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The components of the Wired Spanning Forest are recurrent

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Abstract. We show that a.s. all of the connected components of the Wired Spanning Forest are recurrent, proving a conjecture of Benjamini, Lyons, Peres and Schramm. Our analysis relies on a simple martingale involving the effective conductance between the endpoints of an edge in a uniform spanning tree. We believe that this martingale is of independent interest and will find further applications in the study of uniform spanning trees and forests.

1. Introduction

1.1. Uniform Spanning Forests

A fascinating field that has emerged recently is the study of uniform spanning forest measures for infinite graphs, which are weak limits of uniform spanning tree measures on finite subgraphs. We will now give a brief introduction to this topic. The reader should see [BLPS] for a comprehensive study; for a survey, see [L].

Let G be an infinite graph which is exhausted by finite subgraphs G_n. Pemantle [P] showed that the uniform distributions on spanning trees of G_n converge weakly to a distribution which is supported on spanning forests of G. We call this measure the **free spanning forest** (FSF). There is another natural construction in which the exterior of G_n is contracted to a single vertex before taking the limit. This second construction, which is called the **wired spanning forest** (WSF), was implicit in Pemantle's paper and was made explicit by Häggström [H].

In this paper, we prove that a.s. every connected component of the wired spanning forest is recurrent for simple random walk. This was conjectured by [BLPS], who proved the result in the special cases where G is a transitive graph with unimodular automorphism group, where G is a tree, and where G is a graph whose spectral radius $\rho < 1$.

Our proof uses a martingale which involves the effective conductance between the endpoints of an edge in a uniform spanning tree: For a finite graph G, let T^e be a uniform spanning tree which is conditioned to contain a particular edge $e = (s, t)$. Suppose that we contract or delete the edges in G one-by-one, according to whether each is in T^e or not, respectively. This produces a sequence of graphs $\langle G^k \rangle$. Then the sequence $\langle \mathcal{C}_{G^k}(s \leftrightarrow t) \rangle$ of effective conductances between s and t is a martingale.

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This surprising fact can be generalized to infinite graphs, where analogous martingales can be obtained involving uniform spanning forest measures. In the case of the wired spanning forest this leads to an easy proof that all components are recurrent.

1.2. Random walks and electric networks

Our proofs rely on some well-known connections between random walks and electric networks (see e.g. [DS] or [LP]). We will now give some brief background and definitions. A **network** is a pair (G, C) where G is a graph and C is a map from the unoriented edges of G to the non-negative real numbers. The quantity $C(e)$ is called the **conductance** of the edge e. Define the **resistance** $R(e) = 1/C(e)$. Define **network random walk** as the usual random walk on a weighted graph: if the current state is v, traverse edge e incident to v with probability proportional to $C(e)$. By **contracting an edge** we mean identifying its endpoints. This may result in parallel edges and self-loops.

RANDOM SPANNING TREES AND FORESTS ON A NETWORK. For a finite network G, a natural probability measure on spanning trees of G chooses each tree with probability proportional to its weight, where for spanning trees T we define $\text{weight}(T) = \prod_{e\in T} C(e)$. When we say *random spanning tree* we will mean a tree with this distribution. By analogy with the definitions in the case of unweighted graphs, if G is an infinite network, we define the WSF and FSF as weak limits of random spanning tree measures for finite subnetworks.

Let G be a finite network and fix an edge $e = (s, t)$. For unit flows θ from s to t, define the **energy**

$$\mathcal{E}(\theta) = (\theta, \theta)_R = \sum_f \theta(f)^2 R(f),$$

where the sum is over unoriented edges f. **Thomson's principle** states that the unit flow θ that minimizes $\mathcal{E}(\theta)$ is the unit electrical current flow I^e from s to t. Furthermore, $\mathcal{E}(I^e) = \mathcal{R}_{\text{eff}}(s \leftrightarrow t)$, the *effective resistance* between s and t.

Thomson's principle can be used to define electrical currents on an infinite networks (see [S], Chapter 3 or [BLPS] for more details). Let G be an infinite network and let $e = (s, t)$ be an edge in G. By analogy with the finite case, we wish to define the electrical current I^e between the endpoints of e as the unit s–t flow θ that minimizes $\mathcal{E}(\theta)$. But first, we must define what is meant by a "unit s–t flow" for an infinite graph. It turns out there are two natural definitions:

"Free" definition. Say an antisymmetric function θ of the oriented edges of G is a *unit s–t flow* if it is a weighted average of s–t paths, i.e., θ is of the form $\sum_i \alpha_i \mathcal{P}_i$, where $\alpha_i > 0$, $\sum_i \alpha_i = 1$, and the $\mathcal{P}_i$ are s–t paths.

"Wired" definition. Say θ is a *unit s–t flow* if it satisfies *Kirchhoff's node law*, i.e., for any vertex w we have

$$(\text{amount of flow leaving } w) - (\text{amount of flow entering } w) = \begin{cases} 1, & w = s; \\ -1, & w = t; \\ 0, & \text{else.} \end{cases}$$

Note that these definitions are not equivalent, since in the second definition we allow flow to "pass through infinity." (E.g., let $G = \mathbf{Z}$ and suppose that $\theta(j, j-1) = 1 - \delta_1(j)$. Then θ is a unit flow from 0 to 1 in the wired sense but not in the free sense.)

Following [BLPS], we will call the the energy-minimizing "free" unit flow **free current** and the energy-minimizing "wired" flow **wired current** and denote them by I_F^e and I_W^e, respectively. Define free and wired effective resistance by $\mathbf{R}_{\text{eff}}^F(s \leftrightarrow t) = \mathcal{E}\left(I_F^e\right)$ and $\mathbf{R}_{\text{eff}}^W(s \leftrightarrow t) = \mathcal{E}\left(I_W^e\right)$, respectively.

For an infinite network G and a vertex s, define a unit flow from s to infinity as a flow with a source at s and no sink (i.e., excise the case $w = t$ in the wired definition of a unit $s \leftrightarrow t$ flow). Define the resistance $\mathcal{R}(s \leftrightarrow \infty)$ between s and ∞ by $\mathcal{R}(s \leftrightarrow \infty) = \inf_\theta \mathcal{E}(\theta)$, where the infimum is over unit flows from s to ∞. We will need the following easy proposition, which follows from the "minimum energy" definition of resistance.

Proposition 1. *Let G be an infinite network and let s and t be two vertices. Then*

$$\mathcal{R}_{\text{eff}}^W(s \leftrightarrow t) \le 2\left(\mathcal{R}(s \leftrightarrow \infty) + \mathcal{R}(t \leftrightarrow \infty)\right).$$

(By the triangle inequality for resistances, the constant 2 above is not necessary.) We will also use the following well-known (see, e.g., [LP]) relationship between resistance to ∞ and transience of network random walk:

Proposition 2. *Let (G, C) be an infinite network and let s be a vertex in G. Then the component of G containing s is transient for network random walk iff $\mathcal{R}(s \leftrightarrow \infty)$ is finite.*

This has the following easy consequence:

Lemma 3. *Let (G, C) be an infinite network and suppose that the component of G containing v is transient. Let $\{e_i\}_{i=1}^\infty$ be an enumeration of the edges in G, and let G^k be the graph obtained from G by contracting the edges $e_1, \dots, e_k$. Then $\mathcal{R}_{G^k}(v \leftrightarrow \infty) \to 0$ as $k \to \infty$.*

Proof. Since $\mathcal{R}_G(v \leftrightarrow \infty)$ is finite, there is a unit flow θ on G from v to ∞ such that

$$\mathcal{E}(\theta) = \sum_{e \in E} \theta(e)^2 R(e) < \infty.$$

For all k, let θ_k be the unit flow on G^k which is induced by θ, and let $E_k = E - \{e_1, \dots, e_k\}$. Then $\mathcal{R}_{G^k}(v \leftrightarrow \infty) \le \mathcal{E}(\theta_k) = \sum_{e \in E_k} \theta(e)^2 R(e) \to 0$ as $k \to \infty$. □

2. Main Theorem

We will use the following lemma, which was first proved by Kirchhoff [K].

Lemma 4. *Let G be a finite network and let $e = (s, t)$ be an edge in G. Let T be a random spanning tree of G. Then*

$$\mathcal{C}_G(s \leftrightarrow t) = \frac{1}{\mathbf{P}(e \in T)}.$$

We will also need the following easy and well-known proposition. Denote by G/f and $G{-}f$ the graphs obtained from G by contracting f and deleting f, respectively, and write T_G for a random spanning tree of G.

Proposition 5. *Let G be a finite network and let f be an edge in G. Then*

- *the conditional distribution of T_G given $f \in T_G$ is the same as the distribution of $\{f\} \cup T_{G/f}$.*
- *the conditional distribution of T_G given $f \notin T_G$ is the same as the distribution of T_{G-f}.*

We are now ready to prove the existence of the conductance martingales described in the introduction. I thank Russell Lyons for simplifying my original proofs.

Theorem 6. *Let G be a finite network and let T^e be a random spanning tree conditioned to contain $e = (s, t)$. Let $G^0, G^1, \ldots$ be a sequence of networks constructed as follows. Define $G^0 = G$, and for $i > 0$ form G^i from G^{i-1} by choosing an edge e_i which is not parallel to e and contracting or deleting it according to whether it is in T^e or not, respectively. Then the sequence $\mathcal{C}_{G^n}(s \leftrightarrow t)$ of effective conductances between s and t is a martingale.*

Proof. Let T be a random spanning tree of G and let $f = e_1$ be the edge which is contracted or deleted from G to obtain G_1. By Lemma 4 and Proposition 5, the random variable $\mathcal{C}_{G_1}(s \leftrightarrow t)$ is either $1/\mathbf{P}(e \in T|f \in T)$ or $1/\mathbf{P}(e \in T|f \notin T)$, depending on whether f is in T^e or not, respectively. Thus, we have

$$\begin{aligned}
\mathrm{E}(\mathcal{C}_{G_1}(s \leftrightarrow t)) &= \frac{\mathbf{P}(f \in T|e \in T)}{\mathbf{P}(e \in T|f \in T)} + \frac{\mathbf{P}(f \notin T|e \in T)}{\mathbf{P}(e \in T|f \notin T)} \\
&= \frac{\mathbf{P}(f \in T)}{\mathbf{P}(e \in T)} + \frac{\mathbf{P}(f \notin T)}{\mathbf{P}(e \in T)} \\
&= \frac{1}{\mathbf{P}(e \in T)} \\
&= \mathcal{C}_G(s \leftrightarrow t),
\end{aligned}$$

and a similar argument shows that $\mathrm{E}\left(\mathcal{C}_{G_{n+1}}(s \leftrightarrow t)|G_n\right) = \mathcal{C}_{G_n}(s \leftrightarrow t)$ for all n. Hence $\mathcal{C}_{G_n}$ is a martingale. □

To derive corresponding conductance martingales for infinite graphs, we will need the following extension of Lemma 4, which is proved in [BLPS].

Lemma 7. *Let G be an infinite network and let $e = (s, t)$ be an edge in G. Let F be a forest with the WSF distribution. Then*

$$\mathcal{C}_G^W(s \leftrightarrow t) = \frac{1}{\mathbf{P}(e \in F)}.$$

This statement also holds when FSF replaces WSF and free conductance replaces wired conductance.

Now we can prove:

Theorem 8. *Let G be an infinite network and let F^e be a spanning forest with the WSF distribution, conditioned to contain $e = (s, t)$. Let $G^0, G^1, \ldots$ be a sequence of networks constructed as follows. Define $G^0 = G$, and for $i > 0$ form G^i from G^{i-1} by choosing an edge which is not parallel to e and contracting or deleting it according to whether it is in F^e or not, respectively. Then the sequence $\mathcal{C}^W_{G^n}(s \leftrightarrow t)$ of wired effective conductances between s and t is a martingale. If we replace WSF by FSF, then the sequence $\mathcal{C}^F_{G^n}(s \leftrightarrow t)$ of free effective conductances between s and t is a martingale.*

Proof. The proof is nearly the same as the proof of Theorem 6. Consider first the wired case. Let $f = e_1$ and let F be a random forest with the WSF distribution. Using Lemma 7 and an argument similar that in the proof of Theorem 6, we have

$$\begin{aligned} \mathrm{E}(\mathcal{C}^W_{G_1}(s \leftrightarrow t)) &= \frac{\mathbf{P}(f \in F|e \in F)}{\mathbf{P}(e \in F|f \in F)} + \frac{\mathbf{P}(f \notin F|e \in F)}{\mathbf{P}(e \in F|f \notin F)} \\ &= \frac{1}{\mathbf{P}(e \in F)} \\ &= \mathcal{C}^W_G(s \leftrightarrow t), \end{aligned}$$

and a similar argument shows that $\mathrm{E}\left(\mathcal{C}^W_{G_{n+1}}(s \leftrightarrow t)|G_n\right) = \mathcal{C}^W_{G_n}(s \leftrightarrow t)$ for all n. Hence $\mathcal{C}^W_{G_n}$ is a martingale. A similar argument holds when FSF replaces WSF. □

We are now ready to prove the [BLPS] conjecture. We will actually prove the following stronger theorem:

Theorem 9. *Let (G, C) be an infinite network. Then* $\mathrm{WSF}^2(G) = \mathrm{WSF(G)}$*, i.e., almost surely every component tree of the WSF of G is its own WSF.*

Proof. Let (G, C) be an infinite network and let F be a spanning forest with the WSF distribution. Fix an edge $e = (s, t)$. By Lemma 7, it's enough to show that almost surely either $e \notin F$ or the wired effective resistance in F between s and t is 1 (i.e., in F all of the wired current between s and t passes through e).

Let $G^0, G^1, \ldots$ be a sequence of networks constructed as follows. Define $G^0 = G$, and for $i > 0$ form G^i from G^{i-1} by choosing an edge which is not parallel to e and contracting or deleting it according to whether it is in F or not. Suppose also that the edges are chosen exhaustively in the sense that every edge in G is either e_i for some i or parallel to e in one of the G_i.

Let A_e be the event that the wired effective resistance in $F - \{e\}$ between the endpoints of e is finite. Then by Propositions 1 and 2, A_e is the event that in $F - \{e\}$ both the component of s and the component of t are transient. Let H^n be the network obtained from F by contracting the edges in $F \cap \{e_1, \ldots, e_n\}$. On the event A_e, we can apply Lemma 3 twice (once to each side of e), to get

$$\lim_{n\to\infty} \mathcal{R}_{H^n}(s \leftrightarrow \infty) = \lim_{n\to\infty} \mathcal{R}_{H^n}(t \leftrightarrow \infty) = 0.$$

Hence Proposition 1 implies that $\lim_{n\to\infty} \mathcal{R}^W_{H^n}(s \leftrightarrow t) = 0$. Since G_n has the same vertex set as H^n, and at least as many edges, we have $\lim_{n\to\infty} \mathcal{R}^W_{G^n}(s \leftrightarrow t) = 0$ and hence $\lim_{n\to\infty} \mathcal{C}^W_{G^n}(s \leftrightarrow t) = \infty$.

But $\mathcal{C}^W_{G^k}(s \leftrightarrow t)$ has a finite limit a.s. on the event $\{e \in F\}$ since by Theorem 8 it is a non-negative martingale conditional on this event. Hence $\mathbf{P}(e \in F, A_e) = 0$. It follows that almost surely, either $e \notin F$ or the wired effective resistance in F between s and t is 1 and the proof is complete. □

Remark. There is a direct argument that $\mathbf{P}(e \in F, A_e) = 0$ for all e which does not use the conductance martingale. For positive integers k, let F^k be a forest distributed according to the WSF of G^k. Then the distribution of F^k corresponds to the conditional distribution of F, given $\{e_1, \dots, e_k\} \cap F$. Denote by I_k the wired unit current flow between the endpoints of e in the graph G^k. Then $\mathbf{P}(e \in F_k) = I_k(e)$ (see [BLPS]). But

$$R(e)I_k(e)^2 \leq \mathcal{E}(I_k) = \mathcal{R}^W_{G^k}(s \leftrightarrow t) \to 0$$

on A_e. It follows that

$$\mathbf{P}(e \in F | e_1, \dots, e_k) = \mathbf{P}(e \in F_k) = I_k(e) \to 0$$

on A_e, hence $\mathbf{P}(e \in F, A_e) = 0$. □

Our main result now follows as a corollary:

Corollary 10. *Let (G, C) be an infinite network with* $\sup_{e\in E} C(e) < \infty$. *Then the connected components of the WSF are almost surely recurrent for network random walk.*

Proof. It's enough to show that if T is a transient tree with bounded degree then the WSF of T is not almost surely equal to T. Let T be a transient tree with bounded degree. Then by Proposition 2 there is a finite-energy unit flow θ from some vertex to ∞. Let $e = (s, t)$ be an edge in T such that $0 < \theta(e) < 1$. Then in $T - \{e\}$ there is a finite-energy flow from s to ∞ and one from t to ∞. It follows that $\mathcal{R}^W_{\text{eff}}(s \leftrightarrow t) < 1$, and hence e is not a.s. in the WSF of T. □

Remark. The WSF of a recurrent tree is the whole tree itself. Thus, in the setting of graphs with bounded conductance, the class of trees which can be a component of the WSF is exactly the class of recurrent trees. □

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