecommunications [7], and AT&T's Dynamic Non-Hierarchical Routing (DNHR) algorithm [2].

We state three theorems. The first theorem sets the scene: it concerns the case when we do not have indirect links with their own capacities. If the direct link for a call is saturated, then we seek up to d alternative two-link routes as with a general dynamic alternative routing algorithm GDAR, except that these alternative routes must also use 'direct' links. When $d = 0$, we simply route along direct links whenever possible, and never use alternative routes. We use log to mean natural logarithm, and 'with high probability' to mean 'with probability $\to 1$ as $n \to \infty$'. Given two sequences of numbers, a_n and b_n, we write $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$.

Theorem 1.1. *Let $\alpha > 0$, let each direct link $\{u, v\}$ have capacity $D \sim \alpha \frac{\log n}{\log\log n}$, and suppose that there are no indirect links. Let d be a non-negative integer. Suppose that each call is routed along its direct link whenever possible, and otherwise seeks up to d alternative routes.*

(a) *If* $\alpha(d+1) > 2$ *then, whatever version of GDAR we use, with high probability all* M *calls are successful.*

(b) *If* $\alpha(d+1) < 2$*, and* $0 < \delta < 2 - \alpha(d+1)$ *then, whatever version of GDAR we use, with high probability at least* n^{δ} *calls fail.*

Thus when we do not have separate indirect links, we need capacity of order $\log n / \log\log n$ on the links in order to achieve good communication. We shall see that when we reserve separate capacity for indirect links, we do much better. We assume that direct and indirect links have separate capacities D_1 and D_2 respectively, as discussed earlier. Theorem 1.2 shows roughly that, with the first-fit dynamic alternative routing algorithm FDAR, we need capacities of order $\sqrt{\frac{\log n}{\log\log n}}$ on the links in order to achieve good communication; and that FDAR is about as bad as any version of GDAR.

Theorem 1.2. *Let* $\alpha_1, \alpha_2 > 0$*, and suppose that* $D_i = D_i(n) \sim \alpha_i \sqrt{\frac{\log n}{\log\log n}}$ *for* $i = 1, 2$*. Let* d *be a positive integer.*

(a) *If* $\alpha_1\alpha_2 d > 4$ *and we use any GDAR algorithm, then with high probability all* M *calls are routed successfully.*

(b) *Let* $\alpha_1\alpha_2 d < 4$*, and let* $0 < \delta < 2 - \frac{\alpha_1\alpha_2 d}{2}$*. If we use the FDAR algorithm, then with high probability at least* n^{δ} *calls fail.*

Theorem 1.3 shows that the balanced method BDAR succeeds with much smaller capacities. It says that, as long as the capacity D_1 on the direct links is of order $\log\log n$, there is a tight threshold value close to $\log\log n / \log d$ for good communication with BDAR; and that BDAR is as good as any GDAR algorithm. (See also Section 7 below.) We state the theorem in a precise form. We use $\log^{(3)} n$ to denote $\log\log\log n$ and similarly $\log^{(4)} n$ denotes $\log(\log^{(3)} n)$. The proof is based on ideas in [4].

Theorem 1.3. *Let* $d \geq 2$ *be an integer.*

(a) *Let* $D_1 = \Omega(\log\log n)$*. Then there is a constant* c *such that if* $D_2 = D_2(n) \geq \frac{1}{\log d}(\log\log n - \log^{(3)} n - \log^{(4)} n) + c$ *and we use the BDAR algorithm, then with high probability all* M *calls are routed successfully.*

(b) *Let* $D_1 = O(\log\log n)$ *and let* $\delta > 0$*. Then there is a constant* c *such that the following holds. If* $D_2 \leq \frac{1}{\log d}(\log\log n - \log^{(3)} n - \log^{(4)} n) - c$ *and we use any GDAR algorithm, then with high probability at least* $n^{2-\delta}$ *calls fail.*

The plan of the paper is as follows. In the next section we present some preliminary general results concerning things like balls and bins. Then there is a brief section where we introduce some notation. After that we prove Theorem 1.1, then Theorem 1.2, and then Theorem 1.3. Finally we discuss some related results, including one involving a more general network model.

2. Preliminary results

It is convenient for our later proofs to collect together here several stochastic comparison results, some of which are 'folk knowledge'. Let us start with an elementary inequality concerning the moments of the binomial distribution. For $n \in \mathbb{N}$

and $0 \leq p \leq 1$, let $B(n, p)$ denote a binomially distributed random variable with parameters n and p.

Lemma 2.1. *Let d be a positive integer, and let $Z \sim B(n, p)$. Then*

$$\mathbf{E}[Z^d] \leq d^d \ \max\{np, (np)^d\}.$$

Proof. For $j = 1, \ldots, d$ let $\mathcal{P}_j$ be the set of partitions of $\{1, \ldots, d\}$ into j non-empty blocks. Then $\sum_{j=1}^d |\mathcal{P}_j|$ is the total number of partitions of $\{1, \ldots, d\}$, which is at most d^d. Given a partition π of $\{1, \ldots, d\}$, let $S(\pi)$ denote the set of d-tuples $\mathbf{i} = (i_1, \ldots, i_d) \in \{1, \ldots, n\}^d$ such that $i_s = i_t$ if and only if s and t are in the same block of π. Let $X_1, \ldots, X_n$ be independent identically distributed binary random variables, with each $\mathbf{Pr}(X_i = 1) = p$. Let $n_{(j)}$ denote the j-term product $n(n-1)\cdots(n-j+1)$. Then

$$\begin{aligned}
\mathbf{E}[Z^d] &= \mathbf{E}[(\sum_{i=1}^n X_i)^d] \\
&= \sum_{j=1}^d \sum_{\pi\in\mathcal{P}_j} \sum_{\mathbf{i}\in S(\pi)} \mathbf{E}[\prod_{k=1}^d X_{i_k}] \\
&= \sum_{j=1}^d |\mathcal{P}_j| \, n_{(j)} p^j .
\end{aligned}$$

Hence

$$\mathbf{E}[Z^d] \leq \sum_{j=1}^d |\mathcal{P}_j| \, (np)^j \leq d^d \ \max\{np, (np)^d\},$$

as required. □

The next result may be proved by standard coupling methods. It has previously been used in related applications – see for instance [3, 12]. We shall use it to deduce a minor extension, which will be convenient for us later.

Lemma 2.2. *Let $\phi_0 \subseteq \phi_1 \subseteq \ldots \subseteq \phi_n$ be a filter, let $Y_1, Y_2, \ldots, Y_n$ be binary random variables such that each Y_t is ϕ_t-measurable, and let $S_t = \sum_{i=1}^t Y_i$. Let $0 \leq p \leq 1$ and let k be a positive integer. If $\mathbf{Pr}(Y_t = 1|\phi_{t-1}) \leq p$ for each time $t = 1, \ldots, n$, then*

$$\mathbf{Pr}(S_n \geq k) \leq \mathbf{Pr}(B(n, p) \geq k).$$

Similarly, if $\mathbf{Pr}(Y_t = 1|\phi_{t-1}) \geq p$ for each time $t = 1, \ldots, n$, then

$$\mathbf{Pr}(S_n < k) \leq \mathbf{Pr}(B(n, p) < k).$$

The next result is an extension of Lemma 2.2.

Lemma 2.3. *Let $\phi_0 \subseteq \phi_1 \subseteq \ldots \subseteq \phi_n$ be a filter, let $Y_1, Y_2, \ldots, Y_n$ be binary random variables such that each Y_t is ϕ_t-measurable, and let $S_t = \sum_{i=1}^{t} Y_i$. Let $E_0, E_1, \ldots, E_{n-1}$ be events where $E_t \in \phi_t$ for $t = 0, \ldots, n-1$, and let E be an event with $E \subseteq \cap_t E_t$. Let $0 \leq p \leq 1$ and let k be a positive integer.*

(a) If for each $t = 1, \ldots, n$

$$\mathbf{Pr}(Y_t = 1|\phi_{t-1}) \leq p \quad \text{on} \;\; E_{t-1} \wedge (S_{t-1} < k)$$

(whatever else happened before time t, if E_{t-1} holds and $S_{t-1} < k$ then the probability that $Y_t = 1$ is at most p) then

$$\mathbf{Pr}((S_n \geq k) \wedge E) \leq \mathbf{Pr}(B(n, p) \geq k).$$

(b) If for each $t = 1, \ldots, n$

$$\mathbf{Pr}(Y_t = 1|\phi_{t-1}) \geq p \quad \text{on} \;\; E_{t-1} \wedge (S_{t-1} < k)$$

then

$$\mathbf{Pr}((S_n < k) \wedge E) \leq \mathbf{Pr}(B(n, p) < k).$$

Proof. Let A_t denote the event $E_t \wedge (S_t < k)$.

(a) For each $t = 1, \ldots, n$, let $\tilde{Y}_t = \min\{Y_t, \mathbb{I}_{A_{t-1}}\}$, and let $\tilde{S}_t = \sum_{i=1}^{t} \tilde{Y}_i$. Then $\mathbf{Pr}(\tilde{Y}_t = 1|\phi_{t-1}) \leq p$, since by assumption it is at most p on A_{t-1}, and it equals 0 on $\overline{A_{t-1}}$. Hence by Lemma 2.2

$$\mathbf{Pr}((S_n \geq k) \wedge E) = \mathbf{Pr}((\tilde{S}_n \geq k) \wedge E) \leq \mathbf{Pr}(\tilde{S}_n \geq k) \leq \mathbf{Pr}(B(n, p) \geq k).$$

(b) Now, for each $t = 1, \ldots, n$, let $\tilde{Y}_t = \max\{Y_t, 1 - \mathbb{I}_{A_{t-1}}\}$, and let $\tilde{S}_t = \sum_{i=1}^{t} \tilde{Y}_i$. Then $\mathbf{Pr}(\tilde{Y}_t = 1|\phi_{t-1}) \geq p$, since by assumption it is at least p on A_{t-1}, and it equals 1 on $\overline{A_{t-1}}$. Hence by Lemma 2.2 as above

$$\mathbf{Pr}((S_n < k) \wedge E) = \mathbf{Pr}((\tilde{S}_n < k) \wedge E) \leq \mathbf{Pr}(\tilde{S}_n < k) \leq \mathbf{Pr}(B(n, p) < k). \quad \square$$

The next three lemmas involve balls and bins. Suppose that m balls are thrown independently into n bins, all with the same distribution (not necessarily uniform). We may compare the loads (numbers of balls) in the bins to independent Poisson random variables. Special cases of the following result can be found in [12] (the exact original source is unclear). A subset A of $\mathbb{N}^n$ is called an *up-set* if $\mathbf{x} \in A$ and $\mathbf{y} \geq \mathbf{x}$ (component by component) implies that $\mathbf{y} \in A$. Similarly, A is a *down-set* if $\mathbf{x} \in A$ and $\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}$ implies that $\mathbf{y} \in A$.

Lemma 2.4. *Let m and n be positive integers, and let $Z_1, \ldots, Z_m$ be independent identically distributed random variables, taking values in $\{1, \ldots, n\}$. Let $L(i)$ be the (final) load in bin i, that is $L(i) = |\{t : Z_t = i\}|$. Let $Y(1), \ldots, Y(n)$ be independent random variables where $Y(i) \sim Po(E(L(i)))$. Write $\mathbf{L}$ for $(L(1), \ldots, L(n))$ and $\mathbf{Y}$ for $(Y(1), \ldots, Y(n))$. Then for any up-set or down-set A in $\mathbb{N}^n$,*

$$\mathbf{Pr}(\mathbf{L} \in A) \leq 2\,\mathbf{Pr}(\mathbf{Y} \in A).$$

Proof. Let $Y = \sum_i Y(i)$. Then $Y \sim Po(m)$, and so $\mathbf{Pr}(Y \geq m) \geq 1/2$ and $\mathbf{Pr}(Y \leq m) \geq 1/2$, see [8]. Now use the standard fact that the joint distribution of the $Y(i)$ given $Y = m$ is exactly that of the $L(i)$. Hence, if A is an up-set and $k \geq m$, then

$$\mathbf{Pr}(\mathbf{Y} \in A | Y = k) \geq \mathbf{Pr}(\mathbf{Y} \in A | Y = m) = \mathbf{Pr}(\mathbf{L} \in A),$$

and so

$$\mathbf{Pr}(\mathbf{Y} \in A) \geq \sum_{k \geq m} \mathbf{Pr}(\mathbf{Y} \in A | Y = k)\mathbf{Pr}(Y = k) \geq \mathbf{Pr}(\mathbf{L} \in A)\mathbf{Pr}(Y \geq m).$$

There is a similar proof when A is a down-set. □

From the last lemma and Lemma 2.1 we obtain:

Lemma 2.5. *Let m, n and D be positive integers. Throw m balls uniformly at random into n bins, and let Y be the number of bins with at least D balls. Let $p = \mathbf{Pr}(Po(\frac{m}{n}) \geq D)$, and let $Z \sim B(n, p)$. Then for any k,*

$$\mathbf{Pr}(Y \leq k) \leq 2\,\mathbf{Pr}(Z \leq k)$$

and

$$\mathbf{Pr}(Y \geq k) \leq 2\,\mathbf{Pr}(Z \geq k);$$

and for any positive integer d,

$$\mathbf{E}(Y^d) \leq 2\,\mathbf{E}(Z^d) \leq 2d^d \ \max\{np, (np)^d\}.$$

Proof. The first two inequalities follow immediately from Lemma 2.4. By the second of them, $\mathbf{Pr}(Y^d \geq k) \leq 2\,\mathbf{Pr}(Z^d \geq k)$ for all k, and so

$$\mathbf{E}(Y^d) = \sum_{k \geq 1} \mathbf{Pr}(Y^d \geq k) \leq 2 \sum_{k \geq 1} \mathbf{Pr}(Z^d \geq k) = 2\,\mathbf{E}(Z^d).$$

Now we may use Lemma 2.1 to complete the proof. □

We need a generalised balls-and-bins result that will handle the case when the balls are nearly uniformly distributed over the bins, but the probabilities may vary a little depending on the previous history. In particular, the balls can be 'rejected' with some small probability, depending on the previous history, or conversely there may be 'bonus balls'.

Lemma 2.6. *Let m and n be positive integers, and let S be a set of size n. Let $\phi_0 \subseteq \phi_1 \subseteq \ldots \subseteq \phi_m$ be a filter, and let $Z_1, \ldots, Z_m$ be random variables, where each Z_t is ϕ_t-measurable. Let $E_0, \ldots, E_{m-1}$ be events such that $E_t \in \phi_t$ for each $t = 0, \ldots, m-1$, and let E be an event with $E \subseteq \cap_{t=0}^{m-1} E_t$.*

For each $i \in S$, let $L_t(i)$ be the load in bin i at time t, that is $L_t(i) = |\{s \in \{1, \ldots, t\} : Z_s = i\}|$; and let $\mathbf{L}$ be the load vector $(L_m(i) : i \in S)$. (The random variables Z_t may take values outside S, but we are interested only in *the bins $i \in S$.) We compare* $\mathbf{L}$ *with the load vector* $\hat{\mathbf{L}} = (\hat{L}_{\hat{m}}(i) : i \in S)$ *in a balls-and-bins experiment where we throw* $\hat{m}$ *balls independently and uniformly at random into the* n *bins* $i \in S$*. Let* D *be a positive integer. Let*

$$\hat{X} = |\{i \in S : \hat{L}_{\hat{m}}(i) \geq D\}| = \sum_{i \in S} \mathbb{I}_{\hat{L}_{\hat{m}}(i) \geq D}.$$

Let $0 \leq p \leq 1$.

(a) Assume that

$$\mathbf{Pr}(Z_t = i | \phi_{t-1}) \leq p/n \quad \text{on} \quad E_{t-1},$$

for each $t = 1, \ldots, m$ *and each* $i \in S$*. Let* $X = \sum_{i \in S} \mathbb{I}_{L_m(i) \geq D}$*. Then for any* k

$$\mathbf{Pr}(X \geq k) \leq \mathbf{Pr}(\hat{X} \geq k) + \mathbf{Pr}(\overline{E}) + \mathbf{Pr}(B(m, p) > \hat{m}). \tag{1}$$

(b) For each $t = 1, \ldots, m$ *and* $i \in S$ *let* $Y_t(i)$ *be a non-negative* ϕ_t*-measurable random variable; and let*

$$L_t^+(i) = \sum_{s=1}^{t} (\mathbb{I}_{Z_s = i} + Y_s(i)) = L_t(i) + \sum_{s=1}^{t} Y_s(i).$$

(The $Y_t(i)$ *correspond to 'bonus balls' in bin* i*.) Assume that*

$$\mathbf{Pr}(Z_t = i | \phi_{t-1}) \geq p/n \quad \text{on} \quad E_{t-1} \wedge (L_{t-1}^+(i) < D),$$

for each $t = 1, \ldots, m$ *and each* $i \in S$*. Let* $X^+ = \sum_{i \in S} \mathbb{I}_{L_m^+(i) \geq D}$*. Then for any* k

$$\mathbf{Pr}(X^+ \geq k) \geq \mathbf{Pr}(\hat{X} \geq k) - \mathbf{Pr}(\overline{E}) - \mathbf{Pr}(B(m, p) < \hat{m}). \tag{2}$$

Proof. We may assume that S is the set $\{1, \ldots, n\}$, say. Let us prove part (b): part (a) may be proved in a similar way. We use a coupling approach which is mainly standard, to 'tidy up' the random variables Z_t, in four steps. First, without loss of generality, there are ϕ_t-measurable random variables $Z_t^{(1)}$ such that for each $i \in S$

$$\text{if} \;\; (Z_t^{(1)} = i) \wedge E_{t-1} \quad \text{then} \;\; Z_t = i, \tag{3}$$

and

$$\mathbf{Pr}(Z_t^{(1)} = i | \phi_{t-1}) \geq p/n \;\; \text{on} \;\; L_{t-1}^+(i) < D.$$

Next, without loss of generality, there are ϕ_t-measurable random variables $Z_t^{(2)}$ such that for each $i \in S$

$$\text{if} \;\; Z_t^{(2)} = i \quad \text{then} \;\; Z_t^{(1)} = i, \tag{4}$$

and

$$\mathbf{Pr}(Z_t^{(2)} = i | \phi_{t-1}) = p/n \;\; \text{on} \;\; L_{t-1}^+(i) < D.$$

Now we define random variables $Z_t^{(3)}$. We spell out one way of doing this as the step is not quite standard. We may assume that there exists a ϕ_t-measurable random linear order π_t on S, where π_t is independent of ϕ_{t-1} and $Z_t^{(2)}$, and π_t is uniformly distributed over the $n!$ linear orders on S. Let $\mathcal{F}_t = \{i \in S : L_t(i) \geq D\}$. (The letter $\mathcal{F}$ is for full.) On $(Z_t^{(2)} \in S \backslash \mathcal{F}_{t-1})$ we let $Z_t^{(3)} = Z_t^{(2)}$. On $(Z_t^{(2)} \notin S) \wedge (\mathcal{F}_{t-1} = \emptyset)$ we let $Z_t^{(3)} = 0$. Otherwise, we let $Z_t^{(3)}$ be the first bin $i \in \mathcal{F}_{t-1}$ under the order π_t. Note that

$$\mathbf{Pr}(Z_t^{(3)} = 0 | \phi_{t-1}) \leq 1 - p.$$

It is easy to check that for each $i \in S$

$$\text{if } \ (Z_t^{(3)} = i) \wedge (L_{t-1}^{+}(i) < D) \quad \text{then } \ Z_t^{(2)} = i, \tag{5}$$

and

$$\mathbf{Pr}(Z_t^{(3)} = i | \phi_{t-1}) \geq p/n.$$

Finally, without loss of generality, there are ϕ_t-measurable random variables $Z_t^{(4)}$, taking values in $S \cup \{0\}$, such that for each $i \in S$

$$\text{if } \ Z_t^{(4)} = i \quad \text{then } \ Z_t^{(3)} = i, \tag{6}$$

and

$$\mathbf{Pr}(Z_t^{(4)} = i | \phi_{t-1}) = p/n. \tag{7}$$

Observe that, by (3), (4), (5) and (6), for each $i \in S$,

$$\text{if } \ (Z_t^{(4)} = i) \wedge (L_{t-1}^{+}(i) < D) \wedge E \quad \text{then } \ Z_t = i,$$

and so

$$\text{if } \ (L_m^{(4)}(i) \geq D) \wedge E \quad \text{then } \ L_m^{+}(i) \geq D.$$

Hence

$$\mathbf{Pr}(X^{+} \geq k) \geq \mathbf{Pr}((X^{(4)} \geq k) \wedge E) \geq \mathbf{Pr}(X^{(4)} \geq k) - \mathbf{Pr}(\bar{E}).$$

By (7), the random variables $Z_t^{(4)}$ are independent, and the distribution of $X^{(4)}$ is exactly the same as if we choose $B(m, p)$ balls and then throw them uniformly at random into the n bins. Thus

$$\mathbf{Pr}(X^{(4)} \geq k) \geq \mathbf{Pr}(\hat{X} \geq k) - \mathbf{Pr}(B(m, p) < \hat{m}),$$

and (2) follows. □

We shall frequently appeal to the following large deviations result, which is related to the standard Chernoff bounds, see for example [1]. (It also follows for example from Theorem 2.3 in [11].)

Lemma 2.7. *Let $n \in \mathbb{N}$, let $0 \le p \le 1$ and let $\mu = np$. Then*

$$\mathbf{Pr}(|B(n,p) - \mu| \ge \epsilon\mu) \le 2e^{-\epsilon^2\mu/3} \quad \textit{for each} \;\; 0 \le \epsilon \le 1, \tag{8}$$

and

$$\mathbf{Pr}(B(n,p) \ge e\mu) \le e^{-\mu}. \tag{9}$$

From the latter inequality, it follows that if k is a positive integer and $0 \le p \le 1$, and b satisfies $b \ge ekp$ and $b \ge 2e \log n$, then

$$\mathbf{Pr}(B(k,p) \ge b) \le n^{-2}. \tag{10}$$

Finally, for the proof of Theorem 1.1, we use the following version of Talagrand's inequality, see for example Theorem 4.3 in [11]. (In the notation in [11], the function h below is a (c^2r)-configuration function.)

Lemma 2.8. *Let $\mathbf{X} = (X_1, X_2, \dots)$ be a finite family of independent random variables, where the random variable X_j takes values in a set Ω_j. Let $\Omega = \prod_j \Omega_j$.*

Let c and r be positive constants, and suppose that the non-negative real-valued function h on Ω satisfies the following two conditions for each $\mathbf{x} \in \Omega$.

- *Changing the value of a co-ordinate x_j can change the value of $h(\mathbf{x})$ by at most c.*
- *If $h(\mathbf{x}) = s$, then there is a set of at most rs co-ordinates such that $h(\mathbf{x}') \ge s$ for any $\mathbf{x}' \in \Omega$ which agrees with $\mathbf{x}$ on these co-ordinates.*

Let m be a median of the random variable $Z = h(\mathbf{X})$. For each $x \ge 0$

$$P(Z \ge m + x) \le 2\exp\left(-\frac{x^2}{4c^2r(m+x)}\right). \tag{11}$$

3. Some notation

We shall need quite a lot of notation. For convenience we collect the basic notation here. The letter 'L' will indicate 'load', $\mathcal{S}$ will indicate a set of 'saturated' links, and S will be the size of set $\mathcal{S}$. The letter t will stand for 'time', that is the number of calls considered so far, and will be used as a subscript.

Given distinct nodes u and v, $L_t(\{u,v\})$ denotes the load on the direct link $\{u,v\}$ at time t, that is the number of calls amongst the first t calls that are routed along that link. Similarly, $L_t(uv)$ denotes the load on the indirect link uv at time t.

Given a node v, $\mathcal{S}_t^{\text{dir}}(\text{at } v)$ denotes the set of direct links $\{v,w\}$ incident with v which are saturated at time t, that is, are such that $L_t(\{v,w\}) = D_1$ (or $L_t(\{v,w\}) = D$ in the absence of indirect links); and $S_t^{\text{dir}}(\text{at } v) = |\mathcal{S}_t^{\text{dir}}(\text{at } v)|$. Similarly, $\mathcal{S}_t^{\text{ind}}(\text{at } v)$ denotes the set of indirect links vx for calls at v which are saturated at time t, that is, are such that $L_t(vx) = D_2$; and $S_t^{\text{ind}}(\text{at } v) = |\mathcal{S}_t^{\text{ind}}(\text{at } v)|$. Also, $\mathcal{S}_t^{\text{ind}}(\text{via } x)$ denotes the set of indirect links vx for calls at some node v which are saturated at time t; and $S_t^{\text{ind}}(\text{via } x) = |\mathcal{S}_t^{\text{ind}}(\text{via } x)|$.

From the above 'local' sets, we may define 'global' sets $\mathcal{S}_t^{\mathrm{dir}}$ and $\mathcal{S}_t^{\mathrm{ind}}$ as follows. Let $\mathcal{S}_t^{\mathrm{dir}} = \cup_v \mathcal{S}_t^{\mathrm{dir}}(\text{at } v)$, the set of all direct links saturated at time t, and $S_t^{\mathrm{dir}} = |\mathcal{S}_t^{\mathrm{dir}}|$; and let $\mathcal{S}_t^{\mathrm{ind}} = \cup_v \mathcal{S}_t^{\mathrm{ind}}(\text{at } v)$, the set of all indirect links saturated at time t, and $S_t^{\mathrm{ind}} = |\mathcal{S}_t^{\mathrm{ind}}|$.

We let ϕ_t denote the sigma-field generated by all events up to time t. We defined 'with high probability' above to mean 'with probability $\to 1$ as $n \to \infty$'. It is convenient to use 'with very high probability ' to mean 'with probability $1 - e^{-\Omega(\log^2 n)}$ as $n \to \infty$'.

We may assume that, for each time t, whether or not the direct link for call X_t is saturated, we make d choices of possible intermediate node for an alternative route.

4. Proof of Theorem 1.1

For the proof of Theorem 1.1 we use two lemmas. The first gives bounds on the probability that the call X_t at time t fails (that is, fails to be routed successfully), conditional on the previous history of the process, in terms of the quantities $S_{t-1}^{\mathrm{dir}}(\text{at } v)$ and S_{t-1}^{dir}. Note that since there are no indirect links here, $S_t^{\mathrm{dir}}(\text{at } v)$ is the total number of links at v which are saturated at time t, and similarly for S_t^{dir}.

In the proof of the upper bound, part (a) of the theorem, there are three cases. The case $d = 0$ is easy. For the case $d = 1$, we need to consider both S_{t-1}^{dir} and $\Delta_{t-1} = \max_v S_{t-1}^{\mathrm{dir}}(\text{at } v)$. For the case $d \geq 2$, we need consider only Δ_{t-1}. We show that with very high probability, Δ_N/n is at most about $n^{-\alpha}$, and so the probability that a given call fails is at most about $n^{-\alpha(d+1)}$.

In the proof of the lower bound, part (b) of the theorem, we show that with very high probability at time $N/2$ at least about $n^{2-\alpha}$ links are saturated, and then for $t > N/2$ the probability that call X_t fails is at least about $n^{-\alpha(d+1)}$.

Lemma 4.1. *Let d be a positive integer. For each time t,*

$$\mathbf{Pr}(X_t \textit{ fails } |\phi_{t-1}) \leq \frac{2^d}{Nn^d} \sum_v \left(S_{t-1}^{dir}(at\ v)\right)^{d+1} \tag{12}$$

if $n \geq 4d$, and

$$\mathbf{Pr}(X_t \textit{ fails } |\phi_{t-1}) \geq (\frac{(2S_{t-1}^{dir} - n)^+}{n^2})^{d+1}. \tag{13}$$

Proof. Consider some time t, and temporarily let d_v denote $S_{t-1}^{\mathrm{dir}}(\text{at } v)$. Then

$$\begin{aligned}
\mathbf{Pr}(X_t \text{ fails } |\phi_{t-1}) &\leq \frac{1}{N} \sum_{\{u,v\} \in \mathcal{S}_{t-1}^{\mathrm{dir}}} \left(\frac{d_u + d_v - 2}{n-2}\right)^d \\
&\leq \frac{2^{d-1}}{N(n-2)^d} \sum_{\{u,v\} \in \mathcal{S}_{t-1}^{\mathrm{dir}}} \left((d_u - 1)^d + (d_v - 1)^d\right) \\
&\leq \frac{2^{d-1}}{N(n-2)^d} \sum_v d_v^{d+1}.
\end{aligned}$$

In the second inequality above, we used the fact that the function $f(x) = x^d$ is convex, and so $(x+y)^d \leq 2^{d-1}(x^d + y^d)$. For the third inequality, note that in counting by edges we count the term $(d_v - 1)^d$ exactly d_v times. Further, $(1-\frac{2}{n})^d \geq 1 - \frac{2d}{n} \geq \frac{1}{2}$ if $n \geq 4d$, and then $(n-2)^d \geq \frac{1}{2}n^d$. Thus we obtain (12).

Now we consider (13). Let $W = \{v \in V : d_v \geq 1\}$. We may assume that W is non-empty. Note that the function $f(x) = x^{d+1}$ is convex, and so by Jensen's inequality

$$\begin{aligned} |W|^{-1} \sum_{v \in W} (d_v - 1)^{d+1} &\geq \left(|W|^{-1} \sum_{v \in W} (d_v - 1) \right)^{d+1} \\ &= \left(|W|^{-1} (2S_{t-1}^{\text{dir}} - |W|) \right)^{d+1}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{Pr}(X_t \text{ fails} | \phi_{t-1}) &\geq \frac{1}{N} \sum_{\{u,v\} \in \mathcal{S}_{t-1}^{\text{dir}}} \left(\frac{\max\{d_u - 1, d_v - 1\}}{n} \right)^d \\ &\geq n^{-(d+2)} \sum_{\{u,v\} \in \mathcal{S}_{t-1}^{\text{dir}}} \left((d_u - 1)^d + (d_v - 1)^d \right) \\ &\geq n^{-(d+2)} \sum_{v \in W} (d_v - 1)^{d+1} \\ &\geq n^{-(d+2)} \, |W|^{-d} \left(2S_{t-1}^{\text{dir}} - |W| \right)^{d+1} \\ &\geq \left(\frac{2S_{t-1}^{\text{dir}} - |W|}{n^2} \right)^{d+1}. \end{aligned}$$

which yields (13). □

Lemma 4.2. *Let d be a positive integer, let $\alpha > \frac{2}{d+1}$, and let $\delta > 0$. Let $\Delta_t = \max_v\{S_t^{dir}(at\ v)\}$. Then with very high probability,*

$$\Delta_N \leq n^{(1-\alpha)^+ + \delta} \quad \textit{and} \quad S_N^{dir} \leq n^{(2-\alpha)^+ + \delta}.$$

Proof. We shall consider the 'superprocess', which counts up every time a link is mentioned, not just if it is used. Thus at each time t, there are $2d+1$ links mentioned, including multiplicities, and all are counted. Within this proof, we use $L_t(\{u, v\})$, $S_t^{\text{dir}}(\text{at } v)$ and so on to refer to the superprocess.

Fix distinct nodes u and v. Then

$$\mathbf{Pr}(X_t \cap \{u, v\} \neq \emptyset) = 1 - \binom{n-2}{2} \Big/ \binom{n}{2} \leq 4/n.$$

Let A^0 be the event that $|\{t : X_t \cap \{u, v\} \neq \emptyset\}| \leq 3n$. Then by inequality (8) in Lemma 2.7,

$$\mathbf{Pr}(\overline{A^0}) \leq \mathbf{Pr}(B(N, 4/n) > 3n) = e^{-\Omega(n)}. \tag{14}$$

Let T be a fixed set of at most $3n$ times. Let $A(T)$ be the event that $\{t : X_t \cap \{u, v\} \neq \emptyset\} = T$. Condition on the event $A(T)$. Think of $d+1$ trials for each time $t \in T$, which may or may not mention the link $\{u, v\}$. For each trial, conditional on all previous trials, the probability that the link $\{u, v\}$ is mentioned is either 0 or $\frac{1}{2n-3}$ or $\frac{1}{n-2}$, and so is at most $\frac{1}{n-2}$. Hence in the superprocess, the load on that link is stochastically at most $B(3n(d+1), \frac{1}{n-2})$. Thus

$$\begin{aligned}
\mathbf{Pr}(L_N(\{u,v\}) \geq D | A(T)) &\leq \binom{3n(d+1)}{D} \left(\frac{1}{n-2}\right)^D \\
&\leq \left(\frac{3e(d+1)n}{D}\frac{1}{n-2}\right)^D \\
&= e^{-(1+o(1))D\log D} \\
&= n^{-\alpha+o(1)}.
\end{aligned}$$

Since this holds uniformly for each T,

$$\mathbf{Pr}(L_N(\{u,v\}) \geq D | A^0) \leq n^{-\alpha+o(1)},$$

and then by (14) we may drop the conditioning, to obtain

$$\mathbf{Pr}(L_N(\{u,v\}) \geq D) \leq n^{-\alpha+o(1)}.$$

It follows that $\mathbf{E}(S_N^{\text{dir}}(\text{at } v)) \leq n^{(1-\alpha)+o(1)}$ and $\mathbf{E}(S_N^{\text{dir}}) \leq n^{(2-\alpha)+o(1)}$.

We may think of the loads as being determined by N independent trials, where each trial specifies X_t and the d choices of possible intermediate node. Since the expected value of $S_N^{\text{dir}}(\text{at } v)$ is at most $n^{(1-\alpha)+o(1)}$, the median m must also satisfy $m \leq n^{(1-\alpha)+o(1)}$. So $m \leq \frac{1}{2}n^{(1-\alpha)^+ + \delta}$ for n sufficiently large. We can use the Talagrand inequality Lemma 2.8, with $c = 2d+1$ and $r = D$. This gives

$$\mathbf{Pr}(S_N^{\text{dir}}(\text{at } v) \geq m + x) \leq 2\exp\left(-\frac{x^2}{4(2d+1)^2 D(m+x)}\right).$$

Now take x as $\frac{1}{2}n^{(1-\alpha)^+ + \delta}$, so that $x \geq m$ for n sufficiently large. Then

$$\begin{aligned}
\mathbf{Pr}(S_N^{\text{dir}}(\text{at } v) \geq 2x) &\leq 2\exp\left(-\frac{x^2}{4(2d+1)^2 D(2x)}\right) \\
&= 2\exp\left(-\frac{x}{8(2d+1)^2 D}\right) \\
&= \exp\left(-\Omega\left(n^{(1-\alpha)^+ + \delta}/\log n\right)\right).
\end{aligned}$$

Thus $S_N^{\text{dir}}(\text{at } v) \leq n^{(1-\alpha)^+ + \delta}$ with very high probability, and so this holds also for Δ_N.

We may handle the total number of saturated links S_N^{dir} in a similar way. Since the expected value is at most $n^{(2-\alpha)+o(1)}$, the median m must also satisfy $m \leq$

$n^{(2-\alpha)+o(1)}$. So $m \le \frac{1}{2}n^{(2-\alpha)^+ +\delta}$ for n sufficiently large. We can again use Lemma 2.8 with $c = 2d + 1$ and $r = D$. Take x as $\frac{1}{2}n^{(2-\alpha)^+ +\delta}$. This gives

$$\mathbf{Pr}(S_N^{\mathrm{dir}} \ge 2x) \le 2\exp\left(-\frac{x}{8(2d+1)^2 D}\right),$$

and it follows that with very high probability, $S_N^{\mathrm{dir}} \le n^{(2-\alpha)^+ +\delta}$ as required. □

Proof of Theorem 1.1. (a) First we consider upper bounds. Let $p_1 = \mathbf{Pr}(Po(1) \ge D)$, so $p_1 = n^{-\alpha+o(1)}$. We consider three cases, when $d = 0$, $d = 1$ and $d \ge 2$.

Firstly, let $d = 0$. Let $\alpha > 2$. Then by Lemma 2.5, $E(S_N^{\mathrm{dir}}) \le 2Np_1 = n^{2-\alpha+o(1)} = o(1)$, and so

$$\mathbf{Pr}(\text{some call fails}) \le \mathbf{Pr}(S_N^{\mathrm{dir}} > 0) \le E(S_N^{\mathrm{dir}}) = o(1).$$

Next let $d = 1$. Let $\alpha > 1$ and let $0 < \delta < \min\{1/2, (\alpha-1)/2\}$. Let $\Delta_t = \max_v\{S_t^{\mathrm{dir}}(\text{at } v)\}$. Then by (12)

$$\begin{aligned}\mathbf{Pr}(X_t \text{ fails } |\phi_{t-1}) &\le \frac{2}{Nn}\sum_v S_{t-1}^{\mathrm{dir}}(\text{at } v)^2\\ &\le \frac{2}{Nn}\sum_v \Delta_{t-1}S_{t-1}^{\mathrm{dir}}(\text{at } v)\\ &= \frac{4\Delta_{t-1}S_{t-1}^{\mathrm{dir}}}{Nn}.\end{aligned}$$

Let A_t^1 be the event that $S_t^{\mathrm{dir}} \le n^{(2-\alpha)^+ +\delta}$ and $\Delta_t \le n^{\delta}$. Then A_N^1 holds with very high probability, by the last lemma. But by the above

$$\mathbf{Pr}(X_t \text{ fails } |A_{t-1}^1) \le \frac{4}{N}n^{(2-\alpha)^+ -1+2\delta}.$$

Hence

$$\mathbf{Pr}(\text{some call fails} \wedge A_N^1) \le \sum_t \mathbf{Pr}(X_t \text{ fails } |A_{t-1}^1) \le 4n^{(2-\alpha)^+ -1+2\delta} = o(1).$$

Now let $d \ge 2$. Let $\alpha > \frac{2}{d+1}$, and let $\beta = \min\{1, \alpha\}$. Let $\delta > 0$ be such that $(\beta - \delta)(d+1) > 2$. Let A_t^2 be the event that $S_t^{\mathrm{dir}}(\text{at } v) \le n^{1-\beta+\delta}$ for each v, that is, $\Delta_t \le n^{1-\beta+\delta}$. By the last lemma, A_N^2 holds with very high probability. By (12),

$$\mathbf{Pr}(X_t \text{ fails } |A_{t-1}^2) \le \frac{2^d}{Nn^d}n(n^{1-\beta+\delta})^{d+1} = \frac{2^d}{N}n^{2-(\beta-\delta)(d+1)}.$$

Hence

$$\mathbf{Pr}(\text{some call fails} \wedge A_N^2) \le \sum_t \mathbf{Pr}(X_t \text{ fails } |A_{t-1}^2) \le 2^d n^{2-(\beta-\delta)(d+1)} = o(1).$$

(b) Finally consider lower bounds. Let $\alpha(d+1) < 2$, and let $0 < \delta < 2-\alpha(d+1)$. Let A^3 be the event that $S_{N/2}^{\mathrm{dir}} \ge Np_2/2$, where $p_2 = \mathbf{Pr}(Po(1/2) \ge D) =$

$n^{-\alpha+o(1)}$. Note that $Np_2/2 = n^{2-\alpha+o(1)}$. By Lemma 2.5 and by inequality (8) in Lemma 2.7, A^3 holds with very high probability – just consider the calls themselves and ignore alternative routes.

If $d = 0$, then for $N/2 < t \leq N$, on A^3

$$\mathbf{Pr}(X_t \text{ fails } |\phi_{t-1}) = S^{\text{dir}}_{t-1}/N \geq p_2/2 \geq 5n^{-2+\delta},$$

for n sufficiently large. Now let $d \geq 1$, so $\alpha < 1$. For $N/2 < t \leq N$, by (13), on A^3

$$\mathbf{Pr}(X_t \text{ fails } |\phi_{t-1}) \geq n^{-\alpha(d+1)+o(1)} \geq 5n^{-2+\delta},$$

for n sufficiently large. Hence, for each integer $d \geq 0$, by Lemma 2.3,

$$\mathbf{Pr}(< n^{\delta} \text{ calls fail}) \leq \mathbf{Pr}(B(N/2, 5n^{-2+\delta}) < n^{\delta}) + \mathbf{Pr}(\overline{A^3}),$$

and the first inequality of Lemma 2.7 completes the proof. □

5. Proof of Theorem 1.2

The rough plan of the proof is as follows.

(a) First we consider the upper bound. With very high probability, for each node v at most m calls for v need to seek an alternative route, where m is about $nD_1^{-D_1}$. Even if we allow each such call to grab d alternative routes, then still not too many indirect links get saturated. Hence we can show that the probability that a given call fails is very small.

(b) For the lower bound, we consider separately the first $N/3$ calls, the next $N/3$ and the final $N/3$. By time $N/3$, with very high probability, for each node v at least $\frac{1}{2}m$ direct links incident with v are saturated, where again m is about $nD_1^{-D_1}$. Between times $N/3$ and $2N/3$, with very high probability, for each node v at least m calls for v need to seek an alternative route. Next we show something that may seem at first sight as if we were still trying to prove an upper bound: we show that at time 2N/3, with very high probability not too many indirect links are saturated. Hence when $N/3 < t \leq 2N/3$ and we focus on a given indirect link vx, if an alternative route via x is tried for a call $X_t = \{u, v\}$, then there is little chance that the route fails because the 'partner' indirect link ux is saturated. This allows us to show that at time $2N/3$, with very high probability in total at least m' indirect links are saturated, where m' is about $n^{2-\frac{\alpha_1\alpha_2}{2}}$. Finally, the probability that a given call in the last $N/3$ fails is at least about $\frac{1}{2}(m/n)(m'/N)^d$, which is about $\frac{1}{2}n^{-\frac{\alpha_1\alpha_2 d}{2}}$.

We start by stating and proving two lemmas which we shall need in the proof of Theorem 1.2.

Lemma 5.1. *Consider any GDAR algorithm. Focus on a particular indirect link vx. For each time t,*

$$\mathbf{Pr}(vx \textit{ is selected at time } t|\phi_{t-1}) \leq \frac{S^{dir}_{t-1}(\textit{at } v)}{N} \frac{d}{n-2}, \tag{15}$$

and (assuming $n \geq 3$)

$$\mathbf{Pr}(X_t \textit{ fails } |\phi_{t-1}) \leq \frac{2^d}{N(n-2)^{d-1}} \sum_v S_{t-1}^{ind}(\textit{at } v)^d. \tag{16}$$

Proof. For (15), note that, conditional on the history before time t and given that the call X_t is for a pair $\{u, v\} \in \mathcal{S}_{t-1}^{\text{dir}}$, at each of the d trials for an intermediate node, the probability that the indirect link vx is considered is at most $\frac{1}{n-2}$.

Now consider (16). We have

$$\begin{aligned}
\mathbf{Pr}(X_t \text{ fails } |\phi_{t-1}) &\leq \frac{1}{N} \sum_{\{u,v\}\in\mathcal{S}_{t-1}^{\text{dir}}} \left(\frac{S_{t-1}^{\text{ind}}(\text{at } u) + S_{t-1}^{\text{ind}}(\text{at } v)}{n-2} \right)^d \\
&\leq \frac{2^{d-1}}{N(n-2)^d} \sum_{\{u,v\}\in\mathcal{S}_{t-1}^{\text{dir}}} \left(S_{t-1}^{\text{ind}}(\text{at } u)^d + S_{t-1}^{\text{ind}}(\text{at } v)^d \right) \\
&\leq \frac{2^{d-1}}{N(n-2)^d} \left(\max_u S_{t-1}^{\text{dir}}(\text{at } u) \right) \sum_v S_{t-1}^{\text{ind}}(\text{at } v)^d \\
&\leq \frac{2^d}{N(n-2)^{d-1}} \sum_v S_{t-1}^{\text{ind}}(\text{at } v)^d
\end{aligned}$$

assuming $n \geq 3$, since then $\frac{n-1}{n-2} \leq 2$. □

Lemma 5.2. *Consider the FDAR algorithm. For each time t and each pair of distinct nodes u and v, on the event* $vx \notin \mathcal{S}_{t-1}^{ind}$

$$\mathbf{Pr}(vx \textit{ is selected } |\phi_{t-1}) \geq 2n^{-3}(S_{t-1}^{dir}(\textit{at } v) - S_{t-1}^{ind}(\textit{via } x) - 1). \tag{17}$$

When $d \geq 2$, *then for each time t,*

$$\mathbf{Pr}(X_t \textit{ fails } |\phi_{t-1}) \geq n^{-(2d+1)}(\min_u S_{t-1}^{dir}(\textit{at } u))((S_{t-1}^{ind} - n)^+)^d. \tag{18}$$

When $d = 1$, *then for each time t,*

$$\mathbf{Pr}(X_t \textit{ fails } |\phi_{t-1}) \geq n^{-3}(\min_u S_{t-1}^{dir}(\textit{at } u) - 1)S_{t-1}^{ind}. \tag{19}$$

Proof. Pick an ordered triple U_1, U_2, U_3 of distinct nodes uniformly at random. Then on the event $vx \notin \mathcal{S}_{t-1}^{\text{ind}}$

$$\begin{aligned}
&\mathbf{Pr}(vx \text{ is selected at the first attempt } |\phi_{t-1}) \\
&\quad = 2\,\mathbf{Pr}(U_1 = v, \{U_1, U_2\} \in \mathcal{S}_{t-1}^{\text{dir}}, U_3 = x, U_2U_3 \notin \mathcal{S}_{t-1}^{\text{ind}} |\phi_{t-1}) \\
&\quad = \frac{2}{n(n-1)} \mathbf{Pr}(\{v, U_2\} \in \mathcal{S}_{t-1}^{\text{dir}}, U_2x \notin \mathcal{S}_{t-1}^{\text{ind}} |\phi_{t-1}) \\
&\quad = \frac{2}{n(n-1)(n-2)} |\{u \in V \setminus \{v, x\} : \{v, u\} \in \mathcal{S}_{t-1}^{\text{dir}}, ux \notin \mathcal{S}_{t-1}^{\text{ind}}\}|,
\end{aligned}$$

and (17) follows.

Now we prove (18). Temporarily, let d_v denote $S^{\text{ind}}_{t-1}(\text{at } v)$, and let W denote the set of nodes v such that $d_v \geq 1$. We may assume that W is non-empty. Note that

$$|W|^{-1}\sum_{v\in W}(d_v-1)^d \geq \left(|W|^{-1}\sum_{v\in W}(d_v-1)\right)^d = \left(|W|^{-1}(S^{\text{ind}}_{t-1}-|W|)\right)^d.$$

Thus

$$\begin{aligned}
\mathbf{Pr}(X_t \text{ fails}|\phi_{t-1}) &\geq \frac{1}{N}\sum_{\{u,v\}\in\mathcal{S}^{\text{dir}}_{t-1}}\left(\frac{\max\{d_u-1,d_v-1,0\}}{n}\right)^d\\
&\geq \frac{1}{2Nn^d}\sum_{\{u,v\}\in\mathcal{S}^{\text{dir}}_{t-1}}\left(((d_u-1)^+)^d+((d_v-1)^+)^d\right)\\
&\geq \frac{1}{2Nn^d}\left(\min_u S^{\text{dir}}_{t-1}(\text{at } u)\right)\sum_{v\in W}(d_v-1)^d\\
&\geq \frac{1}{2Nn^d}\left(\min_u S^{\text{dir}}_{t-1}(\text{at } u)\right)|W|\left(\frac{S^{\text{ind}}_{t-1}-|W|}{|W|}\right)^d\\
&\geq \left(\frac{\min_u S^{\text{dir}}_{t-1}(\text{at } u)}{n}\right)\left(\frac{S^{\text{ind}}_{t-1}-|W|}{n^2}\right)^d.
\end{aligned}$$

The above holds for any positive integer d, but we need to consider the case $d=1$ separately. In this case,

$$\begin{aligned}
\mathbf{Pr}(X_t \text{ fails}|\phi_{t-1}) &\geq \frac{1}{N}\sum_{\{u,v\}\in\mathcal{S}^{\text{dir}}_{t-1}}\frac{\max\{d_u-\mathbb{I}_{uv\in\mathcal{S}^{\text{ind}}_{t-1}},d_v-\mathbb{I}_{vu\in\mathcal{S}^{\text{ind}}_{t-1}}\}}{n}\\
&\geq \frac{1}{2Nn}\sum_{\{u,v\}\in\mathcal{S}^{\text{dir}}_{t-1}}\left((d_u-\mathbb{I}_{uv\in\mathcal{S}^{\text{ind}}_{t-1}})+(d_v-\mathbb{I}_{vu\in\mathcal{S}^{\text{ind}}_{t-1}})\right)\\
&\geq \frac{1}{2Nn}\sum_v(S^{\text{dir}}_{t-1}(\text{at } v)-1)d_v\\
&\geq \frac{1}{2Nn}(\min_u S^{\text{dir}}_{t-1}(\text{at } u)-1)\sum_v d_v\\
&= \frac{1}{2Nn}(\min_u S^{\text{dir}}_{t-1}(\text{at } u)-1)S^{\text{ind}}_{t-1},
\end{aligned}$$

which proves (19). □

Proof of Theorem 1.2. Let $p_1 = p_1(n) = \mathbf{Pr}(Po(1) \geq D_1)$, so $p_1 = e^{-(1+o(1))D_1\log D_1}$. Let $m = m(n)$ be an integer with $m \sim 2np_1$, so that $m =$

$n^{1+o(1)}$. Let A_t^1 be the event that $S_t^{\rm dir}(\text{at } v) \le m$ for each node v. Then A_N^1 holds with very high probability, since

$$\mathbf{Pr}(\overline{A_N^1}) \le 2n\,\mathbf{Pr}(B(n-1, p_1) > m) = e^{-\Omega(m)} \tag{20}$$

by Lemmas 2.4 and 2.7.

(a) Let $\alpha_1\alpha_2 d > 4$. Consider any GDAR algorithm. We want to show that with high probability no calls fail. We first show that for each v, not too many indirect links vx ever get saturated. In fact we bound $\mathbf{E}[S_N^{\rm ind}(\text{at } v)^d]$, so that we can use (16) in Lemma 5.1. The first task is to obtain a good upper bound for $\mathbf{Pr}(S_N^{\rm ind}(\text{at } v) \ge k)$.

Suppose that we throw $2md$ balls uniformly at random into $n-1$ bins, and let Y be the number of bins receiving at least D_2 balls. Let $p_2 = p_2(n) = \mathbf{Pr}(Po(\frac{2md}{n-1}) \ge D_2)$, so that $p_2 = n^{-\frac{\alpha_1\alpha_2}{2}+o(1)}$. By Lemma 2.5, for all k

$$\mathbf{Pr}(Y \ge k) \le 2\,\mathbf{Pr}(B(n-1, p_2) \ge k). \tag{21}$$

Consider a fixed node v. Think of the $n-1$ indirect links vx for $x \in V-v$ as bins. Let $Z_t = vx$ if vx is selected by the call X_t at time t. Let $p_3 = p_3(n) = (n-1)\frac{m}{N}\frac{d}{n-2}$. Note that $Np_3 \sim md$, so by inequality (8) in Lemma 2.7,

$$\mathbf{Pr}(B(N, p_3) > 2md) = e^{-\Omega(m)}. \tag{22}$$

By Lemma 5.1,

$$\mathbf{Pr}(Z_t = vx|\phi_{t-1}) \le \frac{p_3}{n-1} \quad \text{on } A_{t-1}^1$$

for each time t and each $x \in V-v$. Hence by Lemma 2.6(a) and the inequality (21), for any k

$$\mathbf{Pr}(S_N^{\rm ind}(\text{at } v) \ge k) \le \mathbf{Pr}(Y \ge k) + \varepsilon_n \le 2\,\mathbf{Pr}(B(n-1, p_2) \ge k) + \varepsilon_n,$$

where

$$\varepsilon_n = \mathbf{Pr}(\overline{A_N^1}) + \mathbf{Pr}(B(N, p_3) > 2md) = e^{-\Omega(m)},$$

and where the last equality comes from (20) and (22).

We have now completed the first task. By Lemma 2.1,

$$\begin{aligned}
\mathbf{E}[S_N^{\rm ind}(\text{at } v)^d] &= \sum_{k=1}^{(n-1)^d} \mathbf{Pr}(S_N^{\rm ind}(\text{at } v)^d \ge k) \\
&\le 2 \sum_{k=1}^{(n-1)^d} \mathbf{Pr}(B(n-1, p_2)^d \ge k) + (n-1)^d \varepsilon_n \\
&= 2\,E(B(n-1, p_2)^d) + (n-1)^d \varepsilon_n \\
&\le 2d^d \max\{(n-1)p_2, ((n-1)p_2)^d\} + (n-1)^d \varepsilon_n.
\end{aligned}$$

So by Lemma 5.1,

$$\begin{aligned}\mathbf{Pr}(X_t \text{ fails}) &\le \frac{2^d}{N(n-2)^{d-1}} \sum_v \mathbf{E}[S^{\text{ind}}_{t-1}(\text{at } v)^d] \\ &\le \frac{2^d}{N(n-2)^{d-2}} \left(2d^d \max\{np_2, (np_2)^d\} + n^d \varepsilon_n\right) \\ &\le \frac{2^{d+2}d^d}{Nn^{d-2}} \max\{np_2, (np_2)^d\} + 2^{d+2}\varepsilon_n\end{aligned}$$

for n sufficiently large. Hence

$$\begin{aligned}\mathbf{Pr}(\text{some call fails}) &\le \sum_t \mathbf{Pr}(X_t \ \text{ fails}) \\ &\le 2^{d+2}d^d n^{2-d} \max\{np_2, (np_2)^d\} + \varepsilon'_n\end{aligned}$$

where $\varepsilon'_n = N2^{d+2}\varepsilon_n = e^{-\Omega(m)}$. Now $\alpha_1\alpha_2 d > 4$ and so both $3 - d - \frac{\alpha_1\alpha_2}{2} < 0$ and $2 - \frac{\alpha_1\alpha_2 d}{2} < 0$. Recall also that $p_2 = n^{-\frac{\alpha_1\alpha_2}{2}+o(1)}$. Hence

$$\begin{aligned}\mathbf{Pr}(\text{some call fails}) &\le 2^{d+2}d^d n^{2-d} \max\{np_2, (np_2)^d\} + \varepsilon'_n \\ &\le 2^{d+2}d^d \max\{n^{3-d-\frac{\alpha_1\alpha_2}{2}+o(1)}, n^{2-\frac{\alpha_1\alpha_2 d}{2}+o(1)}\} + \varepsilon'_n \\ &= o(1).\end{aligned}$$

(b) Now let $\alpha_1\alpha_2 d < 4$, and consider the d-FDAR algorithm. We shall show that with high probability many calls get blocked.

Let $p_4 = p_4(n) = \mathbf{Pr}(Po(1/3) \ge D_1)$, so that $p_4 = e^{-(1+o(1))D_1 \log D_1}$. Let $\tilde{m} = \tilde{m}(n)$ be a positive integer, with $\tilde{m} \sim \frac{1}{2}np_4$, so that $\tilde{m} = n^{1+o(1)}$. Let A^2 be the event that for each node v we have $S^{\text{dir}}_{N/3}(\text{at } v) \ge \tilde{m}$. Then A^2 holds with very high probability, since by Lemmas 2.4 and 2.7,

$$\mathbf{Pr}(\overline{A^2}) \le 2n\, \mathbf{Pr}(B(n-1, p_4) < \tilde{m}) = e^{-\Omega(\tilde{m})}.$$

Next we shall show that with very high probability, each value $S^{\text{ind}}_N(\text{via } x)$ is not too big. As we noted in the introductory comments in this section, it may seem here at first sight as if we were still trying to prove an upper bound as in part (a). But we need to argue in this way in order to show that when we consider an indirect link vx, it is rare for an alternative route which seeks to use the indirect link vx to find that it is blocked by the 'partner' indirect link being saturated. Let δ satisfy $0 < \delta < 1$ and $\delta > 1 - \alpha_1\alpha_2/2$. [We may have $1 - \alpha_1\alpha_2/2 < 0$ if $d = 1$, but this does not matter.] Note that $n^\delta \le \tilde{m}/2 - 1$ for n sufficiently large. Let A^3_t be the event that $S^{\text{ind}}_t(\text{via } x) \le n^\delta$ for each node x. We shall show that A^3_N holds with very high probability.

Fix a node x. Think of the $n-1$ indirect links vx as bins, as v runs through $V \setminus \{x\}$. Consider an indirect link vx and a time t. As before, let $p_3 = p_3(n) = \frac{md(n-1)}{N(n-2)}$. Then $p_3 = n^{-1+o(1)}$ and $Np_3 \sim md$. By Lemma 5.1, if A^1_{t-1} holds then whatever else has happened before time t, the probability that vx is used at time t (so we put a 'ball' in the 'bin' vx) is at most $p_3/(n-1)$. Let $m' = m'(n)$ be a positive integer with $m' \sim 2Np_3$, so that $m' \sim 2md$ and $m'/(n-1) = e^{-(1+o(1))D_1 \log D_1}$. Also, let, $p_5 = p_5(n) = \mathbf{Pr}(Po(m'/(n-1)) \geq D_2)$, so that $p_5 = n^{-\frac{\alpha_1\alpha_2}{2}+o(1)}$. Note that $np_5 = o(n^\delta)$. Suppose that we throw m' balls uniformly at random into $n-1$ bins: let Y be the number of bins with load at least D_2. Then by Lemma 2.5,

$$\mathbf{Pr}(Y > n^\delta) \leq 2\,\mathbf{Pr}(B(n-1, p_5) > n^\delta),$$

and by Lemma 2.6,

$$\mathbf{Pr}(S_N^{\mathrm{ind}}(\text{via } x) > n^\delta) \leq \mathbf{Pr}(Y > n^\delta) + \mathbf{Pr}(\overline{A_N^1}) + \mathbf{Pr}(B(N, p_3) > m').$$

But

$$\mathbf{Pr}(\overline{A_N^3}) \leq \sum_x \mathbf{Pr}(S_N^{\mathrm{ind}}(\text{via } x) > n^\delta).$$

The above estimates, together with the first inequality in Lemma 2.7, show that the event A_N^3 holds with very high probability, as desired.

We shall use the fact that for each time $t > N/3$ and each pair v, x of distinct nodes, we have

$$S_{t-1}^{\mathrm{dir}}(\text{at } v) - S_{t-1}^{\mathrm{ind}}(\text{via } x) - 1 \geq \tilde{m}/2 \quad \text{on } A^2 \wedge A_{t-1}^3$$

(for n sufficiently large). So, for each time $t > N/3$, by Lemma 5.2,

$$\mathbf{Pr}(vx \text{ is selected at time } t \mid \phi_{t-1}) \geq \frac{\tilde{m}}{n^3} \quad \text{on } A^2 \wedge A_{t-1}^3 \wedge (vx \notin \mathcal{S}_{t-1}^{\mathrm{ind}}). \quad (23)$$

This is the point at which we use the fact that it is the FDAR algorithm.

Consider times from $N/3+1$ to $2N/3$. Take $n(n-1)$ bins corresponding to the indirect links. Throw $N/3$ balls into these bins as follows: whenever an alternative route is taken, we select uniformly at random one of the two indirect links used, and put a ball in the corresponding bin, independently of the past. We are interested in lower bounds here, so we may indeed select only one ball as above. The non-selected indirect link corresponds to a 'bonus' ball. By (23), for each bin vx, conditional on $A^2 \wedge A_{t-1}^3 \wedge (L_{t-1}(vx) < D_2)$ and anything else before time t, the probability that at time t a ball is put in that bin is at least $\tilde{m}/(2n^3)$. We shall use Lemma 2.6(b) to show that by time $2N/3$, many bins have load at least D_2, and hence many indirect links are saturated.

Let $p_6 = p_6(n) = n(n-1)\,\tilde{m}/(2n^3)$. Note that $(N/3)p_6 \sim \tilde{m}n/12$. Let $\hat{m} = \hat{m}(n) = \lfloor \tilde{m}n/13 \rfloor$. Then $\mathbf{Pr}(B(N/3, p_6) < \hat{m}) = e^{-\Omega(n)}$. Let $p_7 = p_7(n) = \mathbf{Pr}(Po(\frac{\hat{m}}{n(n-1)}) \geq D_2)$, so that $p_7 = n^{-\frac{\alpha_1\alpha_2}{2}+o(1)}$. If we throw $\hat{m}$ balls uniformly at random into $n(n-1)$ bins each of capacity D_2, then by Lemma 2.5 the probability that less than k bins are full is at most $2\,\mathbf{Pr}(B(n(n-1), p_7) < k)$.

Let $\eta > 0$ be such that $2 - \frac{\alpha_1\alpha_2 d}{2} - (d+2)\eta > 0$. Let A^4 be the event that $S_{2N/3}^{\mathrm{ind}} \geq n^{2-\frac{\alpha_1\alpha_2}{2}-\eta}$. Then by Lemma 2.6,

$$\mathbf{Pr}(\overline{A^4}) \leq 2\,\mathbf{Pr}(B(n(n-1), p_7) < n^{2-\frac{\alpha_1\alpha_2}{2}-\eta}) + \mathbf{Pr}(\overline{A^2 \wedge A_N^3}) + \mathbf{Pr}(B(N/3, p_6) < \hat{m}).$$

Thus by the first inequality in Lemma 2.7, A^4 holds with very high probability.

Let $d \geq 2$. Then $2 - \frac{\alpha_1\alpha_2}{2} - \eta > 1$. Hence by Lemma 5.2, for each time $t > 2N/3$, on the event $A^2 \wedge A^4$,

$$\mathbf{Pr}(X_t \text{ fails}|\phi_{t-1}) \geq n^{-2d-1}\tilde{m}\ (n^{2-\frac{\alpha_1\alpha_2}{2}-\eta} - n)^d \geq n^{-\frac{\alpha_1\alpha_2 d}{2}-(d+1)\eta},$$

assuming that n is sufficiently large.

Let $d = 1$. Then by Lemma 5.2, for each time $t > 2N/3$, on the event $A^2 \wedge A^4$,

$$\mathbf{Pr}(X_t \text{ fails}|\phi_{t-1}) \geq n^{-3}(\tilde{m} - 1)\ (n^{2-\frac{\alpha_1\alpha_2}{2}-\eta}) \geq n^{-\alpha_1\alpha_2-2\eta},$$

Then, by Lemma 2.3(b), for any k,

$$\mathbf{Pr}(\leq k \text{ calls fail}) \leq \mathbf{Pr}(B(N/3, n^{-\frac{\alpha_1\alpha_2 d}{2}-(d+1)\eta}) \leq k) + \mathbf{Pr}(\overline{A^2 \wedge A^4}).$$

Hence

$$\mathbf{Pr}(< n^{2-\frac{\alpha_1\alpha_2 d}{2}-(d+2)\eta} \text{ calls fail}) = e^{-\Omega(n^\eta)}. \qquad \square$$

6. Proof of Theorem 1.3

As mentioned earlier, the proof is based on ideas in [4], though there are extra complications to be handled. We defined $L_t(vx)$ as the load (the number of channels used) in the indirect link vx at time t. Let $L_t(v, = h)$ be the number of indirect links vx with $L_t(vx) = h$, and similarly let $L_t(v, \geq h)$ be the number of indirect links vx at v with $L_t(vx) \geq h$. If the call X_t is for v and takes an alternative route via x, we call $L_t(vx)$ the *height* $H_t(v)$ of the call at v. If the call is not for v or it does not take an alternative route, we let $H_t(v) = 0$.

Lemma 6.1. *Let $v \in V$ and let h be a non-negative integer at most $D_2 - 1$. Then*

$$\mathbf{Pr}(H_t(v) \geq h + 1|\phi_{t-1}) \leq \frac{S^{dir}_{t-1}(at\ v)}{N}\left(\frac{2\max_w L_{t-1}(w, \geq h)}{n-2}\right)^d. \qquad (24)$$

Further, let A be the event that $S^{ind}_{t-1}(via\ x) \leq \frac{1}{2}S^{dir}_{t-1}(at\ v) - 1$ for each x. Then

$$\mathbf{Pr}(H_t(v) = h + 1|\phi_{t-1}) \geq \frac{S^{dir}_{t-1}(at\ v)}{2N}\left(\frac{L_{t-1}(v, = h)}{n-2}\right)^d \quad on\ \ A. \qquad (25)$$

Proof. Suppose that the call X_t is for $\{u, v\}$, and the d choices of possible intermediate node are $x_1, \ldots, x_d$. If $H_t(v) \geq h + 1$, then for each $i = 1, \ldots, d$ there must be at least one of ux_i or vx_i with load at least h. Hence,

$$\begin{aligned}
&\mathbf{Pr}(H_t(v) \geq h + 1|\phi_{t-1}) \\
&\leq \frac{1}{N}\sum_{u \in V-v} \mathbb{I}_{\{u,v\}\in\mathcal{S}^{\text{dir}}_{t-1}(\text{at } v)}\left(\frac{L_{t-1}(u, \geq h) + L_{t-1}(v, \geq h)}{n-2}\right)^d \\
&\leq \frac{S^{\text{dir}}_{t-1}(\text{at } v)}{N}\left(\frac{2\max_w L_{t-1}(w, \geq h)}{n-2}\right)^d,
\end{aligned}$$

and (24) follows.

Now consider (25). If the call $X_t = \{u, v\} \in \mathcal{S}^{\text{dir}}_{t-1}(\text{at } v)$, $L_{t-1}(vx_i) = h$ for each of the d choices of possible intermediate node $x_i \in V \setminus \{u, v\}$, and $ux_1 \notin \mathcal{S}^{\text{ind}}_{t-1}$, then $H_t(v) = h + 1$. In order to keep the expressions below reasonably compact, let J_u be the indicator of the event that the direct link $\{u, v\} \in \mathcal{S}^{\text{dir}}_{t-1}(\text{at } v)$. Then

$$
\begin{aligned}
&\mathbf{Pr}(H_t(v) = h + 1|\phi_{t-1}) \\
&\geq \frac{1}{N} \sum_{u \in V-v} J_u \left(\frac{1}{n-2} \sum_{x_1 \in V-u,v} \mathbb{I}_{ux_1 \notin \mathcal{S}^{\text{ind}}_{t-1}} \mathbb{I}_{L_{t-1}(vx_1)=h} \right) \left(\frac{L_{t-1}(v, = h)}{n-2} \right)^{d-1} \\
&= \frac{L_{t-1}(v, = h)^{d-1}}{N(n-2)^d} \sum_{x_1 \in V-v} \mathbb{I}_{L_{t-1}(vx_1)=h} \sum_{u \in V-v,x_1} J_u \mathbb{I}_{ux_1 \notin \mathcal{S}^{\text{ind}}_{t-1}} \\
&\geq \frac{L_{t-1}(v, = h)^{d-1}}{N(n-2)^d} \sum_{x_1 \in V-v} \mathbb{I}_{L_{t-1}(vx_1)=h} \left(S^{\text{dir}}_{t-1}(\text{at } v) - S^{\text{ind}}_{t-1}(\text{via } x_1) - 1 \right) \\
&\geq \frac{S^{\text{dir}}_{t-1}(\text{at } v) L_{t-1}(v, = h)^d}{2N(n-2)^d}
\end{aligned}
$$

on the event A. □

Proof of Theorem 1.3. (a) (upper bound) Let the constant c be as in (29) below, and let

$$D_2 = D_2(n) \geq \frac{1}{\log d} (\log \log n - \log^{(3)} n - \log^{(4)} n) + c + 1.$$

We shall show that with high probability no calls fail.

Let $M_t(\geq h, \leq b)$ denote the event that $L_t(v, \geq h) \leq b$ for each v. Given numbers b_h, let B_h denote the event $M_N(\geq h, \leq b_h)$. The idea of the proof is to choose numbers $b_1, b_2, \ldots$ tending to zero quickly; show that B_1 holds with high probability, and if B_h holds with high probability then so does B_{h+1}; and deduce that with high probability B_h holds for some $h \leq D_2$ with $b_h = 0$. Thus with high probability no indirect link gets saturated, and so no call can fail.

Let $p = p(n) = \mathbf{Pr}(Po(1) \geq D_1)$, so that $p = e^{-(1+o(1))D_1 \log D_1}$. Let $a = a(n)$ be an integer with $a \sim 4np$. Let A^0_t be the event that $S^{\text{dir}}_t(\text{at } v) \leq a/2$ for each node v. Then A^0_N holds with very high probability, by Lemmas 2.5 and 2.7. Let A^1 be the event that for each node v, at most a calls for v find their direct link saturated and thus seek an alternative route. Consider a fixed node v, and let Z_t be the indicator of the event $X_t \in \mathcal{S}^{\text{dir}}_{t-1}(\text{at } v)$. Then $\mathbf{Pr}(Z_t = 1|\phi_{t-1}) \leq \frac{a}{2N}$ on the event A^0_{t-1}. Hence by Lemma 2.3 (a),

$$\mathbf{Pr}(\sum_{t=1}^{N} Z_t \geq a) \leq \mathbf{Pr}(\overline{A^0_N}) + \mathbf{Pr}(B(N, \frac{a}{2N}) > a).$$

But $\mathbf{Pr}(\overline{A^1}) \leq n\mathbf{Pr}(\sum_{t=1}^{N} Z_t \geq a)$, and so A^1 holds with high probability by Lemma 2.7.

We may assume that all calls arrive before any alternative routes are generated, say they arrive by time 0. Let us condition on the entire history of the call process, subject to event A^1 holding. It will suffice for us now to show that, when we consider the alternate routing process, with high probability no indirect link gets saturated.

Now $\frac{1}{n-2} \leq \frac{2}{n}$ for $n \geq 4$ (which we assume in what follows). So by Lemma 6.1, given that the call at time t is for v and is alternative,

$$\mathbf{Pr}(H_t(v) \geq h + 1 | \phi_{t-1}) \leq \left(\frac{4b}{n}\right)^d \quad \text{on } M_{t-1}(\geq h, \leq b). \tag{26}$$

[Of course, $H_t(v) = 0$ if the call at time t is not for v or is direct.] Suppose that we have chosen $0 \leq b_h \leq n/4$, and let $p_h = \left(\frac{4b_h}{n}\right)^d$ for $h = 0, 1, \ldots$. Recall that B_h denotes the event $M_N(\geq h, \leq b_h)$, that $L_N(v, \geq h) \leq b_h$ for each v. Then

$$\begin{aligned} &\mathbf{Pr}(\ (L_N(v, \geq h+1) > b_{h+1}) \ \wedge \ B_h) \\ &\quad \leq \mathbf{Pr}(\ (|\{t : H_t(v) \geq h+1\}| > b_{h+1}) \ \wedge \ B_h) \\ &\quad \leq \mathbf{Pr}(B(a, p_h) > b_{h+1}), \end{aligned}$$

by (26) and Lemma 2.3(a). Thus,

$$\mathbf{Pr}(\overline{B_{h+1}} \wedge \ B_h) \leq n \ \mathbf{Pr}(B(a, p_h) > b_{h+1}),$$

and so

$$\mathbf{Pr}(\overline{B_{h+1}}) \leq \mathbf{Pr}(\overline{B_h}) + n \ \mathbf{Pr}(B(a, p_h) > b_{h+1}). \tag{27}$$

Let $\rho = \rho(n) = 4ea/n$. Then $\rho = (1 + o(1))16ep = e^{-(1+o(1))D_1 \log D_1}$. Thus $\rho \leq 1$ for n sufficiently large, which we assume. We now choose $b_1, b_2, \ldots$ by setting $p_0 = 1$ and for $h = 0, 1, \ldots$ setting $b_{h+1} = eap_h$ and $p_{h+1} = \left(\frac{4b_{h+1}}{n}\right)^d = (\rho p_h)^d$. Note that $p_{h+1} \leq \rho^{d^h}$ for each h. By (27) and the inequality (10), for $h = 1, 2, \ldots$

$$\mathbf{Pr}(\overline{B_{h+1}}) \leq \mathbf{Pr}(\overline{B_h}) + n^{-1}, \tag{28}$$

as long as $b_h \geq 2e \log n$.

Let $h^* = h^*(n)$ be the least positive integer h such that $b_h < 2e \log n$. We claim that if the constant c is sufficiently large, then

$$h^* \leq (\log d)^{-1}(\log^{(2)} n - \log^{(3)} n - \log^{(4)} n) + c. \tag{29}$$

We may check this as follows. To start, note that

$$b_{h+2} = eap_{h+1} \leq np_{h+1} \leq n\rho^{d^h},$$

and hence $b_{h+2} < 2e \log n$ if $n\rho^{d^h} \leq 1$. Taking logs twice we see that this holds if $h \log d + \log\log \frac{1}{\rho} \geq \log\log n$. But $\log\log \frac{1}{\rho} = \log D_1 + \log\log D_1 + o(1)$, and (29) follows.

By our choice of c we have $D_2 \geq h^* + 1$. Since $\mathbf{Pr}(\overline{B_1}) = 0$ (conditional on A^1), by summing the inequality (28) over $h = 1$ to $h^* - 2$ we obtain

$$\mathbf{Pr}(\overline{B_{h^*-1}}) \leq \frac{h^*}{n}.$$

Now increase b_{h^*} to $2e \log n$ and let $b_{h^*+1} = 0$. Then by (27) and by inequality (9) in Lemma 2.7,

$$\mathbf{Pr}(\overline{B_{h^*}}) \leq \mathbf{Pr}(\overline{B_{h^*-1}}) + n\mathbf{Pr}(B(a, p_{h^*-1}) > 2e \log n) \leq \frac{h^* + 1}{n}.$$

Also, by (27),

$$\mathbf{Pr}(\overline{B_{h^*+1}}) \leq \mathbf{Pr}(\overline{B_{h^*}}) + n\mathbf{Pr}\left(B(a, \left(\frac{8e \log n}{n}\right)^d) \geq 1\right),$$

and since $d \geq 2$ the last term is at most $na(\frac{8e \log n}{n})^2 = o(1)$. Hence with high probability there is no indirect link with load at least $h^* + 1$, and so no indirect link ever gets saturated. This completes the proof of part (a).

(b) (lower bound) Let $\tilde{p} = \tilde{p}(n) = \mathbf{Pr}(Po(\frac{1}{3}) \geq D_1)$, so that $\tilde{p} = e^{-(1+o(1))D_1 \log D_1}$. Let $\tilde{a} = \tilde{a}(n)$ be an integer with $\tilde{a} \sim n\tilde{p}/2$, so that $\tilde{a} = n^{1+o(1)}$. Let A^2 be the event that for each node v, $S^{\text{dir}}_{N/3}(\text{at } v) \geq \tilde{a}$. Then A^2 holds with very high probability, by Lemmas 2.5 and 2.7. We condition on A^2 throughout the rest of the argument.

Let A^3_t be the event that $S^{\text{ind}}_t(\text{via } x) \leq \tilde{a}/2 - 1$ for each $x \in V$. Then arguing as in the proof of Theorem 1.2 , or using Lemma 2.8, we see that A^3_N holds with very high probability.

Consider times t with $N/3 < t \leq 2N/3$. We split these times into j equal periods, where $j = j(n) = \lfloor \log \log n / \log d \rfloor$. We shall choose D_2 shortly, and it will be less than j. Let $s = \lfloor \frac{N}{3j} \rfloor$. Let $n_h = \lfloor N/3 \rfloor + hs$ for $h = 0, 1, \ldots, j$. Note that $n_j \leq 2N/3$. We shall consider the j time periods $T_i = \{n_{i-1} + 1, \ldots, n_i\}$ for $i = 1, \ldots, j$.

Let $\rho = \rho(n) = \frac{\tilde{a}s}{8(n-1)N}$, so that $\rho = e^{-(1+o(1))D_1 \log D_1}$. Let $0 < \delta < 2/d$. It is straightforward to check that if the constant c is sufficiently large, then

$$D_2 = D_2(n) \leq \frac{1}{\log d}(\log \log n - \log^{(3)} n - \log^{(4)} n) - c,$$

and then we have $\rho^{d^{D_2}} \geq (n-1)^{-\delta}$. We shall show that with D_2 as above, with high probability many calls fail, more specifically, at least $n^{2-\delta d+o(1)}$ calls fail.

Let $M_t(\geq h, \geq b)$ denote the event that for each v we have $L_t(v, \geq h) \geq b$, that is, there are at least b indirect links vx which have load at least h at time t. We shall choose positive numbers $b_1, b_2, \ldots$ below, which decrease rapidly. These will satisfy $b_{h+1} \leq b_h(1 - 2^{-1/d})$, and so $(b_h - b_{h+1})^d \geq b_h^d/2$. Let F_h be the event $M_{n_h}(\geq h, \geq b_h)$. We want to show that F_h holds with high probability. The first task is to obtain an upper bound on $\mathbf{Pr}(\overline{F_{h+1}} \wedge F_h)$.

By Lemma 6.1, for each time $t \in T_{h+1}$, on the event $F_h \wedge A^3_{t-1} \wedge (L_{t-1}(v, \geq h+1) < b_{h+1})$ we have $L_{t-1}(v, = h)^d > (b_h - b_{h+1})^d \geq b_h^d/2$, and so

$$\mathbf{Pr}(H_t(v) = h + 1 | \phi_{t-1}) \geq \frac{\tilde{a}}{2N} \left(\frac{b_h - b_{h+1}}{n-2} \right)^d \geq q_h$$

where $q_h = \frac{\tilde{a}}{4N}(\frac{b_h}{n-1})^d$. Now if $H_t(v) = h + 1$ for at least b_{h+1} values $t \in T_{h+1}$, and this holds for each node v, then the event F_{h+1} holds. Hence by Lemma 2.3,

$$\mathbf{Pr}(\overline{F_{h+1}} \wedge F_h \wedge A^3_N) \leq n\mathbf{Pr}(B(s, q_h) < b_{h+1}),$$

and so

$$\mathbf{Pr}(\overline{F_{h+1}} \wedge A^3_N) \leq \mathbf{Pr}(\overline{F_h} \wedge A^3_N) + n\mathbf{Pr}(B(s, q_h) < b_{h+1}). \tag{30}$$

We now choose the values b_h. We let $b_0 = n - 1$, and $b_{h+1} = sq_h/2$ for $h = 0, 1, \ldots$. Let $\beta_h = \frac{b_h}{n-1}$. Then $\beta_0 = 1$, and for $h = 0, 1, \ldots$

$$\beta_{h+1} = \frac{s}{2(n-1)} \frac{\tilde{a}}{4N} \beta_h^d = \rho \beta_h^d.$$

Hence for $h \leq D_2$,

$$\beta_h = \rho^{1+d+\cdots+d^{h-1}} \geq \rho^{d^h} \geq (n-1)^{-\delta},$$

and so

$$b_{D_2} = (n-1)\beta_{D_2} \geq (n-1)^{1-\delta}.$$

Also, for n sufficiently large, $b_{h+1} \leq b_h(1 - 2^{-1/d})$ as required.

Since $(n-1)^{1-\delta} \leq b_{h+1} = sq_h/2$ for each $h = 0, 1, \ldots, D_2 - 1$, each corresponding value $\mathbf{Pr}(B(s, q_h) < b_{h+1})$ in (30) is very small. Since the event F_0 must hold, it follows from (30) and the fact that A^3_N holds with high probability that F_{D_2} holds with high probability. We condition on the event F_{D_2} holding from now on.

Finally consider the calls X_t with $2N/3 < t \leq N$. Then

$$\mathbf{Pr}(X_t \text{ fails} | \phi_{t-1}) \geq n^{-\delta d}.$$

Hence by Lemma 2.3, with high probability at least $n^{2-\delta d + o(1)}$ calls fail, which completes the proof. □

7. Concluding remarks

In this final section, we describe some results related to those above, which may be proved by very similar methods.

(a) Theorems 1.2 and 1.3 above still hold, with only minor alterations to the proofs, if instead of having exactly $N = \binom{n}{2}$ calls we allow $M = \Theta(n^2)$ calls.

(b) There may be some interest in balanced routing without using direct links, in particular see note (c) below. By Theorem 1.2, for the first fit method FDAR both the direct link capacity D_1 and the indirect link capacity D_2 are important. This is not the case with the balanced method BDAR. If we set $D_1 = 0$ then the threshold in Theorem 1.3 moves, but only very slightly. Let us state this as a theorem.

Theorem 7.1. *Let $d \geq 2$ be an integer. Let $D_1 = 0$, so that we do not use direct links. Suppose that there is a random sequence of $M = \Theta(n^2)$ calls.*

(a) There is a constant c such that if $D_2 \geq \log\log n / \log d + c$ and we use the BDAR algorithm, then with high probability all the calls are routed successfully.
(b) Let $\delta > 0$. There is a constant c such that if $D_2 \leq \log\log n / \log d - c$ and we use any GDAR algorithm, then with high probability at least $n^{2-\delta}$ calls fail.

(c) Let us consider a more general network model. For simplicity we shall ignore direct links. Let $G_1, G_2, \ldots$ be a sequence of graphs, where G_n has n nodes. Suppose that for each edge $\{u, v\}$ in G_n, there are indirect links uv and vu each with capacity $D_2 = D_2(n)$. A call specifies an unordered pair of distinct nodes of G_n. If the call at time t is for $\{u, v\}$, we pick d possible intermediate nodes uniformly at random from the common neighbours of u and v, and proceed as before. We call the number of common neighbours the *co-degree* of u and v. Some of our earlier results transfer easily to this more general case. In particular, there is a version of Theorem 1.3 (a) for this case, with essentially the same proof, as long as the co-degrees grow linearly with n. Let us state it as a theorem.

Theorem 7.2. *Let $d \geq 2$ be an integer. Let $G_1, G_2, \ldots$ be a sequence of graphs, where G_n has n nodes and the minimum co-degree is $\Omega(n)$. Suppose that in G_n there is a random sequence of $M = O(n^2)$ calls, and we use the BDAR algorithm. Then there is a constant c such that the following holds: if $D_2 \geq \log\log n / \log d + c$, then with high probability all the calls are routed successfully.*

Note that co-degrees grow linearly with n with high probability if G_n starts as the complete graph K_n, and links fail independently with some fixed probability $q < 1$.

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