Felix Klein's projective representations of the groups S_6 and A_7

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Abstract

This paper addresses an article by Felix Klein of 1886, in which he generalized his theory of polynomial equations of degree 5—comprehensively discussed in his *Lectures on the Icosahedron* two years earlier—to equations of degree 6 and 7. To do so, Klein used results previously established in *line geometry*. I review Klein's 1886 article, its diverse mathematical background, and its place within the broader history of mathematics. I argue that the program advanced by this article, although historically overlooked due to its eventual failure, offers a valuable insight into a time of crucial evolution of the subject.

For geometry not only illustrates and facilitates, it also has the privilege of invention in these investigations. Felix Klein (1879b, p. 253)

1 Introduction

The early work of Felix Klein is today known for two main achievements: First, his *Erlangen Program*, whose credo

Given a manifoldness and a group of transformations of the same; to develop the theory of invariants relating to that group. (Klein [1872] 1893, p. 219)[1]

is well known to historians and mathematicians alike. And second, his *Lectures on the Icosaehdron* (Klein 2019b) in which half a century of mathematical research on the general polynomial equation of degree 5—the general *quintic*—is summarized under

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[1] Here and below, italics are always Klein's. Translations into English are mine, except where (as in this case) a published English translation exists. Years in square brackets refer to the original (usually German) publication.

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the geometrical banner of the *Icosahedron*. Two years later, Klein published a today forgotten paper entitled "On the theory of general equations of degree six and seven" ("Zur Theorie der allgemeinen Gleichungen sechsten und siebenten Grades", 1886) in which he extended his geometrical interpretation of the general quintic to equations of degree 6 and 7:

> The theory of the equations of the fifth degree, which I brought into coherent presentation in my "*Lectures on the Icosahedron etc.*" (Teubner 1884), allows not only, as I have indicated at various points, a transmission to equations of degree four, but also an extension to equations of degree six and seven. It is the purpose of the following lines to define the main features of this extension. Its aim subsumes under the general ideas which I have set out in Klein 1879b for the solution of arbitrary algebraic equations. It differs from them, however, by the concrete form of the geometric-algebraic process to be used, which uses individual moments present only at $n = 6$ and $n = 7$. (Klein 1886, 499–500)

These "general ideas" can be summarized as the principle to find for any class of polynomial equations some canonical *geometrical equations*[2] to which it can be reduced, and subsequently to solve all polynomial equations of that class by the solution of the geometrical one. The interesting cases are those polynomials that cannot be solved directly by algebraic means. In the language of Galois theory, these are the equations whose associated Galois group (over some specified domain of rationality) is unsolvable. For those equations, also the corresponding canonical geometrical equation cannot be solved algebraically, but once a solution is found by other means (borrowing from analysis, differential equations, …), the solution of the original equation easily follows. This idea extends the theory of algebraic theory beyond the "limits" of Galois theory; and Paul Gordan, Klein's closest collaborator in the early stage of research on this topic, therefore called this enterprise jocularly the *Hypergalois Theory*.[3] Emphasizing the extensive social dimension of Klein's theory—Klein entrusted many of his students with particular problems arising from his "general idea", and promoted his theory on many occasions—the term *Hypergalois Program* might be appropriate to describe the broader implications.[4] The principles of the Hypergalois Program were most explicitly stated in the above-mentioned (Klein 1879b), and repeated in Klein (2019a). However, Klein's motivation for a geometrical treatment of algebraic equa-

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[2] The term *geometrical equation* to describe this historical interest was to my knowledge first used in Hölder(1899, p. 518). Independently from Hölder, the term was used in Lê (2015) to describe the same historical phenomenon.

[3] The term appeared in printed form in Klein's *Collected Works* (Klein 1922, p. 261) and in *The Development of Mathematics in the 19th Century* (Klein 1926, p. 90), and was rediscovered in (Gray, [1986] 2008, p. 77). Among the unpublished material, the term appeared in Klein's manuscript for his algebra lecture in Summer 1886 in Göttingen: "Error [to believe], as if the whole question were exhausted with the Galois scheme. The 'Hypergalois' Theory must find on least one month's time." ([UBG], Cod. Ms. F. Klein 14J). Digitally available at https://resolver.sub.uni-goettingen.de/purl?DE-611-HS-3206860.

[4] The term *Program* is chosen not only for the analogy with the *Erlangen Program* (and many other mathematical programs) that are staged around a common methodological, sometimes even metaphysical, basis, but also to indicate the possibility of a socio-historical analysis in terms of a *scientific research program* (Lakatos 1978), or more specifically, a *mathematical research program* (Corfield 2012).

tions was already present at a much earlier stage (Klein 1871), during which even the core idea of Klein's 1886 article was already developed.

In this sense, Klein's 1886 article can be said to have three historical roots: Klein (1871) sets the most important *mathematical* background; Klein's successful treatment of the general quintic (which was mathematically completed already in Klein (1877), but only popularized in the *Lectures on the Icosahedron*) provides the motivation for Klein's renewed interested in the equations of degree 6 and 7; and Klein (1879b) sets the programmatic framework of this and subsequent works. The 1886 article itself was not very influential in mathematical terms, mainly because a "better" geometrical interpretation for equations of degree 6 was unexpectedly found soon after. At the same time, its importance for the Hypergalois Program cannot be overestimated in terms of the renewal of interest in equations of higher degree. The Hypergalois Program eventually failed, but the research questions it opened paved the way for a number of discoveries and conceptual novelties that were indispensable for the transformation that mathematics (especially group theory) underwent at the turn of the 20th century.

In the present paper, my aim is to take Klein's article of 1886 as a testimonial and driving force of these profound transformations. This is achieved in two different ways: in Sect. 2, I locate the paper within the broader historical landscape of the Hypergalois Program: I first discuss the three historical roots mentioned above in Sects. 2.1, 2.2, and 2.3. Then, I outline the position of the paper within the theory of the equation of degree 6 (Sect. 2.4), and finally present some more general implications (Sect. 2.5). Section 3 is devoted to a more detailed reconstruction of the mathematical content of the paper itself, and follows more or less its original setup: From the presentation of Klein's general idea (Sect. 3.1), via the construction of the representations of A_7 and S_6 (Sects. 3.2, 3.3), to some additional elaborations on accessory irrationalities (Sect. 3.4), covariants (Sect. 3.5), and possible generalizations to higher degrees (Sect. 3.6). A summary (Sect. 3.7) compares the results to Klein's earlier achievements with respect to the quintic equation. As literature especially on the topic of line geometry is scarce, a special attention is given to it throughout the section. A short conclusion follows in Sect. 4. Sections 2 and 3 approach the same topic from two different angles, and can be read more or less independently from each other. However, I believe that our understanding of Klein's mathematical thinking can benefit if both the "macro"/historical and the "micro"/mathematical content of his publication are understood side by side.

2 The Hypergalois Program

2.1 The beginnings

The 19th century was the century of classical algebraic geometry, with their practitioners interested in certain geometrical configurations, and in the corresponding equations whose solutions would yield these configurations. Hölder (1899, p. 518), and more recently, François Lê (2015) calls them the *geometrical equations*:

> For a given geometrical situation, one may say that the corresponding geometrical equation is the algebraic equation ruling the configuration. For instance,

for the nine inflection points of a cubic curve, we have the nine inflection points equation, which is the algebraic equation of degree 9 having the abscissas of the inflection points for its roots. (Lê 2015, pp. 317–318)

It was, however, only in 1871 that Felix Klein first uttered the idea to interpret the general polynomial equation geometrically:

[Klein] went on and announced his general and fundamental principle: to conceive every algebraic equation as a geometrical equation, embodying the roots of an equation in geometrical objects and replacing the substitutions of the roots by transformations of the space. [... T]his articulation between algebra and geometry revealed two main leitmotivs of Klein: to bring geometry to the fore because of the intuition it allowed and to stress the importance of transformation groups. (ibid., 336)

Particularly, Klein perceived the solutions of a polynomial equation of degree n as n points in $(n-2)$-dimensional projective space, $\mathbb{P}^{n-2}$, which allowed him to consider the permutations of the roots as transformations of space.[5] In $\mathbb{P}^{n-2}$, there exists a unique linear transformation of space for every permutation of n given points—thus the choice of dimension. In a modern formulation, the depiction of a finite group as a group of transformation of space is called a *finite group representation*, and the particular representation in $\mathbb{P}^{n-2}$ is nothing but the projectivization of the so-called *standard representation*

$$S_n \rightarrow \mathrm{PGL}_{n-1}(\mathbb{C}).$$

The group S_n was defined above as the permutation group of the solutions of a polynomial equation, i.e., as the Galois group of the equation. Klein thus achieved an interpretation of the Galois group of a degree n polynomial as a group of linear transformations of projective space of dimension $(n-2)$. This allowed him to bring together two notions that are considered typical representatives of nineteenth century mathematics: *resolvents* in the theory of algebraic equations on the one hand, and *covariants* as a central concept of invariant theory and geometry on the other. Klein interpreted the solutions x_i of an algebraic equation of degree n as points in space $\mathbb{P}^{n-2}$, and the Galois group of the equation as the corresponding group of space transformations which permute these points. Applying the transformations S_n to any given point $y_0 \in \mathbb{P}^{n-2}$ yields in general $n!$ points $y_0, y_1, \ldots, y_{n!}$. Its geometrical equation

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[5] About this terminology: Given a vector space V, we denote its projectivization by $\mathbb{P}(V)$. If $V = \mathbb{C}^{n+1}$, its projectivization $\mathbb{P}(\mathbb{C}^{n+1})$ is abbreviated by $\mathbb{P}^n$, and we talk about n-dimensional projective complex space. This diction is justified because we can think about $\mathbb{P}^n$ as an affine space $\mathbb{C}^n$ extended by additional elements at infinity. We typically write the elements of $\mathbb{P}^n$ as homogeneous coordinates $z = (z_1 : z_2 : \ldots : z_n)$ which are only defined up to a scalar. For example, the projectivization of $V = \mathbb{C}^2$ is the *complex projective line* $\mathbb{P}^1$ which can be identified with the usual complex numbers extended by a point at infinity:

$$\mathbb{P}^1 = \{(z_1 : z_2) \mid z_{1,2} \in \mathbb{C}, (z_1, z_2) \neq (0,0)\} = \{(z_1/z_2 : 1) \mid z_{1,2} \in \mathbb{C}, z_2 \neq 0\} \cup \{(1:0)\} \cong \mathbb{C} \cup \{\infty\}$$

Thus, we can freely switch between homogeneous coordinates $(z_1 : z_2)$ and non-homogeneous $z = z_1/z_2$.

$$g(y) = (y - y_0)(y - y_1) \cdots (y - y_{n!}) = 0$$

is invariant with respect to the configuration of roots x_i, meaning that it does not change its value when the x_i are permuted. This makes $g(y)$ a covariant of $f(x) = 0$, because any transformation of x (meaning a symmetric transformation in $x_0, \ldots, x_n$) leads to a "compatible" transformation of y, and one can easily calculate the effect of the one to the other. Algebraically speaking, $g(y)$ is a Galois resolvent of $f(x)$ (see Footnote 6). It now might happen that we chose y_0 in a way that some of the y_i coincide. Then, the polynomial $g(y)$ becomes the power of a polynomial of lower degree. It then corresponds to a special resolvent of $f(x) = 0$, namely one which can be used to algebraically simplify the initial equation $f(x) = 0$.[6] Therefore, we can just consider the resolvents of an algebraic equation $f(x) = 0$ as the covariants of $f(x) = 0$ when viewed as a geometrical equation. For the cases $n = 3, 4$, this treatment yields an intuitive geometrical interpretation of the resolvents that were already known for centuries. For the case $n = 6$, a further simplification can be made by representing the roots x_i not as points, but as the so-called *linear line complexes*. This line-geometrical account was brought to perfection in Klein (1886), as is discussed in Sect. 3, especially Sect. 3.3.

The central idea just outlined could be interpreted as the "birth" of the Hypergalois Program, at least of those parts of it that bring together geometry, algebra, and invariant theory. However, the 1871 paper only establishes a principal connection between the two concepts of resolvents and covariants, and does not yet describe how an algebraic equation can practically be turned into a geometric configuration. It also does not give any hints as to how to proceed after such a geometric configuration would be achieved.

2.2 The icosahedron and the general quintic equation

In the years following the 1871 publication, these questions were taken up for the general quintic (Klein 1875, 1877), for which the *icosahedron* played a crucial role. In fact, if we are to believe the closing paragraph of Klein (1875), it was Klein's study of the covariants of the icosahedron that accidentally led him to spot a connection to the theory of the quintic equation, not the other way around. This approach is also manifested in the *Lectures on the Icosahedron*, whose first half considers the invariant theory of the icosahedron without any reference to the quintic equation at all. However, during the second half of the book, Klein's attitude seems to shift in the other direction. In one passage, he remarked that the "proper" approach to study the quintic equation might have to abandon the primacy of the icosahedron:

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[6] In general, a resolvent of a polynomial $f(x) = 0$ is a polynomial $g(y) = 0$ whose coefficients rationally depend on the coefficients of f, and whose solutions yield the solutions of $f(x) = 0$ by some auxiliary equation. Algebraically, if $f(x) = 0$ has Galois group G, then the auxiliary equation has Galois group $H \triangleleft G$, and the resolvent itself has Galois group isomorphic to G/H. The above Galois resolvent is thus an extreme case where H is the trivial group; it therefore does not simplify the problem algebraically. But as we saw, it is also the most general case, in that it subsumes the "special" resolvents.

I believe that we shall be enabled to develop the general theory of form-problems[7] algebraically, and in such wise that our reduction of equation of the fifth degree to the icosahedron appears as a mere corollary, and does not need to be established in a special manner. (Klein, [1884] 2019b, pp. 281–282)[8]

Klein strengthened this idea in the 1886 article:

The fact that the group in question is identical with the otherwise known group of *icosahedral substitutions* appears to be coincidental and insignificant. In fact: if the group of linear transformations of z were not already known elsewhere, all its properties could be taken from the definitions of the group given in (1), (2). (Klein 1886, p. 501)

In the following outline of Klein's theory of the icosahedron, I follow this approach and focus on the application of Klein's *icosahedron* to the solution of the quintic equation only. I also restrict myself to those considerations that find analogues in Klein's 1886 paper, and omit the others.[9]

We are given a quintic equation in which the coefficients of x^4 and of x^3 vanish

$$x^5 + \alpha x^2 + \beta x + \gamma = 0;$$

Klein called such an equation a *principal equation*, which has the advantage that its roots $x_1, \ldots, x_5$ satisfy two additional identities

$$x_0 + x_1 + x_2 + x_3 + x_4 = 0 \quad \text{and} \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$$

Let us consider an ordered list of solutions $(x_1, x_2, x_3, x_4, x_5)$ as a point in homogeneous 4-space, $(x_0 : x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^4$. In this sense, solving the equation means to find the yet "hidden" coordinates of this point.[10] The first of the two identities above says that this point lies on a hypersurface

$$\mathcal{H} = \left\{ (x_0 : x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^4 \mid \sum_{i=0}^{4} x_i = 0 \right\} \cong \mathbb{P}^3.$$

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[7] The so-called *form problem* considers the reduction of a polynomial equation to some homogeneous polynomial equation, where the polynomial is called a *form* and can be studied be methods of invariant theory.

[8] Page numbers refer to the second English edition of 1913, which are provided in square brackets in the 2019 edition.

[9] A short completion of the theory is provided in Sect. 3.7. Among the excellent reconstructions of Klein's icosahedron are the classic (Fricke 1926), the modern (Slodowy 1986), the more elementary (Shurman 1997), the lighthearted (Eschenburg and Hefendehl-Hebeker 2000; Eschenburg 2017), and the historically most complete (Tobies 2021), especially pp. 286–291. Slodowy's account is repeated in the introduction of his edition of Klein's *Lectures on the Icosahedron* (Klein 1993), and translated into English in Klein (2019b).

[10] This looks analogous to the approach in Klein (1871), but here an ordered solution is represented by only one point in $\mathbb{P}^3$, not five points.

We introduce a new coordinate system $\pi_0, \ldots, \pi_4$ with $\pi_k = \frac{1}{5}(1, \varepsilon^{4k}, \varepsilon^{3k}, \varepsilon^{2k}, \varepsilon^k)$ (and ε being a fifth root of unity), with the effect that $\mathcal{H}$ consists of exactly those points for which its first coordinate π_0 vanishes.[11] The second identity says that within $\mathcal{H}$, our point lies on a quadratic surface, which we can directly write in terms of our new coordinates π_i

$$M_2^{(2)} = \{p_1\pi_1 + p_2\pi_2 + p_3\pi_3 + p_4\pi_4 \mid p_1p_4 + p_2p_3 = 0\} \subset \mathcal{H};$$

Klein called this surface, defined by the condition $p_1p_4 + p_2p_3 = 0$ (in π-coordinates), the *principal surface* ("Hauptfläche"). Geometrically, such a surface is commonly known as a (complex) *hyperboloid*.[12] As in the case of the hyperboloid in real space (see Fig. 1), also a complex hyperboloid contains two families of lines, or *reguli* (Sg. *regulus*); let us call them $\mathcal{L}_1$ and $\mathcal{L}_2$. Two lines from the same regulus are parallel in $M_2^{(2)}$, while two lines from different reguli meet each other in a point in $M_2^{(2)}$.[13] If we parametrize $\mathcal{L}_1$ and $\mathcal{L}_2$ with one (homogeneous) parameter each, say $\lambda = (\lambda_1 : \lambda_2)$ and $\mu = (\mu_1 : \mu_2)$, we get an isomorphism that maps a point $p = (p_1 : p_2 : p_3 : p_4)$ (in π-coordinates) to the parameters of the two lines on $M_2^{(2)}$ it is incident with (see, again Fig. 1)

$$F : M_2^{(2)} \xrightarrow{\sim} \mathbb{P}^1_{(1)} \times \mathbb{P}^1_{(2)}$$
$$p \mapsto (\lambda, \mu).$$

We can choose a parametrization λ, μ, such that $(\lambda, \mu) = ((-p_1 : p_2), (p_3 : p_4))$; its inverse F^{-1}, today called a *Segre embedding*, accordingly maps (λ, μ) to $p = (\lambda_1\mu_1 : -\lambda_2\mu_1 : \lambda_1\mu_2 : \lambda_2\mu_2)$.

We now want to understand what happens under permutation of the roots: If we interpret, as above, a listing of the roots $(x_1, x_2, x_3, x_4, x_5)$ as the homogeneous coordinates of $\mathbb{P}^4$, then a permutation of the roots amounts to a permutation of coordinates. Thus, in a representation-theoretic formulation, we obtain the projectivization of the standard representation of S_5 in $\mathbb{P}^4$ (see Sect. 2.1). Since the equations $\sum_{i=0}^4 x_i = 0$ and $\sum_{i=0}^4 x_i^2 = 0$ are invariant under such permutation of coordinates, S_5 acts on $M_2^{(2)}$ as a whole. Linear transformations map lines to lines; therefore, S_5 merely permutes the lines of $M_2^{(2)}$. Specifically, one can show that even permutations $\sigma \in A_5$ permute the lines within their families, and odd permutations $\sigma \in S_5 \backslash A_5$ interchange the lines of $\mathcal{L}_1$ with those of $\mathcal{L}_2$. We consider $\mathbb{P}^1_{(1)}$ as a *Riemann sphere*; this is a sphere whose points are identified with complex numbers. This can be done by stereographically

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[11] In the underlying affine space $\mathbb{C}^5$, this is but a rotation of the coordinate system.

[12] The notation *hyperboloid* is unambiguous, because in complex (hyper-)space, all non-degenerate quadratic surfaces of equal dimension are equivalent. We see in Sects. 3.1 and 3.6 that in general, the root space of a reduced equation of degree n is a quadratic manifold of dimension $n - 3$, i.e., some $M_{n-3}^{(2)}$. Finally, let me emphasize that the notation $M_d^{(k)}$ to indicate a surface of dimension d and order k is due to Klein, and still in use today.

[13] From a purely incidence-geometric point of view, such a geometry is called a "generalized quadrangle", see Beutelspacher and Rosenbaum (2013), p. 156.

Fig. 1 Families of lines on a hyperboloid. F maps a point p (marked orange) to the coordinates (λ, μ) of the lines incident with p (marked red). The code for this graphic was partly taken from the user maphy-psd of the website *texwelt.de*, see https://texwelt. de/fragen/18796
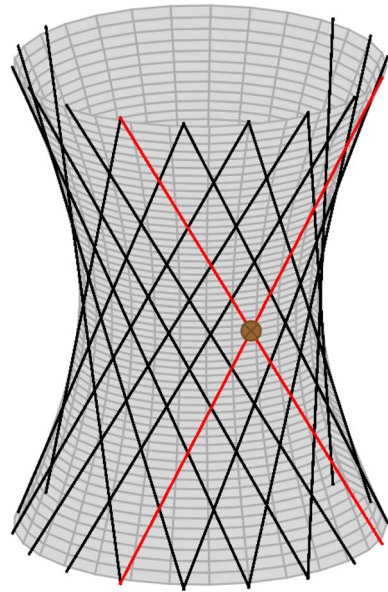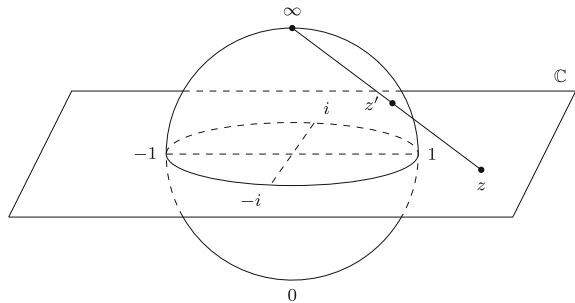

Fig. 2 The stereographical projection between the (usual) representation of complex numbers z on a plane, and its representation z' on the Riemann sphere. The code for this graphic was partly taken from the users Subhajit Paul and Torbjørn T of the website *stackexchange.com*; see https://tex.stackexchange. com/questions/538970

projecting the complex numbers (viewed as a plane) onto a sphere, where the "north pole" of the sphere is identified with ∞ (see Fig. 2).

It can then be shown that the 60 permutations of lines $\mathcal{L}_1$, when viewed as complex numbers, and thus as points of the Riemann sphere, are realized by rotations of the sphere. As rotations of the Riemann sphere are special kinds of projectivities of $\mathbb{P}^1$ (which are generally called *Möbius transformations*), we could say that we found a representation[14]

$$A_5 \rightarrow \mathrm{PGL}_2(\mathbb{C}).$$

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[14] In abstract terms, one can say that the existence of this representation is due to the *exceptional isomorphism*

$$\mathrm{O}_3(\mathbb{R}) \cong \mathrm{PSU}_2(\mathbb{C}),$$

which allows us to view rotations of Euclidean three-space, $\mathrm{O}_3(\mathbb{C})$, as a special kind of Möbius transformations, $\mathrm{PSU}_2(\mathbb{C}) \subset \mathrm{PGL}_2(\mathbb{C})$. I come back to this point in Sect. 2.5

Now, the icosahedron comes in: If we inscribe an icosahedron into the sphere, it can be shown that this group is exactly the group of rotations which leave the icosahedron invariant![15] We can therefore call the above group of rotations the *icosahedral group*.

To summarize: The map $F : M_2^{(2)} \to \mathbb{P}^1$ which sends points of the quadratic surface $M_2^{(2)} \subset \mathbb{P}^4$ to points of the Riemann sphere is such that a permutation of the coordinates of the argument $x \in M_2^{(2)}$ results in the application of an icosahedral rotation of the image $\lambda = F(x)$. In a modern formulation, we would say that F is A_5-*equivariant* with respect to the (projectivized) standard representation in $\mathbb{P}^4$ and the *icosahedral representation* in $\mathbb{P}^1$; or, that the following diagram commutes:

$$
\begin{array}{ccc}
M_2^{(2)} & \xrightarrow{\ \sigma\ } & M_2^{(2)} \\
\Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle F} \\
\mathbb{P}^1_{(1)} & \xrightarrow{\ S\ } & \mathbb{P}^1_{(1)}
\end{array}
$$

We can also note that the permutations of the coordinates x_i act *diagonally* on $\mathbb{P}^1_{(1)}$ and $\mathbb{P}^1_{(2)}$; thus, we only need to consider the above action on $\mathbb{P}^1_{(1)}$ and the other one follows by a multiplication by a constant.

We see in Sect. 3.7 how this representation can be used to reduce a principal quintic equation to some canonical *icosahedral equation*, but let us for conclude with a remark on the *general* quintic equation: It was known already for centuries that a general equation can be brought into principal form by a linear and quadratic *Tschirnhaus transformation*. Today, a Tschirnhaus transformation of a polynomial $f(x)$ is usually considered as a polynomial transformation of the variable x to some new variable y, leading to a new polynomial $g(y)$ with "favorable" properties such as the vanishing of some coefficients. In Klein's time, a Tschirnhaus transformation was more generally considered as a parallel transformation of the roots x_i of $f(x) = 0$. These are spanned by the following basis transformation:[16]

$$
x_i \mapsto x_i - s_1/n, \quad x_i \mapsto x_i^2 - s_2/n, \quad x_i \mapsto x_i^3 - s_3/n, \quad \dots,
$$

where s_j are the elementary symmetric polynomials and thus deducible from the coefficients a_i of the given equation. By the way, taken alone, the first basis transformation makes the coefficient a_1 vanish, the second one makes the coefficient a_2 vanish, and so forth. To make both coefficients vanish, a combination of both basis transformations is necessary, which can be calculated by some quadratic equation. Therefore, every general quintic equation can easily be reduced to a principal equation. If a quintic equation is already in *reduced* form $x^5 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 = 0$, we can interpret its ordered solution as a point in $\mathcal{H}$, and the Tschirnhaus transformation becomes a map that moves this point onto $M_2^{(2)}$. The square root that appears when solving

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[15] Klein inscribes the icosahedron, such that its 12 vertices lie on the south pole 0, the north pole ∞, and 10 other specified points on two different latitudes. The calculations of the symmetries of the icosahedron in terms of transformations of the Riemann sphere are an elementary, but little revealing task. Historically, it was already known to Hamilton (1856) that the symmetry group of the icosahedron is isomorphic to A_5.

[16] I omit the details here which can be read in Kraft and Procesi (2000); Kraft (2006).

the quadratic equation above finds a geometrical interpretation in the fact that any geometrical construction (which happens parallel in all coordinates) performing such a map cuts the quadratic surface $M_2^{(2)}$ at two points. We therefore do not arrive at a uniquely defined map from any point in $\mathcal{H}$ to one point on $M_2^{(2)}$ and after F to one point in $\mathbb{P}_{(1)}^1$, but only a *correspondence*. The analogy of this idea for the higher degree equations is discussed in Sect. 3.5.

All these considerations, as will be seen in the next section, were successfully transferred to the theory of equations of degree 6 and 7 in Klein's 1886 article. Specifically, Klein achieved both:

- the construction of some representation of S_6 and A_7 in three-dimensional space $\mathbb{P}^3$, analogous to the one-dimensional $\mathbb{P}_{(1)}^1$ here (see Sects. 3.1–3.3); and
- the calculation of a covariance between a solution $(x_0: \cdots : x_{n-1})$ of a reduced equation and a point in $\mathbb{P}^3$, analogous to the Tschirnhaus transformation followed by the map F here (see Sect. 3.5).

What Klein could not transfer was the second half of the icosahedral theory, in which the above considerations were used to bring a given quintic equation into a canonical form, called the *icosahedral equation*. For this reason, I omit these considerations in this section. However, they are shortly addressed within the broader historical framework of Klein's Hypergalois Theory in the following subsection.

2.3 Modular equations

After the theory of the general equation of degree 5 was virtually completed in the late 1870s, it seemed only natural to ask for analogous theorems for other classes of polynomials, i.e., polynomials with the same degree and the simple Galois group. Of course, such analogs appear more interesting when the resulting canonical equation can actually be solved by one or the other non-algebraic method. For the icosahedral group $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$, this is possible by the local inversion of the icosahedral equation by modular equations. This approach can be generalized to two other cases, the linear group $\mathrm{PSL}_2(\mathbb{F}_7)$ of order 168, and the linear group $\mathrm{PSL}_2(\mathbb{F}_{11})$ of order 660. The special status of these three simple groups was already known to Galois who showed that $\mathrm{PSL}_2(\mathbb{F}_p)$ (p prime) acts non-trivially on a set of p (or less) elements only for $p \leq 11$, and this action can be used in the theory of modular equations.[17] Klein considered the modular equation of degree 7 in Klein (1879a) and shortly after found a geometrical representation of the group $\mathrm{PSL}_2(\mathbb{F}_7)$ in three (complex) dimensions, namely as the automorphism group of the *Klein quartic* (Klein 2001). The Klein quartic is given by the equation

$$z_1^3 z_2 + z_2^3 z_3 + z_3^3 z_1 = 0.$$

As with the icosahedron before, one is interested in the graded $\mathbb{C}$-algebra $\mathbb{C}(z_1: z_2: z_3)^{\mathrm{PSL}_2(\mathbb{F}_7)}$ of invariants with respect to this automorphism group. Here,

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[17] This result was generalized in a study on the subgroup structure of $\mathrm{PSL}_2(\mathbb{F}_p)$ by Gierster (1881), who expressed his gratitude to Klein for his "multiple stimulation and support" for that article.

Klein left the explicit calculations to Gordan who published five papers on the topic in the following 5 years. Klein tackled the task an equation of degree 7 (or degree 8) with Galois group isomorphic to $PSL_2(\mathbb{F}_7)$ to the canonical equation thus established in the aforementioned articles. The theory can be said to have been successfully completed in 1885, although it never received the prominence of the theory of the icosahedron.[18] In the same spirit, also the above-mentioned group $PSL_2(\mathbb{F}_{11})$ was considered (Klein 1879c), but the absence of a low-dimensional representation seems to have hindered further progress (cf. the modern 1995). This completes the list of Galois groups whose equations are solvable by modular equations. Klein, however, had no intention to stop at this point, but instead sought for a generalization of his methods to other Galois groups. This point of view was first made explicit in Klein (1879b) (see Sect. 1), which one can therefore take as the starting point of the Hypergalois Theory as a systematic research program:

> In my presentation, I have given the principles such a form that they not only solve the problem of equations with 168 substitutions, which is the first problem to be considered, but also make it possible to see *how to treat similar problems with any higher equations and, what is more important, how to set them up*. The resulting *general method for treating higher equations* (which will of course still be open of manifold development) includes both the solution of cyclic equations by root symbols and the Kroneckerian treatment of equations of the fifth degree. One can regard my method virtually as a generalization of the latter. (Klein 1879b, p. 252)

As for the first problem—the construction of some canonical Galois resolvent—Klein remarked:

> The general method which I propose for the rational transformation of the algebraic equations consists simply in this, *that I first search for the smallest number μ for which an isomorphism of the desired kind between the permutations of the x and linear substitutions of the $y_1 \ldots y_\mu$ is possible, and that I then replace the equation $f(x) = 0$ with the "problem of the y"*. (ibid., p. 257)

This outline also indicates a shift of attention in Klein's program: The geometrical considerations of the groups $PSL_2(\mathbb{F}_7)$ and $PSL_2(\mathbb{F}_{11})$ were motivated by successful treatment of the case $PSL_2(\mathbb{F}_5) \cong A_5$. What holds these groups together is their special status with respect to modular equations. Therefore, the solvability of some canonical equation by analytic means came to the foreground, and the geometrical interpretations only followed afterward. When the modular equations "did their job" as much as they could (the next subsection shows that this assessment might not be quite correct, though), the picture was reversed, with the geometrical interpretation forming the starting point.

Under this new credo, the overall aim remained the same, but the steps toward achieving this aim changed their order: First came the geometrical interpretation, then the calculation of the invariants (which are covariants with respect to the Galois group

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[18] A number of modern mathematical and historical contributions have been made on various aspects of the group $PSL_2(\mathbb{F}_7)$; some of them published in Levy (2001), among them Gray (1982) which are used here.

of the original equation), and only then the local solutions to them. Klein called the whole of this problem the *form problem*, and already in the *Lectures on the Icosahedron* explained its central position in his theory of equations:

> The formulation of this problem [how to reduce equations with identical Galois groups to one another] has a certain importance for we obtain thereby at the same time a general program for the further development of the theory of equations. Among the form-problems or equation-systems with isomorphic groups, we have already above described as the simplest that which possesses the smallest number of variables. If therefore, any equation $f(x) = 0$ is given, we will first investigate what is the smallest number of variables with which we can construct a group of linear substitutions which is isomorphous with the Galois group of $f(x) = 0$. Then we shall establish the form problem or the equation-system which appertains to this group, and then seek to reduce the solution of $f(x) = 0$ to this form problem or equation-system, as the case may be. (Klein, [1884] 2019b, p. 138)

For a given Galois group, the "simplest" form problem was later called the *normal problem*. One condition for being "simple" is the above-mentioned minimization of its dimension, i.e., we are looking for a faithful representation

$$G \to \mathrm{PGL}_n(\mathbb{C})$$

with n as small as possible. A second condition is that the representation and the resulting invariants and covariants have as easy or intuitive values as possible, which is usually achieved by choosing a suitable coordinate system or by relying on geometrical intuition, or both. I cannot go into the details of the historical achievements here, but it is clear that working on the normal problem starts with finding the minimal-degree faithful representation. The importance of this first step could be seen, for example, at the occasion of the famous *Evanston Colloquium* of 1894, when Klein repeated the principles of the program to a more general mathematical audience:

> Let us consider the very general problem: *a finite group of homogeneous linear substitutions of* n *variables being given, to calculate the values of the* n *variables from the invariants of the group.*
> This problem evidently contains the problem of solving an algebraic equation of any Galois group. [... A]mong the problems having isomorphic groups we consider as the simplest the one that has the least number of variables, and call this the normal problem. This problem must be considered as solvable by series of any kind. The question is to reduce the other isomorphic problems to the normal problem. (Klein 1894, pp. 72–73)

And further:

> The reduction of the equation of the fifth degree to the icosahedron problem is evidently contained in this as a special case, the minimum number of variables being two. (ibid.)

In this spirit, it is only natural that the results of Klein's 1886 paper were considered a major step in the development of the Hypergalois Program. Additionally, the groups A_6 and S_6 were crucial to the general equations of degree 6, and thus provided the first step toward a real generalization of Klein's icosahedron. I show in the next subsection how Klein's paper fits into this story.

2.4 The general sextic equation

The general equation of degree 5 was one of the dominating forces of development both in algebra and analysis for the better of part of the 19th century, that is, at least until the publication of the *Lectures on the Icosahedron* in 1884. The class of equations to naturally consider next, the *sextic* equation, on the other hand, was only of marginal interest throughout the history. We already witnessed in Sect. 2.1 an early interest of Klein into the sextic equation, whose Galois group S_6 was there geometrically realized as a group of permutations of six linear complexes (see Sect. 3.3). What is more, if we assume the sextic equation in principal form like the quintic before

$$x^6 + \alpha x^3 + \beta x^2 + \gamma x + \delta = 0,$$

then the roots x_i again fulfill some relations $\sum x_i = 0$ and $\sum x_i^2 = 0$; and the second relation can be said to define a *quadratic line complex*, whose focal surface is a *Kummer surface*, i.e., a surface of degree four with 16 nodal points lying on 16 planes (see Rowe (2019), p. 8)). The automorphism group of the resulting configuration is a 16-cover of the group S_6. In short, the theory of the sextic can be approached by a closer study of this configuration, which thus plays a similar role to the icosahedron or the Klein quartic. However, being more concerned with the latter two geometrical objects, Klein did not return to the theory for a while.

Instead, it seems to have been due to the early work of the geometer and Klein student Giuseppe Veronese that the group A_6 and the general equation of degree 6 came back into Klein's agenda, as he remarked in a footnote in the *Lectures on the Icosahedron* 13 years later:

> If we wished to treat equations of the sixth degree in an analogous sense, it would be necessary, after adjunction of the square root of the discriminant, to start from that group of 360 linear transformations of space which I have established in Bd. iv of the Math. Ann., l.c., [Klein (1871)] and to which latterly Signor Veronese has returned from the side of geometry [Veronese (1882b)]. (Klein, [1884] 2019b, p. 139n)

Veronese, who studied mathematics in Zurich and Rome, visited Klein in 1880–1881 in Leipzig (2003, p. 100; 2021, p. 154), and wrote the aforementioned geometrical approach during the end of his stay there. He had worked on the topic already in Veronese (1877), when he used Pascal's *Hexagramme Mystique* to approach the group A_6 geometrically. In later works (Veronese 1882a, b), Veronese combined the two available geometrical interpretations of A_6—Pascal's Hexagramme Mystique and Klein's use of the Kummer surface—but did not link them to the theory of equations.

Such a link seems to have been discussed first in Klein's seminars in Leipzig during the years 1884–1886, as at least two of his students recounted: The first student was the American mathematician Frank Nelson Cole, who stayed as a student of Klein in Leipzig between 1883 and 1885 and participated at Klein's seminars in Summer 1884 and Winter 1884/85. After his return to Harvard University, Cole devoted his Ph.D. thesis to *A Contribution to the Theory of the General Equation of the Sixth Degree*. This thesis was effectively supervised by Felix Klein and shortly after published in the *American Journal of Mathematics*. We can read in its introduction:

> The subject of the present article was suggested to me by Prof. Klein, when I was a student in his Seminar at Leipzig, and I wish here to acknowledge my great indebtedness to him for valuable advice and suggestion, which have been of the greatest use to me. The fundamental idea of the entire treatment of the subject is due to him, as I have indicated below, and he might claim many of the particular methods involved as his own, if he should consider them worthy of such recognition. (Cole 1886, p. 265)

Cole started with a short review of Klein's *Lectures on the Icosahedron*, which he believed to have completed the theory of the quintic equation:

> While the theory of the equation of the fifth degree is thus completed in all directions, that of the sixth degree is only just begun. (ibid., p. 266)

His method to treat the sextic followed the proposition of the footnote in Klein's *Lecture on the Icosahedron* (see above), in whose spirit Cole proceeded:

> The method proposed by Klein for the solution of the general equation of any degree is perfectly analogous to this. We have to seek a group of linear substitutions which shall be isomorphic with the group of n! permutations belonging to the equation. Functions of the roots must then be found which undergo these linear transformations when the roots are permuted; and finally, corresponding differential equations must be obtained and their solutions studied. (ibid., p. 269)

As for the first task:

> What is the smallest number of variables for which a group of linear equations is isomorphic with the 720 permutations of six elements? There is no such group for one or two variables. There is, however, such a group for three variables, or, if we write our linear transformation in homogeneous form, for four variables, of which the ratio of three of the fourth will then be transformed by a non-homogeneous transformation. This group of transformations is best known under the geometrical form in which its theory has been treated in connection with the remarkable surface of the fourth order and class known as Kummer's surface. (ibid., pp. 270–271)

Unfortunately, Cole's dissertation ends in a heap of long-winded calculations, and does not permit a conclusion. His closing paragraph contains the announcement of further research on the sextic, together with the promise to calculate some coefficients missing in the dissertation. We can only imagine the hardship Cole faced on the calculations

when reading his complaints in his letter exchange with Klein (also see Parshall and Rowe 1994, pp. 192–197):

> I feel quite exhausted from the arduous work of this year. Especially the calculations were too much for me, as I spent for almost six months ten hours daily on them. ([UBG] Cod. Ms. F. Klein 8: 476; a letter from Cole to Klein, 26 May 1886)

The second student to work on the sextic equation, Wilibald Reichardt was not much luckier. He visited Klein's seminar of the summer term of 1885 in Leipzig (during which time he was also *famulus*, a sort of assistant, of Klein) and reported that Klein proposed to geometrically interpret the group S_6 as some symmetry group of a *Kummer surface*, just as Cole did before (Reichardt 1885, p. 28). Reichardt followed that route and published an extended article on the topic (Reichardt 1886), but seems to have given up on the topic thereafter.[19]

This was the situation in late 1886, when Klein used similar line-geometric ideas to calculate transformation groups of projective complex space that are isomorphic to S_6 and A_7, respectively. In a modern terminology, we would say that Klein was the first to explicitly calculate the generators of some faithful projective representations

$$S_6, A_7 \to \mathrm{PGL}_4(\mathbb{C})$$

(which are furthermore irreducible). In the historical context just described, it does not take much creativity to imagine the real intention behind the publication of this (admittedly, mathematically not very advancing) paper: It is reasonable to assume that Klein wanted to bring a fresh impetus to the theory of the sextic equations, provide some solid ground from which his students could more easily depart, and also demonstrate some personal involvement in the topic, which would certainly boost the motivation of his students. Klein's mentioning of both Reichardt's and Cole's work in a footnote of the 1886 article, together with his announcement to come back to their results at a later stage, support this view (Klein 1886, p. 499n). And finally, it did not seem to cost Klein too much effort to produce his results: He did not start working on the paper before September 1886, and had finished it already in October.[20] In this sense, I believe that Klein's decision to publish a 34 pages strong paper in his *Mathematische Annalen* was guided more by programmatic than by purely mathematical considerations.

A similar conclusion can be drawn with respect to Klein's communication with fellow mathematicians after the publication of his paper, in which he encouraged his students to further study the general sextic. I could find two examples of such

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[19] Reichardt finished his Ph.D. in 1887 under the supervision of Felix Klein and Sophus Lie (who took Klein's position in Leipzig after Klein's departure to Göttingen in 1886). He published a second article on Kant's synthetical *a priori* and its importance to philosophy of mathematics in 1888, but was not academically active thereafter.

[20] In a letter to Hurwitz dated 31 August 1886, Klein announced that he would start working on the "equations of 6th degree" the following day ([UBG] Cod. Ms. Math. Arch. 77: 160; a letter from Klein to Hurwitz, 31 August 1886). The information on the completion in October 1886 is taken from the published paper itself (Klein 1886, p. 543).

engagement: First, Klein naturally sent copies of his article to Cole who in return expressed his wish to work with Klein on the topic in Summer 1887, at which time he saw a chance to visit Germany.[21] However, severe mental conditions prevented him from any kind of mental effort for at least a year.[22] Only in October 1889, did he find the strength to return to mathematical productivity, resulting in an paper on subgroups of $GO_4(\mathbb{C})$ (Cole 1890), an area of interest that could be said to vaguely fit into the scope of finding faithful representations of least degree.[23]

A second mathematician to pick up the topic was Klein's student Heinrich Maschke who spent the spring and summer of 1888 on the attempt to connect Klein's paper with his own research on Borchardt moduli, and thus covered similar terrain as Cole. In a letter to Klein dated to the 16 February 1888, Maschke outlined this connection, concluding:

> The solution of the equation of the 6th degree thus results in the following: Calculation of the Borchardt moduli from a form of 6th order, all of whose [five] invariants having given values. ([UBG] Cod. Ms. F. Klein 10: 936; a letter from Maschke to Klein, 16 February 1888)

Some months later, Maschke managed to simplify Klein's representation of A_7 further. Maschke also recognized that a second, similar, representation can be easily constructed. This anticipated the modern result that $2.A_7$ (the result of the lift from $PGL_4(\mathbb{C})$ to $GL_4(\mathbb{C})$) has two irreducible representations of dimension 4.[24] With the representation thus simplified, it was easier to approach the task of finding its covariants:

> With this, this problem has also been taken so far that I can approach the immediate setting up of the form system. ([UBG] Cod. Ms. F. Klein 10: 938; a letter from Maschke to Klein, 4 July 1888)

Maschke furthermore published a small note on a configuration of 140 lines in space which are permuted by Klein action of A_7 on $\mathbb{P}^3$ (Maschke 1889, 1890), but soon turned to other form-theoretic questions, and a complete treatment of the sextic equation has not been achieved on this way. Thus, at the Evanston Colloquium of 1894 that we discussed already in the last subsection, Klein could only point to his old 1886 article with respect to the achievements on the general equation of degree 6 and 7 (Klein 1894, p. 74).

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[21] See ([UBG] Cod. Ms. F. Klein 8: 477; a letter from Cole to Klein, 12 March 1887).

[22] See ([UBG] Cod. Ms. F. Klein 8: 478; a letter from Cole to Klein, 22 October 1887) and ([UBG] Cod. Ms. F. Klein 8: 479; a letter from Cole to Klein, 22 March 1889). Also, other students of Klein, by name Oskar Bolza, Otto Hölder, and Max Born, reported mental exhaustion as a direct consequence of the workload provided by Klein (Thiele 2018, pp. 84–85). Cole is less direct here, but the exhaustion from working on the coefficients of his covariants is obvious.

[23] This work was (among others) followed by a very enlightening review of the first volume of Klein's *Lectures on the Theory of Elliptic Modular Functions* as well as the production of a complete list of simple (non-Abelian) groups up to order 660. The systematic search for simple groups of small order was mainly advanced by Otto Hölder, and demonstrates an increasing interest in group theory from a completely abstract point of view (see Gallian 1976).

[24] In fact, the representations given in the *Atlas of Finite Groups* (http://brauer.maths.qmul.ac.uk/Atlas/v3/) look almost like Maschke's result.

Unbeknownst to Klein during that time, progress was on the way, and in fact already started in 1889 with the publication of the Danish mathematician Herman Valentiner, who just found a *ternary substitution group* (a subgroup of $PGL_3(\mathbb{C})$) of order 360 (Valentiner 1889). Valentiner recognized that his group contained the icosahedron group A_5 as a subgroup, but only some 6 years later, the Swedish mathematician Anders Wiman (1865–1959) showed that this group is isomorphic to the group of even permutations of six elements, A_6! This came as a surprise even more so as Jordan (1878) had already ruled out any ternary substitution group other than the then-known ones (although the falsity of his claim was already established when Klein showed the existence of a ternary $PSL_2(\mathbb{F}_7)$ that Jordan also missed, cf. (Wiman 1899, p. 529)). Wiman did not doubt that his result would be of interest to Klein, and wrote him on 18 November 1895:

> Hereby I am sending you a treatise "On a simple group of 360 plane collineations" of 33 pages. The group in question is holohedrally isomorphic to the group of even permutations of 6 things, and it seems to be that a representation of this group in the plane has not yet been known. As I believe that the subject offers at least some interest, I am taking the audacity to ask for a place of the treatise in the *Mathematische Annalen*. ([UBG] Cod. Ms. F. Klein 12: 355; a letter from Wiman to Klein, 18 November 1895)

Naturally, Klein met Wiman's (or better: Valentiner's) discovery with enthusiasm and not only approved the result for publication, but also immediately asked Wiman to use it for an advancement of the theory of the sextic equation.[25] One explanation why the representation was so long overlooked lies in the fact that unlike all finite linear groups considered before, the representation $A_6 \rightarrow PSL_3(\mathbb{C})$ does not lift to a double cover in $SL_3(\mathbb{C})$, but to a triple cover

$$3.A_6 \rightarrow SL_3(\mathbb{C})$$

known today as the *Valentiner group*. I might also point out here that Valentiner's and Wiman's discoveries conclude the list of ternary substitution groups, and that Klein's representation of the group of even permutations of *seven* letters, $A_7 \rightarrow PSL_4(\mathbb{C})$, was indeed optimal. Naturally, Wiman's result triggered new hope for a genuine theory of equations of degree 6 (although not of degree 7), which arose immediately: Only months after Wiman's first publication, Robert Fricke, who is today almost exclusively known as Klein's closest collaborator, answered positively the question whether Klein's invariant theory of $PSL_2(\mathbb{F}_7)$ generalizes to A_6.[26] Also, the today unknown Muscovite Leonid Lachtin (today transcribed as Lahtin), unaware of Fricke's results, found a canonical equation of degree 6, solvable by linear differential equations of third order, which he assumed to be the analog of Klein's icosahedral resolvent. (Lachtin 1899, 465n). Finally, Klein, "under the impulses of [his] old friend Mr. Gordan" turned back with new enthusiasm to the theory of the general sextic:

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[25] See ([UBG] Cod. Ms. F. Klein 12: 356; a letter from Wiman to Klein, 28 November 1895).

[26] Wiman (1896) appeared in *Mathematische Annalen* 47(4), which was distributed in 11 August 1896. Fricke dates his publication to 5 September of the same year.

But this is only a beginning; I hope that his continued efforts will succeed in clarifying the subject in every respect as fully as we have been able to do in the past with the theory of equations of the fifth degree. (Klein, [1905] 2019a, p. 2)

However, a number of publications on the topic in the following years[27] could not save the theory of the equation of degree 6 from oblivion, and today, the history of the sextic equation remains forgotten. All in all, the success in treating the general sextic equation came too late, and was already drowned by a new approach to algebra that valued general theories over particular results. The latest contribution to the Hypergalois theory of the sextic was made by Robert Fricke, whose Lehrbuch der Algebra (1924/26/28) might be the historically last treatise to solely employ Klein's approach to algebra.[28]

2.5 The legacy of the Hypergalois Program

There was a second major discovery by Wiman, which ironically might have contributed to the demise of the Hypergalois Program: Already during the Evanston Colloquium, Klein wondered whether the group A_8—the Galois group of the general equation of degree 8 (after adjoining a square root) and thus the next natural "candidate" to consider—possessed a low-dimensional representation similar to the (then-known) cases of $n \leq 7$, or whether "the equation of the eight degree was its own normal problem", i.e., whether no faithful representation in less than seven homogeneous coordinates (i.e., less than six dimensions) existed. Wiman (1897b) soon after showed that the latter was indeed the case, a result that he soon generalized to arbitrary dimensions: for $n \geq 8$, there exists no faithful representation of A_n in less than six dimensions (Wiman 1897a). In other words, the representations

$$A_5 \to \mathrm{PGL}_2(\mathbb{C}) \quad A_6 \to \mathrm{PGL}_3(\mathbb{C}) \quad A_7 \to \mathrm{PGL}_4(\mathbb{C}),$$

which stood at the heart of Klein's theory on the quintic, sextic, and septic, were by no means the beginning of a general series, but rather a list of exceptions that could not be extended to $n \geq 8$. The disappointment connected with this insight is summarized in a letter that Wiman sent to Klein on 2 January 1898:

Here as everywhere everything has to proceed lawfully; to those laws the beautiful $G_{60}, G_{360}, G_{2520}$ [the presentations of $A_{5,6,7}$] must be allowed to subordinate; thus they probably only form the first members of a whole chain of interesting collineation groups, and the corresponding reduction of the general equation of

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[27] Despite the above, Gordan (1905), Gordan (1909), Coble (1911b), Coble (1911a)

[28] In its second volume, Fricke also provided a new representation of A_6 as a principal congruence subgroup of level 3 over some ring of integers:

$$A_6 \cong \mathrm{PSL}_2\left(\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]\right)/\sim, \quad g \sim h\colon \Leftrightarrow g \equiv h \mod 3$$

This representation would potentially enable a solution of the sextic equation in terms of modular equations similar to Klein's quintic (see Sutherland 2019), but it is beyond my capabilities to verify this claim.

degree n to normal equations must be achievable by uniform principles, i.e., yield a general theory. What could be found out famously destroyed this beautiful construction; a law really existed, but the interesting cases, after which the draft was sketched, proved themselves as exceptions. ([UBG] Cod. Ms. F. Klein 12: 359; a letter from Wiman to Klein, 2 January 1898)

Wiman, in close collaboration with Maschke, continued research on the topic for another year, but eventually decided that it was time to move to the more promising area of discontinuous infinite groups, another branch of mathematics invented by Klein. Looking back, he laconically comments:

It is strange that I have almost always had a headache when I was busy with my now terminated work. This may well be due to the fact that I was in spaces of too high a dimension. In a mere four-dimensional space I hope that I will feel at home. ([UBG] Cod. Ms. F. Klein 12: 361; a letter from Wiman to Klein, 27 November 1898)

If we could ask Maschke the same question, he might have given us a different answer though: His involvement in the Hypergalois Program quickly led to a focus of research on the purely group-theoretic considerations of linear substitution groups, i.e., of (linear or projective) group representations. Indeed, Maschke soon made a contribution to the newly emerging *representation theory* which guaranteed remembrance of his name until today: The *Maschke theorem* (1899) states that every given group representation splits is a direct sum of the so-called *irreducible representations*. In this sense, there does exist a "general law" behind the representations of A_5, A_6, A_7 that Wiman was looking for, as they were all irreducible faithful representations of minimal degree. This result could hardly be satisfying in terms of the Hypergalois Program, but it demonstrates that the efforts made throughout the years were not in vain. Also other representation-theoretic results can be traced directly to Klein's program. Among them was the concept of the Schur multiplier (Schur 1904) which was but a pre-mature version of the *second cohomology group* of a group. In two subsequent papers, Schur also showed—using the result from Cole (1893) that $A_6 \cong \mathrm{PSL}_2(\mathbb{F}_9)$—that the Schur multiplier of S_n and A_n is 2 except for A_6, A_7, where it is 6 (Schur 1907, 1911). We can understand this result as the deeper reason for the discrepancy between the Valentiner group and the usual double covers of the groups in question. (Also, A_7 has a triple cover like the Valentiner group.) We should also note that Schur's results stemmed from his interest in projective representations, which were (and still are) much less considered than the usual linear representations. It is not hard to draw the line to Klein's insistence on projective geometry during the whole of the Hypergalois Program.

The general impact of Klein's Program for the development of representation theory is described by Hawkins:

Klein himself and the mathematicians directly associated with him in the execution of his program were concerned with representations by collineation groups and with the representation of specific groups rather than with the creation of a general theory of such representations. But it is not difficult to imagine how suggestive their work might appear, especially to someone aware of the devel-

opments taking place in Lie's theory of groups and the theory of hypercomplex systems. (Hawkins 1972, p. 269)

Hawkins particularly points out the influence of Klein on the work Maschke, but also on William Burnside and Theodor Molien.

A second influence of the Hypergalois Program on modern mathematics can be found in Klein's and his colleagues increasing understanding of *exceptionality* in group theory. This includes the plain discovery of exceptional isomorphisms or behavior, but more importantly the recognition of such objects or properties as exceptional, and the consequential search for root causes of these exceptions. The following list shall only provide an overview of what has been achieved (often as mere "side products") within the Hypergalois Program:

- The isomorphism between A_5 and the symmetry group of the icosahedron was already known to Hamilton. Many other exceptional isomorphisms, such as $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$, $S_5 \cong \mathrm{PGL}_2(\mathbb{F}_5)$, $2.A_5 \cong \mathrm{SL}_2(\mathbb{F}_5)$, were probably first systematically studied in the context of Klein's icosahedron.[29]
- Likewise, the exceptional subgroup structure of the groups $\mathrm{PSL}_2(\mathbb{F}_5)$, $\mathrm{PSL}_2(\mathbb{F}_7)$ and $\mathrm{PSL}_2(\mathbb{F}_{11})$ was already known to Galois, but were consequently employed within the Hypergalois Program. Klein himself did not recognize the set-theoretic identities (not as groups!) with "his" symmetry group of the platonic solids:

$$\mathrm{PSL}_2(\mathbb{F}_5) \cong A_4 \times C_5, \quad \mathrm{PSL}_2(\mathbb{F}_7) \cong S_4 \times C_7, \quad \mathrm{PSL}_2(\mathbb{F}_{11}) \cong A_5 \times C_{11}.$$

  Today, these identities appear as part of the so-called *McKay correspondences*, and continue to be of mathematical interest (Kostant 1995).
- As we saw in Footnote 14 and 38, Klein was the first to systematically use the exceptional isomorphisms of linear groups $\mathrm{PSU}_2(\mathbb{C}) \cong \mathrm{O}_3(\mathbb{R})$ and $\mathrm{PSO}_6(\mathbb{C}) \cong \mathrm{PGL}_4(\mathbb{C})$.
- In his dissertation, Cole recognized that S_6 has two families of subgroups isomorphic to S_5: one family of six subgroups that fixes one element, and one exceptional family of "twisted" $S_5 \subset S_6$:

  As a result of the presence of this exceptional group, all equations of the sixth degree are connected in pairs, the roots of the two equations of each pair belonging respectively to the ordinary and the extraordinary groups of 120 permutations. (Cole 1886, pp. 269–270)

- Wiman showed that the representations of A_5, A_6, A_7 which were central to the Hypergalois Program are in fact exceptional representations.
- Schur showed that the existence of the Valentiner group was the consequence of the exceptional Schur cover of A_6 (together with S_6, A_7, S_7).

Unfortunately, no historical account on the history of exceptional isomorphisms seems to exist (the otherwise very interesting (Stillwell 1998) do not cover them), but it would surely be interesting to obtain certainty on Klein's influence in this respect.

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[29] It seems that Klein was also aware of Schläfli's (then little known) result that the diversity of regular polyhedra in three (and four) dimensions was an exceptional situation, and did not generalize to higher dimensions. He subsequently did not seek to generalize his *Icosahedron* to polyhedra of higher dimensions.

Finally, it should be noted that the Hypergalois Program had a much wider range of influence than the two topics mentioned in this subsection. Their selection was purely based on the direct connection to Klein's 1886 article.

3 Klein's projective representations

Having outlined the historical background and implication of Klein's 1886 article, the current section aims to reconstruct the mathematical content of the article. The article consists of an introduction and three main parts, which are further subdivided into 11 paragraphs; and in the following subsections, my aim is to reconstruct at least the main ideas of all paragraphs from a perspective that mimics Klein's own attempt to present his theory as a natural continuation of his previous *icosahedral mathematics*: In Sect. 3.1, I introduce the general approach that is taken in Klein's paper (reflecting Klein's §1–4); this is followed by a demonstration of Klein's construction of the representations of A_7 (Sect. 3.2) and S_6 (Sect. 3.3) (both §5–6). A short "interlude" on Klein's interest in so-called *accessory irrationalities* (Sect. 3.4, §7) is followed by an analysis of Klein's calculation of the covariants (Sect. 3.5, §8–10). Finally, I briefly consider Klein's thoughts on the possibility of generalizing his theory to equations of arbitrary degree (Sect. 3.6, §11). One can see already that Klein's paper does not attempt to generalize the whole of his previous icosahedral mathematics: The deployment of some canonical invariant, the reduction of the general equation to such invariant, and finally the analytic solutions were not discussed in Klein's 1886 article.

For easy comparison with the original article, I keep Klein's enumeration of equations and most of his variable notation, while at the same time adapting his notation to ease readability.[30]

3.1 The general theory

We are given a general reduced equation of degree 6 or 7, respectively

$$\text{degree 6:} \quad f(x) = x^6 + a_3 x^3 + a_4 x^2 + a_5 x + a_6 = 0$$
$$\text{degree 7:} \quad f(x) = x^7 + a_3 x^4 + a_4 x^3 + a_5 x^2 + a_6 x + a_7 = 0.$$

Then, its solutions x_i fulfill the two additional identities

$$\sum_{i=0}^{n-1} x_i = 0, \qquad \sum_{i=0}^{n-1} x_i^2 = 0. \tag{3}$$

When viewed as a "solution space" $\mathbb{P}^{n-1}$, the first identity singles out a hyperspace of dimension $(n-2)$, while the second identity defines a quadratic manifold $M_{n-3}^{(2)}$ of dimension $n-3$ within that hyperspace. This is completely analogous to the case

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[30] Particularly, I added missing indices, introduced round brackets for homogeneous coordinates, and replaced systems of linear equations with matrix representation.

$n = 5$ (Sect. 2.2). As $M_2^{(2)}$, also the quadratic surfaces $M_3^{(2)}$ and $M_4^{(2)}$ are generated by some family or families of linear subspaces, and Klein notes (without proof), that $M_3^{(2)}$ contains a three-parameter family of lines, while $M_4^{(2)}$ contains two three-parameter families of planes.[31] We want to denote the families of lines by $\mathcal{E}$ and the two families of planes by $\mathcal{E}_1$ and $\mathcal{E}_2$. All of them are isomorphic to $\mathbb{P}^3$, and thus can be parametrized by four homogeneous coordinates

$$(z_1{:}z_2{:}z_3{:}z_4). \tag{4}$$

It is clear that $M_3^{(2)}$ can be represented as a hyperspace of $M_4^{(2)}$, and in fact, the family $\mathcal{E}$ is nothing but the cut of $\mathcal{E}_1$ with the hyperplane that cuts out $M_3^{(2)}$, and coincides with the cut of $\mathcal{E}_2$ with that said hyperplane. In this sense, we can restrict attention for now to the case $n = 7$, and later consider $n = 6$ as a special case thereof.[32] In analogy to the case $n = 5$ (Sect. 2.2), once a parametrization of $\mathcal{E}_1$ and $\mathcal{E}_2$ is defined, the permutations of the seven coordinates x_i permutes the planes in both families, and thus define linear transformations of some $\mathbb{P}^3_{(1)} \times \mathbb{P}^3_{(2)}$. And just as before, the even permutations of A_7 permute planes within one family (say, $\mathcal{E}_1$), while the odd permutations will permute the two families. In a modern notation, this established the desired representation $A_7 \to \mathrm{PGL}_4(\mathbb{C})$. However, it is so far completely unclear how a suitable parametrization of the subspace $\mathcal{E}_1$ should look like, and thus how an explicit matrix representation (or, in Klein's time: a system of linear equations) can be established! (In the case $n = 5$ the map F was quickly found by invariant-theoretic considerations, which I do not cover here.)

This problem is overcome by borrowing some theorem of line geometry, a topic that Klein covered extensively in his Ph.D. thesis and in a couple of subsequent papers, but did not develop much during the late 1870s and 1880s. Especially in the context of the theory of equations, Klein thus seemed to feel obliged to provide a short justification:

> If this way appears strange to some algebrist, it is worth recalling that everything we know about linear spaces in three- and four-times extended quadratic manifolds was originally developed on this exact way. (Klein 1886, pp. 504–505)

To understand exactly how Klein wants to use line-geometric results in this context, we have to go a little into the most important concepts, especially since historico-mathematical literature on this niche topic is scarce.[33] In projective space $\mathbb{P}^3$, points and planes are famously *dual*, meaning that planes form a dual space $(\mathbb{P}^3)^*$ which can

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[31] Klein denotes the maximal linear subspaces of some $M_d^{(2)}$ by R_ν, where ν represents their dimension. In this case: "$M_3^{(2)}$ contains ∞^3 many R_1; $M_4^{(2)}$ contains two families of ∞^3 many R_2" (Klein 1886, p. 503). Today, we would call $\nu + 1$ the *index* of the quadratic manifold. A non-singular quadratic manifold in projective space $\mathbb{P}^d$ has index $d/2$ or $(d + 1)/2$, depending on whether d is even (in which case the quadric is called *parabolic*) or odd (in which case it is called *hyperbolic*. In real projective space, also *elliptic* quadrics of index $(d - 1)/2$ exist for odd d. (cf. Beutelspacher and Rosenbaum 2013, p. 147)

[32] Also in Klein's generalization to arbitrary degrees (§11, see Sect. 3.6), the geometrical treatment of the equation of degree $2k$ is treated as a special case of the one of degree $2k + 1$.

[33] Hawkins (1989), Rowe (1989) and most recently Rowe (2019) treat different aspects of line geometry and therefore all contain an exposition of the most important ideas; the special theorems used in Klein's 1886 paper are however not considered there.

be described by *plane coordinates*.[34] This leaves us with lines (the dual of a line is again a line) for which we might ask if a useful coordinatization of them also exists. Julius Plücker, Klein's former Ph.D. supervisor, answered this question positively: For a given line, he took two distinct points $z = (z_1 : z_2 : z_3 : z_4)$ and $z' = (z'_1 : z'_2 : z'_3 : z'_4)$ lying on it, and defined its six today so called *Plücker coordinates* as follows:

$$\begin{cases} p_1 = z_1 z'_2 - z_2 z'_1, \ p_2 = z_1 z'_3 - z_3 z'_1, \ p_3 = z_1 z'_4 - z_4 z'_1 \\ p_4 = z_3 z'_4 - z_4 z'_3, \ p_5 = z_4 z'_2 - z_2 z'_4, \ p_6 = z_2 z'_3 - z_3 z'_3 \end{cases} \tag{5}$$

Elementary calculations reveal that the Plücker coordinates of a given line are (up to a common scalar) independent from the particular choice of the points z and z'; thus, the term *coordinate* is justified. The coordinates also fulfill the *Plücker equation*

$$\mathcal{P}(p_1 : p_2 : p_3 : p_4 : p_5 : p_6) = p_1 p_4 + p_2 p_5 + p_3 p_6 = 0.$$

Conversely, any sixtuple $p = (p_1 : p_2 : p_3 : p_4 : p_5 : p_6)$ satisfying the Plücker equation form the Plücker coordinates of a line. In other words, there is a one-to-one correspondence between the lines of $\mathbb{P}^3$ and the points of the quadratic surface

$$\mathcal{Q} = \{(p_1 : p_2 : p_3 : p_4 : p_5 : p_6) \mid p_1 p_4 + p_2 p_5 + p_3 p_6 = 0\} \subset \mathbb{P}^5.$$

The quadric $\mathcal{Q}$ later became known as the *Klein quadric*,[35] and the bijection

$$\kappa : \text{lines in } \mathbb{P}^3 \longleftrightarrow \text{points in } \mathcal{Q} \subset \mathbb{P}^5$$

is today called the *Klein correspondence*. In his inaugural dissertation (Klein 1868), Klein made a number of interesting observations about the behavior of Plücker coordinates. One of them concerns *linear transformations* (i.e., projectivities) of Plücker coordinates, concluding that any sixtuple of homogeneous coordinates satisfying some non-degenerate quadratic form can be used as a coordinatization of lines: a projectivity of Plücker coordinates $(p_1, \ldots, p_6) \mapsto (x_0, \ldots, x_5)$ transforms the Plücker equation $\mathcal{P}(p_1, \ldots, p_6) = 0$ to some non-degenerate quadratic equation $\Omega(x_0, \ldots, x_5) = 0$. Conversely, any non-degenerate quadratic form in six homogeneous variables can be obtained from the Plücker equation by a suitable transformation of Plücker coordinates, and therefore defines a coordinatization of lines.

Back to our reduced equation of degree 7, we can also treat its solutions $x_0, \ldots, x_6$ as such coordinates, with the quadratic equation $\Omega(x_0, \ldots, x_6) = \sum_{i=0}^{6} x_i^2 = 0$ holding between them. That we have seven coordinates instead of six poses no problem, because we can eliminate one variable with the equation $\sum_{i=0}^{6} x_i = 0$. But as this

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[34] A plane A in $\mathbb{P}^3$ can be described as the set points $x = (x_1 : x_2 : x_3 : x_4)$ satisfying a linear equation

$$A_1 x_1 + A_2 x_3 + A_3 x_3 + A_4 x_4 = 0$$

It is then only natural to refer to the coefficients $(A_1 : A_2 : A_3 : A_4)$ as the *plane coordinates* of A.

[35] Not to be confused with the *Klein quartic* of Sect. 2.3! The Klein quadric $\mathcal{Q}$ is indeed the *Grassmannian* $\mathbf{Gr}(2, \mathbb{C}^4)$ of 2-dimensional subspaces of $\mathbb{C}^4$.

would disturb the symmetry of the exposition, Klein decided to keep all $x_0, \ldots, x_6$ as superfluous line coordinates between which the additional relation $\sum_{i=0}^{6} x_i = 0$ is assumed. In effect, we can interpret the solution space of the reduced equation of degree 7 as the space of lines in $\mathbb{P}^3$.[36] In this interpretation, the quadratic manifold $M_4^{(2)}$ is nothing but the above Klein quadric $\mathcal{Q}$ in x-coordinates! Most remarkably, this construction already provides a parametrization of our family $\mathcal{E}_1$ above, as one can show that there is an isomorphism between the latter and the original point space $\mathbb{P}^3$. Unfortunately, Klein commented on this astonishing situation rather too concisely:

> Indeed, the points of space, as it is not necessary to explain here, understood as bundles of lines, correspond exactly to the triple infinite R_2 [= planes] of the first kind, which are contained in the $M_4^{(2)}$ defined through (8) (while the planes of spaces, understood as planes of lines, correspond to the triple infinite R_2 of the second kind). (Klein 1886, p. 505)

I therefore want to use the remainder of this section to elaborate a little more on this topic. The Klein correspondence κ between lines in $\mathbb{P}^3$ and points on $\mathcal{Q}$ can be extended to points and planes of $\mathbb{P}^3$ as well: a point $z \in \mathbb{P}^3$ is incident with a two-parameter family of lines, $\mathcal{L}_z$. Under κ, this family maps to a two-dimensional manifold in $\mathcal{Q}$, and it can be verified that this manifold forms in fact a plane. Slightly abusing our notation, we call this plane $\kappa(z)$. Similarly, a plane $A \subset \mathbb{P}^3$ is incident with a two-parameter family of lines $\mathcal{L}_A$, which also maps to a plane in $\mathcal{Q}$, and which we call $\kappa(A)$. In fact, we can show that any plane contained in $\mathcal{Q}$ is the image of either a point or a plane in $\mathbb{P}^3$ under κ. This establishes two families of planes in $\mathcal{Q}$, which are just the above $\mathcal{E}_1$ and $\mathcal{E}_2$ (in new coordinates). Naturally, both families are isomorphic to $\mathbb{P}^3$. We can also convince ourselves that all incidence relations between points, lines, and planes in $\mathbb{P}^3$ can be formulated in terms of lines, and in fact translate to incidence relations in $\mathcal{Q}$.[37] In this sense, the Klein correspondence κ extends to an incidence-preserving isomorphism between the geometry of $\mathbb{P}^3$ and the geometry of $\mathcal{Q}$ as follows:

$$\kappa: \begin{cases} \text{points in } \mathbb{P}^3 \to \text{planes in } \mathcal{E}_1 \\ \text{lines in } \mathbb{P}^3 \to \text{points in } \mathcal{Q} \\ \text{planes in } \mathbb{P}^3 \to \text{planes in } \mathcal{E}_2 \end{cases}$$

In particular, the planes in $\mathcal{E}_1$ can be parametrized by the coordinates of their corresponding point in $\mathbb{P}^3$.

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[36] Thus, an ordered list of solutions of any particular equation of degree 7 can be interpreted as one particular line in $\mathbb{P}^3$. Klein's approach here is more general, in that the coefficients of his equations are completely undetermined.

[37] For example, two points are distinct in $\mathbb{P}^3$, $z_1 \neq z_2$, if and only if they only have one line in common, i.e. if and only if the planes $\kappa(z_1)$ and $\kappa(z_2)$ meet in a single point. (This is possible, because $\mathcal{Q}$ is a quadratic manifold in $\mathbb{P}^5$, not the whole $\mathbb{P}^5$.) The same holds for planes A, B. Similarly, a point z not incident with a plane A, $z \notin A$, have no line in common; thus, the planes $\kappa(z)$ and $\kappa(A)$ do not meet at all. On the other hand, if z is incident with A, $z \in A$, they share a one-parameter family of lines (those lines in A that go through z); thus, $\kappa(z)$ and $\kappa(A)$ will meet in a line. In other words, two planes from the same family $\mathcal{E}_i$ will either coincide or have a single point in common, while two planes from distinct families have either an empty intersection or meet in a line. Similar observations can be made about incidence relations between points and lines, lines and lines, and planes and lines.

3.2 The representation of A_7

Analogous to the case $n = 5$ discussed in Sect. 2.2, we now want to investigate the effect of a permutation of the line coordinates x_i on the planes $\mathcal{E}_1$. With the above isomorphism between $\mathcal{E}_1$ and $\mathbb{P}^3$ in mind, this is the same as asking the effect of a permutation of line coordinates to the underlying space $\mathbb{P}^3$. Here, a second theorem from Klein's inaugural dissertation, reformulated in Klein's 1886 article, is helpful:

> [... E]very linear substitution of line coordinates $\xi_1 \ldots \xi_6$, which transforms the quadratic form Ω [...] into itself, means a collineation or a dualistic transformation of space; namely the first or the second, depending on whether the associated substitution determinant, whose square necessarily equals 1, is equal to $+1$ or -1. (ibid., p. 507)

This result can be translated as follows: The group of projectivities of line coordinates leaving a quadratic form invariant (a special case being: projectivities of Plücker coordinates leaving the Plücker equation invariant) is called the *projective (general) orthogonal group*, because it is the projecitivization of the (general) orthogonal group, $\mathrm{PO}_6(\mathbb{C}) \cong \mathrm{O}_6(\mathbb{C})/\{\pm I_6\}$. The latter $\mathrm{O}_6(\mathbb{C})$ splits into (six-dimensional) rotations $\mathrm{SO}_6(\mathbb{C})$ with determinant $+1$ and reflections with determinant -1. In even dimensions, this distinction is upheld under projectivization of the linear group, i.e., also $\mathrm{PO}_6(\mathbb{C})$ splits into projective "rotations" $\mathrm{PSO}_6(\mathbb{C})$ and "reflections" $\mathrm{PO}_6(\mathbb{C})\backslash\mathrm{PSO}_6(\mathbb{C})$. The above theorem then states that a "rotation" corresponds to a projectivity of $\mathbb{P}^3$ and vice versa, and we can convince ourselves that this correspondence preserves group multiplication and thus forms an isomorphism:[38]

$$\mathrm{PSO}_6(\mathbb{C}) \cong \mathrm{PGL}_4(\mathbb{C})$$

Specifically, the permutations of line coordinates $x_0, \ldots, x_6$ correspond to projectivities and dualities of the underlying space $\mathbb{P}^3$, namely the former for the even permutations A_7, and the latter for the odd permutations. We thus achieve a projective representation

$$A_7 \to \mathrm{PGL}_4(\mathbb{C})$$

Compared to the projectivization of the standard representations, $A_7, S_7 \to \mathrm{PGL}_6(\mathbb{C})$, this is a reduction of dimensionality by 2; the improvement is in this sense as "good" as the one achieved by the icosahedral representation of A_5.

In the remainder of this section, let us reconstruct Klein's explicit calculation of this representation. The first calculation concerns the transformation of our line coordinates x_i to the usual Plücker coordinates p_{jk} (§5). This is necessary, because the relations between $\mathbb{P}^3$ (and thus: $\mathcal{E}_1$) and line coordinates are given in Plücker coordinates, while our intended permutations of line coordinates cannot simply be taken to be transformations of Plücker coordinates, because they would not leave the Plücker equation

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[38] Likewise, the "reflections" correspond to dualities in $\mathbb{P}^3$. Today, this isomorphism is considered *exceptional* (see Footnote 14), because it does not generalize to arbitrary dimensions. I will come back to this point in Sect. 2.5.

invariant. Practically, we are looking for a transformation that maps the quadratic form of line coordinates, $\sum_{i=0}^{6} x_i^2 = 0$, to the Plücker equation $p_1 p_4 + p_2 p_5 + p_3 p_6 = 0$. As for the case $n = 5$ (Sect. 2.2), we start with a rotation of our coordinate system by introducing new coordinates (Eq. 15) $\pi_i = (1, \gamma^i, \gamma^{2i}, \gamma^{3i}, \gamma^{4i}, \gamma^{5i}, \gamma^{6i})$, where γ is a 7th (primitive) root of unity. Again, this makes the first coordinate π_0 vanish for all points on the quadric. We solve this system of equations for x_i, (Eq. 23), $x_i = \left(\gamma^{-i} \pi_1 + \gamma^{-2i} \pi_2 + \cdots + \gamma^{-6i} \pi_6 \right) / 7$, and plug the values into the quadratic equation $\sum_{i=0}^{6} x_i = 0$. This gives us

$$\frac{7}{2} \sum_{i=0}^{6} x_i^2 = \pi_1 \pi_6 + \pi_2 \pi_5 + \pi_3 \pi_4, \tag{24}$$

which looks almost like our Plücker equation! (A great example of Klein's careful preparation that makes almost all actual calculation superfluous.) All we have to do is a small redefinition (Eq. 25) $\pi_6 = p_1, \pi_1 = p_2, \ldots$, which gives us[39]

$$x_i = \gamma^i p_1 + \gamma^{4i} p_2 + \gamma^{2i} p_3 + \gamma^{6i} p_4 + \gamma^{3i} p_5 + \gamma^{5i} p_6. \tag{26}$$

The second step is to use these equations to calculate how the permutations of the x_i effect the Plücker coordinates. To do so, it is enough to calculate the same for some generating elements, Klein took here $S = (0123456)$ and $T = (34)$. For S, the resulting transformation is

$$(p_1 : p_2 : p_3 : p_4 : p_5 : p_6) \mapsto (\gamma p_1 : \gamma^4 p_2 : \gamma^2 p_3 : \gamma^6 p_4 : \gamma^3 p_5 : \gamma^5 p_6)$$

(We cover T below.) The third step consists in finding the corresponding projectivity or duality. As S is even, it will be of the former kind, and "one can easily see" that this is achieved by the representation[40]

[39] As Klein remarked in a footnote, he already considered a similar situation in Klein (1879b), p. 273 where he considered an analogous geometrical theory for equations with Galois group $PSL_2(\mathbb{F}_7)$; I come back to this point in Sect. 3.1.

[40] All Plücker coordinates are mapped to multiples of themselves, so it makes sense to try to achieve such transformation by also mapping projective coordinates z_i to multiples of themselves. As the whole projectivity is only defined up to a scalar, we can take one multiplication at liberty, and choose $z_1 \mapsto z_1$. As $p_1 = z_1 z_2' - z_2 z_1'$ maps to γp_1, we better map $z_2 \mapsto \gamma z_2'$. From $p_2 = z_1 z_3' - z_3 z_1'$ and $p_2 \mapsto \gamma^4 p_2$ we likewise infer $z_3 \mapsto \gamma^4 z_3$, and from $p_3 = z_1 z_4' - z_4 z_1'$ and $p_3 \mapsto \gamma^2 p_3$ we deduce $z_4 \mapsto \gamma^2 z_4$. We then check that these transformations also yield the desired transformations $p_4 \mapsto \gamma^6 p_4$, $p_5 \mapsto \gamma^3 p_5$, and $p_6 \mapsto \gamma^5 p_6$.

$$S: \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma^4 & 0 \\ 0 & 0 & 0 & \gamma^2 \end{pmatrix}. \tag{27}$$

Finally, the fourth step is to normalize the matrix up to a multiple ± 1, the significance of which becomes transparent in Sect. 3.4. In the case above, the matrix already has determinant ± 1, so we are done. The permutation $T = (34)$, is odd and corresponds to a duality. As the formulas for T turn out to be more complicated than the ones for S (so I guess), Klein simplified the case by referring to the "geometrical meaning" (ibid., p. 514) of the permutation: T leaves invariant those lines for which $x_3 - x_4 = 0$. A family of lines determined by one linear equation such as this one is called a *linear (line) complex*.[41] It has some interesting properties: all lines of a linear complex that pass through a fixed point lie in one plane and *vice versa*. Thus, a complex defines a duality between points and planes which is furthermore self-inverse and therefore called a *polarity*. What is more, this polarity leaves all the lines of the complex invariant! As T is a self-inverse ($T^2 = Id$) duality leaving the linear complex $x_3 - x_4 = 0$ invariant, it must be the polarity with respect to that complex. The equation of this polarity is established if we take the defining linear equation, in this case $x_3 - x_4 = 0$, bring it into Plücker coordinates, substitute $p_1 = z_1 z_2' - z_2 z_1', \ldots,$ and order everything by z' and z:

$$z' \cdot \underbrace{\begin{pmatrix} 0 & \gamma^4 - \gamma^3 & \gamma^2 - \gamma^5 & \gamma - \gamma^6 \\ \gamma^3 - \gamma^4 & 0 & \gamma^6 - \gamma^2 & \gamma^5 - \gamma^2 \\ \gamma^5 - \gamma^2 & \gamma - \gamma^6 & 0 & \gamma^3 - \gamma^4 \\ \gamma^6 - \gamma & \gamma^5 - \gamma^2 & \gamma^4 - \gamma^3 & 0 \end{pmatrix}}_{A} \cdot z = 0. \tag{39*}$$

For any given z, the points z' form a plane with plane coordinates A (by definition of the concept of plane coordinates). In this sense, A is the matrix (defined up to a scalar) representing the duality which corresponds to T. All that is left to do is to normalize the matrix. To do so, Klein did not calculate the determinant manually, but plugged in some specific values for z, z' to ease the calculation. It turned out that the scalar $\pm 1/\sqrt{7}$ will do. With the projectivity corresponding to S and the duality corresponding to T established, we have a proper representation (i.e., a group of projectivities) for exactly those products of S's and T's in which T occurs an even number of times. This establishes our $A_7 \to \mathrm{PGL}_4(\mathbb{C})$.

3.3 The special case of S_6

The case $n = 6$ was carried out in principle analogously to the above calculation, but with one additional improvement: the linear equation $\sum_{i=0}^{n-1} x_i = 0$ (which was used for $n = 7$ to interpret the *seven* variables $x_0, \ldots, x_7$ as line coordinates) was not yet "used" in the case $n = 6$. Geometrically, this equation singles out a linear complex

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[41] On the history of the theory of linear complexes, see again (Rowe 2019).

(just as the one defined by T above), which Klein called the *unit complex*. The polarity with respect to this complex can be shown to be given by the equation

$$(-w_2 : w_1 : w_4 : w_3) \leftrightarrow (w_1 : w_2 : w_3 : w_4), \tag{34*}$$

where $w = (w_1 : w_2 : w_3 : w_4)$ can be understood either as a plane which is mapped to a point, or as a point mapped to a plane. Any duality, as we saw, corresponds to a linear transformation of lines, which is in this case given by the self-inverse

$$u : x_i \leftrightarrow x_i - \sum_{i=0}^{n-1} x_i. \tag{13}$$

The latter leaves the lines of the linear complex element-wise invariant. The solutions of the principal equation of degree 6 fulfill the equation $\sum_{i=0}^{5} x_i = 0$, thus are represented by lines on the unit complex, and thus left invariant by the polarity above. As any solution of a reduced sextic is represented by a line at the unit complex, we can at liberty apply the transformation u without changing the solution. In particular, we can concatenate u to any linear transformation which corresponds to a duality, resulting in a product of two dualities, that is, a projectivity. In short, concatenating u is a tool to make permutations $S_6 \backslash A_6$ correspond to projectivities! This establishes a representation of the whole of S_6 as a group of projectivities in $\mathbb{P}^3$

$$S_6 \rightarrow \mathrm{PGL}_4(\mathbb{C}).$$

In a modern formulation, we would speak of a "twist in the sign" of S_6.[42] Klein used this trick for both $S = (012345)$ and $T = (12)$ (which are both odd and therefore correspond to dualities). The corresponding normalizes projectivities are

$$S \mapsto \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma^2 \end{pmatrix},$$

$$T \mapsto \pm \frac{1}{\sqrt{6}} \begin{pmatrix} i\sqrt{2} & 0 & \gamma^4 & \gamma^4 - 1 \\ 0 & -i\sqrt{2} & (\gamma^2 - 1) & \gamma^5 \\ \gamma & \gamma^5 + 1 & -i\sqrt{2} & 0 \\ \gamma + 1 & \gamma^2 & 0 & i\sqrt{2} \end{pmatrix} \tag{31 and 35}$$

with γ being a primitive sixth root of unity.

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[42] As this technique had virtually no impact on the subsequent development of the Hypergalois Program, I decided not to go into the details any further.

3.4 Accessory irrationalities

At the end of the second part, Klein devoted an extra paragraph "About the necessity of the double signs, which arise in the substitution formulas of the z" (§7). Klein's aim in this paragraph was to show that there cannot exist a representation

$$G \to \mathrm{GL}_4(\mathbb{C}), \qquad (G = S_6, A_7)$$

whose projectivization is the above calculated projective representation $G \to \mathrm{PGL}_4(\mathbb{C})$. Put differently, the projective representation of G necessarily lifts to a non-trivial cover in the space of linear transformations. This is shown by reference to the subgroup $V_4 \subset S_6, A_7$, from which we know from the *Lectures on the Icosahedron* that its representation $V_4 \to \mathrm{PGL}_2(\mathbb{C})$ necessarily lifts to a double cover $Q_8 \to \mathrm{GL}_2(\mathbb{C})$. Therefore, also for $S_6, A_7 \subset \mathrm{PSL}_4(\mathbb{C})$, we must pass to some (non-trivial) cover in $\mathrm{SL}_4(\mathbb{C})$. This is probably also the reason why Klein normalized his representations $A \in \mathrm{PGL}_4(\mathbb{C})$ (where A are the above matrices corresponding to $S, T \in S_6, A_7$); as in this way, we directly obtain the representations of the generators of the double covers in $\mathrm{GL}_4(\mathbb{C})$ (where $+A$ and $-A$ are now distinct elements). What he seemed to have missed was the fact that the matrices iA and $-iA$ are distinct elements in $\mathrm{GL}_4(\mathbb{C})$ which also project to the same transformation, we have therefore also the possibility of a 4-cover.[43]

Klein's discussion on these particularities stemmed from his interest in *accessory irrationalities*, of which the above situation is just a geometrical interpretation. These irrationalities occur in the process of solving equations, when the *radical extension* created in the process of solving an equation is strictly larger than the *splitting field* in which the roots lie. A notorious example is the case of an (irreducible) cubic equation with real coefficients. Its roots are always real, but the process of finding them still requires the use of complex numbers. Also the solution of the general quintic equation in terms of the icosahedral equation involves one additional square root. It was to Klein's great pleasure that he could prove what Kronecker only guessed, namely that this irrationality is in fact unavoidable.[44] In this example, the accessory irrationalities find a geometric interpretation in terms of the double cover, because the subsequent problem of solving an equation that is geometrically given necessitates a switch from homogeneous to affine coordinates.[45]

The general problem of accessory irrationalities was more easily treatable with the development of the modern, field-theoretic approach to Galois theory during the 1880s and 1890s by works of Heinrich Weber, Richard Dedekind, and Otto Hölder.

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[43] However, this extension of $G = S_6, A_7$ by $\{Id_4, i \cdot Id_4, -Id_4, -i \cdot Id_4\} \cong \mathbb{Z}_4$ partially splits, for it is already induced by Klein's double cover. I am grateful to Alain Genestier for pointing out and clarifying this detail.

[44] For the more personal story behind Kronecker's and Klein's rivalry, see Tobies (2021, p. 289).

[45] For the quintic, this is shortly explained in Slodowy (1993, p. xvii).

In fact, Hölder was motivated to work in this modern setting precisely because of the better way in which it could treat accessory irrationality, a work that was suggested to him by Felix Klein.[46]

3.5 Tschirnhaus transformations and covariants

We are now turning to the third part of Klein's 1886 paper, which he devoted to the search for covariants between the "solution space" of an equation of degree $n = 6, 7$ and the space $\mathbb{P}^3$, on which S_6, A_7 act as calculated above. We remember this procedure from the case $n = 5$, where first a Tschirnhaus transformation maps a general solution to one that is interpreted as a point on the principal surface $M_2^{(2)}$, from where it is mapped to some $\lambda \in \mathbb{P}^1_{(1)}$ via F.

For $n = 6, 7$, the idea is a bit different because instead of points on $M_2^{(2)}$ and parameters of their generators (i.e., the planes in $\mathcal{E}_1$), we consider lines and points in projective 3-space. As in the case of $n = 5$, Klein here considered not only solutions of a principal equation, but slightly more generally solutions of a reduced equation, i.e., an equation were merely the second coefficient a_1 vanishes, and for whose solutions we thus have

$$\sum_{i=0}^{n-1} x_i = 0. \tag{46}$$

A tuple $(x_0 : \ldots : x_{n-1})$ can then not anymore be understood as the coordinates of a line, but—what is good enough—as the coefficients of a linear equation between line coordinates, i.e., as a three-parameter family of lines, we called a linear complex (Sect. 3.2). If, accidentally, $\sum_{i=0}^{n-1} x_i^2 = 0$ *does* hold, the linear complex is called a *special linear complex* and singles out exactly those lines which meet the line with coordinates $(x_0 : \ldots : x_{n-1})$, and our original intuition is restored.[47]

Thus, for a reduced equation of degree $n = 6, 7$, the task is to find a point $z \in \mathbb{P}^3$ that is covariant with the linear complex $(x_0 : \ldots : x_{n-1})$ in the way that a permutation of coordinates of x_i corresponds to the application of the above representation on z. The principal idea to find such covariant point is to take a small detour, and first to obtain two linear complexes x' and x'' via two Tschirnhaus transformations from x, such that x' and x'' respect the additional identities

$$\sum_{i=0}^{n-1} (x_i')^2 = 0, \qquad \sum_{i=0}^{n-1} x_i' x_i'' = 0, \qquad \sum_{i=0}^{n-1} (x_i'')^2 = 0. \tag{47}$$

The left and the right identity say that both complexes are special linear complexes, and can therefore be interpreted as coordinates of two lines; the middle identity then

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[46] For further discussion, see the fourth chapter of Nicholson (1992), "A Transitional Period for Galois Theory" (p. 37–59), which treats the subject with great care.

[47] We could therefore think about special linear complexes as the dual of lines, and general linear complexes as the dual of some (imaginary) general lines. However, I never found such a line of thought in either Klein's or other contemporaries' work.

states that the two lines are coplanar, i.e., that they intersect in a point which we call z. As both x' and x'' stem from x by Tschirnhaus transformations, they are covariant to x and thus also the point z is. This is not only a beautiful construction of a covariant point z, but also produces, as Klein remarked, "the most general point z covariant to the complex x" (ibid., p. 529).

Unfortunately, the beauty faded when Klein proceeded with the construction of explicit formulas for the complexes x', x'' and the point z (§9, I omit most of the calculations): For the first Tschirnhaus transformation $x_i \mapsto x_i'$, Klein chose a quadratic map

$$x_i' = (x_i^2 - s_2/n) + \lambda x_i \qquad (52)$$

(he doesn't have to write "$x_i - s_1/n$", because already $s_1 = \sum x_i = 0$) for which he set the task to find a λ, such that $\sum x_i'^2 = 0$ will hold. Plugging Eq. (52) into $\sum x_i'^2 = 0$ gives a quadratic equation in λ, which one can solve for λ. Klein denoted the square root that appears in this quadratic equation by W', it represents a first accessory irrationality (see Sect. 3.4). Similarly, by a quite long calculation, a Tschirnhaus transformation $x_i \to x_i''$ can be found, such that $\sum x_i''^2 = 0$ and simultaneously $\sum x_i' x_i'' = 0$ hold. Also here, a coefficient λ has to be calculated from a quadratic equation, and, "oddly enough" (p. 527) the obtained second square root W'' does not involve the previously calculated W'. To summarize:

> It is clear that we cannot avoid the occurrence of two accessory square roots even if we determine λ', λ'' in a different way, provided we stick to the use of ordinary methods. Whether it is at all impossible to satisfy the equations (50) without any involvement of accessory irrationalities remains to be seen. The fact that the accessory irrationalities cannot be completely avoided has already been emphasized in §7. (ibid., p. 527)

Having calculated the two special linear complexes x' and x'', which can be interpreted as coplanar lines, Klein proceeded with the calculation of their intersection. For some reason or another, Klein did not use the elementary formula to calculate the intersection of two lines given in Plücker coordinates Lindemann and Clebsch (1891, p. 49). Instead, he took a third line x (not to be confused with our initial x), set a system of equations which forces x to cut both x' and x'', then replaced the Plücker coordinates of x by the coordinates of two points lying on x ($p_{12} = z_1 z_2' - z_2 z_1'$, ...), ordered the resulting system according to the coordinates of z, and finally asked for a solution z of the system that would hold independently of the choice of the other point, z'. Then, this point z is the intersection of X' and X''. Klein did not finish the calculations (which, according to him would require to give up the symmetry of the calculations), and instead concluded the paragraph on covariants with the observation that the values of z linearly depend on the "three-termed determinants of the Lagrange expression Π, Π'" (ibid., p. 530) (Π and Π' are the Plücker coordinates of X' and X''), compared to the direct linear dependence of the similar analogous expressions in his *Lectures on the Icosahedron*.

3.6 Generalizations

In the last paragraph of his article (§11), Klein discontinued his considerations of equations of degree six and seven, and provided a short outlook on equations of arbitrary degree n. Klein was not very explicit here, but the general idea seems to be the following: We interpret the space $\mathbb{P}^{n-1}$ as the projectivized root space of the general equation of degree n. The roots of the *reduced* equations, i.e., those equations for which the coefficient of x^{n-1} already vanished, lie on a hypersurface isomorphic to $\mathbb{P}^{n-2}$. The permutations of the roots are realized as permutations of the coordinates of $\mathbb{P}^{n-1}$ and thus also act (faithfully) on the hypersurface $\mathbb{P}^{n-2}$. If in the equation not only the coefficient of x^{n-1}, but also of x^{n-2}, vanishes, the solution lies on a quadratic (hyper-)surface of dimension $n-3$

$$M^{(2)}_{n-3} = \left\{ x \in \mathbb{P}^{n-1} \mid \sum_{i=0}^{n-1} x_i = \sum_{i=0}^{n-1} x_i^2 = 0 \right\} \subset \mathbb{P}^{n-2}. \tag{53}$$

Take $\nu = \lfloor n/2 \rfloor$, then $M^{(2)}_{n-3}$ contains a family of linear subspaces of dimension $\nu - 2$, which Klein called $R_{\nu-2}$. If n is even, these are already the maximal linear subspaces; if n is odd, there are two additional families of subspaces of dimension $\nu - 1$, $R^{(1)}_{\nu-1}$ and $R^{(2)}_{\nu-1}$, which meet in $R_{\nu-2}$.[48] The idea is to single out one such space for any given solution $(x_1, \cdots, x_n)$ of the equation. To do so, we again take a number of related Tschirnhaus transformations, namely $\nu - 1$ many. For $n = 4, 5$, we had one single transformation; for $n = 6, 7$, we had two transformations in terms of linear complexes. These relations are, as before, $\sum x_i' x_j'' = 0$ for any two (not necessarily distinct) transformations X' and X''. As Tschirnhaus transformations are covariant with respect to the action of S_n, so is the linear subspace thus singled out.

However, as Klein added, this generalization cannot be fruitfully carried any further, because the actual calculations of the covariants depended on function-theoretic (for $n = 4, 5$, i.e., $\nu = 2$) or line-geometric ($n = 6, 7$, i.e., $\nu = 3$) grounds which do not generalize:

> But for $\nu > 3$ such special tools fail and we will have to answer the question about the most appropriate definition of the resulting linear spaces by parameters, as well as about the behavior of these parameters at the possible permutations of x, in a direct, algebraic way. I would like to reserve the right to come back to this on occasion, and for the time being I will limit myself to referring to Mr. Lipschitz's investigations on orthogonal substitutions (Lipschitz 1880, 1886), which I will have to use in the process. (ibid., pp. 531–532)

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[48] We could also say more efficiently: $M^{(2)}_{n-3}$ has one (for n even) or two (for n odd) families of linear subspaces of maximal dimension $\lfloor (n-3)/2 \rfloor$; also see Footnote 31. Klein also remarked (albeit not using this fact any further) that the dimensionality of such family itself is $\nu(\nu-1)/2$. This gives us for $n = 4$ a one-dimensional space of points; for $n = 5$ two one-dimensional spaces of lines; for $n = 6$ a three-dimensional space of lines; for $n = 7$ two three-dimensional spaces of planes.

3.7 Summary

To summarize the above reconstruction of Klein's 1886 article, we can say that Klein succeeded to construct some "optimal" representations S_6, $A_7 \mapsto \mathrm{PGL}_4(\mathbb{C})$ by use of the exceptional isomorphism between special orthogonal transformations of $\mathbb{P}^5$ and linear transformations of $\mathbb{P}^3$. Klein also outlined the construction of some correspondence between the "hidden" roots of an equation of degree 6 or 7 and a point in $z \in \mathbb{P}^3$, and although the general idea is quite simple, the calculations appear long-winded and could not be finished. What has not been achieved at all was the employment of this geometrical construction for the purpose of actually solving an equation of degree 6 and 7. To understand what such a treatment would look like, let me quickly come back to the icosahedron: We define a polynomial on $\mathbb{P}^1$ that is invariant under the A_5-action by the icosahedral symmetries, i.e., that maps all 60 points that are identified by the A_5-action to one and the same point. This 60-cover of $\mathbb{P}^1$

$$q : \mathbb{P}^1 \to \mathbb{P}^1/A_5 \cong \mathbb{P}^1$$

can be explicitly calculated by considering the image of the *special points*: Under A_5, the 12 vertices, the 20 midpoints of the faces, and the 30 midpoints of the edges of the icosahedron, are permuted among themselves. With the points of the vertices given (Sect. 2.2), it is easy to construct a polynomial f of order 12 whose solutions are thus just these points. Also a polynomial H of order 20 and a polynomial T of order 30 could be calculated elementarily, but Klein more elegantly used results from invariant theory to see that H is in fact just the *Hessian* of f, while T is the *Jacobian* of f and H. (These are just the "special covariants" of Sect. 2.2.) Then, the map

$$q(z_1 : z_2) = \frac{H(z_1 : z_2)^3}{1728 \, f(z_1 : z_2)^5}$$

sends the vertices to ∞, the face midpoints to 0, and the edge midpoints to 1, the latter can be seen by the relation $T^2 = -H^3 + 1728 f^5$ (Klein 2019b, p. 62). Geometrically speaking, the special points above divide the icosahedron into 120 *fundamental triangles*, and it can be observed that q sends 60 of them to the "northern hemisphere" of the Riemann sphere and the other 60 fundamental triangles to the "southern hemisphere" (see Fig. 3), while the edges and among them the special points are sent to the equator.

In a modern formulation, one can say that Klein calculated the graded $\mathbb{C}$-algebra $\mathbb{C}(z_1 : z_2)^{A_5}$ which is generated by the invariants f, T, H between which the above *syzygy* (a term that goes back to Cayley and was popularized by Hilbert) holds.

If we are now given any particular reduced quintic equation $x^5 + \alpha x^2 + \beta x + \gamma = 0$ with an ordered list of "hidden" solutions $(x_0 : x_1 : x_2 : x_3 : x_4)$ and map it via F to some $\lambda \in \mathbb{P}^1$, then $u = q(\lambda)$ will be invariant under the permutations A_5 of the x_i. Thus, u will be symmetric in the solutions x_i, and can thus be calculated from the coefficients α, β, γ and the square root of the discriminant Δ alone![49] We can then pass to affine

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[49] The explicit formulas (which are too long to repeat here) can be found in Eschenburg and Hefendehl-Hebeker (2000), 17f..
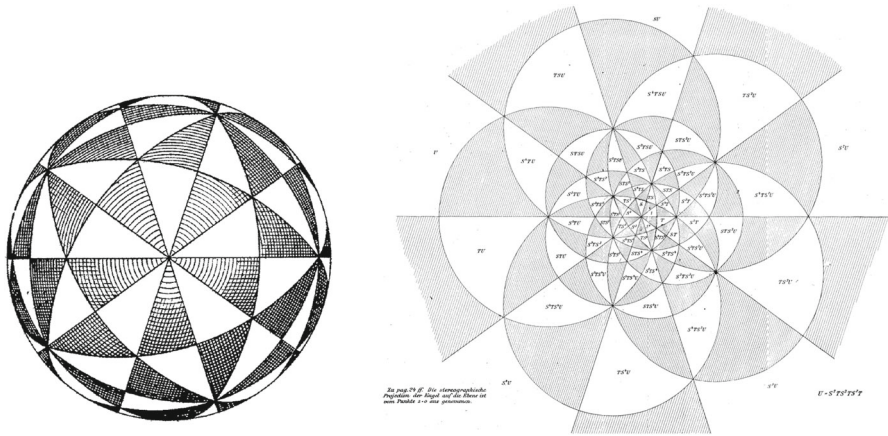
Fig. 3 q maps the 60 colored triangles to the "northern hemisphere" of the Riemann sphere (left), which is the "upper halfplane" $\mathcal{H} = \{x + iy \in \mathbb{C} \mid y > 0$ when complex numbers are viewed as a plane (right)

coordinates $z = z_1$, $z_2 = 1$ and write the above q as the *icosahedral equation*

$$((z^{20} + 1) - 228(z^{15} - z^5) + 494z^{10})^3 + 1728uz^5(z^10 + 11z^5 - 1)^5) = 0,$$

which can be solved by the analytic methods. It is sufficient to note here that these analytic methods use the fact that A_5 is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_5)$ and thus acts on the projective line with six points $(0, 1, 2, 3, 4, \infty)$ (also see Sect. 2.3). The big achievement of this whole complicated construction is that every reduced quintic equation can be transformed to a single equation, with only one variable parameter.[50]

None of this had been achieved for the case $n = 6, 7$, at least not in Klein's 1886 paper. The analogue of the above map q would result in the construction of some $G = S_6$, A_7-invariant

$$Q \colon \mathbb{P}^3 \to \mathbb{P}^3/G \cong \mathbb{P}^3$$

for which a calculation of the field of invariants $\mathbb{C}(z_0 : z_1 : z_2 : z_3)^G$ would be necessary. Later work in the theory of *quarternary* forms shows that attempts in such directions were made; it is however not possible to give closer attention to them within the scope of the present work.

4 Conclusion

This article covered a number of mathematical topics and a comparatively wide range of time, from Klein's inaugural dissertation of 1868 until the turn of the 20th century. Nevertheless, I attempted to restrict attention to those works that are in direct con-

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[50] Klein later described this achievement as analogous to the solution of pure equations $x^n = A$ by logarithms (Klein 2019a, p. 3).

nection to the main historical event of this article, namely Felix Klein's publication "On the theory of general equations of degree six and seven" of 1886. Its roots lie in two branches of mathematics that are today almost forgotten, namely line geometry and Klein's geometrical theory of equations. Likewise, the paper only influenced a relatively small branch of research, and only for a short period of time.

At the same time, the transience of Klein's research ideas was not clear at all during that time ("[The theory] of the sixth degree only just begun." (Cole 1886, p. 266)), and hope in a late success only vanished slowly ("But this is only a beginning." (Klein 2019a, p. 1)). The examination of Klein's article within the context of its time offers the modern reader an authentic insight into algebraic thinking at a time of rapid change in the subject matter. Also, the importance of Klein's program for the development and (maybe more important) for the popularization of Galois theory itself—similarly underestimated in the classical accounts—is worth being discussed (remember Klein's influence on Hölder's modern formulation of Galois theory, Sect. 3.4. In this sense, I can only support Gray's (2019) recent attempt of a "rehabilitation" of Klein as a Galois theorist:

> Klein not only presented a detailed account of the 'Galois theory' of polynomial equations, he argued polemically for his view as the 'right' one. He pushed for an autonomous theory of equations grounded in an analysis of their 'Galois groups' and independent of the theory of elliptic and modular functions (a field he certainly saw as important). This vision contributed to the Göttingen vision of contemporary mathematics, in which Gauss, Riemann, and Galois took pride of place, with its hierarchy of values that emphasized the role of concepts in determining what to calculate. In the 1870s and 1880s, when many regarded abstract group theory as too abstract, Klein's promotion of the subject was louder than anyone else's, and helped create the 20th century definition of Galois theory. (Gray 2019, p. 22).

But not only did Klein's efforts promote his own vision of Galois theory, they additionally (and somewhat ironically) helped in the development of the field-theoretic Galois theory we know today. Also in theory of groups itself, the impact of Klein's Hypergalois Program cannot be underestimated. The particularist approach of the Hypergalois Program, which relies on mathematical "accidents" and low-dimensional geometrical intuition instead of general principles, is often considered either irrelevant or even counter-progressive for the development of "abstract", "axiomatic" or "structural" understanding of group theory.[51] Klein's distaste against a solely axiomatic teaching of group theory and his tireless advocacy for intuition in mathematics and unification of unconnected mathematical branches can easily lead to the impression that Klein was, after all, a hindrance or at least a force of delay in the development of modern algebra.[52] This view is for example taken, at least with respect to the Hypergalois Program, by Wussing (2007), who was interested in the success of group

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[51] For a practice-based account on these notions, see Marquis (2014).

[52] The following reference might be one of the most-quoted statements of Klein: "This abstract formulation [of the group axioms] is excellent for the working out of proofs but it does not help one find one new ideas and methods …In general, the disadvantage of that method is that it fails to encourage thought." (Klein 1926, pp. 335–336), as in (Wussing 2007).

theoretical publications only with respect to their contributions toward an abstraction, or axiomatization of the concept. In this broad-scale narrative, the Hypergalois Program thus played an only marginal role:

> [A]ll these extremely impressive papers [of the Hypergalois Program] had done little to extend and deepen the new conceptual content of a group. This was because Klein and his fellow workers in this area relied on isomorphisms between groups of isometries and permutation groups, and thus stayed within the conceptual environment of (finite) permutation groups. (Wussing, [1969] 2007, pp. 208–209)

In fact, the opposite is the case: Klein's involvement with particular problems, specific interesting groups, and unexpected connections between mathematical theories might not reflect the "spirit" of modern mathematics as we have it today, but were essential ingredients to sharpen the mathematical community's understanding on which algebraic concepts and properties were worth following and which were not. In other words, Klein was an important source (if not, at least for some time, the most important one) to direct his fellow mathematicians' interest in the right direction, which in turn led them to formulate the *right* concepts.[53] Wiman's discovery about the exceptionality of the representations of A_5, A_6, A_7, which led to the invention of the *Schur multiplier* and ultimately (although indirectly) to the concept of cohomology, is only one example of such unintended influence.

Such motivation-based reading of mathematical innovation might be unusual (although it does exist, for example in Schlimm (2008)). However, especially in the case of Klein's Hypergalois *Program* (see Footnote 4), it offers an enriching narrative, and contributes to a better understanding of the becoming of modern mathematics.

Declarations

Conflict of interest The author states that there is no conflict of interest.

Archival material All used archival materials stem from the *Niedersächsische Staats- und Universisätsbibliothek Göttingen, Handschriftenabteilung* [UBG], particularly from the *Klein Nachlass*. (Digitally available at: http://hans.uni-goettingen.de/nachlaesse/Klein.pdf.)

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[53] Mehrtens's characterization of Klein as a proponent of a "counter-modern" vision of mathematics ("Gegenmoderne"), oppositional and at the same time essential for the formation of modern mathematics, goes into a similar direction (Mehrtens 1990, p. 328).

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