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ORIGINAL PAPER

Helmholtz and the geometry of color space: gestation and development of Helmholtz's line element

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Abstract

Modern color science finds its birth in the middle of the nineteenth century. Among the chief architects of the new color theory, the name of the polymath Hermann von Helmholtz stands out. A keen experimenter and profound expert of the latest developments of the fields of physiological optics, psychophysics, and geometry, he exploited his transdisciplinary knowledge to define the first non-Euclidean line element in color space, i.e., a three-dimensional mathematical model used to describe color differences in terms of color distances. Considered as the first step toward a metrically significant model of color space, his work inaugurated researches on *higher color metrics*, which describes how distance in the color space translates into perceptual difference. This paper focuses on the development of Helmholtz's mathematical derivation of the line element. Starting from the first experimental evidence which opened the door to his reflections about the geometry of color space, it will be highlighted the pivotal role played by the studies conducted by his assistants in Berlin, which provided precious material for the elaboration of the final model proposed by Helmholtz in three papers published between 1891 and 1892. Although fallen into oblivion for about three decades, Helmholtz's masterful work was rediscovered by Schrödinger and, since the 1920s, it has provided the basis for all subsequent studies on the geometry of color spaces up to the present time.

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1 Introduction

"[Helmholtz] had picked Young's ideas on a three-receptor mechanism of color vision, imbibed Fechner's ideas on the relationship between physical input and sensory output, and seen Riemann developed his non-Euclidean geometry. In his extraordinary universal mind these ingredients fused into that mathematical expression which, until the present day, formed the basis for any line element developed" (Vos 1979, p. 208).

One of Hermann von Helmholtz's major contributions to color theory was the first description of a line element for dichromatic and trichromatic color spaces. Helmholtz (von Helmholtz after 1883) succeeded, indeed, to produce the first accurate determination of linear sequences through color space combining his knowledge of non-Euclidean geometry and psychophysics. The aim of the present paper is to retrace the path that led Helmholtz toward a Riemannian metric to describe the perceptual color space. Starting from a brief introduction on Helmholtz's experimental findings, which showed that color space could not be uniform, we will introduce Helmholtz's crucial consideration on the geometry of color space contained in his 1868 essay "On the Facts Underlying Geometry". In this memoir, he identified the space of perceived color as the only candidate in common experience (together with the measuring out of the visual field by visual estimation) able to provide empirical evidence on the existence of non-Euclidean geometry, something already recognized by one of the leading mathematicians of the nineteenth century, Bernhard Riemann. In the last paragraphs, we will discuss the content of Helmholtz's three papers published between 1891 and 1892, of paramount importance for the field of color metrics, in which he introduced the final form of the line element as an extension of the one-dimensional Weber–Fechner law of psychophysics. At that time, indeed, Helmholtz was harvesting the fruits of a life-long interest in both mathematics and physiological optics and collecting the pivotal results obtained in collaboration with his students and assistants in Berlin, Arthur König, Conrad Dieterici, and Eugen Brodhun.

Despite long periods of silence, Helmholtz's basic ideas have survived for more than a century, constituting the basis for all subsequent works on color metrics up to the present time.

2 Helmholtz's starting point: experimental findings

The polymath Hermann Helmholtz began his systematic research in the field of color in 1849, as he was appointed Professor of Physiology in Königsberg (now Kaliningrad). Years before, from 1838 to 1842, as student at the Royal Friedrich-Wilhelm Institute of Medicine and Surgery in Berlin, he had encountered Johannes Peter Müller, who was at that time his professor of physiology. Inspired by the work of Müller, Helmholtz started to find connections between physiology and physics, aim which he pursued throughout his life.[1] Together with James Clerk Maxwell and Hermann Günther Grassmann, he

[1] It must be pointed out that Müller was a supporter of vitalism, i.e., the view that life is sustained by an inexplicable non-physical vital force, in the wake of Romanticism and Naturphilosophie. These romantic movements deeply influenced scientific life in Germany until the first half of the nineteenth century. In the late 1840s, in open conflict with vitalistic views, Helmholtz, Carl Ludwig and Müller's pupils Emil

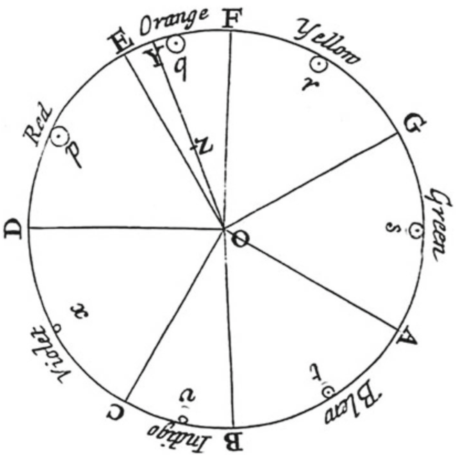

Fig. 1 Newton's color mixing circle showing his theoretical method to determine the outcome of any mixture of seven primary colors, arbitrarily chosen exploiting the analogy between colors and musical tones. The primary or principal colors, red, orange, yellow, green, blue, indigo and violet, are disposed along the arches. At the center of gravity of these sections, Newton depicted a small circle. Being the area of each circle proportional to the number of rays of the color that enter a given mixture, the common center of gravity of these circles can be easily determined (Newton 1979, p. 155)

can be considered the father of modern color science and colorimetry. Among his key contributions to color theory (given independently also by Maxwell), it is worth mentioning the first clarification of the distinction between additive and subtractive color mixing, eliminating the long-standing confusion between the two methods of producing colors, at the center of the debate in the field of light and color especially between the eighteenth century and the first half of the nineteenth century. He adopted Newton's analogy of the center of gravity to predict the outcome of optical mixture of light and eliminated the arbitrary choice of the number of primary colors.

Since Newton's color theory represented the starting point for Helmholtz's research in the field (just as for Maxwell's research), it is worth offering a brief sketch of Newton's method for color mixing. Newton's color circle (Fig. 1) is described in his "Opticks", Book I, Part II, Proposition VI, *In a mixture of primary colours, the quantity and quality of each being given, to know the colour of the compound*. Newton laid down a theoretical method for predicting the compound color which results from a combination of seven principal colors, *violet*, *indigo*, *blue*, *green*, *yellow*, *orange*, and *red*, into which he divided the spectrum in line with the analogy between colors and

Footnote 1 continued
du Bois-Reymond and Ernst Brücke constituted an "anti-vitalistic" circle in Berlin. They supported with strength and authority that life phenomena have to be interpreted as manifestations of physical and chemical laws. On Müller's nativism and Helmholtz's empirical approach in the theory of vision see the contributions of Timothy Lenoir, "The eye as mathematician: Clinical practice, instrumentation, and Helmholtz's construction of an empiricist theory of vision", in Cahan (1993), pp. 109–153, and of Frederic L. Holmes, "The role of Johannes Müller in the Foundation of Helmholtz's Physiological Career", in Krüger (1994), pp. 1–21.

musical harmony.[2] Indeed, the seven parts in which the circumference is divided are proportional to the musical interval contained in an octave (that is, proportional to the numbers 1/9, 1/16, 1/10, 1/9, 1/16, 1/16, 1/9. Newton also used the musical analogy in describing different phenomena, e.g., in defining the thickness of thin films and in explaining the color of thick plates, although the partition he adopted was different for each object of study).

The primary or principal colors are disposed along the arches. At the center of gravity of these sections, Newton depicted a small circle. Being the area of each circle proportional to the number of rays of the color that enter a given mixture, the common center of gravity of these circles can be easily determined. Let that point be Z, then the place of point Y in the circumference will represent the spectral color of the mixture and the distance of point Z from the center will indicate the saturation, "the fulness", of the color (Newton 1979, pp. 154–158). In the example reported by Newton, Z corresponds to an unsaturated orange resulting from a mixture of all the seven primaries in which yellow, orange, and red preponderate. As concluding remark, Newton asserted that "it is such an orange (Z) as may be made by mixing an homogeneal orange with a good white in the proportion of the Line OZ to the line ZY, this Proportion being not of the quantities of mixed orange and white Powders, but of the quantities of the Light reflected from them" (Newton 1979, p. 157). From this excerpt, it can be deduced that Newton supposed that the resulting color could be imitated by a mixture of pigments, as Shapiro first remarked (Shapiro 1994). Indeed, he was aware of the difference between a mixture of lights and a mixture of "powders" but he implicitly admitted the existence of a unique model for describing both processes.

It is perhaps for this reason that Newton's model aroused confusion, until Helmholtz and Maxwell independently clarified, in the 1850s, the distinction between what are now known as additive and subtractive color mixing processes[3] and elaborated Newton's theory in light of their findings. In his 1855 paper titled "Experiments on Colour, as perceived by the Eye, with Remarks on Colour-Blindness" (Clerk Maxwell 1855, pp. 294–295), Maxwell shed light on Newton's method for the composition of colors (at that time often misunderstood) and explained Newton's analogy with the center of gravity for the composition of spectral colors. Clerk Maxwell's (1855) paper is of

[2] Although Newton chose seven principal primary colors, exploiting the musical analogy, it has to be highlighted that he was aware that the number of "rays" composing white light is extremely large if not infinite: "Whether it [white] may be compounded of a mixture of three [colors] taken at equal distances in the circumference I do not know, but of four or five I do not much question but it may. [...] For in all whites produced by Nature, there uses to be a mixture of all sorts of Rays, and by consequence a composition of all Colours" (Newton 1979, pp. 156–157).

[3] Helmholtz offered a clear and complete explanation of the difference between additive and subtractive color mixture in his paper "Über die Theorie der zusammengesetzten Farben", published in 1852 (Helmholtz 1852). On the basis of his observations, Helmholtz underlined that mixture of pigments, *Mischung der Farbstoffe*, and composition of spectral colors, *Zusammensetzung der gefärbten Lichtes*, are different phenomena. The mixture of pigments determines a unique color stimulus that reaches directly the eye and the retina. For what concerns the process of mixing lights, different stimuli come separate and unchanged to the eye and their fusion takes place in the visual system. They are, therefore, phenomena that obey completely different rules and belong to different domains: the mixture of pigments belongs to the *physical*, while the composition of lights belongs to the *physiological* domain. In his "Handbuch der physiologischen Optik" Helmholtz described an effective method to show directly the difference between the two processes using a spinning top (Von Helmholtz 2005, p. 124).

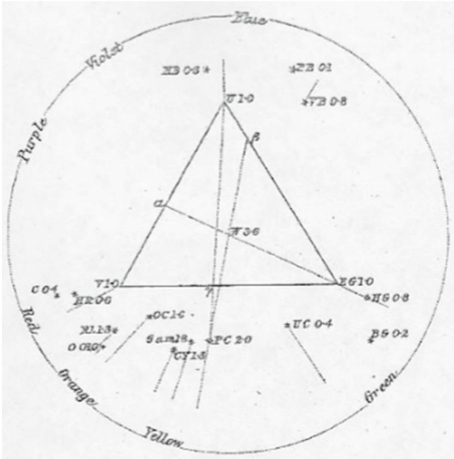

Fig. 2 Maxwell's color diagram, based on Young's color triangle, inside Newton's color circle. Vermillion, ultramarine, and emerald green are taken as primary colors, from which all other colors can be obtained by mixing, and are located at the vertices of an equilateral triangle. The number of primary colors is, thus, reduced from seven to three (Maxwell 1855)

paramount importance for the field of color science, since it contains the first "colorimetric equations", which allowed him to derive his triangular color diagram with red, green, and blue as vertices[4] (see Supplementary Material, section S1) exploiting Newton's barycentric rule (Fig. 2).

So far, we have provided an overview of Helmholtz's achievements, which were not only shared but also discussed with Maxwell, who worked independently on the same subjects. But whereas Maxwell limited his treatment to the so-called "lower color metrics", Helmholtz went further exploring the realm of "higher color metrics", or advanced color metrics. It is worthwhile to mention that the dichotomy "lower-higher color metrics" was first introduced by Erwin Schrödinger in the 1920s. Lower color metric deals indeed with color mixing laws as manifest in the tridimensional color space, and in particular in its bi-dimensional representation, better known as chromaticity diagram (see Supplementary Material, section S2). On the other hand, higher color metrics build on the fundamentals of lower color metrics and describe how distance in the color space translate to difference in color perception implying the definition of line elements in color space. The aim of line elements is that of specifying pairs of color stimuli that present a constant perceived difference, e.g., a just-perceptible difference, by the positions and distance apart in tristimulus space of the points representing these stimuli and, a priori, the space could either be Euclidean or non-Euclidean.

The first experimental evidence which opened the door to Helmholtz's reflections about the geometry of color space dates back to the 1850s. In 1855 indeed, his third paper on color titled "Ueber die Zusammensetzung von Spectralfarben" ("On the composition of spectral colors") was published. It contains the results of a color mixing

[4] Maxwell's color diagram is a bi-dimensional representation of color space, with the two variables being hue (wavelength) and saturation, whereas the brightness is maintained constant. All saturated colors are located on the perimeter of the triangle and any other color inside the triangle can be expressed in terms of the "amount" of light of the three primary colors involved in the mixture and identified in terms of its distance from white, as in Newton's color circle.

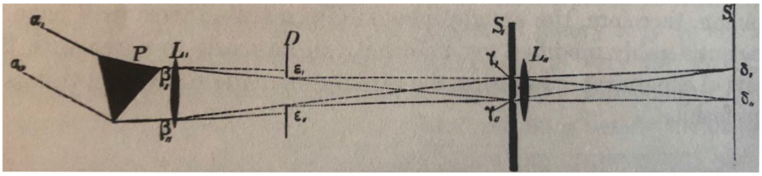

Fig. 3 Sketch of Helmholtz's apparatus for mixing spectral colors, as reprinted in his "Treatise on Physiological Optics" (Von Helmholtz 2005, p. 158). The dashed lines correspond to the more refrangible light and the dotted lines to the less refrangible light

experiment carried out by Helmholtz, which allowed him to obtain a small area covered simultaneously by two spectral colors and, consequently, to study their mixture. By performing this experiment, Helmholtz was able to place Newton's barycentric construction for spectral mixing on an empirical foundation, providing a graphic representation for how the human eye mixes color additively. Since both the experimental findings and their consequent implications were crucial for the subsequent developments of Helmholtz's theory of color, we offer a brief description of the experimental set-up (Fig. 3) reported in the 1855 paper.

A heliostat allowed sunlight to be reflected through a vertical slit into a dark room. The beam, whose width is indicated by $\alpha_{\rm I}\alpha_{\rm II}$, passed through a prism and a lens, denoted by P and $L_{\rm I}$, respectively. A screen S_I was located in the focal plane of this lens, and between the screen and the lens a rectangular diaphragm, D, was placed. Two vertical slits $\gamma_{\rm I}$ and $\gamma_{\rm II}$, located on the screen S_I, were adjustable both in position (to select specific wavelength ranges of the spectrum) and in width (to select the intensities); the screen $S_{\rm I}$ was so constructed to allow the experimenter to vary the hue and the intensity of the mixture gradually. Pairs of spectral colors could, thereby, be combined by another achromatic lens, $L_{\rm II}$, which projected an image of the diaphragm on a second screen, $S_{\rm II}$. The diaphragm could be adjusted to render the two mixing lights uniformly distributed over the image $\delta_{\rm I}$ $\delta_{\rm II}$ projected on the screen $S_{\rm II}$; hence, this area appeared colored by a mixture of the two lights.

Using this relatively simple but effective color mixer, Helmholtz was able to find seven pairs of complementary colors, the wavelengths of which he measured and plotted by pair. By adjusting the respective slit widths, he also compared the relative intensities of the light of each complementary pair and collected the results. The next step was to obtain a graphical representation of his findings by constructing a color diagram. The shape of the curve could be determined from the relative intensities of the complementary pairs.

Adopting Newton's analogy with the calculation of the center of gravity to predict the outcome of optical mixture of light, he observed that the curve for a barycentric representation of spectral mixing was not a circle such as Newton had suggested. He, thus, introduced some corrections to Newton's color circle (see Fig. 1), offering a new version of the gross structure of the bi-dimensional color space.

The first adjustment was the introduction of the so-called *purple line*, a line connecting red to violet. Although different pairs of complementary colors could be detected,

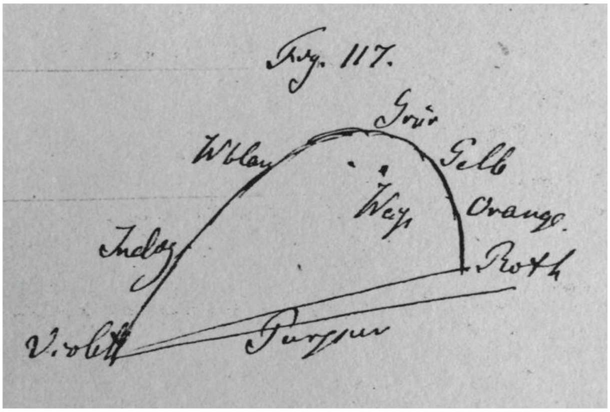

Fig. 4 Helmholtz's original drawing of his color diagram taken from the manuscript version of the "Handbuch der physiologischen Optik" preserved at the Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften (Nachlass Hermann von Helmholtz, 572 Über physiologische Optik, ∫∫20. Die zusammengesetzten Farben, p. 480)

any superposition of green with a simple color did not yield white. The complementary color of green was, according to Helmholtz's observations, purple, i.e., a mixture of violet and red. Furthermore, Helmholtz also departed to the Young's triangular representation, derived from the color models proposed by Mayer and Lambert in the eighteenth century, and subsequently adopted by Maxwell (Fig. 2). Measuring the relative distance from white to the perimeter of the color diagram, he found an asymmetry. Mixing complementary colors, indeed, he recognized that the "quantity" of light required to produce white varied systematically with position around the Newton's color circle. For each pair of complementary colors, different "quantities" (proportional to slit apertures γ_{I} and γ_{II}) of the two colors were required to produce white. Since, for example, the intensity of yellow must be four times larger than that of indigo to make white, indigo must be four times as far away from white on the barycentric curve as yellow. By similar reasoning, Helmholtz was able to sketch a curve for color mixing whose shape is neither a circle nor a triangle, but rather a truncated hyperbola, as shown in Fig. 4.

Having noticed that equal distances did not correspond to equal perceptual differences in his color diagram, Helmholtz realized that the color space, as defined by the color mixing laws, could not be uniform. This was the first experimental evidence that made Helmholtz turn toward non-Euclidean geometry to define a line element in color space.[5]

[5] See the contribution of Richard Kremer, "Innovation through synthesis: Helmholtz and color research", in Cahan (1993), pp. 205–258, for an introduction to Helmholtz's work on color and a detailed analysis of Helmholtz's papers published in the 1850s.

3 Helmholtz's "On the facts" and Riemann's "On the hypotheses"

In the second half of the nineteenth century, Riemann and Helmholtz were the first to conjecture that the perceptual color space is one of the few candidates accessible to experimental investigation to test the existence of geometries, different from the Euclidean one. Considerations of this kind were contained in their essays related to a more general discussion on the foundation of geometry, i.e., "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" ("On the hypotheses which lie at the bases of geometry"), Riemann's 1854 Habilitation dissertation, and "Ueber die Thatsachen, die der Geometrie zum Grunde liegen" ("On the facts underlying geometry") Helmholtz's memoir published in 1868.

It is worthwhile to introduce some key insights related to these works because they were of paramount importance for the development of both geometry and higher color metric. Furthermore, an analysis of their content could help to better understand Helmholtz's line of thinking in the development of the line element.

As common denominator, the two essays are characterized by an empirical approach to the study of geometry and its foundations: according to both scientists, geometry should be founded on our perception and construction of space—which is not necessarily the one constituting the basis for Euclidean geometry. When we are dealing with geometry, in line with their view, we are not simply dealing with immediately evident axioms, with given truths, but rather with hypothetical truths depending on the validity of certain premises and whose truth depends on empirical choices.

One of the most debate topic at that time concerned precisely the foundations of geometry (see Tazzioli 2003). This gave a significant boost to the development of non-Euclidean geometries from the 1820s onwards. After the spread of these ideas and Riemann's and Helmholtz's contributions, mathematicians were free to construct infinitely many geometries and to choose which one could be better applied to physical space. Consequently, Euclidean geometry was not the only possible option. The question, which can be considered as a starting point for both Riemann's and Helmholtz's investigations, was not under which conditions geometrical axioms might be valid, but under which hitherto not explained conditions the knowledge of them can be achieved.

In 1854, Riemann presented the content of his "On the hypothesis" on the occasion of his Habilitation as Privatdozent in Göttingen. The whole text was posthumously published by Richard Dedekind in 1867, 1 year after the author's death. From this work, a deep interrelation of mathematical, physical, and philosophical thought emerges. The *Habilitationsvortrag*, indeed, contains a generalization of physical space. Riemann's aim was to present a deeper analysis of the concept of space and its presupposition, unfastening physical explanation from conceptual limitations. He reported several conditions that limited the property of space, which he called "hypothesis"—in contrast to axioms, word that he introduced in the title of the presentation with the meaning of "empirical facts". This is the reason why Helmholtz used the word "facts" for the title of his essay. The titles of the two dissertations, thus, correspond, in so far as "hypothesis" and "facts" are intended as synonyms. The heart of Riemann's work is the introduction of the concept of "manifolds", which allow to give a precise mathematical formulation to scientific problems. The failure of the traditional geometry was due, according to him, to the lack of a general concept of multiply extended manifolds. This n-ply extended magnitudes can be continuous or discrete, and a comparison is made between the positions of perceived objects and the space of possible colors. Both notions, stated Riemann, form a multiply extended manifoldness: "Notions whose specialisations form a discrete manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a notion in which they are included […], on the other hand so few and far between are the occasions for forming notions whose specialisations make up a continuous manifoldness, that the only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colours" (Jost 2016, original in Riemann 1867, p. 135).

In paragraph entitled "Measure relations of which a manifoldness of n dimensions is capable on the assumption that lines have a length independent of position, and consequently that every line may be measured by every other", Riemann introduced the definition of the line element, which represents the distance between two infinitely close points. The line element, as Tazzioli has elucidated, extends the concept of distance between two infinitely close points to a n-dimensional space, whose curvature is not necessarily equal to zero. Since Riemann indicated the space of color, together with the physical space of sensory objects, as the only example of continuous manifold of several dimensions, he probably saw in it a valid candidate in common experience to prove the existence of non-Euclidean geometry by experimental verification.[6]

This was a pregnant remark whose implications remained hidden until Helmholtz brought them to light in the last decade of the nineteenth century, as we will show in the next sections.

Riemann's pivotal insights on the foundation of geometry were shared by Helmholtz, who exposed the result of his personal reflections in the work "On the facts underlying geometry", written at the time he was professor of physiology at Heidelberg University. Although the publication of Riemann's work canceled the priority of his own essay, as he himself admitted (Von Helmholtz 1977, p. 40), Helmholtz found in it a valid indication that the path he was following was correct, pleased by the fact that "such an outstanding mathematician had honoured the same question with his interest". Helmholtz's starting point for his investigation were his physiological experiences, i.e., his research on spatial intuitions in the visual field. After his experimental research on color, indeed, he realized that color space could not be uniform, as we have already seen. For this reason, following Riemann, he also mentioned the system of color as an example of "continuous manifold" depending on several variables: "Precisely in physical optics, two examples were available to me of other manifolds which

[6] Although another German mathematician, Hermann Grassmann, attempted to define a pure n-dimensional geometry in his masterpiece "Ausdehnungslehre" (Grassmann 1844), which anticipated the modern theory of vector spaces, we cannot be sure that Riemann knew Grassmann's work, as Ferreirós remarked (Ferreirós 2007, p. 45). The same applies for Grassmann's paper on color, published in 1853 (Grassmann 1853). On the other hand, Riemann had strong interest in philosophical issues, as testified by his in-depth study of the work of Johann Friedrich Herbart, who was professor at Königsberg and Göttingen until his death in 1841. Riemann *Nachlass* at Göttingen University library contains manuscripts, notes, correspondence from Riemann's study of Herbart, which give a clear picture of the aspects of Herbart's philosophy that mostly influenced Riemann (see Scholz 1982). In his theory of spatial concept, precisely, Herbart used the example of two continua which fall under the concepts of tone and color, "the "line of sound" and the color triangle with blue, red, and yellow at the corners and the mixing colors in the two-dimensional continuum in between" (Scholz 1982, p. 422).

can be portrayed spatially and are variable in several respects. Namely, the colour system, which Riemann also cites, and the measuring out of the visual field by visual estimation. Both show certain fundamental differences from the metrical system of geometry, and stimulated comparison" (Von Helmholtz 1977, p. 40, original in Von Helmholtz 1868, p. 194).

An exhaustive analysis of Helmholtz's memoir can be found in the volume "Epistemological writings: the Paul Hertz/Moritz Schlick centenary edition of 1921, with notes and commentary by the editors" (Von Helmholtz 1977, pp. 39–71).[7]

Helmholtz delivered many popular lectures on the axioms of geometry from 1870 onwards. Two years after the publication of this treatise, for example, he gave a public lecture, "On the Origin and Significance of Geometrical axioms" (Von Helmholtz 1977, pp. 1–38), in which he discussed the philosophical significance of the latest inquiries concerning geometrical axioms and the possibility of working out analytically other systems of geometry with axioms different from the Euclidean ones. Here, Helmholtz showed that spatial conceptions have a non-intuitive character. If there were living beings with our logic abilities, who are living on a sphere, they would formulate a system of geometric axioms very different from those which could be worked out by the same living beings moving on a plane or on a three-dimensional Euclidean space.

Concurrently with these epistemological reflections, Helmholtz got familiar with the non-Euclidean geometries that were being developed during the second half of the nineteenth century. He could, thus, exploit the acquired knowledge of non-Euclidean metrics for developing the first line element in color space. Though Riemann's intuition on color space remained a mere conjecture, Helmholtz went further, by extending the affine geometry of colors to a metric one.

4 The Weber–Fechner's law: when psychophysics meets the geometry of color space

Besides his deep knowledge of the latest achievements in the field of geometry, Helmholtz could rely on a solid background in experimental psychology and, in particular, in the new born field of psychophysics, from which he borrowed fundamental ideas for the development of his line element in color space. Before proceeding with Helmholtz's intuitions and mathematical expression of the line element, we will give a brief description of the so-called "Weber–Fechner law".

The first well-defined experimental method to quantify relationships between any physical stimulus and the perceived response by individuals was provided by the German physicist and experimental psychologist Gustav Theodor Fechner, who presented in 1860 his "Elemente der Psychophysik" ("Elements of psychophysics"). Fechner realized that it was impossible to measure a sensation directly but what could be measured was the stimuli or stimulus differences that produced equal differences between sensations. He considered then the intensity of sensation as a function of stimulus strength. By varying, in fact, the stimulus strength, i.e., the independent variable, he

7 For an extended account of Helmholtz's approach to geometry see Lenoir, Timothy "Operationalizing Kant: Manifolds, Models, and Mathematics in Helmholtz's Theories of Perception", in Friedman and Nordmann (2006), pp. 141–210.

was able to obtain the relative value of the intensity of the sensation, i.e., the dependent variable. In this way, for the first time, a defined quantitative relation between physical stimuli and sensations was available. Fechner's work was heavily influenced by his mentor and collaborator Ernst Heinrich Weber, a German physiologist and anatomist. Weber conducted experiments asking subjects to compare the intensities of sensations caused by two physical stimuli. One stimulus was kept fixed, while some parameter of the second was systematically varied. In this way, he could find out the smallest amount of physical difference between stimuli that produced a *just noticeable difference* between sensations.

The expression, known as Weber–Fechner law, can be presented in the following form:

$$\mathrm{d}p = k \times \frac{\mathrm{d}S}{S} \tag{1}$$

where dp denotes the perceived difference of the sensation intensity, dS indicates the intensity variation of the stimulus, S refers to the intensity of the physical stimulus, and k denotes a constant.

By integration, the following expression for the sensation intensity p can be obtained:

$$p = kln\left(\frac{S}{S_0}\right) \tag{2}$$

where S_0 stands for the minimum value of the perceived stimulus.

Helmholtz's idea was that of extending this one-dimensional law to the more complicated case of color sensitivity, i.e., to a complex of more than one dimension. Color space is, indeed, bi-dimensional, in the case of dichromatic color vision, and three-dimensional, in the more general case of normal color vision. His intuition is of paramount importance because marked the beginning of a fertile interchange between methodologies of different disciplines aimed to increase scientific knowledge. It was indeed the first attempt to apply methods of physics of perception to investigate the geometry of color space.

5 Toward Helmholtz's line element

Helmholtz's definition of the line element and related considerations are reported in three papers published between 1891 and 1892, which collect the fruits of his lifelong research on the subject. The entire mathematical treatment contained in these masterworks cannot be adequately covered in the available space. Therefore, we will briefly describe, in this section and the next, Helmholtz's line of thought and key results.

The first paper of the series, titled "Versuch einer erweiterten Anwendung des Fechnerschen Gesetzes im Farbensystem" ("An attempt to extend the application of Fechner's law in the color system"), was published in 1891 and contains Helmholtz's first endeavor to extend the Weber–Fechner law to the two-dimensional color space of dichromats. The all treatment was based on the experimental results obtained by his pupil Eugen Brodhun (Brodhun 1887). Brodhun, indeed, was "green-blind", i.e., affected by deuteranopy, and starting from 1887 devoted himself to the study of color perception at the *Physikalische Institut* in Berlin where Helmholtz was appointed Director.

Starting point for Helmholtz's study was a refined version of the one-dimensional Weber–Fechner law, which he had previously introduced in the first edition of his monumental work "Handbuch der physiologischen Optik" (Von Helmholtz 1867), which was a systematic anthology of all knowledge available at that time on color vision. For an increment dx of color x, the *Empfindungsunterschied* (difference in sensation) can be written as:

$$\mathrm{d}E = \frac{k \times \mathrm{d}x}{(a + x) \times F} \tag{3}$$

where F is a function of brightness; a indicates a constant that should assume a small value for violet and must be considered only in the case of low intensities, whereas it could be neglected for higher intensity. The constant indicated by k corresponds to the Weber–Fechner law as appeared in its original formulation.

The first application of the line element in a two-dimensional space can be found in paragraph *Ähnlichste Farben* (*Most similar colors*), where Helmholtz determined pairs of neighboring colors of the greatest similarity. The two colors were expressed in terms of what he called *Quanta zweier beliebig gewählter Elementarfarben*, i.e., certain defined amounts of two arbitrarily chosen elementary colors, indicated by x and y. For the sake of clarity, it should be pointed out that in the second edition of the "Handbuch", Helmholtz used alternatively the terms "Quanta" and "Farbenwerthe" (color values) of the elementary colors referring to the components of the two most similar colors (Von Helmholtz 1885–1896, pp. 448–456). The oscillation between different denominations reflects his constant research of a proper terminology to describe the new field of "advance color theory", which was systematized only in the first half of the twentieth century.[8]

He envisaged a comparison of one color having fixed *color tone* and fixed *intensity* with another color having slightly different color tone and intensity. Components x and y of both colors were, thus, expressed only in terms of color tone and intensity,

[8] Helmholtz extended at first the term "quanta", borrowed from mathematical physics to the study of colors, probably because it recalled the decomposition of a continuum (in this specific case, the continuum of colors) into discrete elements for allowing a mathematical treatment of the phenomena. In the second half of the nineteenth century, the German terms "Größe" and "Quantum" were essentially defined as synonym in mathematical dictionaries and, in this sense, they were employed by research mathematicians. We refer to Hoffmann's dictionary (1858/1867) which offer a precise definition of the German term "Größe": "Größe wird vielfaches erklärt als Dasjenige, welches sich vermehren und vermindern läßt. Größe ist also ein Vielfaches, das aus mehreren Theilen zusammengesetzt ist. Man unterscheidet Größe als Quantum und Größe als Quantitas. Quantum ist die Größe an sich, Quantitas ist die Größe des Quantums, die Menge der Theile, aus denen das Quantum zusammengesetzt ist" (Hoffmann 1858/1867, vol. 3, p. 148; The term "Größe" can be explained in many ways as something that can be increased and decreased. "Größe", then, is a manifold composed of several parts. A distinction is made between "Größe" as "Quantum" and "Größe" as "Quantitas". "Quantum" is the Größe itself, "Quantitas" is the "Größe" of the "Quantum", the quantity of the parts of which the "Quantum" is composed).

denoted by p and r, respectively. To limit the mathematical treatment to a complex of two dimensions, saturation was not considered.

For what concerns components x and y of the first color, Helmholtz wrote the relations:

$$x = r \times p \tag{4}$$

$$y = r \times (1 - p) \tag{5}$$

and relating to the components of the second color, the following equations were laid down:

$$x = (r + \mathrm{d}r) \times (p + \mathrm{d}p) \tag{6}$$

$$y = (r + \mathrm{d}r) \times (1 - p - \mathrm{d}p) \tag{7}$$

As we will see, the terms dr and dp, which appear in Eqs. (6) and (7), will be used to calculate the minimal perceptual difference between the fixed color and the variable one.

According to the Weber–Fechner law in its simplest appearance (thus, neglecting constants F and a), the differences in sensation were expressed as:

$$\mathrm{d}E_1 = k \times \frac{\mathrm{d}x}{x} \tag{8}$$

$$\mathrm{d}E_2 = k \times \frac{\mathrm{d}y}{y} \tag{9}$$

Equations (8) and (9), reproduced here using Helmholtz's original notation, represent the difference in sensation relating separately to coordinates x and y.

And, therefore, the *Empfindungsunterschied* assumed the form:

$$dE = k \cdot \sqrt{\left(\frac{dx}{x}\right)^2 + \left(\frac{dy}{y}\right)^2} \tag{10}$$

Helmholtz calculated dE^2 by inserting Eqs. (4) and (5) into Eq. (10), and obtained the following expressions:

$$\mathrm{d}E^2 = k^2\left[\frac{(p\mathrm{d}r + r \times \mathrm{d}p)^2}{p^2r^2} + \frac{[(1-p)\mathrm{d}r - r \times \mathrm{d}p)]^2}{(1-p)^2r^2}\right] \tag{11}$$

$$= k^2\left[\left(\frac{\mathrm{d}r}{r} + \frac{\mathrm{d}p}{p}\right)^2 + \left(\frac{\mathrm{d}r}{r} - \frac{\mathrm{d}p}{1-p}\right)^2\right] \tag{12}$$

Keeping color tone dp unchanged (i.e., considering fixed value of dp which appears in Eqs. (6) and (7)), dE^2 can be varied by varying dr, i.e., the intensity of the second color.

At this stage of the discussion, Helmholtz introduced one of the key points of the paper. To follow the thread in his reasoning, we report below some excerpts from the original work:

"Ein Minimum wird dE^2, d. h. die beiden Farbenmischungen werden am ähnlichsten, bei Änderung von *dr* allein, wenn (dE^2 will be minimum, i.e., both color mixtures will be most similar, by varying dr alone, if)

$$2\frac{\mathrm{d}r}{r} + \frac{\mathrm{d}p}{p} - \frac{\mathrm{d}p}{1-p} = 0 \tag{13}$$

(obtained by differentiating dE^2 of Eq. (12) with respect to dr)
Oder
(or)

$$\log\left\{r^2 \times p \times (1-p)\right\} = \log(x \times y) = \mathrm{Const} \tag{14}$$

Setzen wir diesen Werth von dr in den Werth von dE^2, und bezeichnen wir dieses Minimum mit dE_0^2, so erhalten wir:

(By substituting the obtained value of dr into the value of dE^2 and denoting this minimum by dE_0^2, we obtain:)

$$\mathrm{d}E_0^2 = \frac{k^2}{2}\left[\frac{\mathrm{d}p}{1-p} + \frac{\mathrm{d}p}{p}\right]^2 \tag{15}$$

$$\mathrm{d}E_0^2 = \frac{k^2}{2}\left[\mathrm{d}\log\left(\frac{x}{y}\right)\right]^2 \tag{16}$$

[…]

Wenn wir in einem rechtwinkeligen Coordinatensystem die Quanta der Farbe x als Ordinaten und die Quanta von y als Abscissen auftragen, so stellt die Gleichung [14], eine Curve dar, in denen die Farben kleinsten Unterschiedes neben einander liegen. Diese Curve ist eine gleichseitige Hyperbel, deren Asymptoten in der Entfernung sich den Coordinataxen anschließen" (Von Helmholtz 1891, pp. 23–24).

(If we plot the quanta of the color x as an ordinate and the quanta of y as an abscissa in a right-angled coordinate system, equation [14] (according to our numbering) represents a curve in which the colors of "the smallest difference" lie next to each other. This curve is an equilateral hyperbola whose asymptotes tend to the coordinate axes).

Colors of the greatest similarity, therefore, are located along a curve, instead of a straight line; this was a valuable clue for the non-Euclidean interpretation of the geometry of color space. In the second edition of the "Handbuch", Helmholtz extended the mathematical treatment to three dimensions (Von Helmholtz 1885–1896, pp. 448–456).

6 Helmholtz's line element definition

The pivotal results obtained for bi-dimensional color space opened the door to a more general investigation on three-dimensional color space. The outcomes of this examination are contained in two papers published in 1892 with the titles "Versuch, das psychophysische Gesetz auf die Farbenunterschiede trichromatischer Augen anzuwenden" ("Attempt to apply the psychophysical law to color differences of trichromatic eyes") and "Kürzeste Linien im Farbensystem" ("The shortest lines in the color system") (see Von Helmholtz 1892a, b). In these memoirs, Helmholtz presented his final definition of the line element in trichromatic space, which we report below:

$$dE^2 = \left(\frac{dx}{x+a}\right)^2 + \left(\frac{dy}{y+b}\right)^2 + \left(\frac{dz}{z+c}\right)^2 \tag{17}$$

$$x = a_1 \times R + b_1 \times G + c_1 \times V \tag{18}$$

$$y = a_2 \times R + b_2 \times G + c_2 \times V \tag{19}$$

$$z = a_3 \times R + b_3 \times G + c_3 \times V \tag{20}$$

Here, x, y, and z are the *physiologische Urfarben* (physiological primary colors) and a, b, and c refer to the self-light constants. The physiological primary colors were expressed as linear homogeneous equations of R (red), G (green), V (violet), the *Elementarfarben*, elementary colors (see Fig. 5), found by his assistants Arthur

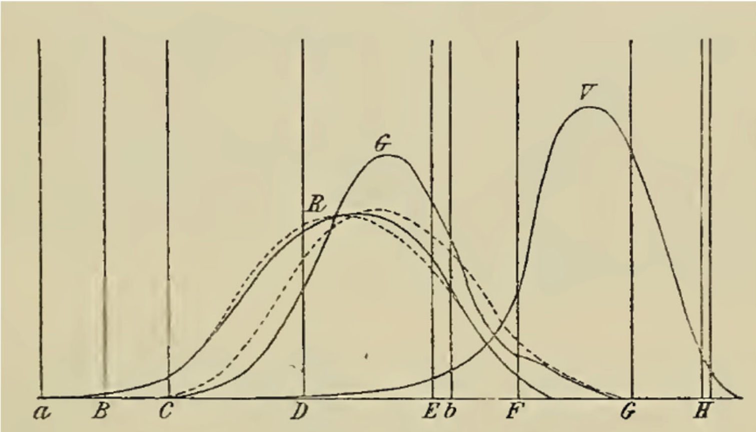

Fig. 5 König' and Dieterici's elementary curves for people with trichromatic vision (normal eye: R, G, and V; eye affected by deficiencies: dotted curves, V). In the abscissa are shown Fraunhofer lines as reference (König and Dieterici 1886, p. 435)

König[9] and Conrad Dieterici,[10] who provided the first good empirical estimates of the chromaticity of Young's fundamentals in 1886.

The choice of the coefficients a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, and c_3, according to Young's theory, was limited only to the fact that the values of R, G, V belonging to the spectral colors must not yield negative values of x, y, and z. This would never be the case, as Helmholtz underlined, if all the coefficients a, b, c had positive values. Once calculated the value of all the coefficients, Helmholtz could introduce the new fundamentals, x, y, and z into the line element formula.

He named the smallest perceptual difference $\mathrm{d}E$ "kürzeste Linie" (shortest line), i.e., the expression describing infinitesimal distances in a three-dimensional non-Euclidean color space. Lines of smallest color difference were taken as geodesic; in other words, as shortest lines between colors as points in color space. Below, we add some brief remarks on Helmholtz's treatment of the line element.

As shown in Eq. (17), Helmholtz included the self-light constants a, b, and c in his formulation. In fact, they played no special role in the applications he studied, as Stiles first pointed out (Stiles 1972). Moreover, as the illumination level grows, the difference between the constant self-light elements becomes increasingly unimportant. At such high levels, the Weber fraction will have the same value for all colors, according to König' and Brodhun's findings:

$$\frac{\Delta x}{x} = \frac{\Delta y}{y} = \frac{\Delta z}{z} \tag{21}$$

König and Brodhun investigated, indeed, the validity of the Weber–Fechner law for light illumination for different colors (i.e., wavelengths, see Fig. 6). They calculated the visual differential threshold with several intensities, from the slightest barely perceptual illuminations up to the intense brightness. The obtained results were of crucial importance and were exposed in the paper titled "Experimentelle Untersuchungen über die psychophysische Fundamentalformel in Bezug auf den Gesichtssinn" ("Experimental investigation on the psycho-physical fundamental formula in relation to the sense of vision"), first published in 1888 for the Königlich Preussischen Akademie der Wissenschaften.

Analyzing the graph, reproduced in Fig. 5, König and Brodhun could state that for a specific intensity interval, 2000–20,000, the curve runs horizontally, i.e., the ratio $\frac{\delta r}{r}$, the Weber–Fechner law, has a constant value. Here r denotes the output intensity and δr the output intensity increment. This is probably the reason why Helmholtz limited his applications to this region. An increase then occurs on both sides of this interval.

[9] Arthur Peter König was Helmholtz's assistant and co-worker in Berlin. Born in Krefeld in 1856, after studying in Bonn and Heidelberg, König moved to Berlin in the fall of 1879, where he remained for the rest of his life. Here, he met Professor Hermann von Helmholtz, who suggested a doctoral dissertation to him. In 1882 König obtained his Ph.D. degree and in the same year became Helmholtz's assistant. He was named first as lecturer and in 1889 Director of the physical division of the Physico-technical Institute, with the special task of teaching physiological optics.

[10] Conrad Dieterici studied physics at the University of Berlin and received his doctorate there in 1882. From 1885 to 1890, he was assistant at the Physical Institute of the University of Berlin and librarian of the Physikalische Gesellschaft. In 1887, Dieterici became *Privatdozent* before receiving, in 1890, a chair at the University of Breslau.

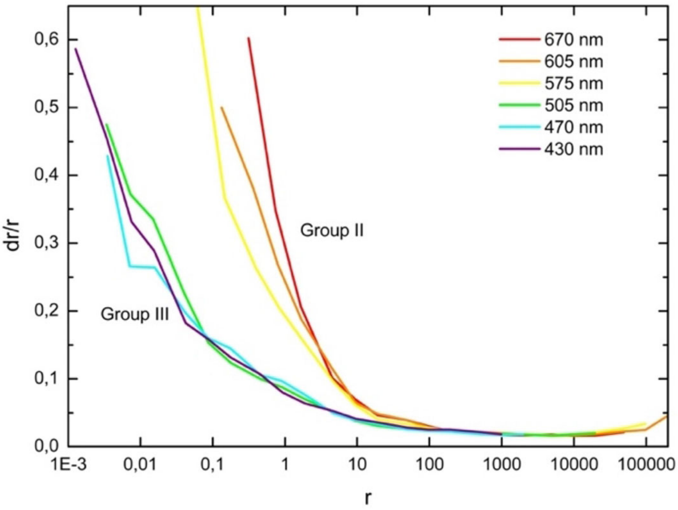

Fig. 6 König's (normal trichromat) curves illustrating Weber fraction (y-axis) as a function of light intensity (x-axis, on a logarithmic scale), re-plotted by the authors using König's original data (König, and Brodhun 1888). Branch II applies to wavelengths 670, 605, and 575 nm, while branch III applies to wavelengths 505, 470, and 430 nm. r denotes the output intensity and δr the output intensity increment. The unit of intensity was the illumination seen by an eye looking through a diaphragm, whose opening measured 1 m^2 at a surface coated with magnesium oxide. The surface was located 1 m far from the eye and reflected the light from a platinum surface, whose area measured 0.1 cm^2 standing parallel to it. The intensity of the illumination was varied for each wavelength upwards from 1 up to 100,000 and downwards up to 0.02 (for certain wavelengths)

The curves increase steadily as the intensity decreases, and, in particular, a more rapid increase occurs for Group II, i.e., for yellow, orange, and red.

To Helmholtz's line element treatment was dedicated prominent space in the second revised edition of his "Handbuch" (1885–1896), where also the pivotal findings of his collaborators were delineated. However, as underlined first by Schrödinger and later by Stiles (Schrödinger 1920; Stiles 1972), the entire mathematical derivation was omitted by his successors from the third edition (1909–1911). This could be probably due to the fact that Helmholtz's pupils had recognized some critical aspects related to Helmholtz's line element. On the other hand, this omission probably contributed to make the outstanding work of Helmholtz fall into oblivion for at least a quarter century, until Schrödinger rediscovered its precious value in the 1920s proposing a modification of Helmholtz's line element. In his memoir on higher color metrics, Schrödinger accurately pointed out some shortcomings in Helmholtz's proposal (Schrödinger 1920). First, he noticed that the new primaries chosen by Helmholtz had to be computed ad hoc to make the line element for just distinguishable color pairs approximately constant. Furthermore, in Helmholtz's treatment, brightness was not an additive property of color and this contradicted the latest results on photometry. The brightness function ($V(\lambda)$ in modern terms) obtained from his model was also in contradiction to experience. Although Helmholtz's system was Riemannian, the color space turned out to be isometric to Euclidean space. With the purpose of solving the above-mentioned deficiencies, Schrödinger built his own line element, entirely based on Helmholtz's first proposal. Since Schrödinger's theoretical work, several attempts to define line elements in color space have been made. The first significant implementation of line element theory was provided by the American physicist and color scientist David MacAdam, (MacAdam 1942, see Supplementary Material, section S3).

An exhaustive collection of the main achievements in the field of color metrics up to 1971 can be found in the Proceeding of the Memorial Symposium on Color Metrics entitled to Helmholtz in honor of the 150th anniversary of his birth (Vos et al. 1972). In that occasion, a whole range of line element' and color difference's formula passed the review. From the examples mentioned above, it emerges clearly the profound influence of the German polymath on the field of color metrics and color perception, influence which persists unaltered to the present time.

7 Conclusion

The space of color is a fascinating space. It is a real vector space but the resulting Euclidean distance does not correspond to human perception of difference between colors. Riemann and Helmholtz were the first to assume, in the second half of the nineteenth century, that the perceptual color space is a n-dimensional space not necessarily Euclidean. Contrary to Riemann, Helmholtz went further proposing the first metrically significant model of color space basing on the Weber–Fechner law. In the science of color vision, indeed, the quantitative dimension is intimately connected with the subjective one. Helmholtz's knowledge in the fields of both differential geometry and psychophysics allowed him to provide a fruitful link between the two disciplines, forming the basis for all subsequent studies on the geometry of color spaces up to the present time.

Systematic research on color space is currently a cutting-edge topic in scientific research as the definition of an appropriate metric for color spaces has become increasingly important also for technological application, relating, in particular, to color synthesis for computer graphics, color printing processes, and medical research. Therefore, refined models and systems need to be evolved to respond to the continuous demand for a practical standard for measuring perceptual color differences accurately, allowing consequently a more precise and comprehensive description of human color perception.

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Declarations

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References

Brodhun, Eugen. 1887. *Beiträge zur Farbenlehre, Inaugural-Dissertation*. Berlin: Niethe. Staatsbibliothek zu Berlin.

Cahan, David, ed. 1993. *Hermann von Helmholtz and the Foundations of Nineteenth-Century Science*. Berkeley, Los Angeles, London: University of California Press.

Clerk Maxwell, James. 1855. Experiments on Colour, as Perceived by the Eye, with Remarks on Colour-Blindness. *Transactions of the Royal Society of Edinburgh* 21: 275–298. https://doi.org/10.1017/S0080456800032117.

Fechner, Gustav Theodor. 1860. *Elemente der Psychophysik: Zweiter Theil*. Leipzig: Breitkopf und Härtel.

Ferreirós, José. 2007. *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics, Second Revised Edition*. Basel: Birkhäuser Verlag AG.

Friedman, Michael, and Alfred Nordmann. 2006. *The Kantian Legacy in Nineteenth-Century Science*. Cambridge, Mass., and London: MIT Press

Grassmann, Hermann Günther. 1844. *Die Lineale Ausdehnungslehre ein neuer Zweig der Mathematik Dargestellt und durch Anwendungen auf die übrigen Zweige der Mathematik, wie auch auf die Statik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert*. Leipzig: Wigand.

Grassmann, Hermann Günther. 1853. Zur Theorie der Farbenmischung. *Annalen der Physik* 165 (5): 69–84. https://doi.org/10.1002/andp.18531650505.

Hoffmann, Ludwig. 1858–1867. *Mathematisches Wörterbuch*. Berlin: Wiegandt und Hempel

Jost, Jürgen., ed. 2016. *Bernhard Riemann: On the Hypotheses Which Lie at the Bases of Geometry*. Basel: Birkhäuser.

König, Arthur, and Conrad Dieterici. 1886. The Modern Development of Thomas Young's Theory of Colour-Vision. *Report of the British Association for the Advancement of Science* 56: 431–439.

König, Arthur, and Brodhun, Eugen. 1888. Experimentelle Untersuchungen über die psychophysische Fundamentalformel in Bezug auf den Gesichtssinn. *Königlich Preussischen Akademie der Wissenschaften zu Berlin, zweiter Halbband, Juni bis December*

König, Arthur, and Dieterici, Conrad. 1892. *Die Grundempfindungen in normalen und anomalen Farbensystemen und ihre Intensitätsverteilung im Spektrum*. Hamburg und Leipzig: Leopold Voss

Krüger, Lorenz, ed. 1994. *Universalgenie Helmholtz. Rückblick nach 100 Jahren*. Berlin: De Gruyter Verlag.

MacAdam, David Lewis. 1942. Visual Sensitivities to Color Differences in Daylight. *Journal of the Optical Society of America* 32 (5): 247–274. https://doi.org/10.1364/JOSA.32.000247.

Newton, I. 1979. *Optiks or a Treatise of the Reflections, Refractions, Inflections and Colours of Light*. New York: Dover Publications.

Riemann, Bernhard. 1867. Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* 13: 133–150. https://doi.org/10.1002/phbl.19540100702.

Scholz, Erhard. 1982. Herbart's Influence on Bernhard Riemann. *Historia Mathematica* 9: 413–440.

Schrödinger, Erwin. 1920. Grundlinien einer Theorie der Farbenmetrik im Tagessehen. *Annalen der Physik* 4(63): 397–426, https://doi.org/10.1002/andp.19203682102; 427–456, https://doi.org/10.1002/andp.19203682103; 481–520, https://doi.org/10.1002/andp.19203682202

Shapiro, Alan E. 1994. Artists' Colors and Newton's Colors. *Isis* 85 (4): 600–630.

Stiles, Walter Stanley. 1972. The Line Element in Colour Theory: A Historical Review. *Color Metrics* 1–25

Tazzioli, Rossana. 2003. Towards a History of the Geometric Foundations of Mathematics. *Revue de Synthèse* 124: 11–41. https://doi.org/10.1007/BF02963399.

Von Helmholtz, Hermann. 1852. Über die Theorie der zusammengesetzten Farben. *Annalen der Physik* 163 (9): 45–66.

Von Helmholtz, Hermann. 1855. Ueber die Zusammensetzung von Spectralfarben. *Annalen der physik* 170 (1): 1–28. https://doi.org/10.1002/andp.18551700102.

Von Helmholtz, Hermann. 1867. *Handbuch der physiologischen Optik*. Leipzig: Leopold Voss.

Von Helmholtz, Hermann. 1868. Ueber die Thatsachen, die der Geometrie zum Grunde liegen. *Nachrichten von Der Königlichen Gesellschaft der wissenschaften zu Göttingen* 15 (9): 193–221.

Von Helmholtz, Hermann. 1885–1896. *Handbuch der physiologischen Optik. 2. umgearbeitete Auflage*. Hamburg und Leipzig: Leopold Voss

Von Helmholtz, Hermann. 1891. Versuch einer erweiterten Anwendung des Fechnerschen Gesetzes im Farbensystem. *Zeitschrift für psychologie und Physiologie der sinnesorgane* 2: 1–30.

Von Helmholtz, Hermann. 1892a. Versuch, das psychophysische Gesetz auf die Farbenunterschiede trichromatischer Augen anzuwenden. *Zeitschrift für Psychologie und Physiologie der Sinnesorgane* 3: 1–20.

Von Helmholtz, Hermann. 1892b. Kürzeste Linien im Farbensystem: Auszug aus einer Abhandlung gleichen Titels in Sitzgsber. der Akademie zu Berlin. 17. Dezember 1891. *Zeitschrift für psychologie und physiologie der sinnesorgane* 3: 108–122.

Von Helmholtz, Hermann. 1909–1911. *Handbuch der physiologischen Optik, Dritte Auflage*. Hamburg und Leipzig: Leopold Voss

Von Helmholtz, Hermann. 1977. *Epistemological Writings: The Paul Hertz/Moritz Schlick Centenary Edition of 1921, with Notes and Commentary by the Editors*, 37. Dordrecht: D. Reidel Publishing Company

Von Helmholtz, Hermann. 2005. *Treatise on Physiological Optics*, vol. 2, ed. James P. C. Southall. Mineola: Dover Publications, Inc

Vos, Johannes J. 1979. Line Elements and Physiological Models of Color Vision. *Color Research and Application* 4: 208–216. https://doi.org/10.1002/col.5080040406.

Vos, Johannes J., L. F. C. Friele, and P. L. Walraven. 1972. Color Metrics: Proceedings of the Helmholtz Memorial Symposium on Color Metrics, September 1–3, Driebergen 1971. Holland, Soesterberg: AIC.