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A Note on Internal Partitions: The 5-Regular Case and Beyond

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Abstract
An internal or friendly partition of a graph is a partition of the vertex set into two nonempty sets so that every vertex has at least as many neighbours in its own class as in the other one. It has been shown that apart from finitely many counterexamples, every 3, 4 or 6-regular graph has an internal partition. In this note we focus on the 5-regular case and show that among the subgraphs of minimum degree at least 3 of 5-regular graphs, there are some which have small intersection. We also discuss the existence of internal partitions in some families of Cayley graphs, notably we determine all 5-regular Abelian Cayley graphs which do not have an internal partition.

Keywords Internal partitions · Friendly partition · Cohesive set · Satisfactory partition

1 Introduction

An *internal* or *friendly* partition of a graph is a partition of the vertices into two nonempty sets so that every vertex has at least as many neighbours in its own class as in the other one. The problem of finding or showing the existence of internal partitions

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in graphs has a long history. The same concept was introduced by Gerber and Kobler [13] under the name of *satisfactory partitions*, while Kristiansen et al. [17] considered a related problem on *graph alliances*. A survey of Bazgan et al. [4] describes early results on the area and discusses the complexity of the problem as well as how to find such partitions. Let us denote by $d_G(v)$ the degree of vertex v in graph G. For a set $U \subset V(G)$, $d_U(v)$ denotes the number of neighbours of v in U.

Stiebitz [24] proved that for every pair of functions $a, b : V \to \mathbb{N}^+$ such that $d_G(v) \geq a(v) + b(v) + 1,\ \forall v \in V$, there exists a partition of the vertex set $V(G) = A \cup B$, such that $d_A(v) \geq a(v),\ \forall v \in A$ and $d_B(v) \geq b(v),\ \forall v \in B$. This confirms a conjecture of Thomassen [25] in a strong form. Kaneko proved [15] that if G is triangle-free, then $d_A(v) \geq a(v),\ \forall v \in A$ and $d_B(v) \geq b(v),\ \forall v \in B$ can be satisfied even with $a, b : V \to \mathbb{N}^+$ such that $d_G(v) \geq a(v) + b(v),\ \forall v \in V$. This also implies that triangle-free Eulerian graphs have internal partitions, and reveals that the difficulty of the problem is fairly different for regular graphs having odd or even valency (which means the common degree of the vertices in a regular graph). The condition $d_G(v) \geq a(v) + b(v),\ \forall v \in V$ cannot be assumed in general, since there are graphs, e.g. K_{2n} which has no partition satisfying $d_A(v) \geq a(v), \forall v \in A$ and $d_B(v) \geq b(v), \forall v \in B$. Likewise, there exist infinitely many graphs having no internal partitions, e.g. K_{2n} and $K_{2n+1,2n+1}$. However, several large classes of graphs have been shown to have internal partitions. Diwan proved [8] that if a graph of girth at least 5 has minimum degree at least $a+b-1$, then its vertex set has a suitable partition $A \cup B$ with minimum degrees $\delta_{G|_A} \geq a$ and $\delta_{G|_B} \geq b$, on the graph induced by A and B, respectively. Moreover, Ma and Yang [20] showed that in the last statement of the theorem, it suffices to assume that G is C_4-free. Note however that graphs having internal partitions do not have a forbidden subgraph characterization [23].

The main goal of this paper is to make a contribution in the case of regular graphs. DeVos posed the following problem.

Problem 1 [6] Is it true that all but finitely many r-regular graphs have friendly (internal) partitions?

For certain small values of r, this was confirmed.

Theorem 1 (Shafique–Dutton [23], Ban–Linial [3]) *Let $r \in \{3, 4, 6\}$. Then apart from finitely many counterexamples, all r-regular graphs have internal partitions. The list of counterexamples is as follows.*

- *for $r = 3$, K_4 and $K_{3,3}$ do not have an internal partition* [23].
- *for $r = 4$, K_5 does not have an internal partition* [23].
- *for $r = 6$, every graph on at least 12 vertices has an internal partition, thus counterexamples have at most 11 vertices (and this bound is tight)* [3].

In fact, Shafique and Dutton conjectured that in the r even case only the complete bipartite graph does not admit an internal partition, but this was disproved by Ban and Linial [3] who constructed $2k$-regular graphs on $3k + 2$ vertices which do not have such partitions.

There are several directions in which partial results have been achieved recently concerning Problem 1. A natural weakening of the requirement is to show that a randomly chosen r-regular graph admits an internal partition. One may also pose some restrictions to obtain an affirmative answer for a large class of graphs. Another variant is to allow a small proportion of the vertices to have fewer neighbours than required. In these directions, impressive breakthrough results have been achieved lately.

Linial and Louis [18] proved that for every positive integer r, asymptotically almost every $2r$-regular graph has an internal partition. Very recently, Ferber et al. [11] resolved a conjecture of Füredi by proving that with high probability, the random graph $G(n, 1/2)$ admits a partition of its vertex set into two parts whose sizes differ by at most one, in which $n - o(n)$ vertices have at least as many neighbours in their own part as across.

We propose a new direction in the spirit of a lemma of Ban and Linial. For short, they introduced the term k-cohesive for vertex sets spanning a graph of minimum degree at least k.

Proposition 2 (Ban and Linial [3]) *Every n-vertex d-regular graph has a $\lceil d/2 \rceil$-cohesive set of size at most $\lceil n/2 \rceil$ for d even and of size at most $n/2 + 1$ for d odd.*

Problem 1 aims for two disjoint $\lceil d/2 \rceil$-cohesive sets A and B in d-regular graphs, provided that n is large enough. Indeed, let us add the vertices from the complement of $A \cup B$ one by one to A, provided that they have at least $\lceil d/2 \rceil$ neighbours in A. After the procedure stops, we add the remaining vertices to B, and it is straightforward that the resulting partition is internal.

Since there are d-regular graphs without two disjoint $\lceil d/2 \rceil$-cohesive sets, it is a natural goal to obtain a good universal upper bound on the intersection size of well-chosen pairs of $\lceil d/2 \rceil$-cohesive sets in d-regular graphs. This leads to

Problem 2 Let $\mathcal{G}_{n,d}$ denote the set of d-regular n-vertex graphs. Determine

$$\Phi(n, d) := \max_{G \in \mathcal{G}_{n,d}} \min \left\{ \frac{|V(H_1) \cap V(H_2)|}{n} \ : \ H_i \subseteq G, \delta(H_i) \geq \lceil d/2 \rceil \ \forall i \in \{1, 2\} \right\}.$$

If the answer for Problem 1 is affirmative, then clearly $\Phi(n, d) = 0$ holds for fixed d and $n > n_0(d)$. Note also that the lower bound on n is necessary since the choice $G = K_{d+1}$ and $G = K_{d,d}$, d odd, shows that the function Φ can admit values equal to at least $\frac{2}{d+1}$ and $\frac{1}{d}$ for $n = d + 1$ and $2d$, respectively. On the other hand, if $n \geq 12$ and $d \in \{3, 4, 6\}$, then $\Phi(n, d) = 0$ holds due to Theorem 1.

Our main result is

Theorem 3 $\Phi(n, 5) \leq 0.2456 + o(1)$.

We also show a slightly weaker statement, which on the other hand provides an exact result: in each n-vertex 5-regular graph, the intersection of 3-cohesive sets with minimum size contains at most $n/4 + 1$ vertices.

We also prove that there are exactly three Cayley graphs of valency 5 over finite Abelian groups which do not admit an internal partition.

Our paper is organized as follows. In Sect. 2. we briefly summarize the main definitions and notations that we will use throughout the paper and state some results which will serve as a starting point. Then we discuss how results concerning the bisection width relate to our problem, and point out that random-like or expander-like graphs are those in which internal partitions are hard to find. Indeed, as Bazgan, Tuza and Vanderpooten remark [4], one can find an internal partition via a simple local vertex-switching algorithm if there is a bisection of size at most $n/2$. On the other hand, the bisection width of almost all d-regular graphs of order n is at least $n(\frac{d}{4} - \frac{\sqrt{d \ln 2}}{2})$ according to the bound of Bollobás [5] and in fact this lower bound is not far from the upper bound $n\frac{d}{4} - \Theta(n\sqrt{d})$ due to Alon [1].

In Sect. 3 we prove the main result in a slightly weaker form first. Then we extend the theorem of Kostochka and Melnikov on the bisection width of a sparse graph, making it applicable to non-regular graphs as well. This enables us to prove the main result of the paper Theorem 3. We also discuss a different approach which relies on finding dense enough subgraphs with maximum degree constraints, which may be of independent interest.

Motivated by the expander-like property of graphs not having internal partitions, we study some families of Cayley graphs in Sect. 4, and characterize those graphs in these families that do not admit such partitions, including the 5-regular Cayley graphs over finite Abelian groups. Finally, we discuss further open problems in the area in the last section.

2 Preliminaries and Connections to Bisection Width

We begin this section by setting the main notations and definitions. Then we discuss the connection between the existence of internal partitions and the minimum size of bisection.

A bisection and a near-bisection of a graph with n vertices is a partition of its vertices into two sets whose sizes are the same, and whose sizes differ by at most one, respectively. The bisection size is the number of edges connecting the two sets. Note that finding the bisection of minimum size, in other words, the *bisection width* is NP-hard and only very weak approximations are known in general (see e.g. [10]).

Consider a regular graph G on vertex set V. A set $U \subseteq V$ is k-cohesive if G restricted to U has minimum degree at least k. $G|_U$ denotes the graph induced by the subset U. $N(v)$ denotes the set of neighbours of vertex v while $d(v)$ denotes the degree of vertex v, i.e., $d(v) = |N(v)|$. If we consider the degrees with respect to a certain induced subgraph or another graph, the respective graph is indicated in the index. $N[v]$ denotes the closed neighbourhood $N[v] := N(v) \cup \{v\}$.

Notation 4 *We use the notion $n_{(k)} = \binom{n}{k} \cdot k!$ for the falling factorial.*

Let us introduce two lemmas from the paper of Ban and Linial [3].

Claim [3] *An $(n-3)$-regular graph G has an internal partition if and only if its complementary graph G has at most one odd cycle. Furthermore, this partition is a near-bisection.*

Claim [3] *For even n, every $(n-2)$-regular graph has an internal bisection.*

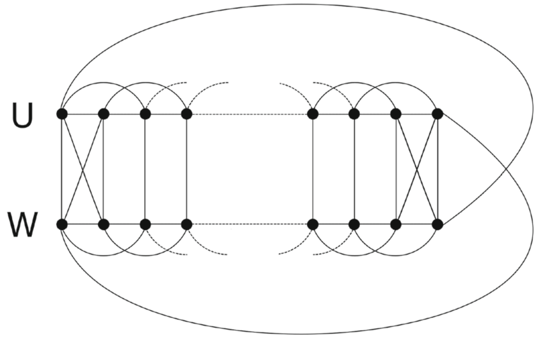


Fig. 1 A 5-regular graph with a relatively small bisection (U, W) where the local switching algorithm fails

As it was mentioned in [4], relatively small cuts give evidence to the existence of internal partitions. We present the proof for the case of 5-regular graphs as this class is the focus of our work and opt to extend it to the case of arbitrary regular graphs in the next step.

Proposition 5 *If there exists a bisection of a* 5*-regular simple graph* G *of size at most* $n/2 + 5$, *then there exists an internal partition for* G.

Proof Let us call a vertex *bad* if it has fewer neighbours in its own partition class than in the other one. Let us successively move bad vertices from their class to the other class. The number of edges is decreasing between the partition classes in each move. If no bad vertices remain after some moves, we ended up at an internal partition or one of the partition classes became empty. However, the latter case cannot happen. Suppose by contradiction that one of the classes could become empty at the end of the procedure. Then, after at least $(n/2 - 2)$ moves, we reached a phase where one of the partition classes has size 2. The number of edges between the two classes is at most $n/2 + 5 - (n/2 - 2) = 7$ at this point, but this contradicts to the valency of G. □

Proposition 5 is sharp, as the bound $n/2 + 5$ cannot be improved according to the result below.

Proposition 6 *For every even* $n \geq 8$, *there exists a* 5*-regular graph admitting a bisection of size* $n/2 + 6$ *in which the algorithm that successively put bad vertices to the other partition class ends with a trivial partition* (*consisting of the whole vertex set and an empty set*).

Proof Let $V(G) := \{u_1 \dots u_n\} \cup \{w_1, \dots w_n\}$, while $E(G) := \{u_i u_{i+1}, w_i w_{i+1} : i \in 1 \dots n-1\} \cup \{u_i w_i : i \in 1 \dots n\} \cup \{u_i u_{i+2}, w_i w_{i+2} : i \in 1 \dots n-2\} \cup \{u_1 w_2, u_1 w_n, u_2 w_1, u_{n-1} w_n, u_n w_1, u_n w_{n-1}\}$, see Fig. 1. The bisection is $U \cup W$. After moving u_1 to W as a first step, a chain of moves begins with moving u_i to W in the i-th step. □

Theorem 7

7.1 *If there exists a bisection in a* $(2k + 1)$*-regular graph* G *of size at most* $n/2 + k(k + 1) - 1$, *then there exists an internal partition for* G.

7.2 *If there exists a bisection in a $2k$-regular graph G of size at most $n + k(k-1) - 1$, then there exists an internal partition for G.*

Remark 1 For every integer $k > 0$ and even $n \geq 4k$, there exists a $2k+1$-regular graph admitting a bisection of size $n/2 + k(k+1)$ and a $2k$-regular graph admitting a bisection of size $n + k(k-1)$ in which the algorithm that successively put bad vertices to the other partition class ends with a trivial partition (consisting of the whole vertex set and an empty set).

Proof of Theorem 7, odd (even) valency We follow the spirit of the proof of case $k = 2$. After at least $n/2 - k$ moves, we reached a phase where one of the partition classes has size k. Since each move decreases the number of edges between the partition classes by one (and in the even case: two), the number of edges between the two classes is at most $n/2 + k(k+1) - 1 - (n/2 - k) = k(k+2) - 1$ (and in the even case: $n + k(k-1) - 1 - 2(n/2 - k) = k(k+1) - 1$) at this point, but this contradicts to the valency of G. Indeed, at most $\binom{k}{2}$ edges are induced by k points, thus there should be at least $k(2k+1) - 2\binom{k}{2}$ (and in the even case: $k \cdot 2k - 2\binom{k}{2}$) edges going between the two sets. □

Proof of Remark 1, odd (even) valency Let $V(G) := \{u_1 \dots u_n\} \cup \{w_1, \dots w_n\}$, and let $\{u_i u_j : 1 \leq i, j \leq n, 0 < |i - j| \leq k\} \cup \{w_i w_j : 1 \leq i, j \leq n, 0 < |i - j| \leq k\} \cup \{u_1 w_n, u_n w_1\}$ (and in the even case: $\{u_i u_j : 1 \leq i, j \leq n, 0 < |i - j| \leq k - 1\} \cup \{w_i w_j : 1 \leq i, j \leq n, 0 < |i - j| \leq k - 1\} \cup \{u_1 w_n, u_n w_1\}$) be part of the edge set. In order to obtain a regular graph of valency $2k + 1$ (and in the even case: $2k$) we complete the edge set which is possible due to the Gale–Ryser theorem (see [21, chapter 6]) on solving the bipartite realization problem. Consider the bisection $U \cup W$. After moving u_1 to W as a first step, a chain of moves begins with moving u_i to W in the i-th step. □

A result of Díaz et al. [7] proves that in fact, random 5-regular graphs indeed have small bisection width. Almost the same bound was obtained by Lyons [19] via a different method, namely using local algorithms.

Theorem 8 (Díaz et al. [7]) *The bisection width of random* 5*-regular graphs is asymptotically almost surely below* $0.5028n$.

As a consequence, we note that a tiny improvement on the result of Diaz, Serma and Wormald would imply the existence of internal partitions for almost all 5-regular graphs, in view of Proposition 6.

3 Finding Cohesive Sets with Small Intersection

Erdős, Faudree, Rousseau and Schelp proved the following.

Theorem 9 (Erdős et al. [9]) *Every graph G on $n \geq k - 1$ vertices with at least $(k-1)n - \binom{k}{2} + 1$ edges contains a subgraph with minimum degree at least k.*

Corollary 10 *Specializing to $k = 3$, this yields that an n-vertex graphs on $2n-2$ edges have 3-cohesive sets.*

This result has been strengthened in the following two directions.

Theorem 11 (Alon et al. [2]) *Let p be a prime power and G be a graph having average degree $\bar{d} > 2p - 2$ and maximum degree $\Delta(G) \leq 2p - 1$. Then G has a p-regular subgraph.*

This celebrated result was obtained by a clever application of the Combinatorial Nullstellensatz. Observe that for $k = p$, a dense enough graph G contains not only a k-cohesive set but also a subgraph which is k-regular.

Sauermann recently proved the following strengthening of the theorem of Erdős et al.

Theorem 12 (Sauermann [22]) *For every k there exists an $\varepsilon := \varepsilon_k > 0$ such that for every graph G on n vertices with at least $(k-1)n - \binom{k}{2} + 2$ edges contains a subgraph on at most $(1 - \varepsilon)n$ vertices with minimum degree at least k.*

Remark 2 Note that these results imply that if one finds a small enough k-cohesive set U in a $2k - 1$-regular graph, then the theorem of Erdős, Faudree, Rousseau and Schelp is applicable to $G \backslash G|_U$.

In order to prove our main result, the strategy is similar. Once we obtain a k-cohesive set U of minimum size in a $2k - 1$-regular graph $G = G(V, E)$, we wish to delete a set E^* of edges such that

- $|E \backslash E^*| \geq (k - 1)n - \binom{k}{2} + 1$ and
- $G^*(V, E^*)$ has as many vertices $v \in U$ of degree at least k as possible.

This would in turn provide a pair of k-cohesive sets with small intersection, due to Theorem 9. We discuss further only the case $k = 3$, however the methods below can be generalized.

3.1 Proof of the Main Result

First, we reiterate the lemma of Ban and Linial.

Proposition 13 (Ban and Linial [3]) *Every n-vertex d-regular graph has a $\lceil d/2 \rceil$-cohesive set of size at most $\lceil n/2 \rceil$ for d even and of size at most $n/2 + 1$ for d odd.*

We consider a result which may count on independent interest as well. The problem is to find a subgraph of fixed order with maximum number of edges which fulfills an extra constraint on a maximum degree. Some related work can be found in [12, 14].

Proposition 14 *If H is a 3-cohesive graph on n vertices with maximum degree 5, then for each $1 \leq k \leq n$ there exists a subgraph H' such that $|V(H')| = k$, $|E(H')| \geq k-1$ and the maximum degree $\Delta(H')$ of H' is at most 3.*

Proof We will show later that the statement holds in a stronger form if $n = k$, i.e., in this case a subgraph H' with $|E(H')| \geq k$ also exists. Thus we may assume that H is connected. First, we show that the statements holds for $k \leq 0.88n$. It clearly does hold for $k = 1$. Suppose, by contradiction, that there exists a number k less than $|V(H)|$ for which the statement fails for a certain graph H and let us choose the smallest k with that property. Hence, we get that for each subgraph $H' \subset H$ on $k-1$ vertices with maximum degree 3, $|E(H')| \leq k-2$. Indeed, otherwise we could add an isolated vertex to obtain a subgraph on k vertices with the prescribed property. Moreover, since k is the smallest such number, there exists a subgraph H' on $k-1$ vertices with maximum degree 3, $|E(H')| = k-2$.

Let us consider such a subgraph H' on $k-1$ vertices and maximum number of edges, and denote by t_i the number of vertices of degree i in H'. We in turn obtain that

$$t_0 + t_1 + t_2 + t_3 = |V(H')| = k - 1. \tag{1}$$

$$t_1 + 2t_2 + 3t_3 = 2|E(H')| = 2(k-2) = 2(t_0 + t_1 + t_2 + t_3) - 2. \tag{2}$$

Consider now the edges in $E(H)\backslash E(H')$. Since H' was maximal with respect to the number of edges, if $uv \in E(H)\backslash E(H')$ and $v \in V(H')$ is of degree $d_{H'}(v) < 3$, then $d_{H'}(u) = 3$. Indeed, $d_{H'}(u) < 3$ with $u \in V(H')$ contradicts to the maximality of the H' with respect to the number of edges, while $u \in V(H)\backslash V(H')$ would imply that u together with the edge uv can be added to H' to obtain a subgraph with the prescribed property. Thus by double counting the edges from $E(H)\backslash E(H')$ between vertices $v \in V(H')$ having degree $d_{H'}(v) < 3$ and vertices $u \in V(H')$ having degree $d_{H'}(u) = 3$, we obtain

$$3t_0 + 2t_1 + t_2 \leq 2t_3. \tag{3}$$

Here we also used that $3 \leq d_H(v) \leq 5$ for all $v \in V(H)$. However, Inequalities (2) and (3) together yield

$$3t_0 + 2t_1 + t_2 \leq 2t_3 \leq 4t_0 + 2t_1 - 4, \tag{4}$$

and this is in turn a contradiction unless $t_0 \geq 4$. The maximality of H' with respect to the number of edges however implies also that $t_0 \geq 2$ can only occur if there is no $e \in E(H)\backslash E(H')$ joining two vertices from $V(H)\backslash V(H')$. In other words, each of these vertices must be connected to the set of vertices having degree $d_{H'}(v) = 3$. Hence if $t_0 \geq 2$, then Inequality (4) can be improved as follows.

$$3(n-k+1) + 3t_0 + 2t_1 + t_2 \leq 2t_3 \leq 4t_0 + 2t_1 - 4. \tag{5}$$

We have $3t_0 \leq 2t_3$ from Inequality (3), thus (1) implies $t_0 \leq k - 1 - t_3 \leq k - 1 - 1.5t_0$, so we get $t_0 \leq \frac{2}{5}(k-1)$.

Putting all together, we obtain

$$3(n-k+1) \leq t_0 - 4 \leq \frac{2}{5}(k-1) - 4, \tag{6}$$

which is a contradiction for $k < \frac{3n+7.4}{3.4}$.

In the case $k \geq 0.88n$ we apply the probabilistic method. Let m_i denote the number of vertices of degree i in our 3-cohesive graph H. We have $n = m_3 + m_4 + m_5$. Let us choose uniformly at random a set Z of λn distinct vertices from $V(H)$. Moreover, let X denote the random variable counting the number of edges in Z. To obtain a suitable edge set, we must delete an edge from each vertex of degree 4 and delete a pair of edges from each vertex of degree 5 in Z. (Note that we may suppose that there are no edges joining vertices of degree larger than 3 in H.) Let Y denote the random variable which counts the number of edges which we should delete to obtain a graph on Z of maximum degree 3. We will use the alteration method and determine the expected value of $X - Y$, which gives a lower bound on the number of edges in a suitable subgraph of H on λn vertices with maximum number of edges. This will be carried out by decomposing the expected value into the sum of indicator variables corresponding to edges (for X) and vertices of degree four or five (for Y). Thus we have

$$\begin{aligned}\mathbb{E}(X-Y) = &\sum_{e \in E(H)} \mathbb{P}(\{x,y\} \subset Z : xy = e) - \sum_{v \in V(H), d(v)=4} \mathbb{P}(N[v] \subset Z)\\ &- \sum_{v \in V(H), d(v)=5} 2\mathbb{P}(N[v] \subset Z) - \sum_{v \in V(H), d(v)=5} \mathbb{P}(v \in Z, |N(v) \cap Z| = 4).\end{aligned} \tag{7}$$

Calculating the expressions above, we obtain

$$\begin{aligned}\mathbb{E}(X-Y) &= \frac{1}{2}(3m_3 + 4m_4 + 5m_5)\frac{\binom{\lambda n}{2}}{\binom{n}{2}} - m_4\frac{\binom{\lambda n}{5}}{\binom{n}{5}} - 2m_5\frac{\binom{\lambda n}{6}}{\binom{n}{6}}\\ &\quad -5m_5\frac{(\lambda n)_{(5)}(1-\lambda)n}{n_{(6)}}\\ &\geq \frac{3}{2}\lambda^2 \cdot \left(\frac{n-\frac{1}{\lambda}}{n-1}\right) \cdot n + m_4\left(\frac{\lambda^2}{2}\left(\frac{n-\frac{1}{\lambda}}{n-1}\right) - \frac{\binom{\lambda n}{5}}{\binom{n}{5}}\right)\\ &\quad +m_5\left(\lambda^2\left(\frac{n-\frac{1}{\lambda}}{n-1}\right) - 2\frac{\binom{\lambda n}{6}}{\binom{n}{6}} - 5\frac{(\lambda n)_{(5)}(1-\lambda)n}{n_{(6)}}\right)\\ &\geq \frac{3}{2}\lambda^2 \cdot \left(\frac{n-\frac{1}{\lambda}}{n-1}\right) \cdot n + m_4\left(\frac{\lambda^2}{2}\left(\frac{n-\frac{1}{\lambda}}{n-1}\right) - \lambda^5\right)\\ &\quad +m_5\left(\lambda^2\left(\frac{n-\frac{1}{\lambda}}{n-1}\right) - 2\lambda^6 - 5\lambda^5(1-\lambda)\frac{n}{n-5}\right).\end{aligned} \tag{8}$$

Indeed, $3m_3+4m_4+5m_5$ is the number of edges, and each edge is counted in X with probability $\frac{\binom{\lambda n}{2}}{\binom{n}{2}}$, while the probability that a vertex of degree i and its neighborhood is in Z is $\frac{\binom{\lambda n}{i+1}}{\binom{n}{i+1}}$.

Suppose first that $\lambda = 1$. Then $\mathbb{E}(X-Y) \geq n + \frac{1}{2}(m_3 - m_5)$. Since we have $3m_3 \geq 4m_4 + 5m_5$, $\mathbb{E}(X-Y) \geq n + \frac{1}{2}(m_3 - m_5) > n$ in turn follows. This proves that in Proposition 14 we could indeed assume that the graph is connected.

Suppose now that $n \geq 15$ and $\lambda \geq 0.8$.

We bound from below the expression (8) above by taking the minimum of $\frac{n-\frac{1}{\lambda}}{n-1}$, which is at $n = 15$ and $\lambda = 0.8$. This yields

$$\begin{aligned}
\mathbb{E}(X-Y) &= \frac{1}{2}(3m_3+4m_4+5m_5)\frac{\binom{\lambda n}{2}}{\binom{n}{2}} - m_4\frac{\binom{\lambda n}{5}}{\binom{n}{5}} - 2m_5\frac{\binom{\lambda n}{6}}{\binom{n}{6}} \\
&\quad -5m_5\frac{(\lambda n)_{(5)}(1-\lambda)n}{n_{(6)}} \\
&\geq \frac{3}{2}\lambda^2 \cdot \frac{13.75}{14} \cdot n + m_4\left(\frac{\lambda^2}{2}\frac{13.75}{14} - \lambda^5\right) \\
&\quad +m_5\left(\lambda^2\frac{13.75}{14} - 2\lambda^6 - 5\lambda^5(1-\lambda)\frac{n}{n-5}\right). \qquad (9)
\end{aligned}$$

Then both m_4 and m_5 have negative coefficients, moreover, their ratio is smaller than $7/8$. This means that the minimum of the expression with respect to the inequality $3m_3 \geq 4m_4 + 5m_5$ takes its value when $m_4 = 0$ and $m_5 = \frac{3}{8}n$. However,

$$\frac{41.25}{28}\lambda^2 \cdot n + \frac{3}{8}n\left(\frac{13.75}{14}\lambda^2 - 2\lambda^6 - 5\lambda^5(1-\lambda)\frac{n}{n-5}\right) > \lambda \cdot n \qquad (10)$$

holds, since the left side can be bounded below by taking $\frac{15}{10}$ instead of the term $\frac{n}{n-5}$, which concludes to the verification of positivity and monotonicity of the polynomial $\frac{41.25}{28} \cdot x^2 + 3/8(\frac{13.75}{14}x^2 - 2x^6 - 5x^5 \cdot 1.5 \cdot (1-x)) - x$ in the interval $[0.8, 1]$.

Thus, there exists a subgraph of size $k = \lambda \cdot n$ with at least k edges and each vertex has degree at most 3 in this case. □

Note that the constraint on the maximum degree of G was essential to obtain a linear bound on the edge cardinality. Indeed, a biregular complete bipartite graph with one class consisting of vertices of degree 3 shows that if one omits that constraint, only a constant number of edges can be guaranteed in the subgraphs for each order.

Now we are ready to prove the weaker form of our main result.

Theorem 15 *Suppose that G is a 5-regular graph on n vertices. Then there are two distinct internal sets $V_1, V_2 \subset V(G)$ such that $|V_1 \cap V_2| \leq n/4 + 1$.*

Proof Due to Proposition 13 we have a 3-cohesive subgraph $H \subset G$ on at most $n/2+1$ vertices. Our goal is to determine an edge set E^* of size at most $n/2+2$ such that after deleting it we can use Theorem 9 to find another 3-cohesive set with a intersection of size at most $n/4 + 1$.

First we use Proposition 14 in order to find a subgraph $H' \subset H$ such that $|V(H')| = n/4$, $|E(H')| \geq n/4 - 1$ and the maximum degree $\Delta(H')$ of H' is at most 3. Suppose that $|E(H')| = n/4 - 1 + t$ with $t \geq 0$. It is easy to see that we can add $3|V(H')| - 2|E(H')| = n/4 + 2 - 2t$ edges to $E(H')$ from $E(H)$ to increase the degree of each vertex $v \in V(H')$ to at least 3. Thus, we obtain an edge set of cardinality $|E^*| \leq n/2 + 1$, for which there are at least $n/4$ vertices which are incident to at least 3 edges of E^*.

Finally, we apply Theorem 9 to the graph obtained by deleting the edges of E^* from G. This graph has n vertices and at least $2n - 1$ edges, thus the theorem provides a 3-cohesive subgraph G' that does not contain $n/4$ vertices of H, hence $|V(G') \cap V(H)| \leq n/4 + 1$ holds. ◻

3.2 Improvement Via the Result of Kostochka and Melnikov

In order to improve Theorem 15, our aim is to strengthen Proposition 14 by pointing out the existence of a denser subgraph with the same constraints on the maximum degree. We proceed by applying a generalized version of a theorem of Kostochka and Melnikov.

Theorem 16 (Kostochka and Mel'nikov [16]) *For any given natural number $d \geq 2$ and for any connected d-regular graph G on n vertices, the bisection width $bw(G)$ fulfils*

$$bw(G) \leq \frac{d-2}{4}n + O(d\sqrt{n}\log n).$$

The proof of Theorem 16 consists of two main steps. First the authors cluster the vertex set of the graph to even number of equal clusters (of size roughly $\sqrt{n}$), apart from a small set of remainder vertices, in such a way that all the clusters contain at least roughly $\sqrt{n}$ edges. Then they randomly distribute the clusters into two large clusters of equal size, and they do the same with the remainder vertices as well. It is easy to verify that the generalization below also follows from their proof.

Theorem 17 (A generalization of the Kostochka–Melnikov bound) *For any given rational number $d \geq 2$, positive constant $c \geq 1$ and for any n-vertex connected graph G of average degree d and maximum degree at most cd, its bisection width fulfils*

$$bw(G) \leq \frac{d-2}{4}n + O(d\sqrt{n}\log n).$$

To show Theorem 17, one have to observe that in the paper of Kostochka and Melnikov, Lemma 1 provided a clustering of any n-vertex tree with maximum degree q to k-vertex forests with at least $k - 1 - \log_{\frac{q-1}{q-2}} k$ edges. Then while proving their main Theorem 16 on q-regular graphs, the estimation on the number of edges in between the random partition of the clusters actually uses only the average degree, and in addition the fact that q being the maximum degree as well, the missing number of edges $\log_{\frac{q-1}{q-2}} k$ affects only the error term. However, the latter statement equally holds if the tree is of bounded degree.

If we are interested in a dense subgraph of given order, the same approach can be applied. Indeed, by making clusters of size $\sqrt{n} + O(1)$ via Lemma 1 which contain at least $\sqrt{n} - O(\log n)$ edges each, distribute randomly the clusters to class A and class B in such a way that the number of clusters in A compared to the number of all clusters is proportion to $\alpha + o(1)$. Since the edges within clusters will contribute to edge set with probability roughly α while other edges will contribute with probability roughly α^2, we obtain the following theorem by averaging.

Theorem 18 *Let $d \geq 2$ be a rational number, and $\alpha \in (0, 1)$, $c \geq 1$ positive constants. For any n-vertex connected graph G of average degree d and maximum degree at most cd, there exists a subgraph G' on $\lfloor \alpha n \rfloor$ vertices, which has at least $(\alpha + o(1))n + \frac{d-2}{2}\alpha^2 n$ edges.*

Let $\mu \in (0, 1)$ be the real root of $36x^5 - 45x^4 + 8$. Note that $\mu \approx 0.88$. Now we are ready to make an improvement on Proposition 14.

Proposition 19 *Let H be an n vertex 3-cohesive graph with maximum degree 5 where $n > 12$. Then for each $0 \leq k \leq n$, there exists a subgraph H' such that $|V(H')| = k$, $|E(H')| \geq f(k) + o(n)$ and $\Delta(H') \leq 3$, where*

$$f(k) = \begin{cases} k + 0.1355k^2/n & \text{if } k \leq \mu n \\ 1.875k^2/n - 1.875k^5/n^4 + 1.125k^6/n^5 & \text{if } k > \mu n. \end{cases}$$

Proof We apply the probabilistic method of Proposition 14 together with a probabilistic clustering of the graph in the spirit of the Kostochka-Melnikov bound. Let m_i denote again the number of vertices of degree i in our 3-cohesive graph H, which implies $n = m_3 + m_4 + m_5$. We may suppose that each edge is incident to a vertex of degree 3, otherwise erasing the edge would still result a 3-cohesive graph. Let us choose uniformly at random a set Z of $c_1 n$ distinct vertices from $V(H)$. The constant $c_1 = c_1(k)$ is chosen later on in order to obtain an optimized bound. Let X denote the random variable counting the number of edges in $H|_Z$. To obtain a suitable subgraph with maximum degree at most 3, we must delete an edge from each vertex of degree 4 and delete a pair of edges from each vertex of degree 5 in $H|_Z$. Let Y denote the random variable which counts the number of edges which we should delete to obtain a graph on Z of maximum degree 3 as described above. Then we have

$$\begin{aligned} \mathbb{E}(X - Y) = &\sum_{e \in E(H)} \mathbb{P}(\{x, y\} \subset Z : xy = e) - \sum_{v \in V(H),\, d(v)=4} \mathbb{P}(N[v] \subset Z) \\ &- \sum_{v \in V(H),\, d(v)=5} 2\mathbb{P}(N[v] \subset Z) - \sum_{v \in V(H),\, d(v)=5} \mathbb{P}(v \in Z, |N(v) \cap Z| = 4). \end{aligned} \tag{11}$$

Calculating the expressions above we obtain

$$\mathbb{E}(X-Y) \geq \frac{3}{2}c_1^2 \cdot n + m_4\left(\frac{c_1^2}{2} - c_1^5\right) \\ + m_5\left(c_1^2 - 2c_1{}^6 - 5c_1^5(1-c_1)\frac{n}{n-5}\right) - O(1). \tag{12}$$

Here we followed the calculations of (8) and used the fact that $\frac{\binom{\lambda n}{t}}{\binom{n}{t}} = \lambda^t - O_\lambda(1/n)$, provided that t is smaller than $\lambda n/2$. The condition will clearly hold for us as t is small constant in each term. This implies the existence of a dense enough subgraph on a set V_1 of $c_1 n$ vertices, which has maximum degree at most 3.

Now, we use Theorem 18 to find a set $V_2 \subseteq V_1$ with $|V_2| = k = c_2 n$, i.e., $\alpha = c_2/c_1$. Then $e := e(G[V_2]) \geq (c_2 + o(1))n + (\mathbb{E}(X-Y) - c_1 n)\left(\frac{c_2}{c_1}\right)^2$. Thus we get

$$\frac{e}{c_2 n} \geq (1+o(1)) + \frac{\mathbb{E}(X-Y)c_2}{c_1^2 n} - \frac{c_2}{c_1} \\ \geq (1+o(1)) + \frac{c_2}{c_1^2 n}\left(\frac{3}{2}c_1^2 \cdot n + m_4\left(\frac{c_1^2}{2} - c_1^5\right) + m_5\left(c_1^2 - 2c_1{}^6 - 5c_1^5(1-c_1)\right)\right) \\ - \frac{c_2}{c_1}. \tag{13}$$

Our aim is to determine $c_1 = c_1(k)$ which provides the best universal lower bound for the right hand side of (13). In order to do this, we have to find the extremum with restrictions $n = m_3 + m_4 + m_5$, $m_i \geq 0$. We know that the extremum is admitted at a point where at least one of the variables m_3, m_4, m_5 equals to zero. Furthermore, $m_4 \leq \frac{3}{7}n$ and $m_5 \leq \frac{3}{8}n$ hold since vertices of degree 5 are joint to vertices of degree 3 according to our assumption.

Case 1. $m_5 = 0$ and $m_4 = \lambda n$, $\lambda \in [0, 3/7]$.

$$\frac{e}{c_2 n} \geq 1 + \frac{3c_2}{2} - \frac{c_2}{c_1} + \frac{m_4 c_2}{n}\left(\frac{1}{2} - c_1^3\right) = 1 + \frac{3c_2}{2} - \frac{c_2}{c_1} + \lambda c_2\left(\frac{1}{2} - c_1^3\right).$$

This expression is linear in λ, so the minimum of $\frac{e}{c_2 n}$ is taken at $\lambda = 0$ or $\lambda = \frac{3}{7}$. If $\lambda = 0$, then the right hand side is maximal at $c_1 = 1$, which yields the lower bound $1 + \frac{c_2}{2}$.

If $\lambda = \frac{3}{7}$, then the maximum of $1 + \frac{3c_2}{2} - \frac{c_2}{c_1} + \frac{3c_2}{7}(\frac{1}{2} - c_1^3)$, with respect to c_1, is at $c_1 = \sqrt[4]{7}/\sqrt{3}$ if $c_2 \leq \sqrt[4]{7}/\sqrt{3}$, otherwise at $c_1 = c_2$. This follows from the monotonity properties of $1/x + \frac{3}{7}x^3$. Consequently, the universal lower bound can be bounded from below by taking simply $c_1 = 1$ which means that the minimum of $\frac{e}{c_2 n}$ is at least $1 + \frac{2c_2}{7}$ once we apply the optimal choice of c_1.

Therefore $e \geq (1 + \frac{2c_2}{7})c_2 n$.

Case 2. $m_4 = 0$ and $m_5 = \lambda n$, $\lambda \in [0, 3/8]$.

$$\frac{e}{c_2 n} \geq 1 + \frac{3c_2}{2} - \frac{c_2}{c_1} + \frac{m_5 c_2}{n}(1 - 5c_1^3 + 3c_1^4) = 1 + \frac{3c_2}{2} - \frac{c_2}{c_1} + \lambda c_2(1 - 5c_1^3 + 3c_1^4).$$

Let $f(c_2, c_1, \lambda)$ denote the expression on the right hand side. Since it is linear in λ, the minimum of $f(c_2, c_1, \lambda)$ is at $\lambda = 0$ or at $\lambda = \frac{3}{8}$.

For $\lambda = 0$ we get back again the bound $1 + \frac{c_2}{2}$. For $\lambda = \frac{3}{8}$, we determine the maximum value of $1 + \frac{3c_2}{2} - \frac{c_2}{c_1} + \frac{3c_2}{8}(1 - 5c_1^3 + 3c_1^4)$ with respect to c_1, with partial differentiation:

$$\frac{d\left(1 + \frac{3c_2}{2} - \frac{c_2}{c_1} + \frac{3c_2}{8}(1 - 5c_1^3 + 3c_1^4)\right)}{dc_1} = \frac{(36c_1^5 - 45c_1^4 + 8)c_2}{8c_1^2}.$$

So the maximum point μ is the feasible solution of $36c_1^5 - 45c_1^4 + 8$, that is $\mu \approx 0.88$. In this case the minimum of $\frac{e}{c_2 n}$ is $f(c_2, \mu, 3/8)$ for $c_2 \leq \mu$. Otherwise, since $c_2 \leq c_1$ the minimum of $\frac{e}{c_2 n}$ is $f(c_2, c_2, 3/8)$ at $c_1 = c_2$.

This concludes to $e \geq (1 + 0.1355c_2)c_2 n$ for $c_2 \leq \mu$, and

$$e \geq \left(\frac{15c_2}{8} - \frac{15c_2^4}{8} + \frac{9c_2^5}{8}\right) c_2 n = (1.875c_2 - 1.875c_2^4 + 1.125c_2^5)c_2 n$$

for $c_2 > \mu$ in Case 2.

Finally, by comparing the results of Case 1 and 2, we have the following conclusions. If $c_2 \leq \mu$, then the minimum of $\frac{e}{c_2 n}$ is $f(c_2, \mu, 3/8)$, which is approximately $1 + 0.1355c_2$. Otherwise, it is $f(c_2, c_2, 3/8)$, which gives the expression $1.875c_2 - 1.875c_2^4 + 1.125c_2^5$. □

Theorem 20 *Suppose that G is a 5-regular graph on n vertices. Then there are two distinct internal sets $A, B \subset V(G)$ such that $|A \cap B| \leq (0.2456 + o(1))n$.*

Proof We follow the proof of Theorem 15. Due to Proposition 2 we have a 3-cohesive subgraph $H \subset G$ on at most $n/2 + 1$ vertices.

First we use Proposition 19 in order to find a subgraph $H' \subset H$ such that $|V(H')| = k$, $|E(H')| \geq f(k)$ and $\Delta(H') \leq 3$. Then we can add t edges to H' from $E(H)$ to increase the degree of each vertex in H' to at least 3, such that $t \leq 3k - 2f(k)$. Thus we obtain an edge set of cardinality $|E^*| \leq 3k - f(k)$.

To apply Theorem 9 to the graph obtained by deleting the edges of E^* from G, we need $3k - f(k) \leq \frac{n}{2} + 2$ to hold. This implies the choice

$$k = \frac{2n - \sqrt{3.729n^2 - 1.626n}}{2 \cdot 0.1355} \approx 0.2456n + o(n),$$

which satisfies these conditions. □

4 Internal Partitions in Cayley Graphs

As we could see in Sect. 2, the existence of internal partition follows if the bisection width is not large, or in general, if there is an almost balanced vertex cut of relatively small size. A celebrated theorem of Bollobás [5] proves that random r-regular graphs provide good expanders in the sense that the isoperimetric number is large compared to r, thus these vertex cut sizes are relatively large. Hence to seek graphs without internal partitions, it is natural to investigate well-structured expander graphs.

The first observation is derived by a computer-aided search.

Definition 21 (*Paley graph*) Let q be a prime power such that $q \equiv 1 (\text{mod } 4)$, let $V = \mathbb{F}_q$ and let $E = \left\{\{a, b\} : a - b \in \left(\mathbb{F}_q^\times\right)^2\right\}$. Then $G = (V, E)$ is the Paley graph of order q.

Claim *There exists an internal partition in every Paley graph of order less than* 500.

Paley graphs are special Cayley graphs. Next we study the existence of internal partitions in 5-regular Cayley graphs. Note that all previously known 5-regular graphs which do not admit an internal partition belong to this graph class. We will show that there is no further example in this class without internal partition.

Definition 22 Let G be a finite group and let S be a subset of G satisfying $0 \notin S$, and $S = -S$, i.e., $s \in S$ if and only if $-s \in S$. Then define the *Cayley graph* on group G with connection set S, denoted $Cay(G; S)$, to have its vertices labelled with the elements of G and x adjacent to y if and only if $y = x + s$ for some $s \in S$.

Definition 23 G is called an (additive) cyclic Cayley graph with a generating set $(i_1, \ldots, i_t)$ if $G = Cay(K, S)$, where K is a cyclic group and $S = \{\pm i_1, \ldots, \pm i_t\}$. If $K \simeq \mathbb{Z}_n$, then we denote G by $\langle i_1, \ldots, i_t \rangle_n$.

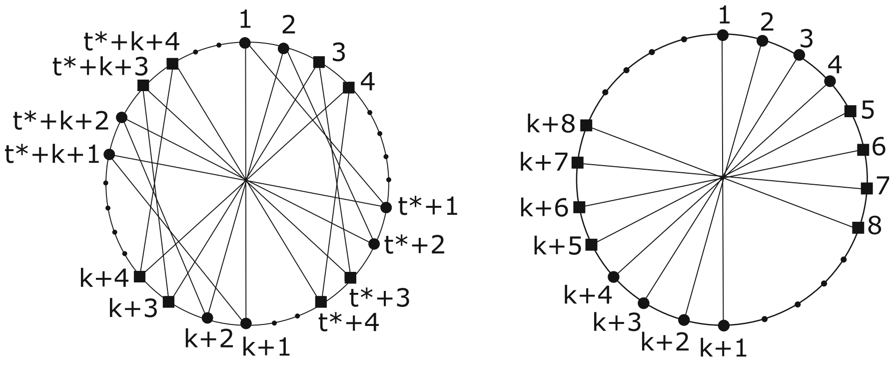
4.1 Cyclic Cayley Graphs

Theorem 24 *Every* 5*-regular cyclic Cayley-graph has an internal partition except for* K_6, $K_{5,5}$, *and* $\langle 1, 2, 5 \rangle_{10}$.

Observe first that the order of the group must be even, $n = 2k$. Furthermore, if the cyclic Cayley graph has odd valency, then k must be one of the generators, and we may suppose that other generators are less than k. We begin with some auxiliary lemmas.

Claim *Suppose that* $(r, 2k) = 1$ *holds for positive integers* r, k. *Then* $\langle r, t, k \rangle_{2k}$ *is isomorphic to* $\langle 1, t^*, k \rangle_{2k}$, *where* $r \cdot t^* \equiv t \pmod{2k}$.

Proof Let v_1 be an element of the vertex set of $\langle 1, t^*, k \rangle_{2k}$, v_1 is labeled with $g_1 \in (\mathbb{Z}_n, +)$ and v_2 an element of the vertex set of $\langle r, t, k \rangle_{2k}$, v_2 is labeled with $g_2 \in (\mathbb{Z}_n, +)$. Let v_2 be assigned to v_1, if and only if $g_2 = r \cdot g_1$. It is a bijection, because $(r, 2k) = 1$ and $g_1 - g_2 \in \{\pm 1, \pm t^*, k\}$ if and only if $r \cdot g_1 - r \cdot g_2 = r \cdot (g_1 - g_2) \in \{\pm r, \pm t, k\} \pmod{2k}$, therefore it is an isomorphism between $\langle 1, t^*, k \rangle_{2k}$ and $\langle r, t, k \rangle_{2k}$. □



(a) Subfigure 1 (b) Subfigure 2

Fig. 2 Types of disjoint internal subset pairs

Claim *If* $(t, k) \neq 1$ *then* $\langle r, t, k\rangle_{2k}$ *has an internal partition.*

Proof Consider the congruence classes of $\{1, 2, \ldots, 2k\}$ modulo (t, k). It is easy to check that they are internal subsets of $\langle r, t, k\rangle_{2k}$: if the distance of two elements is t or k, then they will be in the same class, hence every vertex degree is at least 3.

Since $1 < (t, k) < k$ (according to $0 < t < k$), thus $\frac{2k}{(t,k)} > 2$. Therefore, we find two disjoint internal subsets, which completes the proof. □

Proof of Theorem 24 If $(r, k) \neq 1$ or $(t, k) \neq 1$, then we are done by Claim 4.1. In the remaining case, r and t are even integers or without loss of generality we can assume that $(r, 2k) = 1$. In the first case, the vertices with even index will define an internal partition set. In the second case, by Claim 4.1 it is enough to examine the graphs $\langle 1, t^*, k\rangle_{2k}$.

First, we assume that $k \geq 8$. It is easy to check that

$$\{1, 2, t^* + 1, t^* + 2, k + 1, k + 2, t^* + k + 1, t^* + k + 2\} \quad \text{and}$$
$$\{3, 4, t^* + 3, t^* + 4, k + 3, k + 4, t^* + k + 3, t^* + k + 4\}$$

will be a pair of disjoint internal subsets for $t^* \in \{4, \ldots, k - 4\}$ (see Fig. 2a), and similarly,

$$\{1, 2, 3, 4, k + 1, k + 2, k + 3, k + 4\} \quad \text{and} \quad \{5, 6, 7, 8, k + 5, k + 6, k + 7, k + 8\}$$

will be a pair of disjoint internal subsets for $t^* \in \{2, 3, k-3, k-2, k-1\}$ (see Fig. 2b).

$\langle 1, 2, 3\rangle_6$ is K_6, $\langle 1, 2, 5\rangle_{10}$ is $P_{2,5}$, $\langle 1, 3, 5\rangle_{10}$ is $K_{5,5}$, so the list of Table 1 summarizes the remaining cases. □

Based on the cyclic Cayley graph $P_{2,5}$, it is natural to ask whether there exist cyclic Cayley graphs for each valency r, which are different from K_{r+1} and $K_{r,r}$, furthermore which do not admit an internal partition.

Table 1 Table of small 5-reg. Cayley graphs with internal partitions

Example	Internal sets
$\langle 1, 2, 4\rangle_8$	$\{1, 3, 5, 7\}, \{2, 4, 6, 8\}$
$\langle 1, 3, 4\rangle_8$	$\{1, 2, 5, 6\}, \{3, 4, 7, 8\}$
$\langle 1, 4, 5\rangle_{10}$	$\{1, 2, 6, 7\}, \{3, 4, 8, 9\}$
$\langle 1, 2, 6\rangle_{12}$	$\{1, 2, 3, 7, 8, 9\}, \{4, 5, 6, 10, 11, 12\}$
$\langle 1, 3, 6\rangle_{12}$	$\{1, 4, 7, 10\}, \{2, 5, 8, 11\}$
$\langle 1, 4, 6\rangle_{12}$	$\{1, 3, 5, 7, 9, 11\}, \{2, 4, 6, 8, 10, 12\}$
$\langle 1, 5, 6\rangle_{12}$	$\{1, 2, 7, 8\}, \{3, 4, 9, 10\}$
$\langle 1, 2, 7\rangle_{14}$	$\{1, 2, 3, 8, 9, 10\}, \{4, 5, 6, 11, 12, 13\}$
$\langle 1, 3, 7\rangle_{14}$	$\{1, 4, 5, 8, 11, 12\}, \{3, 6, 7, 10, 13, 14\}$
$\langle 1, 4, 7\rangle_{14}$	$\{1, 4, 5, 8, 11, 12\}, \{3, 6, 7, 10, 13, 14\}$
$\langle 1, 5, 7\rangle_{14}$	$\{1, 2, 3, 8, 9, 10\}, \{4, 5, 6, 11, 12, 13\}$
$\langle 1, 6, 7\rangle_{14}$	$\{1, 2, 3, 8, 9, 10\}, \{4, 5, 6, 11, 12, 13\}$

Proposition 25 *For every even $n > 2$ there exists a $(n - 3)$-regular cyclic Cayley graph on n vertices which does not contain an internal partition if and only if n is not a power of* 2.

Proof If n is not a power of 2, then it can be written of the form $n = l \cdot m$, where $l > 1$ is odd. Consider the graph $\langle m\rangle_n$. It is the union of $m > 1$ pieces of cycles of length l. Hence, by Claim 2, there is no internal partition in complementary of this graph. Therefore, we found a $(n - 3)$-regular cyclic Cayley graph on n vertices, such that it does not contain an internal partition.

Consider a $(n - 3)$-regular cyclic Cayley graph on n vertices, such that it does not contain an internal partition. The complementary of this graph (denoted by $\langle s\rangle_n$) is the union of cycles with the same length, and n is divisible by this common length. According to Claim 2, there are at least 2 cycles with odd length in $\langle s\rangle_n$. Hence, n has an odd divisor, therefore n is not a power of 2. $\square$

4.2 Cayley Graphs on the Group $\mathbb{Z}_2^t$

Let $G = Cay(\mathbb{Z}_2^t; \{g_1, \ldots, g_k\})$. Now for all $1 \le i \le k$, the edges generated by g_i determine a perfect matching, because all elements of $\mathbb{Z}_2^t$ is the negative of himself.

Theorem 26 *Let $G = Cay(\mathbb{Z}_2^t; \{g_1, \ldots, g_5\})$. Then G has an internal partition.*

Proof If $t = 3$, then the complementary of G is the union of two perfect matchings, so it is a two-regular and bipartite graph (with the vertex sets A and B). A and B induce a K_4 in the graph G, so they determine an internal partition.

If $t > 3$, then consider three generators g_1, g_2, g_3. We can assume that $g_3 \neq g_1 + g_2$, otherwise we change g_3 and g_4. Then $0, g_1, g_2, g_3, g_1 + g_2, g_1 + g_3, g_2 + g_3$ and $g_1 + g_2 + g_3$ are distinct element, and they are connected as shown in Fig. 3. G thus can be tiled by its subgraph $G' = Cay(\mathbb{Z}_2^t; \{g_1, g_2, g_3\})$, so we found 2^{t-3} disjoint internal sets which implies the existence of an internal partition in G. $\square$

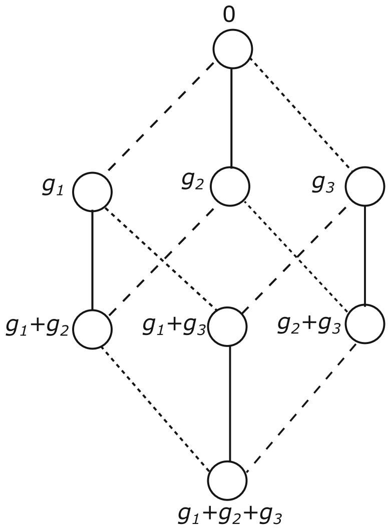


Fig. 3 Graph generated by three elements

4.3 Cayley Graphs of Finite Abelian Groups

We apply the structure theorem of finite Abelian groups and deduce that apart from some Cayley graphs arising from the small cyclic groups, every 5-regular Cayley graph over a finite Abelian group has an internal partition. First, we prove a special case and extend Theorem 24.

Proposition 27 *Suppose that $p > 1$ is a positive integer. Then every 5-regular Cayley graph on the group $\mathbb{Z}_2 \times \mathbb{Z}_{2p}$ has an internal partition.*

Proof Let $G = Cay(\mathbb{Z}_2 \times \mathbb{Z}_{2p}; S)$, $S = \{g_1, \ldots, g_k\}$. Let $T = \{(1, 0); (0, p); (1, p)\}$. If $g_i \in T$, then the edges generated by g_i determine a perfect matching, because they are their own negatives. If $g_i \notin T$, then the edges generated by g_i determine the union of disjoint cycles. So either $|S \cap T| = 3$ and $|S \setminus T| = 1$ or $|S \cap T| = 1$ and $|S \setminus T| = 2$ hold.

Suppose that $|S \cap T| = 3$. Then for all $0 \le q < p$, the set $\{(*, q); (*, q + p) : * \in \mathbb{Z}_2\}$ induces a complete graph K_4, i.e., a 3-cohesive set. These are disjoint subgraphs for all choices of q, so we found two disjoint internal sets.

Now suppose that $|S \cap T| = 1$ and $|S \setminus T| = 2$ hold. In the first case, let $g_1 = (1, 0)$ and denote the second coordinate of g_2 and g_3 by q and r with $q \le r < p$ and $* \in \mathbb{Z}_2$. If $q = r$, we obtain again induced K_4 graphs in the Cayley graph, thus we are done. Otherwise, consider the graph $G' = Cay(\mathbb{Z}_{2p}; \{q; r\})$. It is 4-regular and it is not the complete graph K_5 as G' has $2p$ vertices, so it has an internal partition in view of Theorem 1. We denote this partition by $A' \cup B'$. Let $A = \{(*, a) \mid * \in \mathbb{Z}_2; a \in A'\}$ and $B = \{(*, b) \mid * \in \mathbb{Z}_2; b \in B'\}$. Then $A \cup B$ is an internal partition of G.

This method works similarly in the other case $g_1 \in \{(0, p); ((1, p)\}$ after considering $G' = Cay(\mathbb{Z}_{2p}; \{q; r; p\})$. □

Theorem 28 *Every 5-regular Cayley graph arising from an Abelian group admits an internal partition except for three graphs as described in Theorem 24.*

Proof Let $\mathcal{G}$ be a finite Abelian group. Consider a 5-regular Cayley graph $Cay(\mathcal{G}, S)$ of $\mathcal{G}$ and let $\mathcal{G}_{(2)}$ denote its subgroup generated by the elements of order at most two.

In the first case, suppose that $|S \cap \mathcal{G}_{(2)}| \geq 3$. This implies we have 3 distinct generators $g_1, g_2, g_3 \in S$ of order 2. Then each coset of $\langle g_1, g_2, g_3 \rangle$ induces a 3-regular subgraph on at most 8 vertices, thus we are done provided that $|\mathcal{G}| > 8$. Groups of smaller order are already considered above.

In the second case, we have $|S \cap \mathcal{G}_{(2)}| < 3$, which in turn implies $|S \cap \mathcal{G}_{(2)}| = 1$ by the parity of the valency of $Cay(\mathcal{G}, S)$. Then there exists $g_1 \in S \backslash \mathcal{G}_{(2)}$.

$\langle g_1 \rangle = \mathcal{G}$ would imply that $\mathcal{G}$ is cyclic, which case is covered already in Theorem 24. Now suppose that $|\langle g_1 \rangle| = |\mathcal{G}|/2$. Then $\mathcal{G}$ must be either a cyclic group or a direct product of $\mathbb{Z}_2$ and a cyclic group. These subcases are already covered by Theorems 24 and 27. Finally, suppose that $|\langle g_1 \rangle| < |\mathcal{G}|/2$. The cosets of $\langle g_1 \rangle$ determine cycles in the Cayley graph. Let us take $g_2 := S \cap \mathcal{G}_{(2)}$-t and consider $\langle g_1, g_2 \rangle$. The cosets of this subgroup induce 3-regular graphs, moreover $|\langle g_1, g_2 \rangle| \in \{|\langle g_1 \rangle|, 2|\langle g_1 \rangle|\}$. As a consequence, we find at least two disjoint 3-regular subgraphs. □

5 Concluding Remarks

We presented an approach for how to show the existence of cohesive sets which have rather small intersection. Although our main Theorem 3 does not provide a bound close enough to the desired result $o(1)$, the applied technique pinpoints several subproblems of independent interest in which any breakthrough would imply an improvement for the bound of main Theorem 3 as well.

We pose this list of problems below.

Problem 3 Improve the bound of Ban and Linial, Lemma 2 by showing the existence of $\lceil r/2 \rceil$-cohesive sets in r-regular n-vertex graphs on much less than $n/2$ vertices, subject to $n \gg r$.

Note for example that if one could show the existence of a 3-regular H subgraph of the 5-regular graph G on less than $n/3 = |V(G)|/3$ vertices, that would provide a straightforward application of the Alon–Friedland–Kalai theorem. It would be really interesting to find an analogue of the Alon–Friedland–Kalai Theorem 11 with a restriction on the size of the subgraph as well. That would enable us to easily find dense subgraphs with the prescribed maximum degree, at least for certain values of the maximum degree d.

In a more general form, we formalize

Problem 4 Determine the best possible $\lambda_{r,t}$ constant, depending on r and t, for which the following holds. Let G be a r-regular graph on n vertices. Then there is a subgraph $H \subset G$ on at most $(\lambda_{r,t} + o(1))n$ vertices with minimum degree $\delta(H) \geq t$.

Problem 5 Prove a common generalisation of Theorems 11 and 12 which fixes the degree sequence of the subgraph and guarantee many 0-degrees.

It would be interesting to obtain a general lower bound $f(k)$ on the edge cardinality which can be guaranteed in at least one k-vertex subgraph of n-vertex graphs with a prescribed maximum degree condition. We showed that under the conditions of Proposition 14, $f(k) \geq k-1$ holds for every $k \leq n$, and Sect. 3.2 presents a possible way for how to improve that bound at least when k is not small compared to n.

Problem 6 Improve and generalize the result of Proposition 14 by obtaining a lower bound function on the cardinality of the edges of k-vertex subgraphs having a given bound on the maximum degree.

Problem 7 Prove that every Paley graph has an internal partition.

Problem 8 Prove that almost all 5-regular graphs have an internal partition via improving the algorithmic approach of Proposition 5 and applying Theorem 8.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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