Graphs and Combinatorics (2019) 35:1405–1432
https://doi.org/10.1007/s00373-019-02096-1

ORIGINAL PAPER

Cyclic Permutation Groups that are Automorphism Groups of Graphs

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Received: 30 May 2018 / Revised: 30 May 2018 / Published online: 13 September 2019

Abstract
In this paper we establish conditions for a permutation group generated by a single permutation to be an automorphism group of a graph. This solves the so called concrete version of König's problem for the case of cyclic groups. We establish also similar conditions for the symmetry groups of other related structures: digraphs, supergraphs, and boolean functions.

Keywords Graph automorphism · Automorphism group · Permutation group · Cyclic group

1 Introduction

Frucht's theorem, conjectured by Dénes König states that every abstract finite group is isomorphic to the automorphism group of a graph [30]. On the other hand it is known that not every permutation group is an automorphism group of a graph. For example, there is no graph on n vertices whose automorphism group is the cyclic group C_n generated by an n-element cycle. The problem asking which permutation groups can be represented as automorphism groups of graphs is known as the concrete version of König's problem [29].

This problem turned out much harder and was studied first for regular permutation groups as the problem of *Graphical Regular Representation*. There were many partial results (see for instance [15–17,22–24,26–28]) until the full characterization was obtained by Godsil [6] in 1979. In [2], Babai uses the result of Godsil to prove a similar characterization in the case of directed graphs.

In [20,21], Mohanty et al., consider permutation groups generated by a single permutation (they call them *cyclic permutation groups*) whose order is a prime or a

Supported in part by Polish NCN Grant 2012/07/B/ST1/03318.

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power of a prime. In [21, Theorem 3], they described those cyclic permutation groups of prime power order greater than 5 that are automorphism groups of graphs. However, the results contained some gaps. The authors made a false claim that there are no such groups of prime order 3 or 5. Also the proof of the main result contained a gap. All this has been corrected in [9]. Our aim is to generalize these results to cyclic permutation groups of arbitrary order.

When comparing the results in [7,8,10,14,25] one may observe that usually formulations of theorems concerning graphical representability are more natural and nicer when the problems are considered for edge-colored graphs rather than for simple graphs. In [13] we provide a relatively simple characterization of those cyclic permutation groups that are automorphism groups of edge-colored graphs. We also prove that each such permutation group is an automorphism group of a 3-colored graph.

In fact, the problem for edge-colored graphs has been considered already by H. Wielandt in [29]. Permutation groups that are automorphism groups of edge-colored digraphs were called 2-closed, and those that are automorphism groups of colored graphs were named 2^*-closed. In [18], A. Kisielewicz introduced the so-called *graphical complexity of permutation groups*. By $GR(k)$ we denote the class of automorphism groups of *k-colored graphs*, by which we mean the graphs whose edges are colored with at most k colors. By GR we denote the union of all classes $GR(k)$, which is the class of 2^*-closed groups. Moreover, we put $GR^*(k) = GR(k)\setminus GR(k-1)$, and for a permutation group A, we say that A has a *graphical complexity* k if $A \in GR^*(k)$. Then, $GR(2)$ is the class of automorphism groups of simple graphs.

In this paper we fully characterize those cyclic permutation groups that are automorphism groups of simple graphs. In the last section we consider the same problem for other structures: digraphs, supergraphs, and boolean functions.

In Sect. 2, we recall some definitions concerning edge-colored graphs and permutation groups. We recall two results from [10], and prove their generalizations we need in the sequel. In Sect. 3, we recall results concerning cyclic permutation groups of prime order. In Sect. 4, we present some minimal (in a sense) permutation groups that do not belong to $GR(2)$, while in Sect. 5, we present minimal cyclic permutation groups which do belong to $GR(2)$. These will be used in the proof of the main results. In Sect. 6, we prove another auxiliary result which we call the extension lemma. The main results of the paper are given in Sect. 7. They include the results of [13]. Our approach is a little different than that in [13], and therefore we obtain, by the way, another proof of the results of [13]. The last section presents the corresponding results for digraphs, supergraphs, and boolean functions.

2 Definitions and Basic Facts

We assume that the reader has the basic knowledge in the areas of graphs and permutation groups. In fact, the terminology is standard and the reader is referred to [1,30]. The permutation groups are considered up to permutation isomorphism.

We need to refer to some results on the automorphism groups of k-colored graphs, so we recall here related terminology. A *k-colored graph* (or more precisely *k-edge-colored graph*) is a pair $G = (V, E)$, where V is the set of vertices, and E is an *edge-color function* from the set $P_2(V)$ of two elements subsets of V into the set of colors $\{0, \dots, k-1\}$ (in other words, G is a complete simple graph with each edge colored by one of k colors). In some situations it is helpful to treat the edges colored 0 as missing. In particular, the 2-colored graph can be treated as a usual graph. Also, if no confusion can arise, we omit the adjective "colored". By a (sub)graph of G *spanned* on a subset $W \subseteq V$ we mean $G' = (W, E')$ with $E'(\{v, w\}) = E(\{v, w\})$, for all $v, w \in W$.

Let $v, w \in V$ and $i \in \{0, \dots, k-1\}$. If $E(\{v, w\}) = i$, then we say that v and w are i*-neighbors*. Moreover, for a set $X \subseteq \{0, \dots, k-1\}$, we say that a vertex w is a X-neighbor of a vertex v if there is a color $i \in X$ such that w is i-neighbor of v. By $d_i(v)$ (i*-degree* of a vertex v) we denote the number of i-neighbors of v. For $X \subseteq \{0, \dots, k-1\}$, we say that G is X*-connected*, if for every $v, w \in V$ there is a path $v = v_0, v_1, \dots, v_n = w$ in G such that the color of each edge $\{v_i, v_{i+1}\}$ belongs to X. Obviously, for a k-colored graph $G = (V, E)$, and for the sets $X, Y \subseteq \{0, \dots, k-1\}$ such that $X \cup Y = \{0, \dots, k-1\}$, G is either X-connected or Y-connected. In particular, there is always a color p such that G is $(\{0, \dots, k-1\} \setminus \{p\})$-connected.

An automorphism of a colored graph G is a permutation σ of the set V preserving the edge function: $(E(\{v, w\}) = E(\{\sigma(v), \sigma(w)\})$, for all $v, w \in V)$. The group of automorphisms of G will be denoted by $Aut(G)$, and considered as a permutation group $(Aut(G), V)$ acting on the set of the vertices V.

Permutation groups are treated up to permutation isomorphism. Generally, a permutation group A acting on a set V is denoted (A, V) or just A, if the set V is clear or not important. By S_n, we denote the symmetric group on n elements, and by I_n the one element group acting on n elements (consisting of the identity only, which in all the cases is denoted by id). By C_n we denote a regular action of $\mathbb{Z}_n$. In particular, $S_2 = C_2$. By D_n we mean the group of symmetries of n-cycle i.e., the group of automorphisms of a graph $G = (V, E)$ with $V = \{v_0, \dots, v_{n-1}\}$, $E(\{v_i, v_{(i+1 \mod n)}) = 1$ for all i, and $E(v_i, v_j) = 0$, otherwise. This is clear that $C_n < D_n$ with index two. Every element of $D_n \setminus C_n$ has order two and is called a *reflection*. If n is odd, every reflection fixes exactly one point; if n is even, the half of reflections fix two points, and the other half fix no point.

Let W be a subset of V that is preserved by (A, V). By a *restriction* of A to the set W, we mean a permutation group (B, W) that is permutation isomorphic with the quotient group $A/Ker_W(A)$ acting naturally on the set W, where $Ker_W(A) = \{a \in A; a(w) = w$ for every $w \in W\}$.

The permutation groups considered in this paper are *cyclic* as abstract groups, i.e., generated by a single permutation: $A = \langle \sigma \rangle$. If σ has a decomposition $c_1 \cdots c_n$ on cycles with disjoint notions, then A has n orbits $O_1, \dots, O_n$ such that $|O_i| = |c_i|$, and A restricted to the orbit O_i is equal to $C_{|O_i|}$. A restriction of A to the set $W = O_{i_1} \cup \dots \cup O_{i_m}$ is a permutation group generated by a permutation $\tau = c_{i_1} \cdots c_{i_m}$. We say also that τ is a restriction of σ to the set W.

Later, we will use two kinds of products of permutation groups:

Direct sum. For permutation groups (A, V), (B, W), by a direct sum of A and B we mean a permutation group $(A \oplus B, V \cup W)$ with the action given by

$$(a, b)(x) = \begin{cases} a(x) & \text{for } x \in V, \\ b(x) & \text{for } x \in W. \end{cases}$$

Parallel product. For a permutation group (A, V), the parallel product $A^{\parallel n}$ is a permutation group $(A, V \times \{1, \ldots, n\})$ with the following natural action.

$$a((v_1, k)) = (a(v_1), k).$$

Thus, $A^{\parallel n} \approx A \times I_n$.

Now, we recall two theorems which are proved in [10] and will be used later.

Theorem 2.1 [10, Corollary 3.5] *Let $A = A_1 \oplus A_2$ be a directed sum. Then, $A \in GR$ if and only if each of A_1 and A_2 belongs to GR or A is equal to $I_2 = I_1 \oplus I_1$.*

Theorem 2.2 [10, Lemma 3.1 and Theorem 4.1] *Let $A_1, A_2 \in GR(k)$, for some $k \geq 2$. Then,*

(1) $A_1 \oplus A_2 \in GR(k + 1)$.
(2) *If* $A_1 \neq A_2$, *then* $A_1 \oplus A_2 \in GR(k)$.
(3) $A_1 \oplus I_n \in GR(k) \cup \{I_2\}$.

Later on, we will need one more lemma.

Lemma 2.3 *Let $k \geq 1$ and $B \notin GR(k)$ be a permutation group such that for every k-colored graph G, with the property $B \subseteq Aut(G)$, there is a permutation $f \in Aut(G) \backslash B$ preserves all the orbits of B. Then, $B \oplus C \notin GR(k)$ for every permutation group C.*

Proof Let $B = (B, V)$ and G' be a k-colored graph such that $B \oplus C \subseteq Aut(G')$. Then, obviously, the graph G, spanned on the set V, has the mentioned property. Let $f \in Aut(G) \backslash B$ be a permutation which preserves all the orbits of B. By f' we denote a permutation which acts as f on V and fixes all other vertices of G'. Obviously, $f' \notin B \oplus C$. We show that $f' \in Aut(G')$.

We have to show that the colors of the edges of the graph G' are preserved by f'. If an edge e is contained in the graph G, then $f'(e) = f(e)$ and $E(f'(e)) = E(f(e)) = E(e)$ as required. If neither of the ends of e belongs to V, then $f'(e) = e$ and the statement is still true. The only nontrivial case is the edges of the form $e = \{v, w\}$, where $v \in V$ and $w \notin V$. Then, $f'(e) = \{f(v), w\}$. Since f preserves all orbits of B, there exists $b \in B$ such that $f(v) = b(v)$. Consequently, $f'(e) = b'(e)$, where $b' = (b, Id) \in B \oplus C$. Hence $E(f'(e)) = E(b'(e)) = E(e)$, as required. This shows that $f' \in Aut(G')$ and completes the proof of lemma. □

This is well known (see [10], for instance) that $I_1 \in GR(0)$, $I_2 \notin GR$, $I_n \in GR^*(3)$, for $n \in \{3, 4, 5\}$, and $I_n \in GR(2)$, otherwise. This completes the case of permutation group of order one (which, as we see, is not quite trivial). In future consideration, we assume that the order of a cyclic permutation group is at least two.

3 Earlier Results

In this section we recall the results completing the description of the graphical complexity of cyclic permutation groups of prime order started in [20,21]. Those groups have the form $C_p{}^{\wr r} \oplus I_q$. The result from [20] can be written as follows.

Theorem 3.1 [20, Theorem 3] *Every permutation group of order two belongs to* $GR(2)$.

Theorem 3.2 [20, Theorem 2] *Let* $p > 5$ *be a prime. Then,* $C_p{}^{\wr r} \in GR(2)$ *if and only if* $r \geq 2$.

Our complete results are the following.

Theorem 3.3 *Let* p *be a prime,* $r \geq 1$, $q \geq 0$ *and* $A = C_p{}^{\wr r} \oplus I_q$. *Then,*

(1) $A \notin GR$, *for* $r = 1$ *and* $p \neq 2$,
(2) $A \in GR^*(3)$, *for* $r = 2$ *and* $p \in \{3, 5\}$,
(3) $A \in GR(2)$, *otherwise.*

Theorem 3.4 *Let* A *be a cyclic permutation group of order* p^n. *Let* k_i, $i \in \{1, \ldots, n\}$ *denotes the number of orbits of* A *of cardinality* p^i. *If* $p \neq 2$, *then*

(1) *if* $\sum_{i=1}^{n} k_i = 1$, *then* $A \notin GR$,
(2) *if* $\sum_{i=1}^{n} k_i = 2$, *then*

- $A \in GR^*(3)$, *for* $k_1 \in \{1, 2\}$ *and* $p \in \{3, 5\}$,
- $A \in GR(2)$, *otherwise,*

(3) *if* $\sum_{i=1}^{n} k_i > 2$, *then* $A \in GR(2)$.

A situation is a little different when $p = 2$.

Theorem 3.5 *Let* A *be a cyclic permutation group of order* 2^n. *Let* k_i, $i \in \{1, \ldots, n\}$ *denotes the number of orbits of* A *of cardinality* 2^i. *Then,*

(1) *if* $\sum_{i=2}^{n} k_i = 1$, *then* $A \notin GR$,
(2) *if* $\sum_{i=2}^{n} k_i = 2$, $k_1 = 0$ *and* $k_2 \in \{1, 2\}$, *then* $A \in GR^*(3)$,
(3) $A \in GR(2)$, *otherwise.*

4 Permutation Groups Outside *GR*(2)

In this section, we show a few one-generated permutation groups that have a small number of orbits, and are not automorphism groups of a 2-colored graph. In the general case, for every one-generated permutation group (A, V), there is a subset $W \subseteq V$ such that A restricted to W is one of the permutation groups of this form.

As it was mentioned before, for $n > 2$ and for every k-colored graph G, if $Aut(G) \supseteq C_n$, then $Aut(G) \supseteq D_n$. Hence, $C_n \notin GR$, for $n > 2$. We prove the similar statement for some other cases.

Theorem 4.1 *Let A be a one-generated permutation group with two orbits O_1 and O_2 such that* $\gcd(|O_1|, |O_2|) \in \{3, 4, 5\}$. *Then,* $A \notin GR(2)$.

Proof Let $(A, O_1 \cup O_2) = \langle \sigma \rangle$, $O_1 = \{v_0, \ldots, v_{|O_1|-1}\}$, $O_2 = \{w_0, \ldots, w_{|O_2|-1}\}$, $\sigma(v_i) = v_{(i+1 \mod |O_1|)}$, and $\sigma(w_i) = w_{(i+1 \mod |O_2|)}$. We consider the action of A on the set of edges. There are three types of orbits in this action. Type one is when the orbits consist of some edges $\{v, w\}$, where $\{v, w\} \subseteq O_1$. Type two is when the orbits consist of some edges $\{v, w\}$, where $\{v, w\} \subseteq O_2$. Type three is when the orbits consist of some edges $\{v, w\}$, where $v \in O_1$ and $w \in O_2$. As it was mentioned above, it is not any matter what are the colors of the edges of the orbits of the type one and two. The group of automorphisms of a graph spanned on O_i, $i \in \{1, 2\}$ will contain $D_{|O_i|}$. This can change when we color the orbits of the type three. However, such a situation does not take place.

If $\gcd(|O_1|, |O_2|) = x$, then there are exactly x orbits of the type three. For $x = 3$, there are two kinds of coloring:

(a) all orbits are colored in one color,
(b) one orbit is colored in some color and two orbits in another.

In the case (a), the group of automorphisms of the graph contains $D_{|O_1|} \oplus D_{|O_2|}$. In the case (b), we may exchange the names of the vertices (in a cyclic way) such that a reflection

$$\begin{aligned} f_1 = {} & (v_1, v_{|O_1|-1})(v_2, v_{|O_1|-2}) \cdots (v_{\lfloor (|O_1|-1)/2 \rfloor}, v_{\lceil (|O_1|+1)/2 \rceil}) \circ \\ & \circ (w_1, w_{|O_2|-1})(w_2, v_{|O_2|-2}) \cdots (w_{\lfloor (|O_2|-1)/2 \rfloor}, w_{\lceil (|O_2|+1)/2 \rceil}) \end{aligned}$$

will be an automorphism of the graph.

For $x = 4$ and $x = 5$ we have four kinds of coloring:

(a) all orbits are colored in one color,
(b) one orbit is colored in different color than the rest of the orbits,
(c) the orbits which contain the edges $\{v_0, w_0\}$ and $\{v_0, w_2\}$ are colored in one color and the rest of the orbits are colored in the second color,
(d) the orbits which contain the edges $\{v_0, w_0\}$ and $\{v_0, w_1\}$ are colored in one color and the rest of the orbits are colored in the second color.

In the case (a), we have the same situation as in the case (a) for $x = 3$. For $x = 5$, in the remaining cases, we have the same situation as in the case (b) for $x = 3$. The same is true in cases (b) and (c) for $x = 4$. In the case (d) for $x = 4$, the situation is a little different. There is no automorphism that acts as a fixing point reflection on every orbit but still there is an automorphism that acts as a reflection on every orbit. After exchanging the names of the vertices (in a cyclic way) the permutation

$$\begin{aligned} f_2 = {} & (v_1, v_{|O_1|-1})(v_2, v_{|O_1|-2}) \cdots (v_{|O_1|/2-1}, v_{|O_1|/2+1}) \circ \\ & \circ (w_0, w_{|O_2|-1})(w_1, v_{|O_2|-2}) \cdots (w_{|O_2|/2-1}, w_{|O_2|/2}) \end{aligned}$$

is an automorphism of the graph. This permutation fixes two points in the orbit O_1; v_0 and $v_{|O_1|/2}$ but fixes no point in the orbit O_2. This is clear that we may also find and automorphism of the graph that fixes two point in the orbit O_2 and fixes no point in the orbit O_1.

Since in every case, we have an automorphism of a graph that does not belong to A, we have $A \notin GR(2)$. □

We note that in the case, where $\gcd(|O_1|, |O_2|) \in \{3, 5\}$, and for every graph G, if $Aut(G) \supseteq A$, then $Aut(G)$ contains a permutation that acts as a fixing point reflection on each orbit. Observe also that if $|O_i|$ is divided by 2 for some $i \in \{1, 2\}$, then there is an automorphism f_3 of a graph that acts as no fixing point reflection of the orbit O_i and as a fixing point reflection on the other orbit.

Using Theorem 4.1, and observations from the proof, we prove the following three theorems.

Theorem 4.2 *Let A be an one-generated permutation group with three orbits O_1, O_2 and O_3 such that* $\gcd(|O_1|, |O_2|) \in \{3, 4, 5\}$*,* $\gcd(|O_1|, |O_3|) \in \{3, 4, 5\}$*, and* $\gcd(|O_2|, |O_3|) = 1$ *. Then, $A \notin GR(2)$.*

Proof Let $(A, O_1 \cup O_2 \cup O_3) = \langle \sigma \rangle$, $O_1 = \{v_0, \ldots, v_{|O_1|-1}\}$ $O_2 = \{w_0, \ldots, w_{|O_2|-1}\}$, $O_3 = \{u_0, \ldots, u_{|O_3|-1}\}$, $\sigma(v_i) = v_{(i+1 \mod |O_1|)}$, $\sigma(w_i) = w_{(i+1 \mod |O_2|)}$, and $\sigma(u_i) = u_{(i+1 \mod |O_3|)}$. We consider the action of A on the set of edges. There are few types of orbits in this action. We color these orbits to obtain a graph such that $Aut(G) \supseteq A$. Any such coloring of the orbits consisting of some edges $\{v, w\}$, where $\{v, w\} \subseteq O_i$ permits to an action of every permutation that acts on O_i as an element of $D_{|O_i|}$. Any such coloring of the orbits consisting of some edges $\{v, w\}$, where $v \in O_i$, $w \in O_{i+1}$, is like in the proof of Theorem 4.1, and permits to an action of every permutation that acts on O_i and O_{i+1} either as f_1 of as f_3 for $\gcd(|O_1|, |O_3|) \in \{3, 5\}$ and either as f_1 of as f_2 for $\gcd(|O_1|, |O_3|) = 4$. There is still and orbit consisting of all edges $\{v, w\}$, where $v \in O_1$, $w \in O_3$. Any coloring of this orbit permits to an action of every permutation that preserves the orbits O_1 and O_2.

This shows that if $\gcd(|O_1|, |O_2|) \in \{3, 5\}$ and $\gcd(|O_2|, |O_3|) \in \{3, 5\}$, then, after exchanging the names of the vertices, the permutation

$$\begin{aligned}
&(v_1, v_{|O_1|-1})(v_2, v_{|O_1|-2}) \cdots (v_{\lfloor(|O_1|-1)/2\rfloor}, v_{\lceil(|O_1|+1)/2\rceil}) \circ \\
&\circ(w_1, w_{|O_2|-1})(w_2, v_{|O_2|-2}) \cdots (w_{\lfloor(|O_2|-1)/2\rfloor}, w_{\lceil(|O_2|+1)/2\rceil}) \circ \\
&\circ(u_1, u_{|O_3|-1})(u_2, u_{|O_3|-2}) \cdots (u_{\lfloor(|O_3|-1)/2\rfloor}, u_{\lceil(|O_3|+1)/2\rceil})
\end{aligned}$$

belongs to $Aut(G) \backslash A$. If $\gcd(|O_1|, |O_2|) \in 4$ and $\gcd(|O_2|, |O_3|) \in \{3, 5\}$, then, after exchanging the names of the vertices, the permutation

$$\begin{aligned}
&(v_1, v_{|O_1|-1})(v_2, v_{|O_1|-2}) \cdots (v_{|O_1|/2-1}, v_{|O_1|/2+1}) \circ \\
&\circ(w_0, w_{|O_2|-1})(w_1, v_{|O_2|-2}) \cdots (w_{|O_2|/2-1}, w_{|O_2|/2}) \\
&\circ(u_1, u_{|O_3|-1})(u_2, u_{|O_3|-2}) \cdots (u_{(|O_3|-1)/2}, u_{(|O_3|+1)/2})
\end{aligned}$$

belongs to $Aut(G) \backslash A$. This completes the proof. □

Theorem 4.3 *Let A be an one-generated permutation group with four orbits O_1, O_2, O_3 and O_4 such that* $\gcd(|O_1|, |O_2|) \in \{3, 4, 5\}$*,* $\gcd(|O_2|, |O_3|) \in \{3, 4, 5\}$*,* $\gcd(|O_3|, |O_4|) \in \{3, 4, 5\}$*, and* $\gcd(|O_1|, |O_3|) = \gcd(|O_1|, |O_4|) = \gcd(|O_2|, |O_4|) = 1$*. Then, $A \notin GR(2)$.*

Theorem 4.4 *Let A be an one-generated permutation group with four orbits O_1, O_2, O_3, O_4 that* $\gcd(|O_1|, |O_2|) = 4$*,* $\gcd(|O_1|, |O_3|) = 3$ *and* $\gcd(|O_1|, |O_4|) = 5$*. Moreover,* $\gcd(|O_2|, |O_3|) = 1$*,* $\gcd(|O_3|, |O_4|) = 1$*, and* $\gcd(|O_2|, |O_4|) = 1$*. Then, $A \notin GR(2)$.*

Proof of Theorems 4.3 and 4.4. The same proof works in both cases and is similar to the previous one. Let $(A, O_1 \cup O_2 \cup O_3 \cup O_4) = \langle \boldsymbol{\sigma} \rangle$, where $O_1 = \{v_0, \ldots, v_{|O_1|-1}\}$, $O_2 = \{w_0, \ldots, w_{|O_2|-1}\}$, $O_3 = \{u_0, \ldots, u_{|O_3|-1}\}$, and $O_4 = \{t_0, \ldots, t_{|O_4|-1}\}$. Moreover, $\boldsymbol{\sigma}(v_i) = v_{(i+1 \mod |O_1|)}$, $\boldsymbol{\sigma}(w_i) = w_{(i+1 \mod |O_2|)}$, $\boldsymbol{\sigma}(u_i) = u_{(i+1 \mod |O_3|)}$, and $\boldsymbol{\sigma}(t_i) = t_{(i+1 \mod |O_4|)}$. All the possibilities that occur in Theorem 4.3 are similar. Therefore, we may assume $\gcd(|O_1|, |O_2|) = 4$, $\gcd(|O_2|, |O_3|) = 3$, and $\gcd(|O_3|, |O_4|) = 5$ in this case.

Any coloring of the graph G, such that $Aut(G) \supseteq A$, permits to an action of a permutation that, after exchanging the names of the vertices, is equal either to

$$\begin{aligned}
&(v_1, v_{|O_1|-1})(v_2, v_{|O_1|-2}) \cdots (v_{|O_1|/2-1}, v_{|O_1|/2+1}) \circ \\
&\circ(w_1, w_{|O_2|-1})(w_2, v_{|O_2|-2}) \cdots (w_{|O_2|/2-1}, w_{|O_2|/2+1}) \circ \\
&\circ(u_1, u_{|O_3|-1})(u_2, u_{|O_3|-2}) \cdots (u_{(|O_3|-1)/2}, u_{(|O_3|+1)/2}) \\
&\circ(t_1, t_{|O_4|-1})(t_2, t_{|O_4|-2}) \cdots (t_{(|O_4|-1)/2}, t_{(|O_4|+1)/2})
\end{aligned}$$

or to

$$\begin{aligned}
&(v_1, v_{|O_1|-1})(v_2, v_{|O_1|-2}) \cdots (v_{|O_1|/2-1}, v_{|O_1|/2+1}) \circ \\
&\circ(w_0, w_{|O_2|-1})(w_1, v_{|O_2|-2}) \cdots (w_{|O_2|/2-1}, w_{|O_2|/2}) \\
&\circ(u_1, u_{|O_3|-1})(u_2, u_{|O_3|-2}) \cdots (u_{(|O_3|-1)/2}, u_{(|O_3|+1)/2}) \\
&\circ(t_1, t_{|O_4|-1})(t_2, t_{|O_4|-2}) \cdots (t_{(|O_4|-1)/2}, t_{(|O_4|+1)/2})
\end{aligned}$$

None of these permutations belong to A. This completes the proof. □

A little more complicated proof is in the situation, where A has three orbits such that 3 divides $|O_1|$ and $|O_2|$ but not $|O_3|$, 5 divides $|O_2|$ and $|O_3|$ but not $|O_1|$, and 2 divides $|O_1|$ and $|O_3|$ but not $|O_2|$. However, the statement is the same.

Theorem 4.5 *Let A be an one-generated permutation group with three orbits O_1, O_2, O_3 such that* $\gcd(|O_1|, |O_2|) = 3$, $\gcd(|O_2|, |O_3|) = 5$ *and* $\gcd(|O_1|, |O_3|) \in \{2, 4\}$*. Then, $A \notin GR(2)$.*

Proof There are four cases depending on if 4 divides $|O_1|$ and if 4 divides $|O_3|$. At the beginning, we consider the three cases where 4 divides at most one of $|O_1|$ and $|O_3|$. Similarly as in the proof of Theorem 4.2 (case $\gcd(|O_1|, |O_2|) \in \{3, 5\}$ and $\gcd(|O_1|, |O_3|) \in \{3, 5\}$), it is not any matter how we color the edges which are not of the form $\{v, w\}$, where $v \in O_1$ and $w \in O_3$. In every coloring, there is a permutation

$\boldsymbol{\sigma}$ which acts as a fixing point reflection on every of three orbits. Consider the orbits of the group A (in action on the set of edges) consisting of some edges $\{v, w\}$, where $v \in O_1$ and $w \in O_3$. There are only two such orbits. Those orbits are preserved by $\boldsymbol{\sigma}$. Hence, the group of automorphisms of the graph is not equal to A. Moreover, it contains a permutation that acts as a fixing point reflection on every of the three orbits.

The remaining case is where 4 divides both $|O_1|$ and $|O_3|$. We consider the restriction of A to the set $O_2 \cup O_3$ (we denote it $B = \langle\tau\rangle$). By Theorem 4.1, for every graph G such that $Aut(G) \supseteq B$, there exists a permutation $\boldsymbol{\sigma}$ that acts as a fixing point reflection on both orbits O_2 and O_3. We consider the action of the group A on the set of the edges. Let $R = \{\{v, w\} : v \in O_1, w \in O_2\}$ and $T = \{\{v, w\} : v \in O_1, w \in O_3\}$. Then, A has three orbits which are contained in R and four orbits which are contained in S. The only three nontrivial colorings of those orbits (up to symmetries) are when we color one orbit which is contained in R in color 1 and other two in color 0 and we color either one or two orbits which are contained in T in color 1 and other in color 0 (as in Theorem 4.1 in case $x = 4$). It is easy to verify that in all those three cases there is an automorphism of a graph which acts as reflection on O_1 and as $\tau^n\boldsymbol{\sigma}$ on the set $O_2 \cup O_3$, for some n. Hence, the group of automorphisms of the graph is not equal to A. □

5 The First Step of Induction

In this section we study one-generated permutation groups $(A, V) \in GR(k)$ such that whenever we remove one of the orbits (say O), then a restriction of A to the set $V \setminus O$ does not belong to $GR(k)$.

In [21], it is proved the following.

Theorem 5.1 [21, Theorem 1] *Let A be a one-generated permutation group with two orbits O_1, O_2 with the property $|O_1| > 5$ and $|O_1|$ divides $|O_2|$. Then, $A \in GR(2)$.*

At first we generalize this theorem and prove the following.

Lemma 5.2 *Let A be a one-generated permutation group with two orbits O_1, O_2 such that* $\gcd(|O_1|, |O_2|) > 5$. *Then, $A \in GR(2)$.*

Proof Let $(A, V) = \langle\sigma\rangle$. Let $\gcd(|O_1|, |O_2|) = x$, $|O_1| = xy$ and $|O_2| = xz$. We may assume $O_1 = \{v_0, \ldots, v_{xy-1}\}$, $O_2 = \{w_0, \ldots, w_{xz-1}\}$, where $\sigma(v_i) = v_{(i+1 \mod xy)}$ and $\sigma(w_i) = w_{(i+1 \mod xz)}$. By σ_i, we denote the restriction of σ to the set O_i. Then, $A \subseteq \langle\sigma_1\rangle \oplus \langle\sigma_2\rangle$ and $A = \{\sigma_1^i\sigma_2^j : \text{ such that } i \equiv j \mod x\}$.

We consider the case when either $y \neq 1$ or $z \neq 1$. By $C(xy, xz)$ we denote the graph $G = (V, E)$ defined as follows. $V = O_1 \cup O_2$,

$$E(\{v, w\}) = \begin{cases} 1 & \text{for } v = v_i, w = v_j \text{ and } j \equiv i + 1 \mod xy \\ 1 & \text{for } v = w_i, w = w_j \text{ and } j \equiv i + 1 \mod xz, \\ 1 & \text{for } v = v_i, w = w_j \text{ and } (i - j \mod x) \in \{0, 1, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

This is easy to verify that A preserves the colors of the edges of $C(xy, xz)$ and therefore, $A \subseteq Aut(C(xy, xz))$. We prove the opposite inclusion. The 1-degree of a vertex which belongs to O_1 is equal to $3z+2$. The 1-degree of a vertex which belongs to O_2 is equal to $3y+2$. Since $y \neq z$, every automorphism of $C(xy, xz)$ preserves the partition of V on O_1 and O_2. Consequently, $Aut(C(xy, xz)) \subseteq D_{xy} \oplus D_{xz}$.

We show that reflections are forbidden. Since A is transitive on O_1 and on O_2, and moreover, $A \subseteq Aut(C(xy, xz))$, it is enough to exclude one reflection on O_1 and one reflection on O_2. We show that an element $ab \in D_{xy} \oplus D_{xz}$, where $b \in D_{xz}$ and

$$a = (v_1, v_{xy-1})(v_2, v_{xy-2}) \cdots \left(v_{\lfloor \frac{xy-1}{2} \rfloor}, v_{\lceil \frac{xy+1}{2} \rceil}\right),$$

does not belong to $Aut(C(xy, xz))$. For $v \in V$, by $N(v)$, we denote the set of 1-neighbors of v in the opposite orbit. Then,

$$\begin{aligned} N(v_0) &= \{w_i : (i \mod x) \in \{0, x-1, x-3\}\}, \\ N(v_1) &= \{w_i : (i \mod x) \in \{0, 1, x-2\}\}, \\ N(v_3) &= \{w_i : (i \mod x) \in \{0, 1, 3\}\}, \\ N(v_{xy-1}) &= \{w_i : (i \mod x) \in \{x-1, x-2, x-4\}\}, \\ N(v_{xy-3}) &= \{w_i : (i \mod x) \in \{x-3, x-4, x-6\}\}. \end{aligned}$$

Since $a(v_0) = v_0$, $a(v_1) = v_{xy-1}$, $a(v_3) = v_{xy-3}$, we have $a(w_0) \in N(v_0) \cap N(v_{xy-1}) \cap N(v_{xy-3})$. Since $x > 5$, this intersection is empty. In the similar way, one may exclude a reflection on O_2. Hence, $Aut(C(xy, xz)) \subseteq C_{xy} \oplus C_{xz}$.

We show that if $ab \in Aut(C(xy, xz))$, where $a = id$ and $b \in C_{xz}$, then $b = \sigma_2^{xl}$ for some l. We know that ab fixes v_0, v_1 and v_3. Hence, the image of w_0 has to belong to the intersection $N(v_0) \cap N(v_1) \cap N(v_3)$ which is equal to $\{w_i : i \equiv 0 \mod x\}$. Consequently, $b = \sigma_2^{xl}$, as required. In similar way one may show that if $ab \in Aut(C(xy, xz))$, where $b = id$ and $a \in C_{xy}$, then $a = \sigma_1^{xl}$ for some l. Since $A \subseteq Aut(C(xy, xz))$, this implies the inclusion $Aut(C(xy, xz)) \subseteq A$.

Now, let $y = z = 1$. Then, by $C(x, x)$, we denote the graph $G = (V, E)$ defined as follows. $V = O_1 \cup O_2$,

$$E(\{v, w\}) = \begin{cases} 1 & \text{for } v = v_i,\ w = v_j \text{ and } j \equiv i+1 \mod x, \\ 1 & \text{for } v = v_i,\ w = w_j \text{ and } (i-j \mod x) \in \{0, 1, 3\}, \\ 0 & \text{otherwise,} \end{cases}$$

Again, it is easy to verify that A preserves the colors of the edges of $C(x, x)$. Hence, we have $A \subseteq Aut(C(x, x))$. We prove the opposite inclusion. The 1-degree of a vertex which belongs to O_1 is equal to 5. The 1-degree of a vertex which belongs to O_2 is equal to 3. Hence, every automorphism of $C(x, x)$ preserves the partition of the set V on the orbits O_1 and O_2. The graph spanned on O_1 is isomorphic with a x-cycle. This implies that $Aut(C(x, x))$, restricted to the set O_1, is contained in D_x. Let $a \in Aut(C(x, x))$ be a permutation such that $a(v_0) = (v_0)$. We consider the possibilities on $a(w_0)$. We obtain that $a(w_0) \in N(v_0) = \{w_0, w_{x-1}, w_{x-3}\}$.

Assume first that a does not act trivially on O_1. Then, for $i > 1$, we have $a(v_i) = v_{x-i}$. Since $w_0 \in N(v_0) \cap N(v_1) \cap N(v_3)$, we know that

$a(w_0) \in N(v_0) \cap N(v_{x-1}) \cap N(v_{x-3})$. However, the set $N(v_0) \cap N(v_{x-1}) \cap N(v_{x-3})$ is empty. Consequently, if $a \in Aut(C(x, x))$ and it fixes v_0, then it fixes every vertex which belongs to O_1. This implies that $a(w_0) \in N(v_0) \cap N(v_1) \cap N(v_3) = \{w_0\}$. Hence, a fixes w_0. Since $C(x, x)$ is preserved by σ, we have immediately that a fixes every vertex in O_2. Hence a is a trivial permutation. Consequently, $Aut(C(x, x)) = A$. □

Now, we consider the cases where $\gcd(|O_1|, |O_2|) \in \{3, 4, 5\}$. We prove the following two lemmas.

Lemma 5.3 *Let A be a one-generated permutation group with two orbits O_1 and O_2 such that* $\gcd(|O_1|, |O_2|) \in \{3, 4, 5\}$. *Then, $A \in GR(3)$.*

Proof In the case $y = z = 1$, this is a consequence of Lemma 3.3 in [9]. We have also defined the graph $C(x, x)$ there. In the other cases, we use the same symbols as in the proof of Lemma 5.2. We define $C(xy, xz)$ as follows. $G = O_1 \cup O_2$.

$$E(\{v, w\}) = \begin{cases} 1 & \text{for } v = v_i, w = v_j \text{ and } j \equiv i + 1 \mod xy, \\ 1 & \text{for } v = w_i, w = w_j \text{ and } j \equiv i + 1 \mod xz, \\ 1 & \text{for } v = v_i, w = w_j \text{ and } i \equiv j \mod x, \\ 2 & \text{for } v = v_i, w = w_j \text{ and } i \equiv j + 1 \mod x, \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of $C(xy, xz)$, we have immediately that A preserves the colors of the edges of $C(xy, xz)$. Hence, $A \subseteq Aut(C(xy, xz))$. We prove that $Aut(C(xy, xz))$ preserves the sets O_1 and O_2. Indeed, the 1-degree of a vertex, which belongs to O_1 is equal to $z+2$. The 1-degree of a vertex, which belongs to O_2 is equal to $y + 2$. These numbers are different.

The remaining part of the proof is similar to the appropriate part of the proof of Lemma 5.2. By $N_1(v)$, we denote the set of 1-neighbors of the vertex v in the set O_2, and by $N_2(v)$, the set of 2-neighbors of the vertex v in the set O_2. We show that permutations that acts as reflections on some of the sets O_1 or O_2 are forbidden. We take $a \in Aut(C(xy, xz))$ that fixes v_0. Observe that $N_2(v_0) = N_1(v_1)$. Moreover, since $x > 2$, $N_2(v_0) \cap N_1(v_{xy-1}) = \emptyset$. Therefore, a fixes v_1. Since a subgraph of $C(xy, xz)$ spanned on O_1 is a $|O_1$-cycle a acts trivially on O_1. Since $A \subseteq Aut(C(xy, xz))$, no reflection on O_1 is possible. Since the role of O_1 and O_2 are symmetric, the same is true for O_2.

We have to show that if $a \in Aut(C(xy, xz))$ fixes v_0, then a acts as $\boldsymbol{\sigma}_2^{xl}$ on O_2 for some l. We have $N_1(v_0) = \{w_{xl}; l \in \{0, \ldots, z - 1\}\}$. Therefore, $a(w_0) = w_{xl}$ for some l. Since the subgraph of $(C(xy, xz))$ spanned on O_2 is a $|O_2|$-cycle, we have that a acts as $\boldsymbol{\sigma}_2^{xl}$ on O_2. Again the role of O_1 and O_2 are symmetric, therefore, the same is true for O_1 (a permutation that fixes w_0 acts as $\boldsymbol{\sigma}_1^{xl}$ on O_1 for some l. Thus, $A = Aut(C(xy, xz))$. □

Lemma 5.4 *Let A be a one-generated permutation group with three orbits O_1, O_2 and O_3 such that* $\gcd(|O_1|, |O_2|, |O_3|) \in \{3, 4, 5\}$. *Then, $A \in GR(2)$.*

Proof We denote $x = \gcd(|O_1|, |O_2|, |O_3|) \in \{3, 4, 5\}$. Moreover, $|O_1| = xy$, $|O_2| = xzt_2$, $|O_3| = xzt_3$, where y, z, t_2, t_3 are positive integers such that $\gcd(y, zt_2) = 1$, $\gcd(y, zt_3) = 1$ and $\gcd(t_2, t_3) = 1$. We may assume that either $t_2 = t_3 = 1$ or $t_2 < t_3$. By changing the names of orbits we may exclude the situation when $y = 1, z = 1, t_2 = 1$ and $t_3 > 1$. Let $(A, V) = \langle\sigma\rangle$ and $O_i = \{v^i_0, \ldots, v^1_{|O_i|-1}\}$. We may assume that $\sigma(v^i_j) = v^i_{(j+1 \mod |O_i|)}$. Let σ_i be the restriction of σ to the set O_i. Then, $A = \{\sigma_1^n\sigma_2^m\sigma_3^k$, where $n \equiv m \mod x$, $n \equiv k \mod x$ and $m \equiv k \mod xz\}$.

By $C(xy, xzt_2, xzt_3)$, we denote the graph $G = (V, E)$ defined as follows. $V = O_1 \cup O_2 \cup O_3$.

$$E(\{v, w\}) = \begin{cases} 1 & \text{for } v = v^1_i, w = v^1_j \text{ and } (j - i \mod x) \in M, \\ 1 & \text{for } v = v^l_i, w = v^l_j, (j - i \mod x) \in N_l \text{ and } l \in \{2, 3\}, \\ 1 & \text{for } v = v^1_i, w = v^2_j \text{ and } (j - i \mod x) \in \{0, 1\}, \\ 1 & \text{for } v = v^1_i, w = v^3_j \text{ and } i \equiv j \mod x, \\ 1 & \text{for } v = v^2_i, w = v^3_j \text{ and } i \equiv j \mod xz, \\ 0 & \text{otherwise}, \end{cases}$$

where $M = \{1, xz - 1\}$, if neither $y = (2z - 1)t_2 + zt_3$ nor $y = zt_1 + \frac{(z-1)}{2}t_2$ and $M = \{2, \ldots, xz - 2\}$, otherwise, $N_l = \emptyset$, if $y = z = t_1 = t_2 = 1$ and $N_l = \{1, |O_l| - 1\}$, otherwise. In addition, we put $N_2 = \emptyset$ if $y = 3, z = 1, t_2 = 1, t_3 = 6$.

From the definition of G, we have immediately $Aut(G) \subseteq A$. We prove the opposite inclusion. At the beginning, we show that $Aut(G)$ preserves the partition of V on the sets O_1, O_2 and O_3. We count the 1-degrees of the elements which belong to these orbits. By d_i, we denote here the 1-degree of the vertices that belong to O_i. If $y = z = t_1 = t_2 = 1$, then $d_1 = 5$, $d_2 = 3$ and $d_3 = 2$. This implies that the partition is preserved by $Aut(G)$. Similarly, if $y = 3, z = 2, t_2 = 1, t_3 = 2$, then we have $d_1 = 10, d_2 = 8, d_3 = 5$.

Otherwise, $d_2 = 2+2y+t_3, d_3 = 2+y+t_2$. Moreover, if neither $y = (2z-1)t_2+zt_3$ nor $y = zt_1 + \frac{(z-1)}{2}t_2$, then d_1 is equal to $D_1 = 2 + 2zt_2 + zt_3$ and d_1 is equal to $y - 2 + 2zt_2 + zt_3$, otherwise. Since $t_2 \leq t_3$, we have $d_2 > d_3$. If $d_1 = D_1$, then obviously, $d_1 \notin \{d_2, d_3\}$. If $y = (2z - 1)t_2 + zt_3$, then $d_1 = (4z - 1)t_2 + 2zt_3 - 2$, $d_2 = (4z-2)t_2+(2z+1)t_3+2$ and $d_3 = 2zt_2+zt_3+2$. Since $t_2 < t_3$, these numbers are different.

If $y = zt_1 + \frac{z-1}{2}t_2$, then $d_1 = 3zt_2 + \frac{z+1}{2}t_3 - 2$, $d_2 = 2zt_2 + zt_3 + 2$ and $d_3 = (z + 1)t_2 + \frac{z-1}{2}t_3 + 2$. Assume that $d_1 = d_2$. Then, $zt_2 + \frac{zt_3}{2} = 4$. This is possible only if either $y = 3, z = 2, t_2 = 1$ and $t_3 = 2$ or $y = 1, z = 1, t_2 = 1$ and $t_3 = 6$. The former case was considered earlier. The other case, we have excluded.

Now, let $d_1 = d_3$. Then, $(2z - 1)t_2 + t_3 = 4$. This is possible only if $y = 1, z = 1, t_2 = 1, t_3 = 3$. This case was also excluded. As a consequence we have that, in every case, $Aut(G)$ preserves the partition of the set V on the sets O_1, O_2 and O_3.

We show that $Aut(G) = A$. When $G = C(3x, 2x, 4x)$, one may check it directly. However, the fact that, when A has the orbits of cardinality $3x, 2x, 4x$ respectively, is immediate consequence of Lemmas 5.2 and 6.1. In proofs of those two lemmas, we do not use Lemma 5.4. We consider the case when $G = C(x, x, x)$. This is easy to verify that $A \subseteq Aut(C(x, x, x))$. We have to prove the opposite inclusion. The graph spanned on O_1 is a $|O_1|$-cycle. Moreover, every vertex, which belongs to O_3 has exactly one 1-neighbor in the set O_1 and every vertex, which belongs to O_2 has exactly one 1-neighbor in the set O_3. This implies that $Aut(C(x, x, x))$ is equal either to A or to $D_x{}^{\wr\wr 3}$. We have to exclude the second case. Assume that $a \in Aut(C(x, x, x))$ fixes v_0^1. Then, a fixes v_0^2. The vertex v_0^2 has only one 1-neighbor that belong O_1 and is not v_0^1. This is the vertex v_{x-1}^1. This implies that a fixes v_{x-1}^1. Hence, $a = id$ and $A = Aut(C(x, x, x))$.

We consider the remaining cases. The graphs spanned on the sets O_i, $i \in \{1, 2, 3\}$, are either $|O_i|$-cycles or their complements. This implies that $Aut(G) \subseteq D_{|O_1|} \oplus D_{|O_2|} \oplus D_{|O_3|}$. We have to exclude the reflections and unwanted elements of $C_{|O_1|} \oplus C_{|O_2|} \oplus C_{|O_3|}$. By $N_i(v)$, we denote here the set of 1-neighbors of the vertex v in the set O_i.

Assume that $a \in Aut(G)$ fixes v_0^1. We have

$$N_2(v_0^1) = \{v_i^2 : i \mod x \in \{0, 1\}\},$$
$$N_3(v_0^1) = \{v_i^3 : i \equiv 0 \mod x\},$$
$$N_2(v_i^3) = \{v_j^2 : j \equiv i \mod xz\}.$$

This implies that $a(v_0^2) = v_i^2$, for $i \equiv 0 \mod x$, and moreover, $a(v_0^3) = v_j^3$, for $j \equiv 0 \mod x$. Assume that the action of a on O_1 is nontrivial. We have $v_0^2 \in N_2(v_{xy-1}^1)$. Obviously, $a(v_{xy-1}^1) = v_1^1$. Hence, $a(v_0^2) \in N(v_1^1)$. Since $N(v_1^1) \cap \{v_i^2 : i \equiv 0 \mod x\} = \emptyset$, this is impossible. Consequently, there is no element in $Aut(G)$ that acts on O_1 as a reflection. Moreover, if $\sigma_1^n \sigma_2^m \sigma_3^k$, then n, m, k satisfy demanded properties. Similarly, this implies immediately that there is no element in $Aut(G)$ that acts as a reflection on O_3. Finally, in the same way, there is no element in $Aut(G)$ that acts as a reflection on O_2. Hence, $Aut(G) = A$. □

In the remaining part of the paper, we will use not only the statements of Lemmas 5.2, 5.3 and 5.4 but also the constructions of the graphs $C(n, m)$ and $C(n, m, k)$.

On the end of this section, we write one more theorem.

Theorem 5.5 *Let A be a one-generated permutation group with three orbits O_1, O_2 and O_3 such that* $\gcd(|O_1|, |O_2|) = 4$ *and* $|O_3| = 2$. *Then,* $A \in GR(2)$.

Proof Theorem 5.5 is an immediate consequence of more general Lemma 6.4. In proof of Lemma 6.4, we do not use Theorem 5.5. □

6 Extension Lemmas

We prove here the extension lemmas that we will use in a proof of general case.

Lemma 6.1 *Let $(B, V) = \langle \tau \rangle$ acts without fix points. Let O be one of the orbits of B with following properties.*

- *There is an orbit* $O' \neq O$ *such that* $\gcd(|O'|, |O|) \neq 1$. *Moreover, if* $|O| > 2$, *then* $\gcd(|O'|, |O|) > 2$.
- *If* $|O| = 2l$, *with* $l > 1$, *then there is no orbit of cardinality 2.*
- *A restriction of* B *to the set* $V \setminus O$, *belongs to* $GR(k)$.

Then, $B \in GR(k)$.

Proof Obviously, $k \geq 2$. Let C be a restriction of B to the set $V \setminus O$. Let $O_1, O_2, \ldots, O_s$ be a list of those orbits of C that $\gcd(|O|, |O_i|) > 1$ for every $i \in \{1, \ldots, s\}$. Let $c_0 = |O|$, and $c_i = |O_i|$ for $i \in \{1, \ldots, s\}$. We denote $O = \{v_0^0, \ldots, v_{c_0-1}^0\}$ and $O_i = \{v_0^i, \ldots, v_{c_i-1}^i\}$. We may assume that $\tau(v_j^i) = v_{(j+1 \mod c_i)}^i$. Moreover, we denote $x_i = \gcd(c_0, c_i)$. We assume that, for $i < j$, we have $x_i \leq x_j$. This implies that if $x_i = 2$, for some i, then $x_1 = 2$ and $c_0 = 2$.

Let $G = (V \setminus O, E)$ be a k-colored graph such that $Aut(G) = C$. We construct a finite number of graphs $G^* = (V, E^*)$, $G^{**} = (V, E^{**})$, $G_0 = (V, E_0)$, $G_1 = (V, E_1), \ldots$. For one of the graphs G_i, with $i \geq 0$, we will have $Aut(G_i) = B$.

Let $G^* = (V, E^*)$, where

$$E^*(\{v, w\}) = \begin{cases} E(\{v, w\}) & \text{if } \{v, w\} \subseteq V \setminus O, \\ 1 & \text{if } v = v_i, w = v_j \text{ and} \\ & (i - j \mod c_0) \in \{1, c_0 - 1\}, \\ 1 & \text{if } v = v_i^0, w = v_j^h, (i \mod x_h) = (j \mod x_h), \\ & \text{and } h = 1 \text{ or } x_h \neq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $G^* = (V, E^{**})$, where

$$E^{**}(\{v, w\}) = \begin{cases} E^*(\{v, w\}) + 1 \mod 2 & \text{if } v \in O, w \in O_h \text{ and } c_0 > c_h, \\ E^*(\{v, w\}) & \text{otherwise.} \end{cases}$$

We define $E_0 = E^{**}$, for $s = 1$ and for $c_0 \notin \{2, 4\}$. For $s = 1$ and $c_0 \in \{2, 4\}$, we define

$$E_0(\{v, w\}) = \begin{cases} (E^*(\{v, w\}) + 1 \mod 2) & \text{if } \{v, w\} \subseteq O, \\ E^*(\{v, w\}) & \text{otherwise.} \end{cases}.$$

Assume now that we have constructed a graph G_k, for $k \geq 0$. Since O is the orbit in V, every element of O has the same 1-degree (say m_k) in G_k. By P_k we denote the set of those elements of $V \setminus O$ that have 1-degree (in G_k) which is equal to m_k. Obviously, P_k is a union of orbits of the group C.

We will construct a graph G_{k+1} in the situation when either $|P_k| > |O|$ or $O_i \subseteq P_k$, for some $i \in \{1, \ldots, s\}$. In the other cases G_k is the final step of our construction.

If $|P_k| > |O|$ and no set O_i is a subset of P_k, then we put

$$E_{k+1}(\{v, w\}) = \begin{cases} 1 & \text{if } v \in O \text{ and } w \in P_k, \\ E_k(\{v, w\}) & \text{otherwise.} \end{cases}$$

The 1-degree of the elements which belong to $O \cup P_k$ is increased. The 1-degree of remaining elements does not change. Obviously, after this change, the 1-degree of the elements which belong to O is larger than of the 1-degree of those elements which belong to P_k.

Let $i \in \{1, \ldots, s\}$ be such that $O_i \subseteq P_k$. We consider three situations.
Case 1. $s = 1$ and $c_1 = c_0$. Then, $C = D \oplus C_{c_1}$. By Theorem 2.1, $C \in GR(k)$, implies $c_1 = 2$. Then, by assumption $c_0 = 2$. Consequently, $C = D \oplus C_2{}^{\setminus\setminus 2} \in GR(k)$.
Case 2. Either $s = 1$, $c_1 \neq c_0$ and $c_0 > 2$ or $s > 1$ and $x_1 \neq 2$. We put

$$E_{k+1}(\{v, w\}) = \begin{cases} (E_k(\{v, w\}) + 1 \mod 2) & \text{if } v \in O \text{ and } w \in O_{(i \mod s)+1}, \\ E_k(\{v, w\}) & \text{otherwise.} \end{cases}$$

Since $x_1 \neq 2$, in every case, the 1-degree of the elements that belong to O is increased. The 1-degree of the elements that belong O_i is increased at most as 1-degree of the elements that belong to O. Remaining elements have the same 1-degree as before. In particular, 1-degree of the elements which belong to O is larger than the 1-degree of those elements which belong to P_k.
Case 3. Either $s = 1$, $c_0 = 2$ and $c_1 > 2$ or $s > 1$ and $x_1 = 2$. Obviously, $c_0 \notin \{3, 5\}$. We Put

$$E_{k+1}(\{v, w\}) = \begin{cases} (E_k(\{v, w\}) + 1 \mod 2) & \text{if } v \in O, w \in O_{i+1} \text{ and } i \neq s, \\ (E_k(\{v, w\}) + 1 \mod 2) & \text{if } v, w \subseteq O \text{ and } i = s, \\ E_k(\{v, w\}) & \text{otherwise.} \end{cases}$$

If $i = s$, then since $c_0 \notin \{3, 5\}$, the 1-degree of the elements which belong to O is increased. Remaining elements have the same 1-degree as before. If $i < s$, then the 1-degree of the elements which belong to O is increased. The 1-degree of the elements which belong to O_{i+1} is increased at most as 1-degree of the elements which belong to O. Remaining elements have the same 1-degree as before. In both cases, the 1-degree of the elements which belong to O is larger than the 1-degree of those elements which belong to P_k.

Since the group C has a finite number of orbits, the procedure will finish after a finite number of steps. If k is the final step, then $|P_k| \leq |O|$ and there is no $i \in \{1, \ldots, s\}$ such that $O_i \subseteq P_k$. We show that $Aut(G_k) = B$.

First, we prove that every $a \in Aut(G_k)$ stabilizes O. By the construction of the graph G_k, if there is a vertex $v \in V \setminus O$ with the same 1-degree as the elements which belong to O, then $|O| \notin \{2, 4\}$. This implies that the graph spanned on O is 1-connected. We know that every vertex $v \in V \setminus O$ with the same 1-degree as the elements of O is a 0-neighbor of every vertex that belong to O. Moreover, the number of those elements is not greater than $|O|$. Hence, if some vertex that belongs to O is moved out of O, then it is true for all elements which belong to O. This implies that the graph spanned on the set $a(O)$ has to be either O-cycle or its complement. Since $a(O)$ consists of at least two orbits of B, this is impossible. Consequently, the set O is stabilized by every $a \in Aut(G_k)$.

The graph spanned on the set $V \setminus O$ is equal to G. Hence, the group $Aut(G_k)$, restricted to $V \setminus O$, is equal to C. The graph spanned on the set O is equal either to

$|O|$-cycle or to the complement of $|O|$-cycle. Hence, the group $Aut(G_k)$, restricted to O, is equal either to C_{c_0} or to D_{c_0}. For $c_0 > 2$, there is an orbit O' such that $\gcd(|O|, |O'|) > 2$. In similar way as in proof of Lemma 5.2, a reflection on O implies a reflection on O'. Hence, the second case is excluded.

This is immediately from the definition of G^* that τ preserves the colors of G^*. Moreover, this is clear that the changes which are done in the graphs $G^{**}, G_0, \ldots, G_k$ do not break it down. Consequently, every $b \in B$ preserves the colors of the edges of G_k. Hence, $B \subseteq Aut(G_k)$.

Let S be the set of elements $f \in Aut(G_k)$ which acts on the set $V \backslash O$ as τ. Since $B \subseteq Aut(G_k)$, such elements exist (τ is one of them). Since f acts on $V \backslash O$ as τ, the group generated by f, restricted to the set $V \backslash O$, is equal to C. This implies that $|Aut(G_k)| = |C||S|$. We check the possible actions of f on the set O. For $i > 0$, we have $f(v^i_j) = v^i_{(j+1 \mod c_i)}$. By definition of the graph G_k, all the 1-neighbors (or alternatively all 0-neighbors) of v^i_0 in the set O are vertices v^0_h, where $h \equiv 0 \mod x_i$. All the 1-neighbors (or alternatively all 0-neighbors) of v^i_1 in the set O are vertices v^0_g, where $g \equiv 1 \mod x_i$. This implies that $f(v^0_j) = v^0_{(j+1 \mod x_i)}$. Consequently, $f(v^0_j) = v^0_{(j+1 \mod \gcd(|C|,|O|))}$. Hence, the set S has at most $|O|/\gcd(|C|, |O|)$ elements. We have $|Aut(G_k)| = |C||S| \leq |C||O|/\gcd(|C|, |O|) = |B|$. Since $B \subseteq Aut(G_k)$, we have $B = Aut(G_k)$. □

Lemma 6.2 *Let $(B, V) = \langle\tau\rangle$ acts without fix points. Let O_1 and O_2 be orbits of B with the following properties.*

- $\gcd(|O_1|, |O_2|) > 2$.
- *There is an orbit $O \notin \{O_1, O_2\}$ such that $|O| > 2$, $\gcd(|O|, |O_1|) = 2$ and $\gcd(|O|, |O_2|) \leq 2$,*
- *For every orbit $O' \notin \{O_1, O_2\}$, we have $\gcd(|O'|, |O_i|) \leq 2$, for $i \in \{1, 2\}$,*
- *A restriction of B to the set $V \backslash (O_1 \cup O_2)$ belongs to $GR(k)$.*

Then, $B \in GR(l)$, where $l = k$, for $\gcd(|O_1|, |O_2|) > 5$ and $l = \max\{3, k\}$, otherwise.

Proof Obviously, $k \geq 2$. Let C be a restriction of the group B to the set $V \backslash (O_1 \cup O_2)$. We may denote $x = \gcd(|O_1|, |O_2|)$, $|O_1| = xy$, $|O_2| = xz$, $|O| = t$, $O_1 = \{v_0, \ldots, v_{xy-1}\}$, $O_2 = \{w_1, \ldots, w_{xz-1}\}$, $O = \{u_0, \ldots, u_{t-1}\}$ and $\tau(v_i) = v_{(i+1 \mod xy)}$, $\tau(w_i) = w_{(i+1 \mod xz)}$, $\tau(u_i) = u_{(i+1 \mod t)}$. Since $C \in GR(k)$, there is at least one orbit different from O_1, O_2, and O.

Let $G_1 = (V \backslash (O_1 \cup O_2), E_1)$ be a k-colored graph such that $Aut(G_1) = C$. Since $k \geq 2$, without loss of generality we may assume that, for a vertex $v \in O$, the 1-degree of the vertex v (in a graph G_1) is greater than 0. Let $G_2 = (O_1 \cup O_2, E_2) = C(xy, xz)$. We define a graph $G = (V, E)$ as follows.

$$E(\{v, w\}) = \begin{cases} E_1(\{v, w\}) & \text{if } \{v, w\} \subseteq V \backslash (O_1 \cup O_2), \\ E_2(\{v, w\}) & \text{if } \{v, w\} \subseteq O_1 \cup O_2, \\ 1 & \text{when } v = v_i, w = u_j \text{ and } i \equiv j \mod 2, \\ 1 & \text{when } v \in O_1 \text{ and } w \in V \backslash (O_1 \cup O_2 \cup O), \\ 0 & \text{otherwise.} \end{cases}$$

We show that $Aut(G) = B$. The inclusion $B \subseteq Aut(G)$ is an immediate consequence of the definition of the graph G.

We prove the opposite inclusion. We show that $Aut(G)$ preserves the set $O_1 \cup O_2$. We count the 1-degree of the vertices which belong to V. If $v \in O_2$, then according to the constructions of $C(xy, xz)$ in Lemmas 5.2 and 5.3, $d_1(v) \leq 3y+2$ for $x > 5$, and $d_1(v) \leq y+2$, otherwise. If $v \in O$, then since the 1-degree of the vertex v (in the graph G_1) is greater than 0, we have $d_1(v) > xy/2$. If $v \notin O_1 \cup O_2 \cup O$, then $d_1(v) > xy$. In the case when $x > 6$, we have $3y + 2 < xy/2$. If $x = 5$, then $y + 2 < xy/2$. In these two cases, the only vertices which have the same 1-degree as the vertices which belong to O_2, could be those which belong to O_1. If this situation take place, then obviously $O_1 \cup O_2$ is preserved by $Aut(G)$. Otherwise, O_2 is preserved by $Aut(G)$ and O_1 is a set consisting of all those vertices which have at least one 1-neighbor in O_2. Hence, O_1 is preserved by $Aut(G)$.

If $x = 6$, then $3y + 2 < xy$. If $x \in \{3, 4\}$, then $y + 2 < xy$ In both this cases the vertices which belong to O_2 have smaller 1-degree than the vertices which belong to $V \setminus (O_1 \cup O_2 \cup O)$. If, in addition, the vertices which belong to O_2 have different 1-degree than those which belongs to O, then we have the same situation as above. Assume that this is not true. The vertices which belong to O_2 and the vertices which belong to O have the same 1-degree (say d). We have to consider two cases according to 1-degree of the vertices which belong to O_1. Assume $v \in O_1$. If $d_1(v) = d$, then O_2 consists of all these vertices of 1-degree equal to d which do not have 1-neighbor outside $O_1 \cup O_2 \cup O$. Hence, O_2 is preserved by $Aut(G)$. As above, this implies that O_1 is preserved by $Aut(G)$. If $d_1(v) \neq d$, then $O_2 \cup O$ is preserved by $Aut(G)$. Moreover, there is no edge in color 1 between O_2 and O. The graph spanned on O_2 is 1-connected. The graph spanned on O is either 1-connected or has no edge in color 1. This implies that, if for $v \in O_2$ and $a \in Aut(G)$, we have $a(v) \in O$, then $a(O_2) \subseteq O$ and $a(O) \subseteq O_2$. Since $|O_2| \neq |O|$, this is not possible. Hence, the set O_2 is preserved by $Aut(G)$. Once again, this implies that O_1 is preserved by $Aut(G)$.

We have: the set $O_1 \cup O_2$ is preserved by $Aut(G)$, the graph spanned on $O_1 \cup O_2$ is equal to G_2 and the graph spanned on $V \setminus (O_1 \cup O_2)$ is equal to G_1. Hence, $Aut(G) \subseteq C \oplus Aut(C(xy, xz))$. Moreover, since for every orbit $O' \notin \{O_1, O_2\}$, we have $\gcd(|O'|, |O_i|) \leq 2$ for $i \in \{1, 2\}$, and $\gcd(|O|, |O_1|) = 2$, the group B has index 2 in $C \oplus Aut(C(xy, xz))$. Since the vertices v_0 and v_1 have the different sets of 1-neighbors in $V \setminus (O_1 \cup O_2)$, the group $Aut(G)$ has index at least 2 in the group $C \oplus Aut(C(xy, xz))$. Consequently, since $B \subseteq Aut(B)$, we have $B = Aut(G)$. □

Lemma 6.3 *Let $(B, V) = \langle \tau \rangle$ acts without fix points. Let O_1, O_2, O_3 be orbits of B with following properties.*

- $\gcd(|O_1|, |O_2|, |O_3|) = \gcd(|O_1|, |O_2|) = \gcd(|O_1|, |O_3|) \in \{3, 4, 5\}$,
- *there is an orbit $O \notin \{O_1, O_2, O_3\}$ such that $|O| > 2$, $\gcd(|O|, |O_1|) = 2$, $\gcd(|O|, |O_2|) \leq 2$ and $\gcd(|O|, |O_3|) \leq 2$,*
- *for every orbit $O' \notin \{O_1, O_2, O_3\}$, we have $\gcd(|O'|, |O_i|) \leq 2$, for $i \in \{1, 2, 3\}$,*
- *a restriction of B to the set $V \setminus (O_1 \cup O_2 \cup O_3)$ belongs to $GR(k)$.*

Then, $B \in GR(k)$.

Proof Obviously, $k \geq 2$. Let C be a restriction of the group B to the set $V \backslash (O_1 \cup O_2 \cup O_3)$. We denote $x = \gcd(|O_1|, |O_2|, |O_3|)$, $|O_1| = xy$, $|O_2| = xzt_2$, $|O_3| = xzt_3$, where y, z, t_2, t_3 are positive integers such that zt_2 and zt_3 are not divided by 2. Moreover, $\gcd(y, zt_2) = 1$, $\gcd(y, zt_3) = 1$ and $\gcd(t_2, t_3) = 1$. In the same way as in proof of Lemma 5.4, we assume that either $t_2 = t_3 = 1$ or $t_2 < t_3$, and as there, by changing the names of the orbits, we exclude the situation when $y = 1$, $z = 1$, $t_2 = 1$ and $t_3 > 1$. Let $O_i = \{v_0^i, \ldots, v_{|O_i|-1}^1\}$, $i \in \{1, 2, 3\}$, $O = \{w_0, \ldots, w_{t-1}\}$. We may assume that $\sigma(v_j^i) = v_{(i+1 \mod |O_i|)}^i$, $i \in \{1, 2, 3\}$ and $\tau(w_i) = w_{(i+1 \mod t)}$. By A, we denote the restriction of B to the set $O_1 \cup O_2 \cup O_3$.

We consider the case $|O_1| = 3x$, $|O_2| = 2x$, $|O_3| = 4x$. In proof of Lemma 5.4, we have remained the proof that $Aut(C(3x, 2x, 4x)) = A$ to the reader. In this case we prove that $B \in GR(k)$ in another way than in other cases. By Lemma 6.2, the group B restricted to the set $V \backslash O_1$ belongs to $GR(k)$. Consequently, by Lemma 6.1, $B \in GR(k)$.

Further, we consider those cases for which we have proved in Lemma 5.4 (without using this lemma) that $Aut(C(xy, xzt_2, xzt_3)) = A$. Let $G_1 = (V \backslash (O_1 \cup O_2 \cup O_3), E_1)$
be a k-colored graph such that $Aut(G_1) = C$. Let $G_2 = (O_1 \cup O_2 \cup O_3, E_2) = C(xy, xzt_2, xzt_3)$. We define a graph $G = (V, E)$ as follows.

$$E(\{v, w\}) = \begin{cases} E_1(\{v, w\}) & \text{if } \{v, w\} \subseteq V \backslash (O_1 \cup O_2 \cup O_3), \\ E_2(\{v, w\}) & \text{if } \{v, w\} \subseteq O_1 \cup O_2 \cup O_3, \\ 1 & \text{when } v = v_i^1, w = u_j \text{ and } i \equiv j \mod 2, \\ 1 & \text{when } v \in O_1 \text{ and } w \in V \backslash (O_1 \cup O_2 \cup O_3 \cup O), \\ 1 & \text{when } v \in O_2 \text{ and } w \in V \backslash (O_1 \cup O_2 \cup O_3), \\ 0 & \text{otherwise.} \end{cases}$$

We show that $Aut(G) = B$. The inclusion $B \subseteq Aut(G)$ is an immediate consequence of the definition of the graph G.

We prove the opposite inclusion. We show that $Aut(G)$ preserves the set $O_1 \cup O_2 \cup O_3$. We count the 1-degree of the vertices which belong to V. If $v \in O_3$, then according to the constructions of $C(xy, xzt_2, xzt_3)$ in Lemma 5.4, $d_1(v) \leq y + t_2 + 2$. If $v \in O_2$, then $d_1(v) > y + t_3 + 2$. If $v \in O$, then $d_1(v) \geq xy/2 + xzt_2$. If $v \notin O_1 \cup O_2 \cup O_3 \cup O$, then $d_1(v) \geq xy + xzt_2$. The number $y + t_2 + 2$ is the smallest of them. The only vertices, which have the same 1-degree as the vertices that belong to O_3 could be those that belong to O_1. If the situation take place, then obviously $O_1 \cup O_3$ is preserved by $Aut(G)$. If $xy = xzt_3$, then for $v \in O_1$, we have $d_1(v) > y + zt_2 + 2$, and this is not this case. Otherwise, the vertices which belong to O_1 and the vertices which belong to O_3 have different numbers of 1-neighbors in the set $O_1 \cup O_3$. Hence, O_3 is preserved by $Aut(G)$. If there is no vertices outside O_3 with 1-degree equal to $y + zt_2 + 2$, then this is also true. The set $O_1 \cup O_2$ consists of all the vertices which have at least one 1-neighbor in the set O_3. This implies that the set $O_1 \cup O_2$ is preserved by $Aut(G)$.

As in the previous lemma $Aut(G) \subseteq A \oplus C$ and the group B has an index 2 in $A \oplus C$. Since v_0^1 and v_1^1 have the different sets of 1-neighbors in $V \backslash (O_1, \cup O_2 \cup O_3)$, the group

$Aut(G)$ has index at least 2 in the group $A \oplus C$. Consequently, since $B \subseteq Aut(G)$, we have $B = Aut(G)$. □

Lemma 6.4 *Let $(B, V) = \langle\tau\rangle$ acts without fix point. Let O_1, O_2 be orbits of B with the following properties.*

- $\gcd(|O_1|, |O_2|) = 4$,
- *there is an orbit $O \notin \{O_1, O_2\}$ such that* $\gcd(|O|, |O_1|) = 2$, *and moreover,* $\gcd(|O|, |O_2|) = 2$,
- $\gcd(|O'|, |O_1|) \le 2$ *and* $\gcd(|O'|, |O_2|) \le 2$ *for every orbit $O' \notin \{O_1, O_2\}$,*
- *a restriction C of the group B to the set $V \backslash (O_1 \cup O_2)$ belongs to $GR(k)$,*
- *there is a graph G such that $Aut(G) = C$ and its subgraph spanned on the orbit O is colored by two colors.*

Then, $B \in GR(k)$.

Proof Let $|O_1| = 4x$, $|O_2| = 4y$, $|O| = 2z$. We may assume that either $x = y = 1$ or $x < y$. Let $O_1 = \{v_0, \ldots, v_{4x-1}\}$, $O_2 = \{w_0, \ldots, w_{4y-1}\}$, $O = \{u_0, \ldots, u_{2z-1}\}$, and break $\tau(v_i) = v_{(i+1 \mod 4x)}$, $\tau(w_i) = w_{(i+1 \mod 4y)}$, $\tau(u_i) = u_{(i+1 \mod 2z)}$. Let $G = (V \backslash (O_1 \cup O_2), E)$ be a graph as in the assumption. Let t be the number of 1-neighbors of u_0 in O. Since $|O|$ is even, we may assume (eventually exchanging the colors of G) that z and t have a different parity. We define the graphs $G_j = (V, E_j)$ inductively.

$$E_0(\{v, w\}) = \begin{cases} E(\{v, w\}) & \text{if } \{v, w\} \in V \backslash (O_1 \cup O_2), \\ 1 & \text{if } v = v_i \text{ and } w = v_{(i+1 \mod 4x)}, \\ 1 & \text{if } v = w_i,\ w = w_j \text{ and } i - j \in M, \\ 1 & \text{if } v = v_i,\ w = w_i \text{ and } (i - j \mod 4) \in \{0, 1\}, \\ 1 & \text{if } v = u_i,\ w \in \{v_j, w_j\} \text{ and } i \equiv j \mod 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $M = \{1, 4y - 1\}$, if $y > x$ and $M = \emptyset$, if $y = x$.

For a vertex $v \in V$, by $d_{G_i}(v)$ we denote here the number of 1-neighbors of v is the graph G_i. Obviously, $d_{G_0}(v_0) > d_{G_0}(w_0)$. We will construct the graph G_{j+1} in the case when there exists a vertex $v' \notin O_1 \cup O_2 \cup O$ such that $d_{G_j}(v') = d_{G_j}(w_0)$. Assume that we have constructed a graph G_j and there exists such a vertex v'. Then, v' belongs to the orbit $O' \notin \{O_1, O_2, O\}$. We construct a graph G_{j+1} by modifying the graph G_j.

$$E_{j+1}(\{v, w\}) = \begin{cases} 1 & \text{if } v \in O_1 \text{ and } w \in O', \\ E_j(\{v, w\}) & \text{otherwise.} \end{cases}$$

We do not change the 1-degree of the vertices which belong to O_2 and we increase the 1-degree of v'. This is easy to see that in every graph G_j, we have $d_{G_j}(v_0) > d_{G_j}(w_0)$. Moreover, if $v \notin O_1 \cup O_2 \cup O$ and $E_j(\{v, v_0\}) = 1$, then $d_{G_j}(v) > d_{G_j}(w_0)$. Hence,the procedure will finish after a finite number of steps. Let G_k be the last graph in our construction. We show that $Aut(G_k) = B$. The inclusion $B \subseteq Aut(G_k)$ is an immediate consequence of the definition.

We prove the opposite inclusion. First, we show that O_1 and O_2 are preserved by $Aut(G_k)$. We have $d_{G_k}(v) \neq d_{G_k}(w_0)$, for every $v \notin O_1 \cup O$. Assume that $d_{G_k}(w_0) \neq d_{G_k}(u_0)$. Then, O_2 is preserved by $Aut(G_k)$. The set $O_1 \cup O$ consists of all the vertices outside O_2 which have at least one 1-neighbor in O_2. Moreover, there are $z + 2$ vertices in $O_1 \cup O$ which are 1-neighbors of v_0 and there are $2x + t$ vertices in $O_1 \cup O$ which are 1-neighbors of u_0. Since z and t have a different parity, these numbers are different. Hence, O_1 is preserved by $Aut(G_k)$. In the case, when $d_{G_k}(u_0) = d_{G_k}(w_0)$, we have that the set $O_2 \cup O$ is preserved by $Aut(G_k)$. Moreover, there are either $z+2$ or z vertices in $O_2 \cup O$ which are 1-neighbors of w_0 and there are $2x + t$ vertices in $O_2 \cup O$ which are 1-neighbors of u_0. Since z and t have a different parity, these numbers are different. Hence, O_2 is preserved by $Aut(G_k)$. Now the set O_1 consists of all the vertices outside $O_2 \cup O$ which have at least one 1-neighbor in O_2. Hence O_1 is preserved by $Aut(G_k)$.

We show that there is no reflection on O_1. Let $\boldsymbol{\sigma} \in Aut(G_k)$ fixes v_0. The set $\{u_i : i \equiv 0 \mod 2\}$ consists of all 1-neighbors of v_0 that belong to O. Hence, $\boldsymbol{\sigma}(u_0) = u_i$, where $i \equiv 0 \mod 2$. Moreover, the set $\{w_i : (i \mod 4) \in \{0, 1\}\}$ consists of all 1-neighbors of v_0 that belong to O_2, and the set $\{w_j : i \equiv j \mod 2\}$ consists of all 1-neighbors of u_i that belong to O_2. This implies that $\boldsymbol{\sigma}(w_0) = w_l$, where $l \equiv 0 \mod 4$. Since w_0 and $\boldsymbol{\sigma}(w_0)$ are 1-neighbors of the vertex v_{4x-1} and are not 1-neighbors of the vertex v_1, we obtain that $\boldsymbol{\sigma}$ fixes every vertex in O_1. In the same way one can show that every $\boldsymbol{\sigma} \in Aut(G_k)$ which fixes w_0, fixes every vertex in O_2, too.

Hence, $Aut(G_k) \subseteq C \oplus D$, where D is a restriction of the group B to the set $O_1 \cup O_2$. Moreover, the group B has index 2 in $C \oplus D$. Since v_0 and v_1 have different sets of 1-neighbors in O, the group $Aut(G_k)$ has index at least 2 in the group $C \oplus D$. Consequently, since $B \subseteq Aut(G_k)$, we have $B = Aut(G_k)$. □

Let NC be the set of cyclic permutation groups consisting of $C_n, n \geq 3$, groups with two orbits O_1 and O_2 such that $\gcd(|O_1|, |O_2|) \in \{3, 5\}$, groups with three orbits O_1, O_2, O_3 such that $\gcd(|O_1|, |O_2|) = 3, \gcd(|O_2|, |O_3|) = 5, \gcd(|O_1|, |O_2|) \leq 2$. By Theorems 4.1, 4.2, and 4.5, if $A \in NC$ and G is a graph such that $A \subseteq Aut(G)$, then there exists a permutation $\boldsymbol{\sigma} \in Aut(G)$ such that $\boldsymbol{\sigma}$ acts as a fixing point reflection on each of orbits of A.

Now, we prove a negative extension lemma.

Lemma 6.5 *Let $(B, V) = \langle \tau \rangle$ acts without fix points. Let $O_1, \ldots, O_t, t \geq 1$, be orbits of B with following properties.*

- *The group B restricted to the set $\bigcup_{i=1}^t O_i$ belongs to NC,*
- *there are orbits $O \in \bigcup_{i=1}^t O_i$ and $O' \notin \bigcup_{i=1}^t O_i$ such that* $\gcd(|O|, |O'|) = 2$,
- *for every pair of orbits O, O' such that $O \in \bigcup_{i=1}^t O_i$ and $O' \notin \bigcup_{i=1}^t O_i$, we have* $\gcd(|O|, |O'|) \leq 2$.

Then, $B \notin GR(2)$.

Proof Assume to the contrary that $B \in GR(2)$. By C we denote the permutation group B restricted to the set $\bigcup_{i=1}^t O_i$. Let $G = (V, E)$ be a graph such that $Aut(G) = B$. Let $O'_1, \ldots, O'_s$ be the remaining orbits of B. Without loss of generality, we can assume that

$\gcd(|O_1|, |O'_1|) = 2$. We denote $O_i = \{v^i_0, \ldots, v^i_{|O_i|-1}\}$ and $\tau(v_i) = v_{(i+1 \mod |O_i|)}$ and $O'_i = \{w^i_0, \ldots, w^i_{|O_i|-1}\}$ and $\tau(w_i) = w_{(i+1 \mod |O'_i|)}$. We show that a permutation α, which acts as a reflection on every orbit $O_i, i \in \{1, \ldots, t\}$ and fixes every vertex in all $O'_i, i \in \{1, \ldots, s\}$, belongs to $Aut(G)$. ($\alpha(v^i_j) = v^i_{(|O_i|-j \mod |O_i|)}$, $\alpha(w^i_j) = w^i_j$.) Obviously, $\alpha \notin B$, which completes the proof in this case.

Let $\{v, w\} \subseteq V$ be the edge of G. We show that $E(\{\alpha(v), \alpha(w)\}) = E(\{v, w\})$. If $\{v, w\} \subseteq \bigcup_{i=1}^s O'_i$, then $\alpha(v) = v$ and $\alpha(w) = w$, and the equality holds trivially. If $\{v, w\} \subseteq \bigcup_{i=1}^t O_i$, then by Lemmas 4.1, 4.2 and 4.5, for every graph G_1 such that $Aut(G) \supseteq C$, we know that α, restricted to the set $\bigcup_{i=1}^t O_i$, belongs to $Aut(G_1)$. Consequently, in this case the equality also holds. In the last case, we assume that $v \in O_i$ and $w \in O'_j$. If $\gcd(|O_i|, |O'_j|) = 1$, then all the edges $\{v_1, w_1\}$, where $v_1 \in O_i$ and $w_1 \in O'_j$, have the same color, and the equality holds. Finally, if $\gcd(|O_i|, |O'_j|) = 2$, then both orbits have an even number of elements and the group B, with action on edges of G, acts transitively on the set $\{\{v^i_k, w^j_l\} : k \equiv l \equiv 0 \mod 2\}$. Hence, all the edges which belong to this set have the same color. The same is true for the set $\{\{v^i_k, w^j_l\} : k \equiv l \equiv 1 \mod 2\}$. Since α preserves these sets, the equality holds. Consequently, α preserves the colors of the edges of G. Hence, $\alpha \in Aut(G)\backslash B$. This is in a contradiction with the assumption $Aut(G) = B$. Hence, $B \notin GR(2)$. □

Corollary 6.6 *Let $(B, V) = \langle\tau\rangle$ acts without fix points. Let O be an orbit such that $|O| > 2$, and $\gcd(|O|, |O'|) \leq 2$ for every orbit $O' \neq O$. Then, $B \notin GR$.*

Proof No matter how many of colors we use to color the edge with both ends in O, if the group of automorphisms of the subgraph spanned on O contains $C_{|O|}$, then it contains $D_{|O|}$. Thus, it is enough to use two colors. Moreover, there are only two orbits consisting of the edges with one end in O and the second in $V\backslash O$. We may use at most two colors to color these edges. Hence, exactly the same proof as above works here. □

7 Main Results

In this section, we prove the theorems that characterize the graphical complexity of one-generated permutation groups. First we give an alternate proof of the result of [13].

Theorem 7.1 *Let (A, V) be a one-generated permutation group. Then, A belongs to GR if and only if for every orbit O of A such that $|O| > 2$, there exists another orbit O' of A such that $\gcd(|O|, |O'|) > 2$.*

Proof We prove the „only if” part. If $A = C_t$ for $t \geq 3$, then $A \notin GR$. We consider the case with at least two orbits. We assume that there exists an orbit O such that $|O| > 2$ and for every other orbit O', we have $\gcd(|O|, |O'|) \leq 2$. In the case where $\gcd(|O|, |O'|) = 1$, for every orbit $O' \neq O$, we denote by $(B, V\backslash O)$ a restriction of A to the set $V\backslash O$. Then, $A = B \oplus C_{|O|}$. By the fact that $C_{|O|} \notin GR$ and by Theorem 2.1, we obtain $A \notin GR$.

The remaining case is when there exists an orbit $O' \neq O$ such that $\gcd(|O|, |O'|) = 2$. By Corollary 6.6, we immediately have $A \notin GR$. This completes the proof of the „only if" part.

The „if" part, we prove by induction on the number of orbits. By Lemma 2.3, we may restrict our proof to the case where there is no orbit of cardinality one. If there are the only two orbits, then the statement holds by Lemmas 5.2 and 5.3. Now, we assume that the statement holds in all the cases where there are less than k orbits, $k > 2$. We prove that this implies that the statement holds for k orbits. We choose an orbit O with the property that the number $s(O)$ of those orbits $O' \neq O$ such that $\gcd(|O|, |O'|) > 2$ is the least possible. We consider a one generated group $(B, V\backslash O)$ which is a restriction of A to the set $V\backslash O$. If $s(O) = 0$, then O has two elements and the orbits of B satisfy the conditions. By assumption, $B \in GR$. By Lemma 6.1, we have $A \in GR$.

Let, $s(O) = 1$. If the orbits of B satisfy the conditions, then we have the same as in the case above. If the orbits of B do not satisfy the conditions, then there exists an orbit O' such that $\gcd(|O|, |O'|) > 2$ and $\gcd(|O'|, |O''|) \leq 2$ for every orbit $O'' \notin \{O, O'\}$. Let $(C, V\backslash(O \cup O'))$ be a restriction of A to the set $V\backslash(O \cup O')$. Let $(D, O \cup O')$ be a restriction of A to the set $O \cup O'$. Then, the orbits of C and D satisfy the conditions. Hence, $C \in GR$ and $D \in GR$. If $\gcd(|O|, |O''|) = 1$ and $\gcd(|O'|, |O''|) = 1$ for every orbit $O'' \notin \{O, O'\}$, then $A = C \oplus D$ and by Theorem 2.2, we have $A \in GR$. Otherwise, the conclusion holds by Lemma 6.2.

The remaining cases are when $s(O) > 1$. In those cases the orbits of B satisfy the conditions and we have the same as in the first case. □

Since in this paper, there is no place where we have used more than three colors (the step of induction preserves the number of colors), we have

Corollary 7.2 *Let A be a one-generated permutation group. Then, $A \in GR(3)$ if and only if $A \in GR$.*

When we want to describe one-generated permutation groups that belong to $GR(2)$, the theorem becomes more complicated. We may give a number of conditions that each orbit has to satisfy. However, it will be clearer if we write it in another way.

Let A be a one-generated permutation group of order n. We introduce now a graph $Graph(A)$ with loops which gives an information how the „prime powers parts" of A are joined.

- The vertices of $Graph(A)$ are those primes that divide n.
- The prime 2 is not a vertex. Instead of this we put a vertex 4, if $|A|$ is divided by 4.
- A set $\{p, q\}$, $p \neq q$, is an edge in $Graph(A)$ if and only if there exists an orbit O of A such that $|O|$ is divided by pq.
- For $p > 5$, a set $\{p\}$ is a loop in $Graph(A)$ if and only if there are at least two orbits whose cardinality is divided by p.
- For $p \in \{3, 4, 5\}$, a set $\{p\}$ is a loop in $Graph(A)$ if either there are at least two orbits whose cardinality is divided by pq, for some $q > 1$ or if there are at least three orbits whose cardinality is divided by p.
- Moreover, for $p = 4$ if there are at least three orbits with an even cardinality and 4 divides the cardinality of two of them, then also there is a loop $\{4\}$ in $Graph(A)$.

Observe, that if the order of A is equal to 2, then $Graph(A)$ is empty.

Theorem 7.3 *$A \in GR(2)$ if and only if either A has an order 2 or in every connected component of $Graph(A)$ there is a loop.*

Proof By Lemmas 2.2(3) and 2.3, we may restrict our consideration to the one-generated permutation groups without fix points. First, we prove the „only if" part. Assume that a connected component K of $Graph(A)$ does not include the loop. If K includes no vertices of 3, 4, 5, then there exists an orbit O such that $|O| > 2$ and $\gcd(|O|, |O'|) \leq 2$, for every orbit $O' \neq O$. By Theorem 7.1, we have $A \notin GR$.

Assume now that K includes at least one of the vertices 3, 4, 5. We have the following possibilities:

Case 1. There is an orbit O, such that $|O| > 2$ and $\gcd(|O|, |O'|) \leq 2$ for every orbit $O' \neq O$. This is the same case as above.

Case 2. There are two orbits O_1 and O_2 such that $\gcd(|O_1|, |O_2|) \in \{3, 5\}$ and moreover, $\gcd(|O_1|, |O|) \leq 2$, $\gcd(|O_2|, |O|) \leq 2$ for every orbit $O \notin \{O_1, O_2\}$. By Theorem 4.1, Lemmas 2.3 and 6.5, we have $A \notin GR(2)$.

Case 3. There are two orbits $O_1$and O_2 such that $\gcd(|O_1|, |O_2|) = 4$ and $\gcd(|O_1|, |O|) = 1$, $\gcd(|O_2|, |O|) = 1$ for every orbit $O \notin \{O_1, O_2\}$. By Theorem 4.1 and Lemma 2.3, we have $A \notin GR(2)$.

Case 4. There are three orbits O_1, O_2, and O_3 such that $\gcd(|O_1|, |O_2|) \in \{3, 5\}$, $\gcd(|O_2|, |O_3|) \in \{3, 5\}$, $\gcd(|O_1|, |O_3|) \leq 2$. Moreover, we have $\gcd(|O_1|, |O|) \leq 2$, $\gcd(|O_2|, |O|) \leq 2$, $\gcd(|O_3|, |O|) \leq 2$ for every orbit $O \notin \{O_1, O_2, O_3\}$. By Theorems 4.2 and 4.5, Lemmas 2.3 and 6.5, we have $A \notin GR(2)$.

Case 5. There are three orbits O_1, O_2 and O_3 with the properties $\gcd(|O_1|, |O_2|) = 4$, $\gcd(|O_2|, |O_3|) \in \{3, 5\}$, $\gcd(|O_1|, |O_3|) \in \{1, 3, 5\}$. Moreover, $\gcd(|O_1|, |O|) = 1$, $\gcd(|O_2|, |O|) = 1$, $\gcd(|O_2|, |O|) = 1$ for every orbit $O \notin \{O_1, O_2, O_3\}$. By Theorem 4.2, Theorem 4.5 and Lemma 2.3, we have $A \notin GR(2)$.

Case 6. There are four orbits O_1, O_2, O_3, O_4 such that $\gcd(|O_i|, |O_{i+1}|) \in \{3, 4, 5\}$ and $\gcd(|O_i|, |O_j|) = 1$, otherwise. Moreover, $\gcd(|O_i|, |O|) = 1$ for every orbit $O \notin \{O_1, O_2, O_3, O_4\}$. By Theorem 4.3, and Lemma 2.3, we have $A \notin GR(2)$.

Case 7. There are four orbits O_1, O_2, O_3, and O_4 such that $\gcd(|O_1|, |O_2|) = 3$, $\gcd(|O_1|, |O_3|) = 4$, $\gcd(|O_1|, |O_4|) = 5$. Moreover, we have $\gcd(|O_2|, |O_3|) = 1$, $\gcd(|O_3|, |O_4|) = 1$, $\gcd(|O_2|, |O_4|) = 1$. Also, $\gcd(|O_i|, |O|) = 1$ for every orbit $O \notin \{O_1, O_2, O_3, O_4\}$. By Theorem 4.4, and Lemma 2.3, we have $A \notin GR(2)$. This completes the „only if" part of the proof.

We prove the „if" part. If A has order two, then by Theorem 3.1, the statement holds. Assume that $Graph(A)$ is nonempty. Let K be a connected component of $Graph(A)$. By $O(K)$, we denote the set of those orbits of O that there exists $p \in K$ such that p divides $|O|$. Let $K_1, \ldots, K_t$, $t \geq 0$ be connected components of $Graph(A)$ such that, for every $i \in \{1, \ldots, t\}$, there is no $O \in K_i$ of even cardinality. Let $H_1, \ldots, H_r$, $r \geq 0$ be connected components of $Graph(A)$ such that, for every $i \in \{1, \ldots, t\}$, there exists an orbit $O \in K_i$ of even cardinality.

For every connected component $K \in \{K_1, \ldots, K_t\}$, we apply the following procedure. Let p be a vertex with a loop in a component K_i. Let $\{O_1, \ldots, O_w\}$ be a list of the orbits which belong to $O(K)$ such that, for every $i \in \{2, \ldots w\}$, there exists $j < i$ such that $\gcd(|O_i|, |O_j|) > 2$. Moreover, if $|O_s|$ is divided by p and $|O_r|$ is not divided by p, then $s < r$. In addition, if $p \in \{3, 5\}$, then, whenever it is possible, we chose O_1, O_2 such that $\gcd(|O_1|, |O_2|) > p$. Let A^i_K be the restriction of A to the set $O_1 \cup, \cdots, \cup O_i$. We denote $A_K = A^w_K$. If $p \in \{3, 5\}$ and $\gcd(|O_1|, |O_2|) = p$, then by Lemma 5.4, $A^3_K \in GR(2)$. Otherwise, by Lemma 5.2, $A^2_K \in GR(2)$. Let us assume that $A^i_K \in GR(2)$ is proved. Since $\gcd(|O_{i+1}|, |O_j|) > 2$ for some $j \leq i$, by Lemma 6.1, we have $A^{i+1}_K \in GR(2)$. Hence, by induction, we have $A_K \in GR(2)$.

For the connected components $H_1, \dots, H_r$, we modify a little this procedure. Whenever this is possible we choose H_1 such that 4 does not belong to H_1. Let $H \in \{H_1, \dots, H_r\}$. We choose a vertex p with the loop in H . If this is possible, then $p \neq 4$. We make a list of orbits in $O(H) = O^H_1, \dots O^H_{w(H)}$ in the same way as in previous procedure. In addition, for $p = 4$, whenever this is possible, we choose either O^H_1 and O^H_2 such that $\gcd(|O^H_1|, |O^H_2|) = 4t$ for some $t > 1$ or O^H_1, O^H_2, and O^H_3 such that $\gcd(|O^H_1|, |O^H_2|, |O^H_3|) = 4$. Otherwise, we choose O^H_1 and O^H_2 such that $\gcd(|O^H_1|, |O^H_2|) = 4$. Let $A^l_{H_i}$ be the restriction of A to the set $\bigcup_{j=1}^{i-1} \bigcup O(H_j) \cup \bigcup_{j=1}^{l} O^{H_i}_j$. We denote $A_{H_i} = A^{w(H_i)}_{H_i}$. If 4 is not a chosen vertex in H_1 or we have chosen O^H_1 and O^H_2 such that $\gcd(|O^H_1|, |O^H_2|) = 4t$, or else there are at least three orbits of cardinality divided by 4, then in the same way as in the previous procedure, we show that $A_{H_1} \in GR(2)$.

By one of the Lemmas 6.2, 6.3 or Lemma 6.4, respectively to the situation, we obtain that either $A^2_{H_2} \in GR(2)$ or $A^3_{H_2} \in GR(2)$. As in the previous procedure, using Lemma 6.1, we may show that $A_{H_2} \in GR(2)$. Continuing in the same way, we show that $A_{H_r} \in GR(2)$. Let B be the restriction of A to the set $\bigcup_{j=1}^{r} \bigcup O(H_j) \cup \bigcup O(2)$, where $O(2)$ is the set of orbits of cardinality 2. Then, by Lemma 6.1, we have also $B \in GR(2)$. Since $A = B \oplus \bigoplus A_{K_i}$, by Theorem 2.2, we have $A \in GR(2)$.

This procedure works in the situations where

- $r > 1$,
- there exists a loop in H_1 at a vertex different from 4,
- there are two orbits O_1 and O_2 in $O(H_1)$ such that $\gcd(|O_1|, |O_2|) = 4s$ for some $s > 1$,
- there are at least three orbits of cardinality divided by 4.

Otherwise, there are two orbits O_1 and O_2 in the set $O(H_1)$ for which we have $\gcd(|O_1|, |O_2|) = 4$. Moreover, there is no other orbit in $O(H_1)$ of cardinality divided by 4. Also, there is at least one other orbit in $O(H_1)$ of cardinality divided by 2. We consider two situations.

Case 1. There is an orbit $O_3 \in O(H_1)$ such that $O_3 \notin \{O_1, O_2\}$ and $|O_3| > 2$. Every prime, which divides $|O_3|$ belongs to the same component as the vertex 4. Hence, there is a path from every prime that divides $|O_3|$ to the vertex 4. Since there is no loop in this component, except for the vertex 4, there is no other orbit O in $O(H_1)$ such that p divides $\gcd(O, O_3)$ and p is prime greater then 5. Hence, $|O_3| = 2sw$, where $s \in \{3, 5\}$. Moreover, there is an orbit $O_4 \in O(H_1)$ such that $\gcd(|O_3|, |O_4|) = s$. By similar argument as above, 15 divides $|O_4|$. Moreover, 2 does not divide $|O_4|$ and $15/s$ divides either $|O_1|$ or $|O_2|$. In addition, there is no other orbit in $O(H_1)$. Therefore, in this situation $O(H_1)$ consists of four orbits O_1, O_2, O_3 and O_4 such that $\gcd(|O_1|, |O_2|) = 4$, 15 divides $|O_3|$. Moreover, $|O_1|$ is not divided neither by 3 nor by

5. In addition, we have $\gcd(|O_1|, |O_3|) = 2$, $\gcd(|O_1|, |O_4|) = 1$, $\gcd(|O_2|, |O_4|) = 2$, $\gcd(|O_3|, |O_4|) = q$, where $q \in \{3, 5\}$, $\gcd(|O_2|, |O_3|) = 15/q$. We construct a graph G_1 on the set $O_1 \cup O_2 \cup O_3$ as in Lemma 6.4. Unfortunately, $Aut(G_1)$ contains a reflection on the set O_3. However, if we construct a graph G on the set $O_1 \cup O_2 \cup O_3 \cup O_4$ as in Lemma 6.1, using G_1 and the orbit O_4, this reflection will be eliminated. Hence, also in this situation, $A_{H_1} \in GR(2)$ and we can continue the procedure.

Case 2. There is no orbit of A with cardinality larger than 4, divided by 2 but not by 4. Then, there is an orbit of cardinality 2. We add the orbit of cardinality two as the first element of the list of the orbits which belong to $O(H_1)$. We may use $A^1_{H_1} = C_2$ as in the first step of induction. Then, we use Lemma 6.4 to show that $A^3_{H_1} \in GR(2)$ and continue the procedure. Since there are only two orbits of even cardinality greater then 2, i. e., O_1 and O_2, the conditions in Lemma 6.1 are satisfied and the procedure works.

This completes the proof of the theorem. □

8 Other Structures

In this section, we deal with the same problem for other structures, digraphs, super-graphs and boolean functions.

8.1 Digraphs

We start with the digraphs. A digraph G is a pair (V, E), where V is a set of the vertices of G and $E \subseteq (V \times V)\backslash\{(v, v) : v \in V\}$ is a set of directed edges of G (without loops). A permutation $\boldsymbol{\sigma}$ of the set V belongs to $Aut(G)$ if, for every pair (v, w), we have $(v, w) \in E$ if and only if $(\boldsymbol{\sigma}(v), \boldsymbol{\sigma}(w)) \in E$. For a digraph $G = (V, E)$ and a vertex $v \in V$, we define $d_f(v)$ to be the number of these vertex $w \in V$ that $(w, v) \in E$. We say that a permutation group A belongs to the class $DGR(2)$ if there exists a digraph G such that $Aut(G) = A$. In [18], A. Kisielewicz stayed the following.

Theorem 8.1 [18] *Every one generated permutation group belong to the class* $DGR(2)$.

In [18], Kisielewicz has proved this only for one example. He has written that it can be easily generalized for the general case. Since it may be not so obvious how to do it, we write a more detailed proof.

Proof We prove it by induction on the number of orbits. This is obvious, and many authors have observed it, that $C_n \in DGR(2)$ for every n. Hence, we have the first step of the induction. For the second step, we prove the extension lemma, similar as Lemma 6.1. This is an extension of [10, Theorem 2.1].

Lemma 8.2 *Let* (A, V) *be a one-generated permutation group. Let* O *be an orbit of* A *such that there exists a digraph* $G_1 = (V\backslash O, E_1)$ *with the group* $Aut(G_1)$ *which is equal to the restriction of* A *to the set* $V\backslash O$. *Moreover, assume that for every*

$v \in V \backslash O$, we have $d_f(v) \geq 1$. Then, $A \in DGR(2)$. In addition, there exists a digraph $G = (V, E)$ such $Aut(G) = A$ and for every $v \in V$, we have $d_f(v) \geq 1$.

Proof Let $O, O_1, \ldots, O_t$ be a list of orbits of A. We denote $O = \{v_1, \ldots, v_{|O|-1}\}$ and $O_i = \{w_1^i, \ldots, w_{|O_1|-1}^i\}$. Assume that $A = \langle \boldsymbol{\sigma} \rangle$, $\boldsymbol{\sigma}(v_i) = v_{(i+1 \mod |O|)}$, and $\boldsymbol{\sigma}(w_i^j) = w_{(i+1 \mod |O_j|)}^j$. Let B be the restriction of A to the set $V \backslash O$. Let $G_1 = (V \backslash O, E_1)$ be a digraph such that $Aut(G_1) = B$, and, for every $v \in V \backslash O$, we have $d_f(v) \geq 1$. We define a digraph $G = (V, E)$ as follows. $(v, w) \in E$ if and only if one of the following holds.

- $\{v, w\} \subseteq (V \backslash O)$ and $(v, w) \in E_1$,
- $v = v_i$, $w = w_{(i+1 \mod |O|)}$,
- $v = v_i$, $w = w_k^j$ and $i \equiv k \mod \gcd(|O|, |O_j|)$.

Obviously, $d_f(v) = 1$, for every $v \in O$ and $d_f(v) > 1$, otherwise. This implies that the set O is stabilized by $Aut(G)$. The rest of proof is the same as in the proof of Lemma 6.1. □

We continue the proof of the theorem. If A is of order two, then, by Theorem 7.3, $A \in GR(2) \subseteq DGR(2)$. Let A be a one-generated group of order greater than two. There exists an orbit O of cardinality $n > 2$. Then, the group A restricted to O is equal to C_n. There exists a graph $G = (O, E)$ such that $d_f(v) = 1$ for every $v \in O$. Consequently, using repetitively Lemma 8.2, we have $A \in DGR(2)$. □

8.2 Supergraphs

The supergraphs is another graphical structure. It was introduced in [18] by A. Kisielewicz. This is an extension of a graph. A supergraph G is a pair (V, E), where V is a set of vertices of G. The set of the edges is defined inductively. Every vertex is an edge of order 0. Let E_i be the set of the edges of order i. If $k > 0$, then every edge of order k is a pair $\{v, w\}$, $v \neq w$, such that $v \in E_j$, $w \in E_l$ and $j < k, l < k$. Then, $E = \bigcup E_i$. We say that G is of order k, if $E = E_k$ and $E \neq E_{k-1}$. A permutation $\boldsymbol{\sigma}$ of V belongs to $Aut(G)$, if $\boldsymbol{\sigma}$ preserves the structure of G. We say that $A \in SGR(k)$ if there exists a supergraph G of order at most k such that $Aut(G) = A$. In [18], we can find the following.

Theorem 8.3 [18] *$GR(2) = SGR(1)$ and $DGR(2) \subseteq SGR(2)$.*

An immediate consequence of Theorems 8.1 and 8.3 is:

Corollary 8.4 *Every one-generated permutation group belongs to $SGR(2)$. Moreover, a one-generated permutation group belongs to $SGR(1)$ if and only if it belongs to $GR(2)$.*

8.3 Boolean Functions

By a boolean function, we mean every function f of the form $f : \{0, 1\}^{\{0,\ldots,n-1\}} \to \{0, \ldots, k-1\}$. A permutation $\boldsymbol{\sigma}$ of the set $\{0, \ldots, n-1\}$ belongs to $Aut(f)$ if $\boldsymbol{\sigma}$

preserves the function f. We say that a permutation group A belongs to the class $BGR(k)$ if there exists a boolean function $f : \{0,1\}^{\{0,\ldots,n-1\}} \to \{0,\ldots,k-1\}$. By BGR we denote $\bigcup BGR(k)$. The boolean function can be identifying with n-dimension k-colored simplex, i. e., where every subsimplex is colored one of k-colors. In this sense boolean functions are graphical structures, which is one of the natural generalizations of edge-colored graphs. The theorem stated in [4] and repeated in [19], we can write as follows.

Theorem 8.5 [4] *A one-generated permutation group A belongs to BGR if and only if whenever there exists an orbit O such that $|O| \in \{3,4,5\}$, then there exists an orbit O' such that* $\gcd(|O|, |O'|) > 2$. *Moreover, if $A \in BGR$, then $A \in BGR(2)$.*

In [4], the proof is very complicated. In [19], it is much simpler. However, the proof in [19] contains a gap. At the end of the proof, it is used an extension theorem ([19, Theorem 4.4]) without checking the assumptions. In my opinion, the assumptions were forgotten. Obviously, one can prove that they are satisfied. However, the proof of those conditions is as hard as the proof of whole the theorem. The extension lemma that should be used there is a stronger version of [19, Theorem 4.4] but in less general case.

Lemma 8.6 *Let (A, V) be a one-generated permutation group. Let W be a proper subset of V preserved by A such that A restricted to W belongs to $BGR(2)$ and A restricted to $V \backslash W$ belongs to $BGR(2)$. Then, $A \in BGR(2)$.*

The proof is similar to the proof of Lemma 6.1 and the proof of Lemma 8.2. This is as hard as the proof of Lemma 8.2, easier than the proof of Lemma 6.1 and definitely easier than the proof of [19, Theorem 4.4]. We leave it to the reader.

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