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Modal analysis and insights into damping phenomena of a special vibration chain

Abstract Vibration chains are of interest in many fields of practical applications. In this contribution, a modal analysis of the rather special MIKOTA's vibration chain is performed. Herein, focus is set on the mode shapes of this multibody oscillator, which was firstly introduced by MIKOTA as a solid body compensator in hydraulic systems for filtering out fluid flow pulsations. The mode shapes show interesting properties, e.g. an increase in the polynomial representing the coordinates of each mode shape with an increasing eigenfrequency associated with the respective mode shape. This and other properties are discussed exemplary. Some of these properties still have to be proven, which is the task of future work. Additionally, modal damping of MIKOTA's vibration chain is discussed. Moreover, an approach for determining the damping matrix for given LEHR's damping measures without knowing the mode shapes in advance is introduced. This approach involves the determination of a matrix root.

Keywords MIKOTA'S vibration chain · Solid body compensator · Mode shape · Multibody dynamics · Damping

1 Introduction

In many engineering disciplines such as civil engineering or mechanical engineering, multibody oscillators and in detail vibration chains are an appropriate approach for modelling even complicated structures and to get major insights into the structures' behaviour. Standard text books which deal with the investigation of vibration chains and other multibody oscillators are among others [3,4,6–8,14,15]. Typical examples of vibration chains are torsional vibration of turbines [6] or the dynamic behaviour of layered soil deposits above a bedrock [2]. Besides these apparent applications, multibody oscillators serve as examples for application of advanced methods of investigations. Such methods are—among other things—model order reduction techniques [20] or solver routines for determining eigenvalues. The main advantage of multibody oscillators is that there are configurations which give results known a priori. Consequently, the multibody oscillators serve as reference cases. One such configuration is MIKOTA's vibration chain. For the first time, it was introduced in [11] as a special configuration of a solid body compensator for filtering fluid flow pulsations of hydraulic sources, see also [9,10]. Although this solid body compensator is a rather special vibration chain, it possesses nice properties and gives interesting results when analysing its dynamic behaviour.

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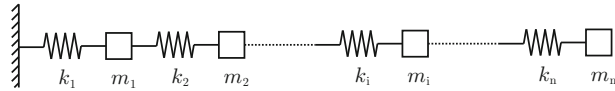E-mail: Wolfgang.Weber@hsu-hh.de

Fig. 1 Undamped vibration chain [16]

The outline of this contribution is as follows: In Sect. 2, the definition of a special vibration chain according to [9] is given. Additionally, this section contains the state of the art of investigations concerning MIKOTA's vibration chain. Based on the findings revisited there, a discussion of the mode shapes of MIKOTA's vibration chain is performed in Sect. 3. Within this section, questions are sketched which are still open and need to be answered in order to better understand the dynamic behaviour of MIKOTA's vibration chain and of multibody oscillators in general. A major topic in vibrational analysis and in control engineering is the investigation of damping phenomena. Thus, in Sect. 4 the related work concerning MIKOTA's vibration chain is revisited and new aspects are discussed. A new approach which allows—for given LEHR's damping measures—the determination of the damping matrix without knowing the mode shapes in advance is introduced. The applicability of this proposed methodology is shown in Sect. 5. The conclusions are provided with Sect. 6.

2 Definition of MIKOTA's vibration chain and literature review

In this contribution, a linear undamped vibration system with n degrees of freedom (DOF) is dealt with. Such a system can be described by

$$\tilde{\mathbf{M}}\ddot{\mathbf{x}}(t) + \tilde{\mathbf{K}}\mathbf{x}(t) = \mathbf{0}, \tag{2.1}$$

with $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n)^{\mathrm{T}}$ and $\ddot{\mathbf{x}} = (\ddot{x}_1, \ldots, \ddot{x}_i, \ldots, \ddot{x}_n)^{\mathrm{T}}$ representing the column matrix of displacements and accelerations, respectively [6]. Herein, $\tilde{\mathbf{M}}$ denote the mass matrix, while the stiffness matrix is given by $\tilde{\mathbf{K}}$.

For a linear vibration chain according to Fig. 1, these matrices are diagonal and tri-diagonal, respectively. In detail, it is [3,6]

$$\tilde{\mathbf{M}} = \mathrm{diag}\,(m_i), \tag{2.2}$$

$$\tilde{\mathbf{K}} = \mathrm{diag}\,\{(k_1 + k_2, -k_2), \ldots, (-k_i, k_i + k_{i+1}, -k_{i+1}), \ldots, (-k_n, k_n)\}. \tag{2.3}$$

MIKOTA introduced a vibration chain with the following evolution equations for the masses and stiffnesses associated with the i-th DOF

$$m_i = \frac{1}{i}m \quad \text{and} \quad k_i = (n - i + 1)\,k \quad \text{where} \quad i = 1, 2, \ldots n \quad, \tag{2.4}$$

cf. [9]. Herein, m is the first mass. As given in Eq. (2.4), the evolution of the single masses is independent of the total number n of DOF, while the stiffness coefficients are related to n in such a manner that the stiffness of the last spring always is $k_n = k$. Using the quantities m and k, Eq. (2.1) can be rewritten as

$$m\mathbf{M}\ddot{\mathbf{x}}(t) + k\mathbf{K}\mathbf{x}(t) = \mathbf{0} \tag{2.5}$$

so that $\mathbf{M}$ and $\mathbf{K}$ are number matrices.

MIKOTA conjectured that this specific vibration chain has the eigenfrequencies

$$\Omega_l = l\Omega = l\sqrt{k/m} \quad \text{with again} \quad l = 1, 2, \ldots n, \tag{2.6}$$

where $\Omega = \sqrt{k/m}$ is the first eigenfrequency, [9]. As can readily be seen, enlarging the system from n DOF to $n + 1$ DOF leads to the following changes in the mass and stiffness matrices

$$\mathbf{M}_{n+1} = \begin{bmatrix} \mathbf{M}_n & \mathbf{0} \\ \hline \mathbf{0}^{\mathrm{T}} & \frac{1}{n+1} \end{bmatrix}, \quad \mathbf{K}_{n+1} = \begin{bmatrix} 2n+1 & -n & \mathbf{0}^{\mathrm{T}} \\ \hline -n & & \\ \mathbf{0} & & \mathbf{K}_n \end{bmatrix}, \tag{2.7}$$

see also [16]. Obviously, there is a counter pattern in the setup of these matrices. Due to this opposite behaviour in constructing the matrices, established methods as given in e.g. [6,15] for analysing the eigenfrequencies, the mode shapes, and the eigenforces for arbitrary n DOF are not suitable. For a discussion of the resulting difficulties for proving MIKOTA's conjecture it is referred to e.g. [12,13]. To overcome these difficulties, two different approaches were proposed by [13,16], each of which leads to a proof of MIKOTA's conjecture. Describing the mode shapes and the eigenforces of MIKOTA's vibration chain was not the scope of MIKOTA's work. However, these are necessary in order to better understand the system's dynamic behaviour. To close this gap, in [13] an approach based on binomial coefficients for determining the mode shapes is provided, while in [18] a modification of the well-known LAGUERRE polynomials is proposed allowing the evaluation of the mode shapes of MIKOTA's vibration chain. The drawbacks of both approaches are that they (i) are quite laborious and (ii) do not reveal an easy structure in order to obtain general formulæ for the mode shapes $\mathbf{u}_l$. Both problems are solved in [19] and some results presented therein shall be used in Sect. 3.

Meanwhile, further investigations of MIKOTA's vibration chain took place. The optimal damping of MIKOTA's vibration chain for the two cases of an (i) absolute damper and (ii) relative damper was exemplary addressed in [17]. For determining the optimal damping for these two cases, the criteria were used which have been presented in [4,14]. A model order reduction in MIKOTA's vibration chain based on the proper orthogonal decomposition according to [5] was performed in [20]. In the latter contribution, two cases were considered: (i) model order reduction of MIKOTA's vibration chain and (ii) model order reduction of MIKOTA's vibration chain with an additional absolute damping element.

3 Mode shapes of MIKOTA's vibration chain and related observations

The coordinates of the eigenvectors of a matrix , i.e. the mode shapes, can be expressed by polynomials $P_l(i)$ in the coordinate i

$$
\begin{aligned}
\mathbf{u}_l = (u_l(i)) &= \left(u_{l,i=1}, u_{l,i=2}, \ldots, u_{l,i=n}\right)^{\mathrm{T}} \\
&= (P_l(i=1), P_l(i=2), \ldots, P_l(i=n))^{\mathrm{T}},
\end{aligned}
\tag{3.1}
$$

see [7]. Based on the proof provided with [19] the mode shapes $\mathbf{u}_l$ of MIKOTA's vibration chain for arbitrary n DOF and $l \leq n$ can be determined in a successive manner. Herein, the polynomial degree of P_l is l. In detail, the first four mode shapes of MIKOTA's vibration chain are given by the following polynomials:

$$
u_1(i) = P_{l=1}(i) = i,
\tag{3.2}
$$

$$
u_2(i) = P_{l=2}(i) = i^2 - \frac{2n+1}{3}i,
\tag{3.3}
$$

$$
u_3(i) = P_{l=3}(i) = i^3 - \frac{3}{5}(2n+1)i^2 + \frac{1}{5}\left[\frac{3}{2}n(n+1)+1\right]i,
\tag{3.4}
$$

and

$$
u_4(i) = P_{l=4}(i) = i^4 - \frac{6}{7}(2n+1)i^3 + \frac{1}{7}[6n(n+1)+5]i^2 \ldots
$$
$$
- \frac{2}{35}(2n+1)[n(n+1)+3]i,
\tag{3.5}
$$

see [19]. Exemplary, for MIKOTA's vibration chain with $n = 10$ DOF the eigenfrequencies Ω_l and the first five mode shapes are given in Fig. 2.

In general, the mode shapes $\mathbf{u}_l$ of a dynamical system to different eigenfrequencies fulfil the relation

$$
\mathbf{u}_l^{\mathrm{T}}\mathbf{M}\mathbf{u}_j \begin{cases} = 0 & \text{for } l \neq j \\ \neq 0 & \text{for } l = j \end{cases}, \quad \mathbf{u}_l^{\mathrm{T}}\mathbf{K}\mathbf{u}_j \begin{cases} = 0 & \text{for } l \neq j \\ \neq 0 & \text{for } l = j \end{cases}.
\tag{3.6}
$$

That means that they are not orthogonal to each other with respect to the (standard) scalar product. However, the mode shapes of MIKOTA's vibration chain show the neat property that the matrix product $\mathbf{U}^{\mathrm{T}}\mathbf{U}$ is tri-diagonal, see also [19], where $\mathbf{U}$ is the modal matrix

$$
\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n].
\tag{3.7}
$$
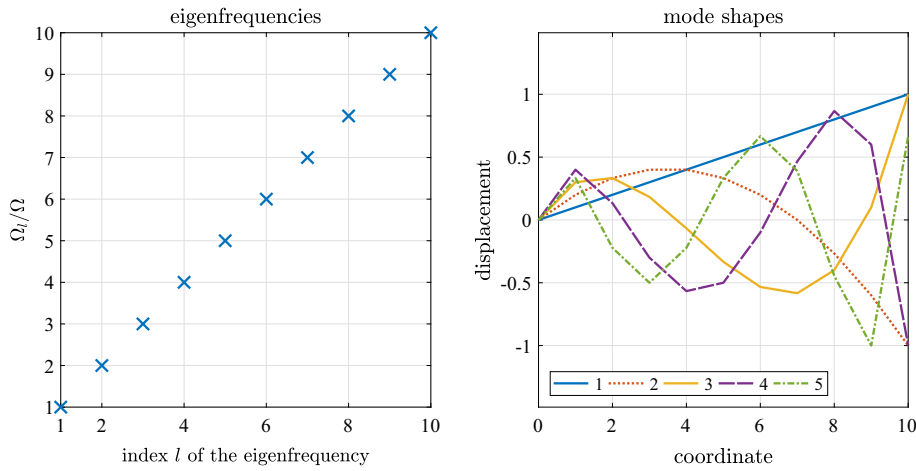
Fig. 2 Eigensolutions of MIKOTA's vibration chain for $n = 10$ DOF, where only the mode shapes $\mathbf{u}_{l=1} \ldots \mathbf{u}_{l=5}$ are shown, cf. [20]. The displacements between the coordinates i are interpolated linearly, where $i = 0$ is at the fixed support according to Fig. 1

(a) $n = 4$ **(b)** $n = 8$
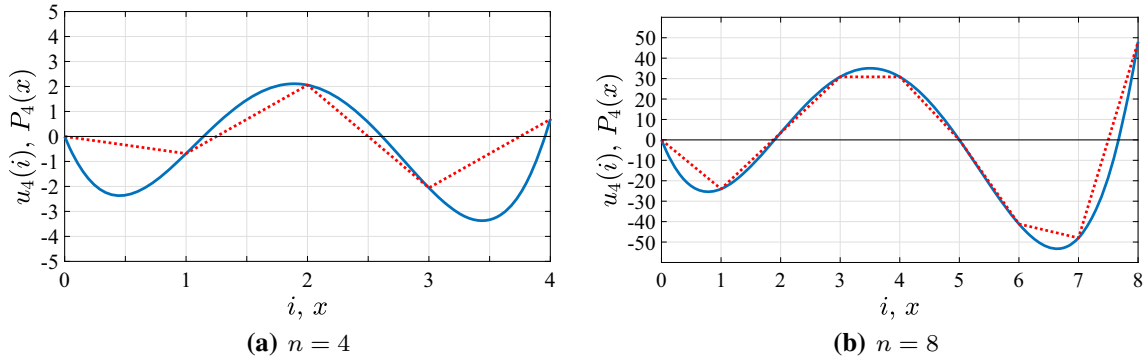
Fig. 3 Mode shape no. 4 represented by $\mathbf{u}_4(i)$ and $P_4(x)$ for different n

Introducing the polynomials $P_l(x)$ with $P_l(i) = u_l(i)$ such that

$$P_{l=1}(x) = x, \tag{3.8}$$

$$P_{l=2}(x) = x^2 - \frac{2n + 1}{3} x, \tag{3.9}$$

$$P_{l=3}(x) = x^3 - \frac{3}{5}(2n + 1)x^2 + \frac{1}{5}\left[\frac{3}{2}n(n + 1) + 1\right]x, \tag{3.10}$$

and

$$P_{l=4}(x) = x^4 - \frac{6}{7}(2n + 1)x^3 + \frac{1}{7}[6n(n + 1) + 5]x^2 \ldots$$
$$- \frac{2}{35}(2n + 1)[n(n + 1) + 3]x \tag{3.11}$$

etc., then $P_l(x)$ approximates the linearly interpolated mode shape $\mathbf{u}_l$, cf. for example Fig. 3.

It is worth noting that only the vectors $\mathbf{u}_l$, that is, the coordinates stemming from evaluating the polynomials $P_l(x)$ at the distinct $x = i$, lead to the tri-diagonality. But the polynomials P_l themselves are not orthogonal in the sense of

$$\int_{x=0}^{n} q(x)P_l(x)P_k(x)\mathrm{d}x \neq 0 \quad \text{with} \quad l, k \in \mathbb{N}, \quad 0 \leq l, k \leq n, \quad x \in \mathbb{R} \tag{3.12}$$

with a certain weighting function $q(x) > 0$. Or—in other words—the $P_l(x)$ do not belong to a classical system of orthogonal polynomials as, e.g. the LEGENDRE- or LAGUERRE polynomials.

Figure 3 contains the 4-th mode shape for $n = 4$ and $n = 8$, respectively. Herein, the representation of the mode shapes is done both by line segments and interpolation polynomials P_l. If these two types of mode shape representation are compared with each other it is apparent that with increasing number of DOF, the difference between the discrete coordinates of the line segments and the corresponding polynomials tends to zero, which is indeed a trivial observation. Some additional observations shall be listed in what follows.

In [3], it has been proved that the mode shapes $\mathbf{u}_l$ represented by line segments show l nodes and l antinodes (loops) including the behaviour at the boundaries, that is, a node at $P_l(0) = 0$ and an antinode at $P_l(n)$. This result has also been quoted in [6, p. 165ff]. Because each antinode is associated with at least one coordinate $u_l(i)$, the polynomial $P_l(x)$ possesses the same number of antinodes and therefore shows the same number l of nodes, too. That means that the polynomials $P_l(x)$ show l real nonnegative zeros $x_j \geq 0$ with $j = 1, \ldots, l$ and $P_l(x_j) = 0$ in $0 \leq x_j < n$. Additionally, the nodes of $P_l(x)$ with $0 \leq x < n$ are separated by the nodes of $P_{l-1}(x)$ for $l = 3, \ldots, n$.

4 Remarks on modal damping of MIKOTA'S vibration chain

The adequate modelling of damping is a major issue in vibrational analysis. Only some works deal with the investigation of damping in the context of MIKOTA's vibration chain. Based on [12], the influence of a single absolute damper and a single relative damper on the vibration behaviour of MIKOTA's vibration chain was investigated in [17] for some n DOF. In [20], MIKOTA's vibration chain with additional dampers was investigated concerning some issues of model order reduction. However, a systematic investigation is still missing, yet. Thus, some general properties shall be discussed in what follows.

The eigenvalue problem of Eq. (2.5) reads

$$(\mathbf{M}\lambda_l + \mathbf{K}) \mathbf{u}_l = \mathbf{0} \tag{4.1}$$

with $\mathbf{M}$, $\mathbf{K}$ according to Eqs. (2.2)–(2.5) and

$$\lambda_l = -\Omega_l^2 = -(l)^2. \tag{4.2}$$

In the present case, for the spectral matrix

$$\boldsymbol{\Omega}^2 = \text{diag}\left(\Omega_l^2\right) = \text{diag}\left(1, 4, \ldots, n^2\right) \tag{4.3}$$

holds. Using the modal matrix as introduced with Eq. (3.7), the following relations hold

$$\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U} = \text{diag}\left(m_1, m_2, \ldots, m_n\right) \tag{4.4}$$

$$\mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U} = \text{diag}\left(\kappa_1, \kappa_2, \ldots, \kappa_n\right), \quad \text{where} \quad \frac{\kappa_i}{m_i} = \Omega_i^2 = i^2 \tag{4.5}$$

$$\boldsymbol{\Omega}^2 = \left(\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}\right)^{-1} \mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U} = \mathbf{U}^{-1}\mathbf{M}^{-1}\mathbf{K}\mathbf{U}. \tag{4.6}$$

In the preceding sections, $\mathbf{M}$ and $\mathbf{K}$ denote number matrices. Thus, for describing modal damping, a number matrix $\mathbf{D}$ is introduced, too:

$$d\mathbf{D} \quad \text{with} \quad \mathbf{D}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{D}. \tag{4.7}$$

where condition (4.7) has been given by [1].

This number matrix $\mathbf{D}$ shall be determined in what follows. Starting point is the equation of motion describing free oscillations

$$\left(\mathbf{M}\mu_l^2 + \mathbf{D}\mu_l + \mathbf{K}\right) \mathbf{u}_l = \mathbf{0}. \tag{4.8}$$

Right-multiplication of $\mathbf{u}_l$ with Eq. (4.7) yields

$$\underbrace{\mathbf{D}\mathbf{M}^{-1}\mathbf{K}\mathbf{u}_l}_{\Omega_l^2 \mathbf{u}_l} = \mathbf{K}\mathbf{M}^{-1}\mathbf{D}\mathbf{u}_l \tag{4.9}$$

and consequently

$$\left(\Omega_l^2 \mathbf{I} - \mathbf{K}\mathbf{M}^{-1}\right)\mathbf{D}\mathbf{u}_l = \mathbf{0}, \tag{4.10}$$

where $\mathbf{I}$ denotes the unit matrix. This can be rewritten

$$\left(\Omega_l^2 \mathbf{M} - \mathbf{K}\right)\mathbf{M}^{-1}\mathbf{D}\mathbf{u}_l = \mathbf{0}, \tag{4.11}$$

showing the eigenvector

$$\mathbf{M}^{-1}\mathbf{D}\mathbf{u}_l = \alpha_l \mathbf{u}_l, \quad \alpha_l \neq 0 \tag{4.12}$$

with an arbitrary scaling factor α_l. Then,

$$\mathbf{D}\mathbf{u}_l = \alpha_l \mathbf{M}\mathbf{u}_l \tag{4.13}$$
$$\mathbf{D}\mathbf{U} = \mathbf{M}\mathbf{U}\boldsymbol{\alpha}, \quad \boldsymbol{\alpha} = \mathrm{diag}\left(\alpha_l\right) \tag{4.14}$$

holds. The eigenvalue problem for the damped MIKOTA's vibration chain can now be formulated as follows:

$$\left(\mu_l^2 + \alpha_l \mu_l + \Omega_l^2\right)\mathbf{M}\mathbf{u}_l = \mathbf{0}. \tag{4.15}$$

As $\mathbf{M}\mathbf{u}_l \neq \mathbf{0}$,

$$\mu_l = -\frac{\alpha_l}{2} \pm \sqrt{\frac{\alpha_l^2}{4} - \Omega_l^2}. \tag{4.16}$$

The α_l can be determined e.g. by setting the LEHR's damping measure D_l to

$$D_l := 1 = \frac{\alpha_l}{2\Omega_l}, \tag{4.17}$$

cf. amongst others [8]. Then,

$$\alpha_l = 2l \quad \text{and with Eq. (4.16)} \quad \mu_l = -l, \tag{4.18}$$

which is a twofold eigenvalue. Equations (4.13), (4.14) then read

$$\mathbf{D}\mathbf{u}_l = 2l\mathbf{M}\mathbf{u}_l \quad \Leftrightarrow \quad \mathbf{D}\mathbf{U} = 2\mathbf{M}\mathbf{U}\boldsymbol{\Omega} \tag{4.19}$$

and thus the sought damping matrix is

$$\mathbf{D} = 2\mathbf{M}\mathbf{U}\boldsymbol{\Omega}\mathbf{U}^{-1}. \tag{4.20}$$

Considering $\mathbf{U}^{\mathrm{T}}\mathbf{D}\mathbf{U}$ which is a diagonal matrix it is obvious that $\mathbf{D}$ is a symmetric positive definite matrix. As can be directly seen, the modal matrix $\mathbf{U}$ and thus all mode shapes $\mathbf{u}_l$ are needed to calculate the damping matrix $\mathbf{D}$. The question arises of how the damping matrix can be determined without knowing the mode shapes in advance. Using the relation

$$\left(\mathbf{U}\boldsymbol{\Omega}\mathbf{U}^{-1}\right)^2 = \mathbf{U}\boldsymbol{\Omega}^2\mathbf{U}^{-1} = \mathbf{U}\mathbf{U}^{-1}\mathbf{M}^{-1}\mathbf{K}\mathbf{U}\mathbf{U}^{-1} \tag{4.21}$$

$$= \mathbf{M}^{-1}\mathbf{K}, \tag{4.22}$$

where Eq. (4.6) was applied, gives

$$\mathbf{U}\boldsymbol{\Omega}\mathbf{U}^{-1} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}. \tag{4.23}$$

Thus, the damping matrix can be determined without knowing the mode shapes by means of the following equation

$$\mathbf{D} = 2\mathbf{M}\sqrt{\mathbf{M}^{-1}\mathbf{K}}. \tag{4.24}$$

However, finding the roots necessitates the eigenvalues of the matrix root to be positive. This preliminary is always fulfilled for the vibration chain dealt with here. But still the roots of matrices in general are ambiguous.
Setting

$$\alpha_l = \Omega_l \tag{4.25}$$

yields

$$\mu_l = -\frac{\Omega_l}{2} \pm \iota\Omega_l \frac{\sqrt{3}}{2} = l\left(-\frac{1}{2} \pm \iota\frac{\sqrt{3}}{2}\right) \tag{4.26}$$

and consequently

$$\mathbf{D}\mathbf{u}_l = \Omega_l\mathbf{M}\mathbf{u}_l \tag{4.27}$$

according to Eq. (4.13). For this case, the damping matrix reads as

$$\mathbf{D} = \mathbf{M}\mathbf{U}\boldsymbol{\Omega}\mathbf{U}^{-1} = \mathbf{M}\sqrt{\mathbf{M}^{-1}\mathbf{K}}. \tag{4.28}$$

As can readily be seen, the damping matrix in the present case is half of the damping matrix as given with Eq. (4.24). The eigenvalues of the present case are

$$\Re(\mu_l) = -\frac{\Omega_l}{2} = -\frac{l}{2}, \quad \Im(\mu_l) = \pm\sqrt{\frac{3}{4}}l \tag{4.29}$$

and LEHR's damping measure follows to

$$D_l = \frac{\Omega_l}{2\Omega_l} = \frac{1}{2}. \tag{4.30}$$

It should be noted that finding the matrix root $\sqrt{\mathbf{M}^{-1}\mathbf{K}}$ again is the major issue for determining the damping matrix $\mathbf{D}$ without using the mode shapes.

5 Example for $n = 3$ DOF

To show the applicability of the proposed methodology, a rather small example shall be looked at in what follows. For $n = 3$, MIKOTA's vibration chain is characterized by

$$\mathbf{M} = \begin{bmatrix} 1 & & \\ & \frac{1}{2} & \\ & & \frac{1}{3} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{5.1}$$

The mode shapes are given by Eqs. (3.2)–(3.4) and yield

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{3}\begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{5}\begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}. \tag{5.2}$$

Therefore, the modal matrix and its inverse are determined as

$$\mathbf{U} = \begin{bmatrix} 1 & -\frac{4}{3} & \frac{3}{5} \\ 2 & -\frac{2}{3} & -\frac{6}{5} \\ 3 & 2 & \frac{3}{5} \end{bmatrix}, \quad \mathbf{U}^{-1} = \frac{1}{12}\begin{bmatrix} 2 & 2 & 2 \\ -\frac{24}{5} & -\frac{6}{5} & \frac{12}{5} \\ 6 & -6 & 2 \end{bmatrix}. \tag{5.3}$$

The damping matrix according to Eq. (4.28) is calculated as

$$\mathbf{D} = \frac{1}{15}\begin{bmatrix} 32 & -7 & -1 \\ -7 & 17 & -4 \\ -1 & -4 & 8 \end{bmatrix}. \tag{5.4}$$

The relation (4.12) is confirmed with $\alpha_l = l$ such that

$$\mathbf{M}^{-1}\mathbf{D}\mathbf{u}_1 = \mathbf{u}_1, \quad \mathbf{M}^{-1}\mathbf{D}\mathbf{u}_2 = 2\mathbf{u}_2, \quad \mathbf{M}^{-1}\mathbf{D}\mathbf{u}_3 = 3\mathbf{u}_3 \tag{5.5}$$

really hold. It should be noted that the most labourious work is related to the determination of $\mathbf{U}^{-1}$.

6 Conclusions

MIKOTA's vibration chain is a multibody oscillator possessing nice properties. Due to these properties, the rather special MIKOTA's vibration chain can serve as reference case and application example for several techniques which allow the solution of practical problems. This special vibration chain has been investigated completely confirming the eigenfrequencies $\Omega_l = l\Omega$ with the first eigenfrequency $\Omega = \sqrt{k/m}$ and determining the eigenvectors $\mathbf{u}_l$ successively by a recursion formula. The mode shapes have been illustrated by means of both line segments from $(i, u_l(i))$ to $(i + 1, u_l(i + 1))$ and by polynomials $P_l(x)$, where the relation $u_l(i) = P_l(i)$ holds. Although the polynomials do not form a classical system of orthogonal polynomials, it seems that there exist similar properties. However, this conjecture has to be proven in future work.

Additionally, the structure of the damping matrix is investigated without knowing the mode shapes in advance by setting LEHR's damping measure to certain values. However, the authors currently have to halt the investigations when the matrix root $\sqrt{\mathbf{M}^{-1}\mathbf{K}}$ has to be evaluated. For the best knowledge of the authors, finding the roots of this expression is an open question. Thus, these investigations concerning the damping matrix have to be continued in future work, too.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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