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A representation of the Belavkin equation via Feynman path integrals

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Abstract. The Belavkin equation, describing the continuous measurement of the position of a quantum particle, is studied. A rigorous representation of its solution by means of an infinite dimensional oscillatory integral (Feynman path integral) defined on the complex Cameron-Martin space is given.

1. Introduction

In the traditional formulation of quantum mechanics, the state of a non relativistic d-dimensional quantum particle is described by a vector ψ in the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$, with $\|\psi\| = 1$, while its time evolution is described by the Schrödinger equation:

$$\dot{\psi} = -\frac{i}{\hbar} H\psi.$$

Here H is the Hamiltonian of the system, which is the self-adjoint[1] operator in $\mathcal{L}^2(\mathbb{R}^d)$ given on smooth functions by:

$$H\psi(x) = -\frac{\hbar^2}{2m}\Delta\psi(x) + V(x)\psi(x);$$

where m is the mass of the particle , $\hbar$ is the (reduced) Planck costant.

The Schrödinger equation is valid if the particle is “undisturbed”, but if it interacts with the outer world, namely if it is submitted to the measurements of one of its physical properties and interacts with the measuring apparatus, the evolution is no longer continuous. Indeed the state of the system in the process of the measurement is the result of a random and discontinuous change and the ordinary Schrödinger equation is no longer valid.

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[1] under suitable assumptions on the potential V, see, e.g., [12]

In 1989 V.P. Belavkin [6] proposed a stochastic partial differential equation giving a mathematical description of this situation. According to this proposal the following stochastic Schrödinger equation describes the dynamics of a quantum particle, whose position is continuously observed:

$$\begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \frac{\lambda|x|^2}{2}\psi dt + \sqrt{\lambda}x\psi\, dW(t) \\ \psi(0,x) = \psi_0(x) \qquad\qquad t \geq 0,\ x \in \mathbb{R}^d \end{cases} \tag{1}$$

where W is an d dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $dW(t)$ denotes the Ito stochastic differential; for each $\omega \in \Omega$, $\psi(\omega) \in C([0,T], \mathcal{H})$, $\mathcal{H} = \mathcal{L}_2(R^d)$ and $\lambda > 0$ is a coupling constant. We denote the $\mathbb{R}^d$ norm with $|\,|$ and the scalar product by $a \cdot b = \sum_{i=1}^d a_i b_i$.

Equation (1) can also be written in the Stratonovich equivalent form:

$$\begin{cases} d\psi = -\frac{i}{\hbar}H\psi dt - \lambda|x|^2\psi dt + \sqrt{\lambda}x\psi \circ dW(t) \\ \psi(0,x) = \psi_0(x) \qquad\qquad t \geq 0,\ x \in \mathbb{R}^d \end{cases} \tag{2}$$

The aim of this paper is to find a representation of the solution of equation (2) by means of an infinite dimensional oscillatory integral, a rigorous version of a Feynman path integral.

The mathematical theory of Feynman path integrals, used to represent the solution of the deterministic Schrödinger equation, can be considered already as a classical topic, we refer to [3, 8, 1, 2, 11, 10] and the bibliography therein.

The representation of the Belavkin equation via a "Feynman Map" has been announced in [4], see also [5]. In the present paper we show that the Feynman path integral corresponds to a stochastic Mehler Kernel, see [13–15]: in those papers an analogous result is obtained for a similar equation.

This approach turns out to be useful in order to show that the function defined pointwise by the Feynman path integral is in fact the solution of the Belavkin equation.

The paper is organized as follows: in section 2 we recall the notion of oscillatory integral in finite and infinite dimension and the Cameron Martin formula. In section 3 we extend, in a suitable way, the definitions given in the previous section to the space of paths in complex spaces and in theorem 1 we compute the Feynman path integral that we will use to represent the solution of (2).

Finally in section 4 we give the proof of theorem 3.

2. Oscillatory integrals and the Cameron Martin formula

In this section we recall for later use some known results, for more details we refer to [3, 8, 1]

2.1. Finite dimensional oscillatory integrals

Let us consider the finite dimensional real Hilbert space $\mathbb{R}^n$, whose elements are denoted by $x, y \in \mathbb{R}^n$ and the scalar product with $\langle x, y\rangle$. Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible and symmetric operator.

Definition 1. *A function* $f : \mathbb{R}^n \to \mathbb{C}$ *is Fresnel integrable with respect to Q if for each* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *such that* $\phi(0) = 1$ *the limit*

$$\lim_{\epsilon \to 0} \int e^{i\langle x, Qx \rangle} f(x)\phi(\epsilon x)\, dx \tag{3}$$

exists and is independent of ϕ. *In this case the limit is called the Fresnel integral of* f *with respect to Q and denoted by*

$$\int e^{i\langle x, Qx \rangle} f(x)\, dx \tag{4}$$

There is an important class, $\mathcal{F}(\mathbb{R}^n)$, of Fresnel integrable functions: those which are Fourier transforms of complex bounded variation measures on $\mathbb{R}^n$, i.e. elements of $\mathcal{M}(\mathbb{R}^n)$:

$$f \in \mathcal{F}(\mathbb{R}^n) \Leftrightarrow f(x) = \int e^{i\langle x, \alpha \rangle} \mu_f(d\alpha), \quad \mu_f \in \mathcal{M}(\mathbb{R}^n).$$

In this case the Parseval equality gives us the following expression for the limit (3):

$$(2\pi i)^{-n/2} \int e^{\frac{i}{2}\langle x, Qx \rangle} f(x) dx = (\det Q)^{-1/2} \int e^{\frac{-i}{2}\langle \alpha, Q^{-1}\alpha \rangle} \mu_f(d\alpha).$$

where det denotes the determinant.

2.2. Infinite dimensional oscillatory integrals

Let us consider an infinite dimensional real Hilbert space of paths $\mathcal{H}$ whose elements are denoted by $\gamma, \eta \in \mathcal{H}$ and the scalar product by $\langle \gamma, \eta \rangle$. Let P_n be a sequence of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \leq P_{n+1}$ and $P_n \to 1$ strongly as $n \to \infty$ (1 being the identity operator in $\mathcal{H}$). Let $f : \mathcal{H} \to \mathbb{C}$ be a function on $\mathcal{H}$ and $Q : D(Q) \subseteq \mathcal{H} \to \mathcal{H}$ be an invertible, densely defined and self-adjoint operator.

Definition 2. *A function* $f : \mathcal{H} \to \mathbb{C}$ *is Fresnel integrable with respect to Q if and only if the finite dimensional approximations of the Fresnel integral of* f *with respect to Q*

$$(2\pi i)^{-n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2}\langle P_n\gamma, QP_n\gamma \rangle} f(P_n\gamma) d(P_n\gamma),$$

are well defined and the limit

$$\lim_{n \to \infty} (2\pi i)^{-n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2}\langle P_n\gamma, QP_n\gamma \rangle} f(P_n\gamma) d(P_n\gamma) \tag{5}$$

exists and is independent on the sequence $\{P_n\}$.

In this case the limit is precisely called the Fresnel integral of f *with respect to Q and is denoted by*

$$\widetilde{\int} e^{\frac{i}{2}\langle \gamma, Q\gamma \rangle} f(\gamma)\, d\gamma$$

One can prove that if $f \in \mathcal{F}(\mathcal{H})$ then $f \circ P_n \in \mathcal{F}(P_n(\mathcal{H}))$ and f is Fresnel integrable. Moreover, if $Q - I$ is trace class, the Cameron Martin formula holds:

$$\widetilde{\int} e^{\frac{i}{2}\langle \gamma, Q\gamma\rangle} f(\gamma)\, d\gamma = (\det Q)^{-1/2} \int_{\mathcal{H}} e^{-\frac{i}{2}\langle \alpha, Q^{-1}\alpha\rangle} \mu_f(d\alpha) \tag{6}$$

where $\det Q = |\det Q| e^{-\pi i \, \mathrm{Ind}\, Q}$ is the Fredholm determinant of the operator Q and $\mathrm{Ind}(Q)$ is the number of negative eigenvalues of the operator Q, counted with their multiplicity.

3. The "analytic continuation" of oscillatory integrals

Let H be the Cameron-Martin space, i.e. the space of absolutely continuous functions $\gamma : [0, t] \to \mathbb{R}^d$, $\gamma(t) = 0$, such that $\int_0^t |\dot\gamma(s)|^2 ds < \infty$. H is endowed with the following scalar product

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot\gamma_1(s) \cdot \dot\gamma_2(s)\, ds.$$

Let us denote by $\mathcal{M}(H)$ the Banach space of the complex bounded variation measures on H, endowed with the total variation norm, that is:

$$\mu \in \mathcal{M}(H), \qquad \|\mu\| = \sup \sum_i |\mu(E_i)|,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of H, such that $\cup_i E_i = H$. $\mathcal{M}(H)$ is a Banach algebra, where the product of two measures $\mu * \nu$ is by definition their convolution:

$$\mu * \nu(E) = \int_H \mu(E - \gamma)\nu(d\gamma), \qquad \mu, \nu \in \mathcal{M}(H)$$

and the unit element is the vector δ_0.
Let $\mathcal{F}(H)$ be the space of complex functions on H which are Fourier transforms of measures belonging to $\mathcal{M}(H)$, that is:

$$f : H \to \mathbb{C} \qquad f(\gamma) = \int_H e^{i\langle \gamma, \beta\rangle} \mu_f(d\beta) \equiv \hat\mu_f(\gamma).$$

$\mathcal{F}(H)$ is a Banach algebra of functions, where the product is the pointwise one; the unit element is the function 1, i.e. $1(\gamma) = 1\ \forall \gamma \in H$ and the norm is given by $\|f\| = \|\mu_f\|$.

We shall now define a linear functional on $\mathcal{F}(H)$, which can be interpreted as the analytic continuation of a "Feynman path integral" to be defined below. Let us consider for this a symmetric trace class operator L_1 in H, such that $I + L_1$ is invertible, and any function $g : H \to \mathbb{C}$, which is the Fourier transform of a corresponding complex bounded variation measure μ_g on H:

$$g(\gamma) = \hat\mu_g(\gamma)$$

Then one can apply the theory of the previous section and compute the Fresnel integral of the function g by the Cameron Martin formula:

$$\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L_1)\gamma\rangle} g(\gamma)\, d\gamma = \det(I+L_1)^{-1/2} \int_H e^{\frac{-i\hbar}{2}\langle\alpha,(I+L_1)^{-1}\alpha\rangle} \mu_g(d\alpha).$$

We shall interpret this as the "Feynman path integral" of the function g. Thus our rigorous definition of the Feynman path integral for the function $g \in \mathcal{F}(H)$ is as the Fresnel integral of Def. 2 (with respect to the operator $Q = (I + L_1)/\hbar$) and with the normalization factor $(2\pi i)^{-n/2}$ replaced by $(2\pi i\hbar)^{-n/2}$.

The following theorem considers the case of function g of a particular form, relevant for the applications we have in mind.

Theorem 1. *Let L_1 and L_2 be two commuting symmetric trace class operators in H, such that $I + L_1$ is invertible and L_2 is nonnegative. Let $f : H \to \mathbb{C}$ be the Fourier transform of a complex bounded variation measure μ_f on H:*

$$f(\gamma) = \hat{\mu}_f(\gamma), \qquad f(\gamma) = \int_H e^{i\langle\gamma,\beta\rangle} \mu_f(d\beta).$$

Then the Fresnel integral (or Feynman path integral) of the function $g(\gamma) := e^{-\frac{1}{2\hbar}\langle\gamma,L_2\gamma\rangle} f(\gamma)$ exists and is given by:

$$\begin{aligned}
&\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L_1)\gamma\rangle - \frac{1}{2\hbar}\langle\gamma,L_2\gamma\rangle} f(\gamma)\, d\gamma \\
&\quad = \det(I+L_1)^{-1/2} \int_H \int_H e^{\frac{-i\hbar}{2}\langle\alpha+\beta,(I+L_1)^{-1}(\alpha+\beta)\rangle} \mu_{L_2}(d\beta)\mu_f(d\alpha) \quad (7)
\end{aligned}$$

where μ_{L_2} is the Gaussian measure on H with covariance operator $L_2/\hbar$.

Remark.

- The left hand side of equation 7 is just an heuristic expression, which is defined by the right hand side, which has a well defined mathematical meaning.
- We remark that the quadratic form on H: $(\ ,\ ) : H \times H \to \mathbb{C}$:

  $$(\gamma_1, \gamma_2) = \langle\gamma_1, (I+L_1)\gamma_2\rangle + i\langle\gamma_1, L_2\gamma_2\rangle,$$

  can be seen as the restriction to the real Cameron Martin space of the quadratic form acting on the complex Cameron Martin space $H_\mathbb{C}$ and defined by

  $$(\gamma_1, \gamma_2) = \int_0^t \frac{d\gamma_1}{ds}(s) \cdot \frac{d}{ds}((I+L)\gamma_2)(s)\, ds, \qquad (8)$$

  where γ_1 and γ_2 are complex paths and L is the operator on $H_\mathbb{C}$ of the following form:

  $$L : H_\mathbb{C} \to H_\mathbb{C} \qquad L = L_1 + iL_2.$$

  A complex path γ can be seen as a couple of real-valued paths (η, ξ), i.e. $\gamma(s) = \eta(s) + i\xi(s)$. A linear operator $A : D(A) \subseteq H \to H$ can be extended to a linear operator denoted again by A on $H_\mathbb{C}$:

  $$A : D(A) \subseteq H_\mathbb{C} \to H_\mathbb{C}, \qquad D(A) = D(A) + iD(A),$$
  $$A\gamma = A(\eta, \xi) = (A\eta, A\xi).$$

Moreover one can easily prove that $(I+L) : H_{\mathbb{C}} \to H_{\mathbb{C}}$ is invertible, if $(I+L_1)$ is invertible ([2]).

With this notation the relation (7) assumes the following form:

$$\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} f(\gamma)d\gamma = \det(I+L)^{-1/2}\int_H e^{\frac{-i\hbar}{2}\langle\alpha,(I+L)^{-1}\alpha\rangle}\mu_f(d\alpha)$$

This can be proved by taking a finite dimensional approximation of both the right and the left hand sides and passing to the limit.

The previous result admits the following generalization, which can be applied in the computation of the representation of a particular type of Schrödinger equations with complex potentials, including equation (2).

Theorem 2. *Let us consider the function* $f : H \to \mathbb{C}$

$$f(\gamma) = e^{-\frac{1}{2\hbar}\langle\gamma,L_2\gamma\rangle} e^{\langle l,\gamma\rangle} g(\gamma)$$

where $l \in H$ *and* $g \in \mathcal{F}(H)$, $g(\eta) = \hat{\mu}_g(\eta)$, $\mu_g \in \mathcal{M}(H)$.

Then f *is the Fourier transform of a complex bounded variation measure* μ_f *on* H, $f = \hat{\mu}_f$, *where* μ_f *is the convolution of* μ_g *and the measure* ν, *with* $\nu(d\gamma) = e^{\frac{\hbar}{2}\langle l,L_2^{-1}l\rangle - i\hbar\langle l,L_2^{-1}\gamma\rangle}\mu_{L_2}(d\gamma)$, *where* μ_{L_2} *is the Gaussian measure on* H *with covariance operator* $L_2/\hbar$. *Moreover the Fresnel (or Feynman path) integral of* f *with respect to the operator* $Q = (I+L_1)/\hbar$ *is well defined and it is given by:*

$$\begin{aligned}\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L_1)\gamma\rangle} f(\gamma)d\gamma &= \widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L_1)\gamma\rangle} e^{-\frac{1}{2\hbar}\langle\gamma,L_2\gamma\rangle} e^{\langle l,\gamma\rangle} g(\gamma)\,d\gamma \\ &= \det(I+L_1)^{-1/2}\int_H e^{\frac{-i\hbar}{2}\langle\gamma,(I+L_1)^{-1}\gamma\rangle}\mu_g * \nu(d\gamma) \\ &= \det(I+L_1)^{-1/2}\int_H\int_H e^{\frac{-i\hbar}{2}\langle\gamma+\eta,(I+L_1)^{-1}(\gamma+\eta)\rangle}\mu_g(d\eta)\nu(d\gamma) \\ &= \det(I+L_1)^{-1/2}\int_H\int_H e^{\frac{-i\hbar}{2}\langle\gamma+\eta,(I+L_1)^{-1}(\gamma+\eta)\rangle} \\ &\quad\times e^{\frac{\hbar}{2}\langle l,L_2^{-1}l\rangle - i\hbar\langle l,L_2^{-1}\gamma\rangle}\mu_{L_2}(d\gamma)\mu_g(d\eta) \end{aligned} \tag{9}$$

Remark. Again the previous expression admits an "analytic continuation": $e^{\frac{i}{2\hbar}\langle\gamma,(I+L_1)\gamma\rangle}e^{-\frac{1}{2\hbar}\langle\gamma,L_2\gamma\rangle}e^{\langle l,\gamma\rangle}$ can be seen as the restriction to the real Cameron Martin space of the function on $H_{\mathbb{C}}$ given by $\gamma \to e^{(\gamma,\gamma)+\langle l,\gamma\rangle}$ where $(\ ,\ )$ is the quadratic form (8) and $\langle l,\ \rangle$ denotes the linear functional on $H_{\mathbb{C}}$ given by $\gamma \to \langle l,\gamma\rangle = \int_0^t \dot{l}(s)\cdot\dot{\gamma}(s)ds$.

With these notations expression (9) assumes the following form:

$$\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l,\gamma\rangle} g(\gamma)d\gamma = \det(I+L)^{-1/2}\int_H e^{\frac{-i\hbar}{2}\langle\alpha-il,(I+L)^{-1}(\alpha-il)\rangle}\mu_g(d\alpha) \tag{10}$$

[2] Notice that $\det(I+L)$ exists as L is trace class

Proof. The first statement is a straightforward calculation. The second can be proved by taking the finite dimensional approximation of both sides of equation (9) and passing to the limit.

As L_1 and L_2 are two commuting symmetric trace class operators on H, they have a common spectral decomposition. Thus there exists a complete orthonormal system $\{e_n\} \subset H$ such that

$$(I+L_1)\gamma = \sum_n a_n \langle e_n, \gamma\rangle e_n, \qquad L_2(\gamma) = \sum_n b_n \langle e_n, \gamma\rangle e_n, \qquad \gamma \in H,$$

with $a_n, b_n \in \mathbb{R}$.

Let $\{P_m\}$ be the family of projectors onto the span of the first m eigenvectors $e_1, \dots, e_m$, namely:

$$P_m(\gamma) = \sum_{n=1}^{m} \langle e_n, \gamma\rangle e_n$$

One can easily see that $P_m \to I$ as $m \to \infty$ and $L_1 P_m(H) \subseteq P_m(H)$, $L_2 P_m(H) \subseteq P_m(H)$. Moreover the Feynman path integral

$$\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle \gamma, (I+L_1)\gamma\rangle} e^{-\frac{1}{2\hbar}\langle \gamma, L_2\gamma\rangle} e^{\langle l, \gamma\rangle} g(\gamma) d\gamma$$

can be computed as

$$\lim_{m\to\infty} (2\pi i\hbar)^{-m/2} \int_{P_m H} e^{\frac{i}{2\hbar}\langle P_m\gamma, (I+L_1)P_m\gamma\rangle} e^{-\frac{1}{2\hbar}\langle P_m\gamma, L_2 P_m\gamma\rangle} e^{\langle l, P_m\gamma\rangle} g(P_m\gamma) d(P_m\gamma)$$

which, from the Cameron Martin formula can be seen to be equal to

$$\begin{aligned}(\prod_{n=1}^{m} a_n b_n)^{-1/2} \left(\frac{2\pi}{\hbar}\right)^{-m/2} \int_{P_m H} \int_{P_m H} & e^{-i\hbar/2 \sum_{n=1}^m a_n^{-1}(\gamma_n+\eta_n)^2 - \hbar/2 \sum_{n=1}^m b_n^{-1}\gamma_n^2} \\ & \cdot e^{+\hbar/2 \sum_{n=1}^m b_n^{-1} l_n^2 - i\hbar \sum_{n=1}^m b_n^{-1} l_n \gamma_n} d(P_m\gamma)(\mu_g \circ P_m)(d\eta) \qquad (11)\end{aligned}$$

where $\gamma_n = \langle \gamma, e_n\rangle$, $\eta_n = \langle \eta, e_n\rangle$, $l_n = \langle l, e_n\rangle$, $d(P_m\gamma)$ being the $m-$dimensional Lebesgue measure on $P_m H$.

The finite dimensional approximation of the right hand side of equation (10) assumes the following form:

$$\left(\prod_{n=1}^{m} (a_n + i b_n)\right)^{-1/2} \int_{P_m H} e^{-i\hbar/2 \sum_{n=1}^m (a_n + i b_n)^{-1}(\gamma_n - i l_n)^2} (\mu_g \circ P_m)(d\gamma) \qquad (12)$$

By direct computation one can verify that expressions (12) and (11) coincide. Now we can pass to the limit and from Lebesgue's dominated convergence theorem we have

$$\begin{aligned}&\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle \gamma, (I+L_1)\gamma\rangle} e^{-\frac{1}{2\hbar}\langle \gamma, L_2\gamma\rangle} e^{\langle l, \gamma\rangle} g(\gamma) d\gamma \\ &\qquad = \det(I+L)^{-1/2} \int_H e^{\frac{-i\hbar}{2}\langle \alpha - il, (I+L)^{-1}(\alpha - il)\rangle} \mu_g(d\alpha) \qquad (13)\end{aligned}$$

□

4. Application to the stochastic Schrödinger equation

In this section we prove the existence of a strong solution for the Stratonovich stochastic differential equation (2). First we give the definition of strong solution in the case of a Schrödinger equation:

Definition 3. *A strong solution for the stochastic equation (2) is a predictable process with values in* $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^d)$*, such that* $\psi(t) \in D(-i/\hbar H - \lambda|x|^2)$ $\mathbf{P}$*-a.s.*

$\mathbf{P}\Big(\int_0^T (\|\psi(t)\|^2 + \|(-i/\hbar H - \lambda|x|^2)\psi\|^2)\, dt < \infty\Big) = 1$

$\mathbf{P}\Big(\int_0^T \||x|\psi(t)\, dt\|^2 < \infty\Big) = 1$ *and*

$\mathbf{P}$ *a.s. for all* $t \in [0, T]$*:*

$$\begin{cases} d\psi = -\frac{i}{\hbar} H\psi dt - \lambda|x|^2\psi dt + \sqrt{\lambda}x \cdot \psi \circ dW(t) & t \geq 0,\ x \in \mathbb{R}^d \\ \psi(0, x) = \psi_0(x) \end{cases} \tag{14}$$

Let us consider again the real Cameron-Martin space H and its complexification $H_{\mathbb{C}}$. Let $L : H_{\mathbb{C}} \to H_{\mathbb{C}}$ be the operator on $H_{\mathbb{C}}$ given by the formula

$$\langle \gamma_1, L\gamma_2 \rangle = -\Omega^2 \int_0^t \gamma_1(s) \cdot \gamma_2(s) ds;$$

where $\Omega^2 = -2i\lambda\hbar$. The $j-$th component of $L\gamma = (L\gamma_1, \ldots, L\gamma_d)$, is given by

$$(L\gamma)_j(s) = 2i\lambda\hbar \int_s^t ds' \int_0^{s'} \gamma_j(s'')ds'' \qquad j = 1, \ldots, d \tag{15}$$

One can verify that $iL : H \to H$ is self-adjoint with respect to the H inner product. Moreover one can compute its eigenvalues and eigenvectors and verify that the operator is of trace class if $t \neq \left(n + \frac{1}{2}\right)\pi/\Omega$, $n \in \mathbb{Z}$, which is always fulfilled here since $\Omega^2 = -2i\lambda\hbar$ has a nonvanishing imaginary part. Moreover the Fredholm determinant of L is given by:

$$\det(I + L) = \cos(\Omega t).$$

For the proof of these results we refer to [8] (the calculations in [8] are still valid even if Ω is complex). In particular we have:

$$[(I + L)^{-1}\gamma]_j(s) = \gamma_j(s) - \Omega \int_s^t \sin[\Omega(s' - s)]\gamma_j(s')ds'$$

$$+ \sin[\Omega(t - s)] \int_0^t [\cos \Omega t]^{-1} \Omega \cos(\Omega s')\gamma_j(s')ds' \qquad j = 1, \ldots, d.$$

Let $l \in H$ be the vector defined by

$$\langle l, \gamma \rangle = -\sqrt{\lambda} \int_0^t \omega(s) \cdot \dot{\gamma}(s)ds = \sqrt{\lambda} \int_0^t \gamma(s) \cdot dW(s), \tag{16}$$

which is given by

$$l(s) = \sqrt{\lambda} \int_s^t \omega(\tau)d\tau.$$

The following result holds:

Theorem 3. *Let V and ψ_0 be Fourier transforms of complex bounded variation measures on $\mathbb{R}^d$. Then there exist a (strong) solution to the Stratonovich stochastic differential equation (2) and it is given by:*

$$\psi(t,x) = e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega(t)} \widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l,\gamma\rangle} e^{-i\int_0^t \Omega^2 x\cdot\gamma(s)ds} \cdot e^{-\frac{i}{\hbar}\int_0^t V(x+\gamma(s))ds}\psi_0(\gamma(0)+x)d\gamma \tag{17}$$

Remark.

1. The result can be extended to general initial vectors $\psi_0 \in L^2(\mathbb{R}^d)$, using the fact that $\mathcal{F}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$.
2. Formula (17) can also be written in the following form:

$$\psi(t,x) = \widetilde{\int} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds} \times e^{\int_0^t \sqrt{\lambda}(\gamma(s)+x)\cdot dW(s)}\psi_o(\gamma(0)+x)d\gamma. \tag{18}$$

The symbol on the right hand side should be understood as the right hand side of (17). The representation of the heuristic integrand in (18) in the form (17) uses the fact that the integrand $\exp(\frac{i}{2\hbar}\Phi)$, where $\Phi(\gamma) \equiv \int_0^t |\dot\gamma(s)|^2 ds + 2i\hbar\lambda\int_0^t |\gamma(s)+x|^2 ds - 2i\hbar\int_0^t \sqrt{\lambda}(\gamma(s)+x)\cdot dW(s)$ can be rigorously defined as the functional on the Cameron Martin space H given by $\Phi(\gamma) = \langle\gamma,(I+L)\gamma\rangle - 2i\hbar\langle l,\gamma\rangle - 2\hbar\int_0^t \Omega^2 x\cdot\gamma(s)ds - \Omega^2|x|^2 t - 2i\hbar\sqrt{\lambda}x\cdot\omega(t)$, where L is the operator (15) and l is the vector (16). We shall call “Feynman path integral” the expression on the right of (18).

Proof. The proof in divided into 3 steps: in the first two we consider the case $V \equiv 0$. First of all we deal with an approximated problem and we find a representation for its solution via a Fresnel (or Feynman path) integral, then we show that the sequence of approximated solutions converges in a suitable sense to the solution of problem (2). In the final step we introduce the potential V and show that the right hand side of (17) is in fact the solution of the equation (2).

1. The solution of the approximated problem. We approximate the trajectory $t \to \omega(t)$ of the Wiener process by a sequence of smooth curves. More precisely we consider the sequence of functions [3]

$$n\int_{t-\frac{1}{n}}^t \omega(s)ds \equiv \omega_n(t), \qquad n\in\mathbb{N}.$$

We have $\omega_n \to \omega$ uniformly on $[0,T]$, indeed

$$\sup_{s\in[0,T]}|\omega_n(s)-\omega(s)| \to 0 \quad \text{as } n\to\infty \qquad \mathbb{P}\text{ a.s.}$$

[3] Here we denote, as usual, the trajectory of the Wiener process $W(t)$ as $\omega(t)$.

Let us consider the sequence of approximated problems:

$$\begin{cases} d\psi_n = -\frac{i}{\hbar} H\psi_n dt - \lambda|x|^2\psi_n dt + \sqrt{\lambda} x \cdot \psi_n dW_n(t) \\ \psi_n(0,x) = \psi_0(x) \end{cases} \tag{19}$$

where $dW_n(t)$ is an ordinary differential, i.e. $dW_n(t) = \dot{\omega}_n(t)dt$, and we can also write:

$$\begin{cases} \dot{\psi}_n = -\frac{i}{\hbar} H\psi_n - \lambda|x|^2\psi_n + \sqrt{\lambda} x \cdot \psi_n \dot{\omega}_n(t) \\ \psi_n(0,x) = \psi_0(x) \end{cases} \tag{20}$$

which can be recognized as a family of Schrödinger equations, with a complex potential, labelled by the random parameter $\omega \in \Omega$.

Now we compute a representation of the solution of (20) by means of a Fresnel integral, under suitable assumptions on the (real) potential V and on the initial data $\psi_n(0,x,\omega) = \psi_0(x)$.

This tecnique has been developed by several authors, for instance [3] and [8].

We can write equation (20) in the following form:

$$\begin{cases} \dot{\psi}_n = -\frac{i}{\hbar}(\frac{-\hbar^2\Delta}{2m} - i\lambda\hbar|x|^2)\psi_n - \frac{i}{\hbar}V\psi_n + \sqrt{\lambda} x \cdot \psi_n \dot{\omega}_n(t) \\ \psi_n(0,x) = \psi_0(x) \end{cases} \tag{21}$$

so that we can recognize in it the Schrödinger equation for an anharmonic oscillator with a complex potential, i.e.

$$\begin{cases} \dot{\psi}_n = -\frac{i}{\hbar}(\frac{-\hbar^2\Delta}{2m} + \frac{\Omega^2}{2}|x|^2)\psi_n - \frac{i}{\hbar}U\psi_n \\ \psi_n(0,x) = \psi_0(x) \end{cases} \tag{22}$$

where $\Omega^2 = -2i\lambda\hbar$ and $U = U(t,x,\omega) = V(x) + i\hbar\sqrt{\lambda} x \cdot \dot{\omega}_n(t)$.

We introduce the sequence of vectors $l_n \in H$ defined by

$$\langle l_n, \gamma\rangle = \sqrt{\lambda}\int_0^t \gamma(s)\cdot\dot{\omega}_n(s)ds = -\sqrt{\lambda}\int_o^t \omega_n(s)\cdot\dot{\gamma}(s)ds,$$

which is given by

$$l_n(s) = \sqrt{\lambda}\int_s^t \omega_n(\tau)d\tau. \tag{23}$$

First of all let us consider equation (2) with H replaced by the free Hamiltonian $H = -\hbar^2\Delta/2$. The following result holds:

Lemma 1. *Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. Then the solution of the Cauchy problem:*

$$\begin{cases} \dot{\psi}_n(t,x) = \frac{i\hbar}{2}\Delta\psi_n(t,x) - \lambda|x|^2\psi_n(t,x) + \sqrt{\lambda} x \cdot \dot{\omega}_n(t)\psi_n(t,x) \\ \psi_n(0,x) = \psi_0(x) \end{cases} \tag{24}$$

is given by:

$$\psi_n(t,x) = \widetilde{\int_H} e^{\frac{i}{2\hbar}\int_0^t|\dot{\gamma}(s)|^2ds - \lambda\int_0^t|\gamma(s)+x|^2ds} e^{\sqrt{\lambda}\int_0^t(\gamma(s)+x)\cdot\dot{\omega}_n(s)ds}\psi_0(\gamma(0)+x)d\gamma$$

(where the right hand side is interpreted as the Fresnel integral of $\psi_0(\gamma(0)+x)e^{\langle l_n,\gamma\rangle}$ relative to $Q = (I+L)/\hbar$, with H the Cameron Martin space, l_n the vector defined by (23) and L the operator defined by (15)).

Proof. We have, by the definition 2 of the Fresnel integral

$$\widetilde{\int_H} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot\dot\omega_n(s)ds}\psi_o(\gamma(0)+x)d\gamma$$
$$= e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega_n(t)}\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle}e^{\langle l_n,\gamma\rangle}\int_{\mathbb{R}^d} e^{i\alpha\cdot x}e^{i\langle b(\alpha,x),\gamma\rangle}\tilde\psi_0(\alpha)d\alpha d\gamma$$

where $b(\alpha, x) \in H$, precisely:

$$b(\alpha, x)(s) = \alpha(t-s) - \frac{x\Omega^2}{2\hbar}(t^2 - s^2).$$

One can directly verify that the function $f(\gamma) \equiv \int_{\mathbb{R}^d} e^{i\alpha\cdot x}e^{i\langle b(\alpha,x),\gamma\rangle}\tilde\psi_0(\alpha)d\alpha$ is the Fourier transform of a measure $\mu \in \mathcal{M}(H)$, that is:

$$\mu(d\gamma) = \int_{\mathbb{R}^d} e^{i\alpha\cdot x}\tilde\psi_0(\alpha)\delta_{b(\alpha,x)}(d\gamma)d\alpha$$

so we can apply theorem 2 and have:

$$\psi_n^{(t,\lambda)} = e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}\cdot\omega_n(t)}$$
$$\times\int_{\mathbb{R}^d} e^{i\alpha\cdot x}\det(I+L)^{-1/2}e^{\frac{-i\hbar}{2}\langle b(\alpha,x)-il_n,(I+L)^{-1}(b(\alpha,x)-il_n)\rangle}\tilde\psi_0(\alpha)d\alpha$$

By simple calculations we get the final result:

$$\psi_n(t,x) = \int_{\mathbb{R}^d} G_n(t,x,y)\psi_0(y)dy$$

where $G_n(t, x, y)$ is given by:

$$G_n(t,x,y) \equiv \frac{1}{\sqrt{2\pi i\hbar}}\sqrt{\frac{\Omega}{\sin(\Omega t)}}e^{\sqrt{\lambda}x\cdot\omega_n(t)-\frac{\sqrt{\lambda}\Omega x}{\sin(\Omega t)}\cdot\int_0^t\omega_n(s)\cos(\Omega s)ds}$$
$$e^{\frac{i\hbar\lambda}{2}\int_0^t|\omega_n(s)|^2ds}e^{\frac{i\hbar\lambda}{2}(-\Omega\int_0^t\omega_n(s)\cdot\int_s^t\omega_n(s')\sin[\Omega(s'-s)]ds'ds)}$$
$$\cdot e^{\frac{i\hbar\lambda}{2}(-\Omega\int_0^t\sin(\Omega s)\omega_n(s)ds\cdot\int_0^t\cos(\Omega s)\omega_n(s)ds-\Omega\cot(\Omega t)|\int_0^t\cos(\Omega s)\omega_n(s)ds|^2)}$$
$$e^{\frac{i}{2\hbar}(\cot(\Omega t)(|x|^2+|y|^2)-\frac{2x\cdot y}{\sin(\Omega t)})}\cdot e^{\Omega\sqrt{\lambda}y\cdot(\cot(\Omega t)\int_0^t\cos(\Omega s)\omega_n(s)ds+\int_0^t\sin(\Omega s)\omega_n(s)ds)} \tag{25}$$

which is, as one can easily directly verify, the foundamental solution to the approximate Cauchy problem (19).

Remark. The result can be extended to general initial data $\psi_0 \in L^2(\mathbb{R}^d)$, using the density of $S(\mathbb{R}^d) \subset \mathcal{F}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$.

2. The convergence of the sequence of approximated solutions. We will prove the following result:

Lemma 2. *The following equation*

$$\begin{cases} d\psi = -\frac{i}{\hbar} H\psi dt - \lambda|x|^2\psi dt + \sqrt{\lambda}x \cdot \psi \circ dW(t) & t > 0 \\ \psi(0,x) = \psi_0(x), \quad \psi_0 \in S(\mathbb{R}^d) \end{cases} \tag{26}$$

has a unique strong solution given by the Feynman path integral (in the sense explained in connection with (18)):

$$\psi(t,x) = \widetilde{\int_H} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot dW(s)} \psi_0(\gamma(0)+x) d\gamma$$

Moreover it can be represented by the process

$$\psi(t,x) = \int_{\mathbb{R}^d} G(t,x,y)\psi_0(y)dy$$

where

$$G(t,x,y) = \frac{1}{\sqrt{2\pi i\hbar}}\sqrt{\frac{\Omega}{\sin(\Omega t)}} e^{\sqrt{\lambda}x\cdot\omega(t) - \frac{\sqrt{\lambda}\Omega x}{\sin(\Omega t)}\cdot\int_0^t \cos(\Omega s)\omega(s)ds}$$

$$e^{\frac{i\hbar\lambda}{2}(-\Omega\int_0^t \omega(s)\cdot\int_s^t \omega(s')\sin[\Omega(s'-s)]ds'ds)} e^{\frac{i\hbar\lambda}{2}\int\limits_0^t |\omega(s)|^2 ds}$$

$$\cdot e^{\frac{i\hbar\lambda}{2}(-\Omega\int_0^t \sin(\Omega s)\omega(s)ds\cdot\int_0^t \cos(\Omega s)\omega(s)ds - \Omega\cot(\Omega t)|\int_0^t \cos(\Omega s)\omega(s)ds|^2)}$$

$$e^{\frac{i}{2\hbar}\left(\cot(\Omega t)(|x|^2+|y|^2) - \frac{2x\cdot y}{\sin(\Omega t)}\right)} e^{\Omega\sqrt{\lambda}y\cdot\frac{1}{\sin(\Omega t)}(\int_0^t \cos[\Omega(s-t)]\omega(s)ds)}$$

Proof. As first we consider the sequence $\psi_n(t,x) = \int_{\mathbb{R}^d} G_n(t,x,y)\psi_0(y)dy$. Using the dominated convergence theorem we have that:

$$\mathbf{P}\left(\lim_{n\to\infty}\int_{\mathbb{R}^d} |\psi_n(t,x) - \tilde\psi(t,x)|^2 dx \to 0\right) = 1 \tag{27}$$

with $\tilde\psi(t,x) = \int_{\mathbb{R}} G(t,x,y)\psi_0(y)dy$, as:

$$\lim_{n\to\infty} |G_n(t,x,y) - G(t,x,y)| \to 0$$

for all $t \in [0,T]$ and $x, y \in \mathbb{R}^d$. Moreover, one can see by a direct computation that $\Omega = \sqrt{-2i\lambda\hbar}$ can be chosen is such a way that:

$$|\int_{\mathbb{R}^d} G_n(t,x,y)\psi_0(y)dy|^2 \le C(t)e^{P(t,x)}\|\psi_0(y)\|^2, \tag{28}$$

where $P(t,x)$ is a second order polynomial with negative leading coefficient and $C(t)$ and $P(t,x)$ are continuous functions of the variable $t \in [0,T]$. Applying the

Itô formula to the limit process $\tilde{\psi}(t)$ we see that it verifies equation (26) for every (t, x, y). Since the kernel $G(t, x, y)$ is $\mathcal{F}_t$ adapted by construction it follows that the solution is predictable. By direct computation and using estimates analogous to (28) one can verify that $\tilde{\psi}$ is a strong solution. On the other hand every $\psi_n(t, x)$ is equal to

$$\begin{aligned}
&\widetilde{\int_H} e^{\frac{i}{2\hbar}\int_0^t |\dot{\gamma}(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot\dot{\omega}_n(s)ds}\psi_0(\gamma(0)+x)d\gamma \\
&\quad = e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega_n(t)}\widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l_n,\gamma\rangle} e^{-i\int_0^t \Omega^2 x\cdot\gamma(s)ds}\psi_0(\gamma(0)+x)d\gamma \\
&\quad = e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega_n(t)}\det(I+L)^{-1/2}\int_H e^{\frac{-i\hbar}{2}\langle\gamma-il_n,(I+L)^{-1}(\gamma-il_n)\rangle}\mu(d\gamma)
\end{aligned}$$

where μ is the measure on H whose Fourier transform is the function $\gamma \to e^{-i\int_0^t \Omega^2 x\cdot\gamma(s)ds}\psi_0(\gamma(0)+x)$.

We have $||l_n - l||_H^2 \to 0$ as $n \to \infty$, where $l(s) = \sqrt{\lambda}\int_s^t \omega(r)dr$. Therefore, by the Lebesgue's dominated convergence theorem, we have that, for every $x \in \mathbb{R}^d$:

$$\begin{aligned}
&\lim_{n\to\infty} e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega_n(t)}\det(I+L)^{-1/2}\int_H e^{\frac{-i\hbar}{2}\langle\gamma-il_n,(I+L)^{-1}(\gamma-il_n)\rangle}\mu(d\gamma) \\
&\quad = e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\omega(t)}\det(I+L)^{-1/2}\int_H e^{\frac{-i\hbar}{2}\langle\gamma-il,(I+L)^{-1}(\gamma-il)\rangle}\mu(d\gamma) \qquad (29)
\end{aligned}$$

Therefore, taking into account the uniqueness of the pointwise limit, we have shown that:

$$\begin{aligned}
\psi(t,x) &= \int_{\mathbb{R}} G(t,x,y)\psi_0(y)dy \\
&= \widetilde{\int_H} e^{\frac{i}{2\hbar}\int_0^t |\dot{\gamma}(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{\int_0^t (\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma. \qquad (30)
\end{aligned}$$

Now the result can be extended to more general $\psi_0 \in \mathcal{L}^2(\mathbb{R}^d)$, using the density of $S(\mathbb{R}^d)$ in $\mathcal{L}^2(\mathbb{R}^d)$. □

3. The proof of Feynman-Kac-Ito formula by means of Dyson expansion. In this subsection we generalize our previous results to the case $H = -\hbar^2\Delta/2 + V$ and complete the proof of theorem 3. We follow here the technique of Elworthy and Truman [8].

We set

$$\begin{aligned}
\Theta(t,0)\psi_0(x) = \widetilde{\int_H} & e^{\frac{i}{2\hbar}\int_0^t |\dot{\gamma}(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds} \\
&\cdot e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot dW(s)}\psi_0(\gamma(0)+x)d\gamma \qquad (31)
\end{aligned}$$

and

$$\begin{aligned}
\Theta_0(t,0)\psi_0(x) = \widetilde{\int_H} & e^{\frac{i}{2\hbar}\int_0^t |\dot{\gamma}(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot dW(s)} \\
&\psi_0(\gamma(0)+x)d\gamma \qquad (32)
\end{aligned}$$

then we have:

$$\Theta(t,0)\psi_0(x) = e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega(t)} \widetilde{\int_H} e^{\frac{i}{2\hbar}\langle\gamma,(I+L)\gamma\rangle} e^{\langle l,\gamma\rangle} e^{-i\int_0^t \Omega^2 x\cdot\gamma(s)ds}$$
$$\cdot e^{-\frac{i}{\hbar}\int_0^t V(x+\gamma(s))ds}\psi_0(\gamma(0)+x)d\gamma \tag{33}$$

Let $\mu_0(\psi)$ be the measure on H such that its Fourier transform evaluated in $\gamma \in H$ is $\psi_0(\gamma(0)+x)$.

For $0 \leq u \leq t$ let $\mu_u(V,x)$, $\nu_u^t(V,x)$ and $\eta_u^t(x)$ be the measures on H, whose Fourier transforms when evaluated at $\gamma \in H$ are respectively $V(x+\gamma(u))$, $\exp\Big(-i\int_u^t V(x+\gamma(s))ds\Big)$, and $\exp\Big(-i\int_u^t \Omega^2 x\gamma(s)ds\Big)$. We shall often write $\mu_u \equiv \mu_u(V,x)$, $\nu_u^t \equiv \nu_u^t(V,x)$ and $\eta_u^t \equiv \eta_u^t(x)$ If $\{\mu_u : a \leq u \leq b\}$ is a family in $\mathcal{M}(H)$, we shall let $\int_a^b \mu_u du$ denote the measure on H given by:

$$f \to \int_a^b \int_H f(\gamma)\mu_u(d\gamma)du$$

whenever it exists.

Then, since for any continuous path γ

$$\begin{aligned}&\exp\left(-\frac{i}{\hbar}\int_0^t V(\gamma(s))ds\right)\\ &\quad= 1-\frac{i}{\hbar}\int_0^t V(\gamma(u))\exp\left(-\frac{i}{\hbar}\int_u^t V(\gamma(s))ds\right)du,\end{aligned} \tag{34}$$

we have

$$\nu_0^t = \delta_0 - \frac{i}{\hbar}\int_0^t (\mu_u * \nu_u^t)du \tag{35}$$

where δ_0 is the Dirac measure at $0 \in H$.

By the Cameron-Martin formula:

$$\Theta(t,0)\psi_0(x) = e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega(t)}\det(I+L)^{-1/2}$$
$$\cdot\int_H e^{\frac{-i\hbar}{2}\langle\alpha-il,(I+L)^{-1}(\alpha-il)\rangle}(\eta_0^t * \nu_0^t * \mu_0(\psi))(d\alpha) \tag{36}$$

Applying to this equality (35) we obtain:

$$\begin{aligned}&\Theta(t,0)\psi_0(x)\\
&= e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega(t)}\det(I+L)^{-1/2}\int_H e^{\frac{-i\hbar}{2}\langle\alpha-il,(I+L)^{-1}(\alpha-il)\rangle}(\eta_0^t * \mu_0(\psi))(d\alpha)\\
&\quad -\frac{i}{\hbar}\int_0^t e^{\frac{-i\Omega^2|x|^2 t}{2\hbar}+\sqrt{\lambda}x\cdot\omega(t)}\det(I+L)^{-1/2}\\
&\qquad\cdot\int_H e^{\frac{-i\hbar}{2}\langle\alpha-il,(I+L)^{-1}(\alpha-il)\rangle}(\eta_0^t * \mu_u(V,x) * \nu_u^t * \mu_0(\psi))(d\alpha)du\\
&= \Theta_0(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t \widetilde{\int_H} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds-\lambda\int_0^t|\gamma(s)+x|^2ds} e^{-\frac{i}{\hbar}\int_u^t V(\gamma(s)+x)ds}\\
&\qquad e^{\sqrt{\lambda}\int_0^t(\gamma(s)+x)\cdot dW(s)}V(\gamma(u)+x)\psi_0(\gamma(0)+x)d\gamma du\end{aligned}$$

By Fubini theorem for oscillatory integrals (see [3, 1]), we get that

$$\widetilde{\int_H} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \lambda\int_0^t |\gamma(s)+x|^2 ds} e^{-\frac{i}{\hbar}\int_u^t V(\gamma(s)+x)ds} e^{\sqrt{\lambda}\int_0^t (\gamma(s)+x)\cdot dW(s)} V(\gamma(u)+x)\cdot$$

$$\psi_0(\gamma(0)+x)d\gamma = \widetilde{\int_{H_{u,t}}} e^{\frac{i}{2\hbar}\int_u^t |\dot\gamma_2(s)|^2 ds - \lambda\int_u^t |\gamma_2(s)+x|^2 ds} e^{-\frac{i}{\hbar}\int_u^{tu} V(\gamma_2(s)+x)ds}\cdot$$

$$e^{\sqrt{\lambda}\int_u^t (\gamma_2(s)+x)\cdot dW(s)} V(\gamma_2(u)+x)\widetilde{\int_{H_{0,u}}} e^{\frac{i}{2\hbar}\int_0^u |\dot\gamma_1(s)|^2 ds - \lambda\int_0^u |\gamma_1(s)+\gamma_2(u)+x|^2 ds}\cdot$$

$$e^{\sqrt{\lambda}\int_0^u (\gamma_1(s)+\gamma_2(u)+x)\cdot dW(s)} \psi_0(\gamma_1(0)+\gamma_2(u)+x)d\gamma_1 d\gamma_2.$$

Here $\gamma_1 \in H_{0,u}$ and $\gamma_2 \in H_{u,t}$ are the integration variables. We denote by $H_{r,s}$ the Cameron-Martin space of paths $\gamma : [r,s] \to \mathbb{R}^d$.
Finally we have:

$$\Theta(t,0)\psi_0(x) = \Theta_0(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t \Theta(t,u)(V\Theta_0(u,0)\psi_0)(x)du \qquad (37)$$

Now the iterative solution of the latter integral equation is the Dyson series for $\Theta(t,0)$, which coincides with the corresponding power series expansion of the solution of the stochastic Schrödinger equation, which converges strongly in $L^2(\mathbb{R}^d)$. The equality holds pointwise. On the other hand, following [9], it is possible to prove that the problem (26) has a strong solution that verifies (37) in the L^2 sense, therefore $\Theta(t,0)\psi_0$ coincides with the solution $\psi(t)$. This concludes the proof of theorem 3. □

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