2-Edge-Colored Chromatic Number of Grids is at Most 9

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Abstract

A 2-edge-colored graph is a pair (G, σ) where G is a graph, and $\sigma : E(G) \rightarrow \{', +', ' - '\}$ is a function which marks all edges with signs. A 2-edge-colored coloring of the 2-edge-colored graph (G, σ) is a homomorphism into a 2-edge-colored graph (H, δ) . The 2-edge-colored chromatic number of the 2-edge-colored graph (G, σ) is the minimum order of H . In this paper we show that for every 2-dimensional grid (G, σ) there exists a homomorphism from (G, σ) into the 2-edge-colored Paley graph SP_9 . Hence, the 2-edge-colored chromatic number of the 2-edge-colored grids is at most 9. This improves the upper bound on this number obtained recently by Bensmail. Additionally, we show that 2-edge-colored chromatic number of the 2-edge-colored grids with 3 columns is at most 8.

Keywords 2-Edge-colored coloring · Grids · Paley graphs

Mathematics Subject Classification 05C15

1 Introduction

In the whole paper we will use standard graph theory notations. A *2-edge-colored graph* is a pair (G, σ) where G is a undirected graph and $\sigma : E(G) \rightarrow \{', +', ' - '\}$. For the vertex $v \in V(G)$ by $N^-(v)$ (resp. $N^+(v)$) we denote the set of neighbors of v such that the edge to v has assignment $' - '$ (resp. $' + '$). Similarly for a set $S \subset V(G)$, we define $N^-(S) = \bigcup_{v \in S} N^-(v)$ and $N^+(S) = \bigcup_{v \in S} N^+(v)$.

A *2-edge-colored coloring* of a 2-edge-colored graph G is a proper coloring ϕ of $V(G)$ such that if there exist two edges $\{u, v\}$ and $\{x, y\}$ with $\phi(u) = \phi(x)$ and $\phi(v) = \phi(y)$, then these two edges have the same sign. The *2-edge-colored*

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chromatic number of the 2-edge-colored graph G , denoted by $\chi_2(G)$, is the minimum number of colors needed for a 2-edge-colored coloring.

Equivalently, the 2-edge-colored chromatic number $\chi_2(G, \sigma)$ of the 2-edge-colored graph (G, σ) is the minimum order of the graph (H, λ) such that (G, σ) admits a 2-edge-colored homomorphism (preserving edges and signs) to (H, λ) . Graph (H, λ) we call a *target graph* or *coloring graph*. The 2-edge-colored chromatic number $\chi_2(G)$ of a graph G is defined as maximum over all 2-edge-colored of G . For a graph class $\mathcal{F}$, we define the 2-edge-colored chromatic number $\chi_2(\mathcal{F})$ as the maximum over 2-edge-colored chromatic number for any members of $\mathcal{F}$. For convenience, in the rest of the paper we will omit the σ and simply write G for a 2-edge-colored graph (G, σ) if there is no ambiguity.

In this paper we focus on 2-edge-colored chromatic number for class of 2-dimensional *grids* $\mathcal{G}$. The grid is defined as the graph being the Cartesian product of two paths. Similarly, by $\mathcal{G}_k$ we denoted the grid with k columns.

The $\chi_2(\mathcal{G})$ was investigated in 2016 by Bensmail [1], who showed that $7 \leq \chi_2(\mathcal{G}) \leq 12$. Recently, in 2019, the same author [2] improved both bounds by showing, that $8 \leq \chi_2(\mathcal{G}) \leq 11$, determine the exact value $\chi_2(\mathcal{G}_2) = 5$ and bounds $7 \leq \chi_2(\mathcal{G}_3) \leq 9$.

In Sect. 2 we define 2-edge-colored Paley graph SP_q and focus on SP_9 as a target graph in 2-edge-colored homomorphisms. That graph was used in this context earlier, for example Montejano et al. [6] show that there exist 2-edge-colored homomorphism from every 2-edge-colored outerplanar graph to SP_9 . Some other application of that target graph we can find in [7].

In Sect. 3 we prove:

Theorem 1 *For every 2-edge-colored grid G there exists a 2-edge-colored homomorphism $h : G \rightarrow SP_9$.*

Corollary 2 $8 \leq \chi_2(\mathcal{G}) \leq 9$.

In Sect. 4 we focus on grids with 3 columns and improve upper bound given by Bensmail. More precisely we show that:

Theorem 3 *For every 2-edge-colored grid G with 3 columns there exists a 2-edge-colored homomorphism $h : G \rightarrow SP_9 \setminus \{0\}$.*

Corollary 4 $7 \leq \chi_2(\mathcal{G}_3) \leq 8$.

2 Graph SP_9

Let q be a prime power such that $q \equiv 1 \pmod{4}$ and $\mathbb{F}_q$ be finite field of order q . The *Paley graph* P_q is undirected graph with vertex set $V(P_q) = \mathbb{F}_q$ and edge set $E(P_q) = \{\{x, y\} : y - x \text{ is a non-zero square in } \mathbb{F}_q\}$. Notice that -1 is a square in $\mathbb{F}_q$. Hence, if $x - y$ is a square, then $y - x$ is a square.

The *2-edge-colored Paley graph* SP_q is 2-edge-colored graph (K_q, σ) , where K_q is the complete graph on vertices $\mathbb{F}_q$ and $\sigma(\{x, y\}) = ' + ' \iff y - x$ is a non-zero square in $\mathbb{F}_q$. In this paper we use 2-edge-colored Paley graph SP_9 build over the

Lemma 5 [8] *SP_q is isomorphic to the 2-edge-colored graph constructed by flipping signs of all edges.*

Lemma 6

- Lemma 7** *The Paley graph SP_9 is vertex-transitive and edge-transitive. Moreover for every two induced subgraphs S_1 and S_2 with $|S_1| = |S_2| = 3$ and the same number of edges marked with ‘-’, there is an automorphism of SP_9 which maps S_1 on S_2 .*

We shall call 3-elements induced subgraph $T \subset V(SP_9)$ a *triangle* if all its edges are marked with the same sign. Furthermore, we shall say that set $S \subset V(SP_9)$ is *triangle free* if the subgraph of SP_9 induced by S does not contain a triangle.

Proof By Lemmas 5 and 7, it is enough to prove this lemma for vertex $v = 0$ and $N^+(v)$. The subgraph of SP_9 induced by $N^+(0) = \{1, 2, x, 2x\}$ contains two disjoint edges signed '+' ($\{1, 2\}$ and $\{x, 2x\}$) and the cycle C_4 signed '-' ($(1, 2x, 2, x)$). $\square$

Lemma 10 Suppose that $S \subset V(SP_9)$, $|S| = 3$ and S is triangle free, then $|N^+(S)| = 8$ and $|N^-(S)| = 8$.

Proof By Lemma 5 and 7, it is sufficient to consider the triangle free set $S = \{0, 1, x\}$. Then $N^+(S) = SP_9 \setminus \{2x + 2\}$ and $N^-(S) = SP_9 \setminus \{0\}$. $\square$

3 Proof of Theorem 1

Consider a path (u, v, w) with arbitrary signs on the edges $\{u, v\}$ and $\{v, w\}$. Suppose that we have: an arbitrary 3-elements triangle free set $S_1 \subset V(SP_9)$ of colors available in u and an arbitrary color $b \in V(SP_9)$ for the vertex w . Then there is a 3-elements triangle free set $S_2 \subset V(SP_9)$ available in v . More precisely:

Lemma 11 Consider a path (u, v, w) with arbitrary signs on the edges $\{u, v\}$ and $\{v, w\}$. For every 3-elements triangle free set $S_1 \subset V(SP_9)$ and every color $b \in SP_9$, there exists a 3-elements triangle free set $S_2 \subset V(SP_9)$ such that for each $s_2 \in S_2$ there exists $s_1 \in S_1$ and a coloring $c : (u, v, w) \rightarrow V(SP_9)$ with $c(u) = s_1$, $c(v) = s_2$, $c(w) = b$.

Proof We will prove the lemma in case when both edges of path (u, v, w) are marked with '+'. In any other case the proof is similar..

By Lemmas 10 and 8, $|N^+(S_1)| = 8$ and $|N^+(b)| = 4$. Hence, there exists 3-elements set $S_2 \subset N^+(s_1) \cap N^+(b)$. By Lemma 9, the set S_2 is triangle free. $\square$

Proof of Theorem 1 We color the 2-edge-colored grid G row by row. It is easy to color first row by SP_9 (in fact, we can do it using only four colors, for example by $N^+(0) = \{1, 2, 2x, 2x + 1\}$). Assume now that, for $k > 1$, the first $k - 1$ rows of G have been already colored and we color k -th row.

Let us denote the vertices in the $k - 1$ -th row by $a_1, a_2, \dots, a_n$ and the vertices in the k -th row by $b_1, b_2, \dots, b_n$. By Lemma 8, the vertex b_1 can be colored by four possible colors. By Lemma 9, any three of these colors form a triangle free set. Let us denote by S_1 any of these sets. Now for each $i = 2, 3, \dots, n$ we define set S_i as a result of applying Lemma 11 for the set S_{i-1} and the color $h(a_i)$.

Now we can color vertices $b_1, b_2, \dots, b_n$ in reverse order. First we choose any color in S_n for $h(b_n)$. For $h(b_{n-1})$ we set the color in S_{n-1} such that the sign of the edge $(h(b_{n-1}), h(b_n))$ in SP_9 equals to the sign of the edge (b_{n-1}, b_n) in the grid G . Notice that for each $s \in S_{n-1}$, the sign of the edge (a_{n-1}, b_{n-1}) in the grid is equals to the sign of the edge $(s, h(a_{n-1}))$ in SP_9 . Consecutive vertices we color in the same way. $\square$

4 Grids with 3 Columns

In the previous section, we show that every 2-edge-colored grid admits 2-edge-colored coloring with 9 colors. This bound can be improved when we consider grids with a small number of columns. Bensmail [2] shows that 2-edge-colored chromatic number of grids with two columns is equal to 5. From the same paper we know that

$7 \leq \chi_2(\mathcal{G}_3) \leq 9$. We can improve this upper bound by showing that there exists a homomorphism from every 2-edge-colored grids with 3 columns to SP_9^{-0} , see Fig. 2. SP_9^{-0} is construct from SP_9 by removing vertex 0. It is easy to see that this graph has 8 automorphisms. Namely, four rotations ($f(y) = ay$ where a is a non-zero square) and four reflection. SP_9^{-0} is also isomorphic to the graph constructed by flipping signs of all edges.

Lemma 12 Consider a path (u, v, w) with arbitrary signs on the edges $\{u, v\}$ and $\{v, w\}$. For every color $a \in SP_9^{-0}$ and every $b \notin \{a, -a\}$ there exists a coloring of path $c : (u, v, w) \rightarrow V(SP_9^{-0})$ such that $c(u) = a$ and $c(w) = b$.

Proof By automorphisms of SP_9^{-0} and the fact that it is isomorphic to the graph constructed by flipping signs of all edges, we can only consider the case when $a = 1$.

- $N^+(N^+(1)) = V(SP_9^{-0}) \setminus \{2\}$
- $N^+(N^-(1)) = V(SP_9^{-0}) \setminus \{1\}$
- $N^-(N^+(1)) = V(SP_9^{-0}) \setminus \{1\}$
- $N^-(N^-(1)) = V(SP_9^{-0})$

□

This observation leads to partition of SP_9^{-0} into four sets: $\{1, 2\}$, $\{x, 2x\}$, $\{x + 1, 2x + 2\}$ and $\{x + 2, 2x + 1\}$. Regardless of the signs of the edges of the path (u, v, w) , if endpoints are colored using colors from two different sets, then we can color the middle point of the path.

Proof of Theorem 3 Let us consider the partition of $V(SP_9^{-0})$ into four sets: $\{1, 2\}$, $\{x, 2x\}$, $\{x + 1, 2x + 2\}$ and $\{x + 2, 2x + 1\}$. Observe that each set contains a pair $\{a, -a\}$ and that for every vertex $v \in SP_9^{-0}$, both $N^+(v)$ and $N^-(v)$ contain an element in at least three sets. We color the grid row by row. Three vertices in the first row we can easily color with SP_9^{-0} . Assume now that, for $k > 1$, the first $k - 1$ rows of G have been already colored and we color k -th row. Let us denote the

Fig. 2 2-edge-colored graph SP_9^{-0} . Solid lines represent edges signed by '+'

vertices in the $k - 1$ -th row by a_1, a_2, a_3 and the vertices in the k -th row by b_1, b_2, b_3 . Now color vertices b_1, b_2, b_3 starting from b_2 . Depending on sign of the edge $\{a_2, b_2\}$, for $c(b_2)$ use color in $N^+(c(a_2))$ or $N^-(c(a_2))$ which does not belong to the pairs: $\{c(a_1), -c(a_1)\}$ and $\{c(a_3), -c(a_3)\}$. By Lemma 12, we can extend this coloring to vertex b_1 (using path (a_1, b_1, b_2)) and to b_3 (using (b_2, b_3, a_3)). $\square$

5 Discussion

In this article, we show that SP_9 colors all 2-edge-colored grids and that SP_9^{-0} colors all 2-edge-colored grids with 3 columns. Combining this with known lower bounds we have $8 \leq \chi_2(\mathcal{G}) \leq 9$ and $7 \leq \chi_2(\mathcal{G}_3) \leq 8$. In our opinion, SP_9^{-0} is the best candidate to lower the upper bound for $\chi_2(\mathcal{G})$. However, we do not know whether SP_9^{-0} colors all grids, and even whether it colors all grids with 4 columns. On the other hand, to increase lower bound for $\chi_2(\mathcal{G})$ we have to prove that there exists a 2-edge-colored grid G such that there is no homomorphism $G \rightarrow SP_9^{-0}$. Finding such a grid also seems to be difficult.

Many authors investigate the relations between 2-edge-coloring and oriented coloring. Oriented coloring (see [9] for a short survey) of an oriented graph G is a homomorphism from G to a tournament H . The oriented chromatic number $\chi_o(G)$ for oriented graph and for undirected graph are defined analogously to 2-edge-colored case. A lot of research on 2-edge-colored and oriented coloring used the same techniques and get similar results. For example, in both cases, authors often use Paley graphs (or Paley tournaments) for the target graph. These may result from the similarity of the definitions: in both cases, edges are in one of two states (one of two signs or one of two directions). Despite these similarities, it is known that $\chi_o(G) - \chi_2(G)$ and $\chi_2(G) - \chi_o(G)$ can be arbitrarily large [3].

For grids, these two colorings seem to behave very similar. For example, for grids with 2 columns, we know that: $\chi_2(\mathcal{G}_2) = 5$ [2] and $\chi_o(\mathcal{G}_2) = 6$ [5]. For grids with 3 columns, we have $7 \leq \chi_2(\mathcal{G}_3) \leq 8$ and $\chi_o(\mathcal{G}_3) = 7$ [10]. The difference between these two chromatic numbers is very small. Therefore one can hope that also for the class of grids, they are close to each other. We know that oriented chromatic number for grids lies between 8 and 11 [4, 5]. Upper bound has not been improved since the first paper on the topic.

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