On the Relation of Separability, Bandwidth and Embedding

Béla Csaba¹ · Bálint Vászárhelyi²

Received: 23 September 2018 / Revised: 30 March 2019 / Published online: 9 September 2019
© The Author(s) 2019

Abstract

In this paper we construct a class of bounded degree bipartite graphs with a small separator and large bandwidth, thereby showing that separability and bandwidth are not linearly equivalent. Furthermore, we also prove that graphs from this class are spanning subgraphs of graphs with minimum degree just slightly above $n/2$, even though their bandwidth is large.

Keywords Extremal graph theory · Bandwidth · Separability

1 Introduction

One of the most basic questions in graph theory is, given graphs H and G , to decide whether H is a subgraph of G . If so, we also say that H can be *embedded* into G , and call G the *host graph*.

In this paper we construct a class of bounded degree bipartite graphs that have small separators and large bandwidth, and prove that the graphs of this class are spanning subgraphs of n -vertex graphs that have minimum degree just slightly larger than $n/2$. We also show that using earlier methods these graphs cannot be embedded in general into host graphs with such small minimum degree.

The structure of the paper is as follows. First we give a brief overview of the area, finishing with a list of our new results. Next we define an infinite class of bounded degree graphs having small separators and very large bandwidth. Finally, we embed such graphs using the Regularity Lemma–Blow-up Lemma method.

Partially supported by Ministry of Human Capacities, Hungary, Grant 20391-3/2018/FEKUSTRAT, NKFIH Fund No. KH 129597 and by the NKFIH Fund No. SNN 117879.

✉ Béla Csaba
bcsaba@math.u-szeged.hu

Bálint Vászárhelyi
mesti@math.u-szeged.hu

¹ Bolyai Institute, Interdisciplinary Excellence Centre, University of Szeged, Aradi vértanúk tere 1, Szeged 6720, Hungary

² Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged 6720, Hungary

2 Overview of the Area and Our New Results

In this paper we consider only simple graphs. We use standard graph theory notation. In particular, if $F = (V, E)$ is a graph then the degree of a vertex $v \in V(F)$ is denoted by $\deg_F(v)$, or just $\deg(v)$, if F is understood from the context. The number of vertices of F is denoted by $|F|$, and we let $e(F) = |E(F)|$. Given $A \subset V(F)$ we let $F[A]$ denote the subgraph of F that is spanned by the vertices in A . If $A, B \subset V(F)$ such that $A \cap B = \emptyset$ then $F[A, B]$ denotes the bipartite subgraph of F which contains precisely the edges with one endpoint in A and the other endpoint in B . Given $S \subset V(F)$ then $N(S)$ denotes the set of those vertices that have at least one neighbor in S . The maximum degree of F is denoted by $\Delta(F)$, the minimum degree of F is denoted by $\delta(F)$.

Perhaps the most cited result in extremal graph theory is the celebrated theorem of Dirac, stating that if the minimum degree of a graph on $n \geq 3$ vertices is at least $n/2$, then the graph contains a Hamilton cycle. This result was generalized in various ways. In a *Dirac-type* embedding problem one gives a lower bound on the minimum degree of the host graph G , and finds conditions for a graph H on the same number of vertices which guarantees that $H \subset G$.

The famous Bollobás–Eldridge–Catlin conjecture [4,8] is a Dirac-type question, in which we have a bound for the maximum degree of the graph to be embedded. It asserts that if G and H are graphs on n vertices, and $\delta(G) \geq (1 - 1/(\Delta(H) + 1))n$ then $H \subset G$. It is still open in general, only some special cases were solved (for large n), e.g., the cases $\Delta(H) = 2$ [2], $\Delta(H) = 3$ [12], and when $\Delta(H)$ is bounded and H is bipartite [10]. Kaul et al. proved an approximation result in [16].

One may impose other kind of restrictions on H and obtain still hard problems. For example, one may upper bound the so-called bandwidth of H , this guarantees that H is “far from being an expander”. The bandwidth of a graph H is denoted by $bw(H)$ and is defined to be the smallest positive integer b such that there exists a labelling of the vertices of $V(H)$ by the numbers $\{1, \dots, n\}$ such that the labels of every pair of adjacent vertices differ by at most b .

Note that a Hamilton path has bandwidth 1, a Hamilton cycle has bandwidth 2. Expander graphs have large, linear bandwidth, a star on n vertices has bandwidth $n/2$, a complete graph has bandwidth $n - 1$.

One of the most important open problems concerning the bandwidth was a conjecture by Bollobás and Komlós [20]. This conjecture was proved by Böttcher et al. [7]¹ using deep tools, in particular the Regularity Lemma and the proof of the celebrated Pósa–Seymour conjecture by Komlós et al. [19].

Theorem 1 *For every $D, k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\beta > 0$ such that the following holds. Every n -vertex graph G having minimum degree $\delta(G) \geq (1 - 1/k + \varepsilon)n$ contains all k -chromatic n -vertex graphs of maximum degree at most D and bandwidth at most βn as subgraphs.*

Böttcher [5] and Böttcher et al. [6] went further and explored relations of bandwidth with other notions, like separability. We say that an n -vertex graph H is γ -separable

¹ We remark, that the case of $k = 2$ was first proved by Abbasi [1], and is also a special case of a result by the author [11]. Some of the ideas of the latter proof are used in the present paper as well.

if there exists a separator set $S \subset V(H)$ with $|S| \leq \gamma n$ such that every component of $H - S$ has at most γn vertices. Böttcher et al. [6] observed that bandwidth and separability are closely related: they proved the Sublinear Equivalence Theorem. This states that, roughly speaking, in bounded degree graphs sublinear bandwidth implies the existence of a sublinear sized separating set and vice versa.

It is easy to see that there are bounded degree graphs having linearly large bandwidth: it is well-known that a random l -regular graph with $l \geq 3$ has large bandwidth with positive probability. However, such random graphs do not have small separators. One of our results shows that when the separating set has small (but not very small) linear size, the bandwidth can be very large even for bounded degree graphs. Hence, there is no linear equivalence of bandwidth and separability, despite the sublinear equivalence of these graph parameters.

In order to state one of the main results of the paper we need a definition: we call a number r *eligible* if there exists a prime p such that $r = p + 1$ and $p \equiv 1 \pmod{4}$.

Theorem 2 *Let $r \geq 38$ be an eligible number and $t \geq 1$ be an integer. Set $\gamma = \gamma(r) = 1/((r+1)2^{r+4})$. Then one can construct an infinite class of bipartite graphs $\mathcal{H}_{r,t}$ such that every element H of $\mathcal{H}_{r,t}$ is γ -separable, has bandwidth at least $0.3|H|/(2t+4)$, moreover, $\Delta(H) < 1/\gamma = (r+1)2^{r+4}$.*

There is a recent interest in embedding graphs with sublinear bandwidth. Staden and Treglown [24] embed graphs on n vertices with sublinear bandwidth into locally dense graphs on n vertices having minimum degree at least $(1/2 + o(1))n$. Condon et al. [9] find an approximate decomposition of a certain class of graphs into graphs that have sublinear bandwidth.

The result of Knox and Treglown [17] is particularly interesting for us. They embedded bounded degree graphs with sublinear bandwidth into so called robust expanders. Let $0 < \nu \leq \tau < 1$. Assume that G is a graph of order n and $S \subset V(G)$. The ν -robust neighborhood $RN_{\nu,G}(S)$ of S is the set of vertices $v \in V(G)$ such that $|N(v) \cap S| \geq \nu n$. We say that G is a robust (ν, τ) -expander if $|RN_{\nu,G}(S)| \geq |S| + \nu n$ for every $S \subset V(G)$ such that $\tau n \leq |S| \leq (1 - \tau)n$.

We will also show that elements of $\mathcal{H}_{r,t}$ (the graph class of Theorem 2) cannot be embedded into arbitrary robust expanders. However, if an n -vertex graph G has minimum degree slightly larger than $n/2$, then it contains the elements of $\mathcal{H}_{r,t}$ as spanning subgraphs. We will prove the following.

Theorem 3 *Let $r \geq 38$ be an eligible number and $t \geq 1$ be an integer. Set $\gamma = \gamma(r) = 1/((r+1)2^{r+4})$. Then there exists an $n_0 = n_0(\gamma)$ such that the following holds. Assume that $n \geq n_0$ and G is an n -vertex graph having minimum degree $\delta(G) \geq (1/2 + 3\gamma^{1/3})n$. If $H \in \mathcal{H}_{r,t}$ is a graph on n vertices, then $H \subset G$.*

Theorem 3 tells us that, from the embedding standpoint, in certain cases the existence of a small separator set is more important than low bandwidth.

A standard example shows that one cannot significantly reduce the minimum degree of G in the above theorem. Let G be the union of two complete graphs on, say, $n/2 + \gamma n/100$ vertices that share $\gamma n/50$ vertices. Clearly, $\delta(G) = n/2 + \gamma n/100 - 1$.

It is an easy exercise to prove² that if $H \in \mathcal{H}_{r,t}$ and $|H| = |G| = n$, then $H \not\subset G$. We leave this proof for the reader.

The proof of Theorem 3 will rely heavily on the proof method of [11] and an important result of Fox and Sudakov [14]. Let us remark that in [11] the size of the separator set was $o(n)$, and therefore, by the Sublinear Equivalence theorem, the bandwidth was also $o(n)$. This time the separator set is quite large compared to previous results.

Finally, let us mention a tightness result for Theorem 1 by Abbasi [1]. He proved that for infinitely many n there exist graphs G and H on n vertices such that H is bipartite, $bw(H) \leq 4\eta n$, $\delta(G) = (1/2 + \eta)n$, still $H \not\subset G$. So in general if one wants to be able to embed every graph with bandwidth ηn , then the minimum degree bound for the host graph must be larger than $n/2 + \eta n/4$.

3 Construction of $\mathcal{H}_{r,t}$ and Proof of Theorem 2

In order to exhibit the infinite family of graphs $\mathcal{H}_{r,t}$ we first need to construct certain kind of bipartite expander graphs. We begin with defining a bipartite graph F with vertex classes V_1 and V_2 such that $|V_1| = |V_2| = k$ and F has relatively good expansion properties. Our construction of F relies on the existence of so called Ramanujan graphs³: an r -regular (non-bipartite) graph U is a Ramanujan graph if $\lambda \leq 2\sqrt{r-1}$, where λ is the second largest in absolute value of the eigenvalues of U (since U is r -regular, the largest eigenvalue is r).

Recall the notion of eligible numbers. Lubotzky et al. [22], and independently Margulis [23], constructed infinite families of r -regular graphs for every eligible number r with second largest eigenvalues at most $2\sqrt{r-1}$. We need a fact about these graphs, a lower bound for the number of edges between subsets of U .

Lemma 1 *Let U be a graph as above. Then for every two subsets $A, B \subset V(U)$ where $|A| = ak$ and $|B| = bk$ we have*

$$|e(A, B) - abrk| \leq 2\sqrt{r-1}\sqrt{abk}.$$

The proof of Lemma 1 can be found for example in [3].

Corollary 1 *Let U be an r -regular Ramanujan graph on k vertices with $r \geq 38$. Let us assume that $A, B \subset V(U)$ with $|A| = |B| = k/3$ and $A \cap B = \emptyset$. Then $e(A, B) \geq 1$.*

Proof It is easy to see that the expression of Lemma 1 gives a lower bound for $e(A, B)$ which is monotone increasing in r . Hence it is sufficient to apply Lemma 1 with $r = 38$ and $a = b = 1/3$. Straightforward computation gives what was desired. $\square$

We are ready to discuss the details of the construction of F . Given an r -regular Ramanujan graph U with $r \geq 38$ the vertex classes V_1 and V_2 of F will be copies

² Of course, one first needs the definition of $\mathcal{H}_{r,t}$ for this proof, which is given in Sect. 3 soon.

³ We remark that here one may as well work with random regular bipartite graphs instead of explicit constructions.

of $V(U)$: for every $x \in V(U)$ we have two copies, $x_1 \in V_1$ and $x_2 \in V_2$. For every $xy \in E(U)$ we include the edges x_1y_2 and x_2y_1 in $E(F)$. Finally, for every $x \in V(U)$ we will also have the edge x_1x_2 in $E(F)$. Observe that F is an $(r+1)$ -regular balanced bipartite graph. The following claims are crucial for the construction of $\mathcal{H}_{r,t}$.

Claim *Let $A \subset V_1$ and $B \subset V_2$ be arbitrary such that $|A| = |B| = k/3$. Then $e_F(A, B) \geq 1$.*

Proof If there exists $x \in V(U)$ such that $x_1 \in A$ and $x_2 \in B$ then we are done since every x_1x_2 edge is present in F . If there is no such $x \in V(U)$ then we can apply Corollary 1 and obtain what is desired. $\square$

Claim *For every $A \subset V_1$ we have $|N_F(A)| \geq |A|$. Analogous statement holds for any subset $B \subset V_2$.*

Proof The claim easily follows from the fact that we included a perfect matching in F when for every $x \in V(U)$ we added the x_1x_2 edge to $E(F)$. $\square$

Observe that we can construct a balanced bipartite graph F with $|V(F)| = 2k$ having the above properties whenever there exists a Ramanujan graph U with $|V(U)| = k$, for the latter we also assume that $r \geq 38$. Thus, there exists an infinite sequence $\mathcal{F} = \{F_i\}_{i=1}^\infty$ of graphs on an increasing number of vertices. We let k_i denote the number of vertices in a vertex class of $F_i \in \mathcal{F}$ has $2k_i$ vertices, so $|F_i| = 2k_i$ for every i .

For technical reasons we discard those F_i graphs from $\mathcal{F}$ for which $2k_i \leq (r+1)2^{r+4} = 1/\gamma$. Clearly, we discard at most $(r+1)2^{r+4}$ graphs. For simplicity we keep the notation $\mathcal{F} = \{F_i\}_{i=1}^\infty$ for the infinite family we obtain this way.

We are ready to define $\mathcal{H}_{r,t}$. Each graph from this class is γ -separable where $\gamma = \gamma(r)$ can be relatively small as we will see soon. Still, the bandwidth of each of them is very large. Hence, $\mathcal{H}_{r,t}$ demonstrates that in spite of sublinear equivalence of separability and bandwidth, there is no *linear* equivalence.

The construction of $\mathcal{H}_{r,t}$ is somewhat specific, we do it with foresight as our goal is not only to further explore the relation of separability and bandwidth but also to be able to embed the elements of $\mathcal{H}_{r,t}$ later.

Definition 1 Let $r \geq 38$ be an eligible number and $t \geq 1$ be an integer. Set $\gamma = \gamma(r) = 1/((r+1)2^{r+4})$, and let $D = \max\{1, (r+1)2^{r+4} - t + 1\}$. Denote $\mathcal{F} = \{F_i\}_{i=1}^\infty$ the infinite family of $(r+1)$ -regular balanced bipartite graphs defined above. For every $i \geq 1$ we let $n_i = 2k_i(t+D-1)$. For every $i \geq 1$ the i th graph $H = (A, B; E) \in \mathcal{H}_{r,t}$ is a balanced bipartite graph on n_i vertices constructed as follows.

1. $V(H) = A \cup B$ and $|H| = |A \cup B| = n_i$,
2. let $S \subset V$ such that $S = S_A \dot{\cup} S_B$ and $|S_A| = |S_B| = k_i$,
3. $E(H[S_A]) = E(H[S_B]) = \emptyset$ and $H[S_A, S_B] = F_i$,
4. $H - S$ has exactly $|S|$ isomorphic components; each component is a tree T which contains a path on t vertices, and one of its endpoints, the *last vertex*, has $D - 1$ leaves attached to it; we call the other endpoint of this path the *first vertex*.

Fig. 1 The separator set S and the way the components of $H - S$ are connected to S

5. every $x \in S$ has a unique neighbor y in $V - S$ which is the first vertex of one of the tree components of $H - S$, moreover, every first vertex has exactly one neighbor in S (Fig. 1).

Claim Every $H \in \mathcal{H}_{r,t}$ is γ -separable.

Proof Clearly, S is a separator set of H with $|S| = 2k_i$, and every component of $H - S$ has exactly $t + D - 1$ vertices. In order to prove that $2k_i$ and $t + D - 1$ are both upper bounded by γn_i , we consider two cases depending on the value of D .

In the first case $D = 1$, which implies that $1 - t + 1/\gamma \leq 1$, hence, $t \geq 1/\gamma$. Since in this case $2k_i t = n_i$, we get that $|S| = 2k_i \leq \gamma n_i$. Recall that $2k_i > (r+1)2^{r+4} = 1/\gamma$ for every i , therefore, $t = n_i/(2k_i) < \gamma n_i$. This in turn implies that H is γ -separable.

Now we consider the case $D \geq 2$. Then by the definition of D we have $D = \frac{1}{\gamma} + 1 - t$, hence, $t + D - 1 = t + \frac{1}{\gamma} + 1 - t - 1 = \frac{1}{\gamma}$. Since $2k_i(t + D - 1) = n_i$, we get that $|S| = 2k_i = \gamma n$. Finally, similarly to the previous case, we have that $t + D - 1 \leq \gamma n$, otherwise $2k_i$ were less than $1/\gamma$, proving what was desired. $\square$

We remark that when $t = 1$ then the first and the last vertex in every path are the same. We have the following claim.

Claim For every $H \in \mathcal{H}_{r,t}$ we have $\Delta(H) = \max\{r + 2, D\} < (r + 1)2^{r+4}$.

Proof First observe, that there are only four possible values for vertex degrees in H . If $x \in S$ then $d(x) = r + 2$. If $y \in V(H) - S$, then there are three possibilities for $d(y)$: either $d(y) = 2$, or $d(y) = 1$, or $d(y) = D$. Since $D = (r + 1)2^{r+4} - t + 1$, and $r + 2 < (r + 1)2^{r+4}$ for every positive integer r , we get what was desired. $\square$

The following lemma is crucial for bounding the bandwidth of $H \in \mathcal{H}_{r,t}$.

Lemma 2 Let H be an element of $\mathcal{H}_{r,t}$ on n vertices. Assume that $X, Y \subset V(H)$ with $|X|, |Y| \geq 0.35n$ and $X \cap Y = \emptyset$. Then there exists an $x \in X$ and a $y \in Y$ such that the distance of x and y is at most $2t + 4$.

Proof Denote the vertices of $H - S$ closer to S_A by A^* , and analogously, the vertices of $H - S$ closer to S_B by B^* . By the construction of H we have $|A^*| = |B^*| \geq (1 - \gamma)n/2$. Note that $\gamma < 0.01$ since $r \geq 35$. Hence we have that $|X - S| \geq 0.34n$ and $|Y - S| \geq 0.34n$. Thus, either $|X \cap A^*| \geq |A^*|/3$ or $|X \cap B^*| \geq |B^*|/3$. Without loss of generality, suppose the former. This also implies that at least $1/3$ of the components of A^* have vertices in X . Denote the first vertices in these components by X^* . We also let Y_A^* denote the set of first vertices of those components in A^* that have at least one vertex from Y , and analogously, Y_B^* denotes the set of first vertices of those components in B^* that have at least one vertex from Y .

Let $X_A = N(X^*) \cap S$, $Y_A = N(Y_A^*) \cap S$ and $Y_B = N(Y_B^*) \cap S$. Using these notations we have that $|X_A| \geq k/3$ and either $|Y_A| \geq k/3$, or $|Y_B| \geq k/3$.

If $|Y_B| \geq k/3$, then by Claim 3 there is an edge $z_1 z_2$ between X_A and Y_B , and therefore we have a path $x v_s \dots v_1 z_1 z_2 u_1 \dots u_q y$ of length $\leq 2t + 3$, where $x \in X$, $y \in Y$, $v_i \in A^*$, $z_1 \in X_A$, $z_2 \in Y_B$, $u_i \in B^*$.

If $|Y_B| < \frac{k}{3}$, then $|Y_A| \geq k/3$. Let $Y_B' = N(Y_A) \cap S$. Claim 3 implies that $|Y_B'| \geq |Y_A| \geq k/3$, so by Claim 3 H has an edge $z_2 z_1$ between Y_B' and X_A . Thus, we have a path $x v_s \dots v_1 z_1 z_2 z_3 u_1 \dots u_q y$ of length $\leq 2t + 4$, where $x \in X$, $y \in Y$, $v_i \in A^*$, $u_i \in B^*$, $z_1 \in X_A$, $z_3 \in Y_A$ and $z_2 \in Y_B'$. $\square$

Corollary 2 *Let H be an element of $\mathcal{H}_{r,t}$ on n vertices. Then the bandwidth of H is at least $\frac{0.3n}{2t+4}$.*

Proof Take an arbitrary ordering $\mathcal{P}$ of the vertices of H . Let X be the first $0.35n$ vertices, and Y be the last $0.35n$ vertices of $\mathcal{P}$. Using Lemma 2 there is an $x \in X$ and a $y \in Y$ such that the distance of x and y is at most $2t + 4$. Their distance in $\mathcal{P}$ is at least $0.3n$. Thus at least one of the edges of the shortest path between x and y must have “length” at least $\frac{0.3n}{2t+4}$, from which the bound for the bandwidth follows immediately. $\square$

With this we proved Theorem 2. Observe that choosing $t = 1$ results in graphs having bandwidth at least $3n/60 = n/20$ while being γ -separable. It is easy to see that, using the above ideas, one can define a much wider class of graphs having very large bandwidth that are also γ -separable. Our main goal, however, is not only to construct but also to be able to embed such graphs.

Recall the notion of robust expanders. As we mentioned in the introduction, Knox and Treglown [17] embedded spanning subgraphs of sublinear bandwidth into robust expanders. The following example shows that graphs of $\mathcal{H}_{r,t}$ not only have very large bandwidth, these graphs are not necessarily subgraphs of robust expanders. Hence, in the theorem of Knox and Treglown one cannot replace sublinear bandwidth by γ -separability, unless γ is very small. In fact the proofs of [6] and [17] works only in case $\gamma \leq 1/\ell$, where ℓ denotes the number of clusters in a sufficiently large graph after applying the Regularity lemma with some small parameter $\varepsilon > 0$. It is known (see [15]) that ℓ is bounded from below by a tower function of $1/\varepsilon$.

Next let us construct a robust expander as follows. Let $G = (V, E)$ be a graph on n vertices such that the vertex set of G is $V = A_1 \dot{\cup} A_2 \dot{\cup} \dots \dot{\cup} A_{100}$, here $|A_i| = \frac{n}{100}$ for every $1 \leq i \leq 100$. The edges of G are defined as follows: $G[A_i, A_{i+1}]$ is a complete bipartite graph for every $1 \leq i \leq 99$, and $G[A_1]$, $G[A_{100}]$ are complete graphs on $n/100$ vertices.

From the following remark one can easily see that G is a (ν, τ) -robust expander, where $1/n < \nu$ and $\tau \geq 100\nu$. Let $1 < i < 100$ and $Q \subset V(G)$ be an arbitrary set. Then every vertex in $A_{i-1} \cup A_{i+1}$ has $|Q \cap A_i|$ neighbors in $Q \cap A_i$. Furthermore, every vertex in $A_1 \cup A_2$ has at least $|Q \cap A_1| - 1$ neighbors in $Q \cap A_1$, unless $|Q \cap A_1| \leq 1$. Analogous statement holds for A_{100} .

Lemma 3 *Let H be a graph from $\mathcal{H}_{r,t}$ on n vertices and let G be as above. Then $H \not\subset G$ if $t \leq 13$.*

Proof Let $X = A_1 \cup \dots \cup A_{35}$ and $Y = A_{66} \cup \dots \cup A_{100}$. If one embeds H into G , then between the vertices that were mapped onto some vertex of X and those that were mapped onto some vertex of Y there is a path having length at most $2t + 4$. In G the shortest path between X and Y has length 31. Since $2t + 4 < 31$ whenever $t \leq 13$, we proved what was desired. $\square$

4 Proof of Theorem 3

The proof of Theorem 3 is very similar to the proof of the main result of [11]. First let us state a special case of the main theorem of [11] for embedding bipartite graphs with small separators.

Theorem 4 [11] *For every $\epsilon > 0$ and positive integer D there exists an $n_0 = n_0(\epsilon, D)$ such that the following holds. Assume that H is a bipartite graph on $n \geq n_0$ vertices which has a separator set S such that $|S| = o(n)$, and every component of $H - S$ has $o(n)$ vertices. Assume further that $\Delta(H) \leq D$. Let G be an n -vertex graph such that $\delta(G) \geq (1/2 + \epsilon)n$. Then $H \subset G$.*

One can observe the similarities with Theorem 3. The main difference is that in Theorem 3 the separator set can be very large compared to the separator set in Theorem 4. This difference requires a new embedding tool.

4.1 Main Tools for the Proof

The Regularity Lemma of Szemerédi [25] and the Blow-up Lemma [18] plays a very important role in the proof. While we assume familiarity with these tools, below we give a brief summary of the necessary definitions and results. The interested reader may consult with the survey paper by Komlós and Simonovits [21] for further details.

Definition 2 Given a graph G and two disjoint subsets $X, Y \subset V(G)$, the *density* between X and Y is

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}. \quad (1)$$

Definition 3 We call a pair (A, B) of disjoint vertex sets in G ϵ -regular, if for every $X \subset A$ and $Y \subset B$ we have $|d(X, Y) - d(A, B)| < \epsilon$, whenever $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$.

We will need the so-called Degree Form of the celebrated Regularity Lemma of Szemerédi:

Lemma 4 *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$, then there is a partition of the vertex set V into $\ell + 1$ clusters $W_0, W_1, \dots, W_\ell$ and there is a subgraph G' of G such that*

- $\ell \leq M$,
- $|W_0| \leq \varepsilon|V|$,
- All clusters W_i , $i \geq 1$ are of the same size m
- $\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)|V|$ for all $v \in V$,
- W_i is an independent set in G' for all $i \geq 1$,
- All pairs (W_i, W_j) ($1 \leq i < j \leq \ell$) are ε -regular, each with density either 0 or at least d in G' .

We call W_0 the *exceptional cluster*, $W_1, \dots, W_\ell$ are the *non-exceptional clusters*.

Definition 4 Apply Lemma 4 to the graph $G = (V, E)$ with parameters ε and d . We construct the *reduced graph* G_r as follows. Its vertices are the non-exceptional clusters, and two vertices are connected if the corresponding clusters form an ε -regular pair with density at least d .

Claim (Corollary 8 in [11]) *Let $G = (V, E)$ be a graph of order n and $\delta(G) \geq cn$ for some $c > 0$. Let G_r be the reduced graph of G' after applying Lemma 4 with parameters ε and d . Then $\delta(G_r) \geq (c - \theta)\ell$, where $\theta = 2\varepsilon + d$.*

Definition 5 We say that the pair of disjoint vertex sets (A, B) is (ε, δ) -super-regular, if it is ε -regular, and $\deg(a) > \delta|B|$ for all $a \in A$, and $\deg(b) > \delta|A|$ for all $b \in B$.

Remark 1 If (A, B) is an ε -regular pair, and we add $\leq 2\varepsilon|A|$ new vertices to A resulting in A' , then the new pair (A', B) is ε' -regular, where $\varepsilon' \leq 2\sqrt{\varepsilon}$.

Remark 2 If (A, B) is an ε -regular pair with density d , then for any $Y \subset B$, $|Y| > \varepsilon|B|$ we have

$$|\{x \in A : |N(x) \cap Y| \leq (d - \varepsilon)|Y|\}| \leq \varepsilon|A|. \quad (2)$$

We need the following very important lemma:

Theorem 5 (Blow-up Lemma [18]) *Given a graph R of order r and positive parameters δ, Δ , there exists a positive $\varepsilon = \varepsilon(\delta, \Delta, r)$ such that the following holds: let $n_1, n_2, \dots, n_r$ be arbitrary positive integers and replace the vertices $v_1, v_2, \dots, v_r$ of R with pairwise disjoint sets $V_1, V_2, \dots, V_r$ of sizes $n_1, n_2, \dots, n_r$ (blowing up). We construct two graphs on the same vertex set $V = \cup V_i$. The first graph F is obtained by replacing each edge $v_i v_j$ of R with the complete bipartite graph between V_i and V_j . A sparser graph G is constructed by replacing each edge $v_i v_j$ arbitrarily with an (ε, δ) -super-regular pair between V_i and V_j . If a graph H with $\Delta(H) \leq \Delta$ is embeddable into F , then it is also embeddable into G .*

Theorem 6 (Strengthening the Blow-up Lemma [18]) *Assume that $n_i \leq 2n_j$ for every $1 \leq i, j \leq r$. Then we can strengthen the lemma: given $c > 0$ there are positive numbers $\varepsilon = \varepsilon(\delta, \Delta, r, c)$ and $\alpha = \alpha(\delta, \Delta, r, c)$ such that the Blow-up Lemma remains true if for every i there are certain vertices x to be embedded into V_i whose images are a priori restricted to certain sets $T_x \subset V_i$ provided that each T_x within a V_i is of size at least $c|V_i|$, and the number of such restrictions within a V_i is not more than $\alpha|V_i|$.*

Another important tool for the proof is the following result by Fox and Sudakov (Theorem 6.1 on page 79 in [14]).

Theorem 7 *Let H be a bipartite graph with n vertices and maximum degree $\Delta \geq 1$. Assume that $\rho \in (0, 1)$ is a real number. If G is a graph with $N \geq 8\Delta\rho^{-\Delta}n$ vertices and at least $\rho\binom{N}{2}$ edges, then $H \subset G$.*

We are going to apply Theorem 7 in the special case $\rho \geq 1/2$.

Finally one more notation: we will write $a \ll b$ for two positive numbers a and b if a is sufficiently smaller than b .

4.2 Proof of Theorem 3

Below we present the embedding algorithm as an itemized list. As we indicated above, certain parts of the proof of Theorem 3 are very similar to the proof of Theorem 4 from [11]. Hence, whenever it is possible we refer to the claims, lemmas of [11].

Step 1 Assume that $H \in \mathcal{H}_{r,t}$ has n vertices. Denote the separator set of H by S , where $|S| \leq \gamma n$. We can apply the result of Fox and Sudakov, Theorem 7 above, for finding a copy of $H[S]$ in G as follows. First observe that ρ , the density of G , is larger than $1/2$, since $\delta(G) > n/2$. Next, recall that $\Delta(H[S]) = r + 1$ and $n = |H| \geq |S|/\gamma = |S|(r + 1)2^{r+4} = |S|8(r + 1)2^{r+1}$. Therefore, the conditions of Theorem 7 are satisfied, $H[S]$ is a subgraph of G . Let us call the uncovered part of G by $\tilde{G}$ after embedding $H[S]$. Note that $\delta(\tilde{G}) \geq (1/2 + 2\gamma^{1/3})n$ and $|\tilde{G}| \geq (1 - \gamma)n$.

Step 2 Next we apply the Degree form of the Regularity lemma for $\tilde{G}$ with parameters ε and d such that $\varepsilon \ll d = \sqrt{\gamma}$. As a result, we have $\ell + 1$ clusters, $W_0, W_1, \dots, W_\ell$. The exceptional cluster W_0 has at most εn vertices, while all other clusters have the same size m . We form the reduced graph $\tilde{G}_r$, for which we have $\delta(\tilde{G}_r) \geq (1/2 + \gamma')\ell$, where $\gamma' = 2\gamma^{1/3} - \sqrt{\gamma} - \varepsilon > \gamma^{1/3}$.

Step 3 Since $\delta(\tilde{G}_r) \geq \ell/2$, we have a perfect (or almost perfect) matching M in $\tilde{G}_r$: at most one cluster will remain uncovered by M , and only if ℓ is odd. If there is such a cluster, we add it into W_0 .

Step 4 Next we transform the edges of M into super-regular pairs. Given a δ with $\varepsilon \ll \delta \ll d$ we have to remove at most εm vertices from a cluster to make a regular pair (ε, δ) -super-regular. We discard the same number of vertices, εm , from every non-exceptional cluster, and place the discarded vertices into W_0 . Note that pairs become ε' -regular with $\varepsilon' < 2\varepsilon$, and the sizes of the clusters are $m' = m - \varepsilon m$. For sake of simplicity we will still use ε and m in the rest of the paper.

Step 5 The next step is to distribute the vertices of the exceptional cluster W_0 among the non-exceptional clusters while maintaining the super-regularity of the edges of

M . We also require that the resulting clusters have about the same size (i.e. we need, that $||W_i| - |W_j|| \leq 3\epsilon m$ for all $1 \leq i, j \leq \ell$). For this, we use the same procedure described in [11], the reader may consult with that paper for the details, here we only sketch the argument. Let us denote the neighbor of a cluster W_i in the matching M by $N_M(W_i)$.

For the distribution of W_0 we define an auxiliary bipartite graph J having vertex classes W_0 and $V(\tilde{G}_r)$. Here $vW_i \in E(J)$ if $\deg_G(v, N_M(W_i)) \geq \delta m$.

Let $\gamma'' = 3(\gamma^{1/3} - 2(\epsilon + d))$. We need the following lemma, which is a special case of Lemma 10 in [11].

Lemma 5 $\deg_J(v) \geq (\frac{1}{2} + \gamma'')\ell$ for every $v \in W_0$. □

Using Lemma 5 we can distribute the vertices of W_0 among the clusters one-by-one, essentially greedily. Assume that we have already found non-exceptional clusters for the first s vertices of W_0 . Then for every $1 \leq i \leq \ell$ we let $P_s(W_i)$ to be the number of W_0 -vertices put into W_i .

When we look for a non-exceptional cluster for the $(s + 1)$ st vertex $v \in W_0$, we take the smallest $P_s(W_i)$ value for every $W_i \in N_J(v)$ (we break ties arbitrarily). It is easy to see by the large minimum degree of J that no non-exceptional cluster will get more than $2|W_0|/\ell$ new vertices from W_0 . Since $|W_0| \leq \epsilon n$, and $n/\ell \approx m$, we get that $||W_i| - |W_j|| < 3\epsilon m$. Using Remark 1 super-regularity of edges of M is maintained, although instead of ϵ we have at most $2\sqrt{2\sqrt{\epsilon}} < 3\sqrt[4]{\epsilon}$ here.

Step 6 Next we assign the components of $H - S$ to the non-exceptional clusters. We do it using a random procedure as in [11] (randomness is not necessary here, but a simple choice), i.e., the components are assigned randomly to edges of M .

The algorithm is as follows. Consider a component A of H , it is a tree, so it is bipartite. Denote its vertex classes by A_1 and A_2 . Pick an edge $Q_1 Q_2$ of M randomly, uniformly. Let π be a uniform random permutation on $\{1, 2\}$, and assign the vertices of A_i to $Q_{\pi(i)}$ for $i = 1, 2$. The following lemma holds (for a proof see [11]).

Lemma 6 *With positive probability the mapping algorithm assigns $n/\ell \pm \epsilon m/\ell$ vertices of H to every edge of $\tilde{G}_r$.* □

For $x \in V(H) - S$ we let $C(x)$ denote the cluster to which x is assigned. The algorithm immediately implies that whenever $x, y \in V(H) - S$ are adjacent, then $C(x)C(y)$ is an edge of M .

Recall that in Step 1 $H[S]$ was embedded using Theorem 7, before even applying the Regularity Lemma. So the vertices of S were not assigned to clusters but directly mapped onto vertices of G . It is clear that adjacent vertices of S were mapped onto adjacent vertices of G .

However, for applying Theorem 6, the Blow-up lemma with restrictions, we need that if $x \in S$ is mapped onto $v \in V(G)$, $y \in V(H) - S$ is assigned to $C(y)$, then v must have many neighbors in $C(y)$.

In such a case we will assign y to another cluster (again, we use a procedure from [11] with a few minor modifications) as follows. Let L denote those clusters in which v has at least $3\gamma^{2/3}m$ neighbors. Simple counting argument shows that $|L| \geq \ell/2$. Let $C(y)W_i$ denote the edge of M to which the component of y was

assigned. Then we locate a cluster $W_j \in L$ such that W_j is adjacent to W_i in $\tilde{G}_r$. Since L has more than $\ell/2$ clusters, using the minimum degree of $\tilde{G}_r$ we have at least $2\gamma^{1/3}\ell$ choices for W_j in L .

Then we change the assignment of y , we let $C(y) = W_j$. This way v will have many neighbors in $C(y)$ and $C(y)$ will be adjacent to the cluster of the neighbors of y in its component. Observe that if we locate the W_j clusters as evenly as possible then we can achieve that at most $|S|/(2\gamma^{1/3}\ell) \leq \gamma n/(2\gamma^{1/3}\ell) \leq \gamma^{2/3}m$ vertices are reassigned to a particular cluster, using that $|S| \leq \gamma n$ and the number of choices for a new cluster is always at least $2\gamma^{1/3}\ell$. Of course, changing the assignment of vertices of H does not affect regularity or super-regularity between clusters of $\tilde{G}_r$.

Step 7 In this step we will achieve that the number of vertices assigned to a cluster is the same as the size of that cluster. As before, we will reassign some of the vertices. While in the previous step we changed the assignment of first vertices of components, this time we will work with the leaves of the components.

Say, that W_s has more vertices assigned to it than its size $|W_s|$. Then there must be a cluster W_i to which we assigned less than $|W_i|$ vertices of H . Let W_j denote the neighbor of W_s in the matching M . If $W_j W_i$ is an edge in $\tilde{G}_r$ then we pick a vertex x such that $C(x) = W_j$ and $\deg_H(x) = D - 1$. This is possible since the number of last vertices (these have degree $D - 1$) is at least $(1 - \gamma)n/(t + D - 1)$, and by Chernoff's inequality with probability $> 1 - 1/n^2$ we assigned more than $(1 - \gamma)m/(2(t + D - 1))$ to every cluster. This implies that the number of leaves that are assigned to a particular cluster is at least $(D - 1)(1 - \gamma)m/(2(t + D - 1))$, which is much larger than $\gamma^{2/3}m$. We reassign some of the *leaves* that are adjacent to x , the necessary number will be assigned to W_i .

If $W_j W_i$ is not an edge, then by the minimum degree of $\tilde{G}_r$ there must exist at least $\gamma^{1/3}\ell$ clusters W_p and W_q such that $W_q W_i$ and $W_j W_p$ are edges in $\tilde{G}_r$, and $W_p W_q$ is an edge in M . Then the above procedure is done in two steps: first we reassign some vertices (always leaves) from W_s to W_p and then the same number of leaves from W_p to W_i . Note that the same computation works as above: at most $\gamma^{2/3}m$ vertices are reassigned at every cluster.

Let us remark again that reassigning the vertices of H during this step does not affect regularity and super-regularity of pairs in $\tilde{G}_r$, unlike when vertices of G were distributed in Step 5.

Step 8 Recall that the density of the regular pairs is at least $\sqrt{\gamma}$, and at most $\gamma^{2/3}m$ vertices are reassigned at every cluster. It is crucial here to look at the so called a priori restrictions for some of the vertices of H before applying the Blow-up Lemma. These are the first vertices of components (since their neighbors belong to S and are already mapped), and the leaves that were reassigned during Step 7. As the number of these vertices is very small compared to m in every cluster, we are able to apply the Blow-up Lemma. This finishes the proof of Theorem 3.

Acknowledgements Open access funding provided by University of Szeged (SZTE).

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Abbasi, S.: How tight is the Bollobás–Kömlos conjecture? *Graphs Comb.* **16**, 129–137 (2000)
2. Aigner, M., Brandt, S.: Embedding arbitrary graphs of maximum degree two. *J. Lond. Math. Soc.* **48**, 39–51 (1993)
3. Alon, N., Spencer, J.: *The Probabilistic Method*, 3rd edn. Wiley, New York (2008)
4. Bollobás, B., Eldridge, S.E.: Packing of graphs and applications to computational complexity. *J. Combin. Theory Ser. B* **25**, 105–124 (1978)
5. Böttcher, J.: *Embedding Large Graphs: The Bollobás–Kömlos Conjecture and Beyond*. Ph.D. thesis, Technische Universität München (2009)
6. Böttcher, J., Pruessmann, K.P., Taraz, A., Würfl, A.: Bandwidth, expansion, treewidth, separators and universality for bounded-degree graphs. *Eur. J. Comb.* **31**, 1217–1227 (2010)
7. Böttcher, J., Schacht, M., Taraz, A.: Proof of the bandwidth conjecture of Bollobás and Kömlos. *Math. Ann.* **343**, 175–205 (2009)
8. Catlin, P.A.: *Embedding Subgraphs and Coloring Graphs Under Extremal Degree Conditions*. Ph.D. thesis, Ohio State Univ., Columbus (1976)
9. Condon, P., Kim, J., Kühn, D., Osthus, D.: A Bandwidth Theorem on Approximate Decompositions [arXiv:1712.04562](https://arxiv.org/abs/1712.04562)
10. Csaba, B.: On the Bollobás–Eldridge conjecture for bipartite graphs. *Comb. Probab. Comput.* **16**(5), 661–691 (2007)
11. Csaba, B.: On embedding well-separable graphs. *Discret. Math.* **308**, 4322–4331 (2008)
12. Csaba, B., Shokoufandeh, A., Szemerédi, E.: Proof of a conjecture of Bollobás and Eldridge for graphs of maximum degree three. *Combinatorica* **23**, 35–72 (2003)
13. Dirac, G.: Some theorems on abstract graphs. *Proc. Lond. Math. Soc.* **2**, 69–81 (1952)
14. Fox, J., Sudakov, B.: Dependent random choice. *Random Struct. Algorithms* **38**, 68–99 (2011)
15. Gowers, T.: Lower bounds of tower type for Szemerédi’s uniformity lemma. *GAF* **7**, 322–337 (1997)
16. Kaul, H., Kostochka, A.V., Yu, G.: On a graph packing conjecture by Bollobás, Eldridge, and Catlin. *Combinatorica* **28**, 469–485 (2008)
17. Knox, F., Treglown, A.: Embedding spanning bipartite graphs of small bandwidth. *Comb. Probab. Comput.* **22**(1), 71–96 (2013)
18. Kömlos, J., Sárközy, G.N., Szemerédi, E.: Blow-up lemma. *Combinatorica* **17**, 109–123 (1997)
19. Kömlos, J., Sárközy, G.N., Szemerédi, E.: On the Pósa–Seymour conjecture. *J. Graph Theory* **29**(3), 167–176 (1998)
20. Kömlos, J.: The blow-up lemma. *Comb. Probab. Comput.* **8**(1–2), 161–176 (1999)
21. Kömlos, J., Simonovits, M.: Szemerédi’s Regularity lemma and its applications in graph theory. *Combinatorics, Paul Erdős is eighty 2*(Keszthely), 295–352 (1993)
22. Lubotzky, A., Phillips, R., Sarnak, P.: Ramanujan graphs. *Combinatorica* **8**(3), 261–277 (1988)
23. Margulis, G.A.: Explicit constructions of graphs without short cycles and low density codes. *Combinatorica* **2**, 71–78 (1982)
24. Staden, K., Treglown, A.: The Bandwidth Theorem for Locally Dense Graphs. [arXiv:1807.09668](https://arxiv.org/abs/1807.09668)
25. Szemerédi, E.: Regular Partitions of Graphs, *Colloques Internationaux C.N.R.S No 260-Problèmes Combinatoires et Théorie des Graphes*, Orsay, pp. 399–401 (1976)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.