CORRECTION

Ugurcan Eroglu · Giuseppe Ruta

Correction to: On inner constraints in plane circular arches

Published online: 26 August 2019
 © Springer-Verlag GmbH Germany, part of Springer Nature 2019

Correction to: *Archive of Applied Mechanics* 74 (2004) 212–222,
<https://doi.org/10.1007/s00419-004-0344-7>

The series expansions in Eq. (16)₁ of [1] for trigonometric functions depending on a small evolution parameter η shall be corrected as

$$\sin \varphi = \eta \dot{\varphi} + \frac{\eta^2}{2} \ddot{\varphi} - \frac{\eta^3}{6} (\dot{\varphi}^3 - \ddot{\varphi}) + o(\eta^3), \quad \cos \varphi = 1 - \frac{\eta^2}{2} \dot{\varphi}^2 - \frac{\eta^3}{2} \dot{\varphi} \ddot{\varphi} + o(\eta^3), \quad (1)$$

where superimposed dots denote η -derivatives evaluated at $\eta = 0$. Accordingly, the formal expansions for strain measures in Eq. (18) of [1] must be replaced with

$$\begin{aligned} \dot{\chi} &= \dot{\varphi}', & \ddot{\chi} &= \ddot{\varphi}', & \ddot{\chi} &= \ddot{\varphi}', \\ \dot{\gamma} &= \dot{v}' - \kappa \dot{u} + \dot{\varphi}, & \dot{\gamma} &= \dot{v}' - \kappa \dot{u} + \dot{\varphi} + 2\dot{\varphi}\dot{\varepsilon}, \\ \ddot{\gamma} &= \ddot{v}' - \kappa \ddot{u} + \ddot{\varphi} - \dot{\varphi}^3 + 3\ddot{\varphi}(\dot{u}' + \kappa \dot{v}) - 3\dot{\varphi}^2(\dot{v}' - \kappa \dot{u}) + 3\dot{\varphi}(\ddot{u}' + \kappa \ddot{v}), \\ \dot{\varepsilon} &= \dot{u}' + \kappa \dot{v}, & \dot{\varepsilon} &= \dot{u}' + \kappa \dot{v} + \dot{\varphi}^2 - 2\dot{\varphi}\dot{\gamma}, \\ \ddot{\varepsilon} &= \ddot{u}' + \kappa \ddot{v} - 3\dot{\varphi}\ddot{\varphi} - 3\dot{\varphi}^2(\dot{u}' + \kappa \dot{v}) - 3\ddot{\varphi}(\dot{v}' - \kappa \dot{u}) - 3\dot{\varphi}(\ddot{v}' - \kappa \ddot{u}). \end{aligned} \quad (2)$$

First-order terms coincide with infinitesimal strains known since mid-nineteenth century [2, 3], and all curvature variations are as in [1]; higher-order elongations and shearing strains in Eq. (19) of [1], on the other hand, shall be replaced by

The original article can be found online at <https://doi.org/10.1007/s00419-004-0344-7>.

U. Eroglu
 Istanbul Technical University, Istanbul, Turkey

Present address:
 U. Eroglu
 DISG, Univ. Roma “La Sapienza”, Rome, Italy

G. Ruta (✉)
 DISG, Univ. Roma “La Sapienza”, Rome, Italy
 E-mail: giuseppe.ruta@uniroma1.it

G. Ruta
 National Group for Mathematical Physics, Rome, Italy

$$\begin{aligned}\dot{\gamma} &= \dot{v}' - \kappa \ddot{u} + \ddot{\varphi} + 2\dot{\varphi}\dot{\varepsilon}, & \ddot{\varepsilon} &= \ddot{u}' + \kappa \ddot{v} + \dot{\varphi}^2 - 2\dot{\varphi}\dot{\gamma}, \\ \ddot{\gamma} &= \ddot{v}' - \kappa \ddot{\ddot{u}} + \ddot{\ddot{\varphi}} - \dot{\varphi}^3 + 3\ddot{\varphi}\dot{\varepsilon} + 3\dot{\varphi}\ddot{\varepsilon} + 3\dot{\varphi}^2\dot{\gamma}, \\ \ddot{\varepsilon} &= \ddot{\ddot{u}}' + \kappa \ddot{\ddot{v}} + 3\ddot{\varphi}\ddot{\varphi} + 3\dot{\varphi}^2\dot{\varepsilon} - 3\ddot{\varphi}\dot{\gamma} - 3\dot{\varphi}\ddot{\gamma}.\end{aligned}\quad (3)$$

The comments in [1] on second-order strains depending on those of the first order, and third-order strains depending on both first order and second order, are still correct.

Inner constraints. Vanishing shearing strain at various η -orders should be reconsidered. Letting $\dot{\gamma} = 0$ in Eq. (2)₂ (or Eq. (18)₂ of [1]) provides the well-known

$$\dot{\varphi} = \kappa \dot{u} - \dot{v}', \quad (4)$$

present also in [1]. Assuming zero second-order shearing strain in Eq. (3)₁ (or Eq. (19)₁ of [1]) for infinitesimal displacements (i.e., $\ddot{u} = \ddot{v} = \ddot{\varphi} = 0$) yields

$$2\dot{\varphi}\dot{\varepsilon} = 0 \Rightarrow \dot{\varepsilon} = 0, \quad (5)$$

present also in [1]. However, for $\ddot{u} = \ddot{v} = \ddot{\varphi} = 0$ the higher-order elongations become

$$\ddot{\varepsilon} = \dot{\varphi}^2, \quad \ddot{\ddot{\varepsilon}} = 3\dot{\varphi}^3. \quad (6)$$

For non-infinitesimal displacements, if $\dot{\gamma} = \ddot{\gamma} = \ddot{\ddot{\gamma}} = 0$ in Eq. (3)₃ we have

$$\ddot{v} - \kappa \ddot{\ddot{u}} + \ddot{\ddot{\varphi}} - \dot{\varphi}^3 + 3\ddot{\varphi}\dot{\varepsilon} + 3\dot{\varphi}\ddot{\varepsilon} = 0. \quad (7)$$

Equation (7) is general; then, it holds also for $\ddot{u} = \ddot{v} = \ddot{\varphi} = 0$ (i.e., for the first possible non-infinitesimal motion in this perturbation approach, quadratic in η), yielding

$$\dot{\varepsilon}\ddot{\varphi} + \ddot{\varepsilon}\dot{\varphi} = \dot{\varphi}^3/3, \quad (8)$$

which is the correct form of Eq. (26) of [1]. Thus, imposing zero shearing strain affects both first- and second-order elongations, and higher-order strains become

$$\begin{aligned}\ddot{\varphi} &= \kappa \ddot{\ddot{u}} - \ddot{v}' + 2(\dot{v}' - \kappa \dot{u})(\dot{u}' + \kappa \dot{v}), & \ddot{\varepsilon} &= \ddot{\ddot{u}}' + \kappa \ddot{\ddot{v}} + (\dot{v}' - \kappa \dot{u})^2, \\ \ddot{\ddot{\chi}} &= \kappa \ddot{\ddot{\ddot{u}}} - \ddot{\ddot{v}}'' + 2[(\dot{v}'' - \kappa \dot{u}')(\dot{u}' + \kappa \dot{v}) + (\dot{v}' - \kappa \dot{u})(\dot{u}'' + \kappa \dot{v}')], \\ \ddot{\ddot{\varphi}} &= -\ddot{v}'' + \kappa \ddot{\ddot{\ddot{u}}} + \dot{\varphi}^3 - 3\ddot{\varphi}\dot{\varepsilon} - 3\dot{\varphi}\ddot{\varepsilon}, & \ddot{\ddot{\varepsilon}} &= \ddot{\ddot{\ddot{u}}} + \kappa \ddot{\ddot{\ddot{v}}} + 3\ddot{\varphi}\ddot{\varphi} + 3\dot{\varphi}^2\dot{\varepsilon}, \\ \ddot{\ddot{\chi}} &= -\ddot{v}'' + \kappa \ddot{\ddot{\ddot{u}}} + 3[\dot{\varphi}^2\dot{\chi} - \ddot{\chi}\dot{\varepsilon} - \dot{\chi}\ddot{\varepsilon} - \ddot{\varphi}(\ddot{u}'' + \kappa \dot{v}') - \dot{\varphi}(\ddot{\ddot{u}}'' + \kappa \ddot{\ddot{v}}' + 2\dot{\varphi}\dot{\chi})].\end{aligned}\quad (9)$$

Adding the non-extensibility constraint to zero shearing (pure flexibility) yields

$$\begin{aligned}\ddot{\varphi} &= \kappa \ddot{\ddot{u}} - \ddot{v}', & \ddot{\ddot{\varphi}} &= \kappa \ddot{\ddot{\ddot{u}}} - \ddot{\ddot{v}}'' + (\kappa \dot{u} + r \dot{u}'')^3, \\ \ddot{v} &= -r \ddot{\ddot{u}}' - r(r \dot{u}'' + \kappa \dot{u})^2, & \ddot{\ddot{v}} &= -r \ddot{\ddot{\ddot{u}}} - 3r(\kappa \dot{u} + r \dot{u}'')(\kappa \dot{u} - \dot{v}'), \\ \ddot{\chi} &= \kappa \ddot{\ddot{u}}' - \ddot{v}'', & \ddot{\ddot{\chi}} &= \kappa \ddot{\ddot{\ddot{u}}} - \ddot{\ddot{v}}'' + 3(\kappa \dot{u}' + r \dot{u}''')(\kappa \dot{u} + r \dot{u}'').\end{aligned}\quad (10)$$

Equations (10)₂, (4), and (2)₃ imply $\dot{\varphi} = 0$ for infinitesimal displacements, see also [1].

An example. For the example in [1], i.e., a semicircular arch clamped at one end, loaded at the other end by a “horizontal” force of given magnitude at first and second orders in η , we present the results for first- and second-order contact actions and displacement fields, keeping the same symbols. First-order contact actions are

$$\dot{N} = -\dot{W} \sin(\kappa s), \quad \dot{Q} = \dot{W} \cos(\kappa s), \quad \dot{M} = (\dot{W}/\kappa) \sin(\kappa s). \quad (11)$$

Henceforth, the subscripts 1, 2 refer to fields for non-shearable and pure flexible arches, respectively. If $\sigma = (\pi - \kappa s)$, the first-order displacement components are

$$\begin{aligned}\dot{u}_1 &= (\dot{W}/2B\kappa^3) [(\lambda^2 + 1)\sigma \sin(\kappa s) - 2 - 2\cos(\kappa s)], \\ \dot{v}_1 &= -[\dot{W}(\lambda^2 + 1)/2B\kappa^3] [\sigma \cos(\kappa s) + \sin(\kappa s)], \\ \dot{\varphi}_1 &= -(\dot{W}/B\kappa^2) [1 + \cos(\kappa s)], & \dot{u}_2 &= (\dot{W}/2B\kappa^3) [\sigma \sin(\kappa s) - 2 - 2\cos(\kappa s)], \\ \dot{v}_2 &= -(\dot{W}/2B\kappa^3) [\sigma \cos(\kappa s) + \sin(\kappa s)], & \dot{\varphi}_2 &= -(\dot{W}/B\kappa^2) [1 + \cos(\kappa s)].\end{aligned}\quad (12)$$

Fig. 1 Second-order tangential displacement (see also Fig. 2 of [1])

The second-order contact actions when shearing strain vanishes are

$$\begin{aligned}\ddot{N}_1 &= (2\dot{W}^2/B\kappa^2) [\cos(\kappa s) - 1] \cos(\kappa s) - \ddot{W} \sin(\kappa s), \\ \ddot{Q}_1 &= (\dot{W}^2/B\kappa^2) [\sin(2\kappa s) - 2\sin(\kappa s)] + \ddot{W} \cos(\kappa s), \\ \ddot{M}_1 &= (\dot{W}^2/2B\kappa^3) [(\lambda^2 - 1) \cos(2\kappa s) + 4\cos(\kappa s) - \lambda^2 - 3] + (\ddot{W}/\kappa) \sin(\kappa s),\end{aligned}\quad (13)$$

while for a purely flexible arch we find

$$\begin{aligned}\ddot{N}_2 &= \ddot{N}_1, \quad \ddot{Q}_1 = \ddot{Q}_2, \\ \ddot{M}_1 &= (\dot{W}^2/2B\kappa^3) [4\cos(\kappa s) - \cos(2\kappa s) - 3] + (\ddot{W}/\kappa) \sin(\kappa s).\end{aligned}\quad (14)$$

The second-order displacement components when shearing strain vanishes are

$$\begin{aligned}\ddot{u}_1 &= \frac{\dot{W}^2}{4B^2\kappa^5} \{ \sigma [6 + 8(1 + \lambda^2) \cos(\kappa s)] + (\lambda^2 - 1) \sin(2\kappa s) + 2\lambda^2 [\sigma \\ &\quad - 4\sin(\kappa s)] \} + \frac{\ddot{W}}{2B\kappa^3} [\sigma (1 + \lambda^2) \sin(\kappa s) - 2 - 2\cos(\kappa s)], \\ \ddot{v}_1 &= \frac{\dot{W}^2}{2B^2\kappa^5} \{ \lambda^2 [4\cos(\kappa s) + 3 + \cos(2\kappa s)] + 4\sigma (1 + \lambda^2) \sin(\kappa s) \} \\ &\quad - \frac{\ddot{W}}{2B\kappa^3} [\sigma (1 + \lambda^2) \cos(\kappa s) - (1 + \lambda^2) \sin(\kappa s)], \\ \ddot{\phi}_1 &= \frac{\dot{W}^2}{4B^2\kappa^4} [2\sigma (3 + \lambda^2) - 8\sin(\kappa s) + (\lambda^2 - 1) \sin(2\kappa s)] - \frac{\ddot{W}}{B\kappa^2} [1 + \cos(\kappa s)],\end{aligned}\quad (15)$$

while for a purely flexible arch we obtain

$$\begin{aligned}\ddot{u}_2 &= \frac{\dot{W}^2}{4B^2\kappa^5} \{ \sigma [6 + 8\cos(\kappa s)] - \sin(2\kappa s) \} + \frac{\ddot{W}}{2B\kappa^3} [\sigma \sin(\kappa s) - 2 - 2\cos(\kappa s)], \\ \ddot{v}_2 &= \frac{2\dot{W}^2}{B^2\kappa^5} \sigma \sin(\kappa s) - \frac{\ddot{W}}{2B\kappa^3} [\sigma \cos(\kappa s) - \sin(\kappa s)], \\ \ddot{\phi}_2 &= \frac{\dot{W}^2}{4B^2\kappa^4} [6\sigma - \sin(2\kappa s) + 8\sin(\kappa s)] - \frac{\ddot{W}}{B\kappa^2} [1 + \cos(\kappa s)].\end{aligned}\quad (16)$$

Numerical evaluations. We consider the same arch as in [1] and compare the two sets of results to quantitatively highlight differences for second-order displacements of extensible and non-extensible arches with vanishing shearing strain. We also perform finite element analyses by the commercial code ABAQUS, since some differences of the FEM results for geometrically nonlinear and linear analyses provide approximated second-order displacement components for small values of the evolution parameter [4]. Figures 1, 2, and 3 show by solid lines the plots of the second-order displacement components along the arc length for vanishing strain [part (a) of the figures] and the differences among these values and those for purely flexible arches [part (b) of the figures]; in the same plots, dashed lines report the results of Figs. 2–4 of [1] and markers denote FEM results.

Fig. 2 Second-order transverse displacement (see also Fig. 3 of [1])

Fig. 3 Second-order cross section rotation (see also Fig. 4 of [1])

It is apparent how the FEM nonlinear results, almost coinciding for two small values of η , are well caught by our amended second-order approximated results, which correct errors in [1].

Final remarks. In [1] there are erroneous formal expansions of trigonometric functions that imply errors in the expressions for the second- and third-order elongation and shearing strain. Here, we provide the correct expansions; however, the consequences of posing inner constraints, which were the essence of the original paper, are even stronger: One of the most important conclusions, the vanishing of first-order rotation in case of second-order pure flexibility, holds true. We also quantitatively show the outcomes of the initial error by examining the same example of [1]. The second-order tangential and radial displacements are substantially different, but the pure flexibility constraint yields very similar outcomes, at least for the considered geometrical and material properties; the cross section rotation is not altered at all. The present results agree with those of FEM and will help the authors in their forthcoming investigations on damaged arches with various geometries.

Acknowledgements U.E. acknowledges the support of DISG at “La Sapienza.” G.R. acknowledges the support of the grants “Progetto di ricerca d’Ateneo” C26A15JLMP of “La Sapienza” (2015), and PRIN 2015TTJN95 “Identification and monitoring of complex structural systems” of the Italian Ministry of University and Research.

References

1. Ruta, G.: On inner constraints in plane circular arches. *Arch. Appl. Mech.* **74**, 212–222 (2004)
2. Bresse, J.A.C.: *Cours de mécanique appliquée Résistance des matériaux et stabilité des constructions*. Gauthier-Villars, Paris (1859) (in French)
3. Love, A.E.H.: *A Treatise on the Mathematical Theory of Elasticity*. Dover, New York (1944)
4. ABAQUS: *ABAQUS Documentation*. Dassault Systemes, Providence (2014)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.