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ORIGINAL PAPER

Bipartization of Graphs

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Abstract
A dominating set of a graph G is a set $D \subseteq V_G$ such that every vertex in $V_G - D$ is adjacent to at least one vertex in D, and the domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. In this paper we provide a new characterization of bipartite graphs whose domination number is equal to the cardinality of its smaller partite set. Our characterization is based upon a new graph operation.

Keywords Bipartite graph · Bipartization · Domination number

Mathematics Subject Classification 05C69 · 05C76 · 05C05

1 Introduction and Notation

For notation and graph theory terminology we in general follow [2]. Specifically, let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G. For a subset $X \subseteq V_G$, the *subgraph induced* by X is denoted by $G[X]$. For simplicity of notation, if $X = \{x_1, \ldots, x_k\}$, we shall write $G[x_1, \ldots, x_k]$ instead of $G[\{x_1, \ldots, x_k\}]$. For a vertex v of G, its *neighborhood*, denoted by $N_G(v)$, is the set of all vertices adjacent to v, and the cardinality of $N_G(v)$, denoted by $\deg_G(v)$, is called the *degree* of v. The *closed neighborhood* of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. In general, the *neighborhood* of $X \subseteq V_G$, denoted by $N_G(X)$, is defined to be $\bigcup_{v \in X} N_G(v)$, and the *closed* neighborhood of X, denoted by $N_G[X]$, is the set $N_G(X) \cup X$. A vertex of degree one is called a *leaf*, and the only neighbor of a leaf is called its *support vertex*

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(or simply, its *support*). A *weak support* is a vertex adjacent to exactly one leaf. Finally, the set of leaves and the set of supports of G we denoted by L_G and S_G, respectively.

A subset D of V_G is said to be a *dominating set* of a graph G if each vertex belonging to the set $V_G - D$ has a neighbor in D. The cardinality of a minimum dominating set of G is called the *domination number of* G and is denoted by $\gamma(G)$. A subset $C \subseteq V_G$ is a *covering set* of G if each edge of G has an end-vertex in C. The cardinality of a minimum covering set of G is called the *covering number of* G and denoted by $\beta(G)$.

It is obvious that if $G = ((A, B), E_G)$ is a connected bipartite graph, then $\gamma(G) \leq \min\{|A|, |B|\}$. In this paper the set of all connected bipartite graphs $G = ((A, B), E_G)$ in which $\gamma(G) = \min\{|A|, |B|\}$ is denoted by $\mathcal{B}$. Some properties of the graphs belonging to the set $\mathcal{B}$ were observed in the papers [1,3–6], where all graphs with the domination number equal to the covering number were characterized. In this paper, inspired by results and constructions of Hartnell and Rall [3], we introduce a new graph operation, called the *bipartization of a graph with respect to a function*, study basic properties of this operation, and provide a new characterization of the graphs belonging to the set $\mathcal{B}$ in terms of this new operation.

2 Bipartization of a Graph

Let $\mathcal{K}_H$ denote the set of all complete subgraphs of a graph H. If $v \in V_H$, then the set $\{K \in \mathcal{K}_H : v \in V_K\}$ is denoted by $\mathcal{K}_H(v)$. If $X \subseteq V_H$, then the set $\bigcup_{v \in X} \mathcal{K}_H(v)$ is denoted by $\mathcal{K}_H(X)$, and it is obvious that $\mathcal{K}_H(X) = \{K \in \mathcal{K}_H : V_K \cap X \neq \emptyset\}$. Let $f : \mathcal{K}_H \to \mathbb{N}$ be a function. If $K \in \mathcal{K}_H$, then by $\mathcal{F}_K$ we denote the set $\{(K, 1), \ldots, (K, f(K))\}$ if $f(K) \geq 1$, and we let $\mathcal{F}_K = \emptyset$ if $f(K) = 0$. By $\mathcal{K}_H^f$ we denote the set of all positively f-valued complete subgraphs of H, that is, $\mathcal{K}_H^f = \{K \in \mathcal{K}_H : f(K) \geq 1\}$.

Definition 1 Let H be a graph and let $f : \mathcal{K}_H \to \mathbb{N}$ be a function. The *bipartization* of H with respect to f is the bipartite graph $B_f(H) = ((A, B), E_{B_f(H)})$ in which $A = V_H$, $B = \bigcup_{K \in \mathcal{K}_H} \mathcal{F}_K$, and where a vertex $x \in A$ is adjacent to a vertex $(K, i) \in B$ if and only if x is a vertex of the complete graph K $(i = 1, \ldots, f(K))$.

Example 1 Figure 1 presents a graph H (for which $\mathcal{K}_H = \{H[a], H[b], H[c], H[d], H[a, b], H[a, c], H[b, c], H[c, d], H[a, b, c]\}$) and its two bipartizations $B_f(H)$ and $B_g(H)$ with respect to functions f, $g : \mathcal{K}_H \to \mathbb{N}$, respectively, where $f(H[a]) = 1$, $f(H[b]) = 1$, $f(H[c]) = 2$, $f(H[d]) = 0$, $f(H[a, b]) = 3$, $f(H[a, c]) = 0$, $f(H[b, c]) = 2$, $f(H[c, d]) = 3$, $f(H[a, b, c]) = 1$, while $g(H[v]) = 0$ for every vertex $v \in V_H$, $g(H[u, v]) = 1$ for every edge $uv \in E_H$, and $g(H[a, b, c]) = 0$. Observe that $B_g(H)$ is the subdivision graph $S(H)$ of H (i.e., the graph obtained from H by inserting a new vertex into each edge of H).

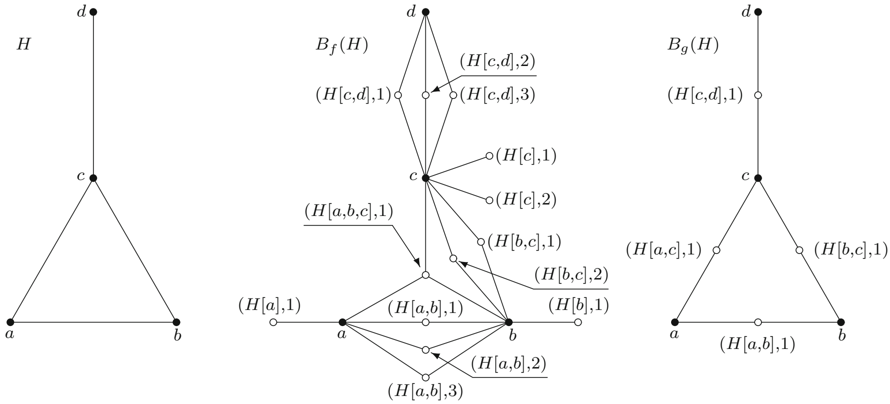

Fig. 1 Graphs H, $B_f(H)$, and $B_g(H)$

3 Properties of Bipartizations of Graphs

It is clear from the above definition of the bipartization of a graph with respect to a function that we have the following proposition.

Proposition 1 *The bipartization of a graph with respect to a function has the following properties:*

(1) *If $B_f(H) = ((A, B), E_{B_f(H)})$ is the bipartization of a graph H with respect to a function $f : \mathcal{K}_H \to \mathbb{N}$, then:*

   (a) $N_{B_f(H)}(v) = \bigcup_{K \in \mathcal{K}_H(v)} \mathcal{F}_K$ *if* $v \in A$.
   (b) $N_{B_f(H)}(X) = \bigcup_{K \in \mathcal{K}_H(X)} \mathcal{F}_K$ *if* $X \subseteq A$.
   (c) $N_{B_f(H)}((K, i)) = V_K$ *if* $(K, i) \in B$ $(i = 1, \ldots, f(K))$.
   (d) $|V_{B_f(H)}| = |V_H| + \sum_{K \in \mathcal{K}_H} f(K)$ *and* $|E_{B_f(H)}| = \sum_{K \in \mathcal{K}_H} f(K)\,|V_K|$.

(2) *If H is a connected graph and $f : \mathcal{K}_H \to \mathbb{N}$ is a function such that every edge of H belongs to a positively f-valued complete subgraph of H, then the bipartization $B_f(H)$ is a connected graph.*

(3) *If H is a graph and $f, g : \mathcal{K}_H \to \mathbb{N}$ are functions such that $f(K) \geq g(K)$ for every $K \in \mathcal{K}_H$, then the graph $B_g(H)$ is an induced subgraph of $B_f(H)$.*

Our study of properties of bipartizations we begin by showing that every bipartite graph is the bipartization of some graph with respect to some function.

Theorem 1 *For every bipartite graph $G = ((A, B), E_G)$ there exist a graph H and a function $f : \mathcal{K}_H \to \mathbb{N}$ such that $G = B_f(H)$.*

Proof We say that vertices x and y of G are *similar* if $N_G(x) = N_G(y)$. It is obvious that this similarity is an equivalence relation on B (as well as on A and $A \cup B$). Let $B_1, \ldots, B_l$ be the equivalence classes of this relation on B, say $B_i = \{b_1^i, b_2^i, \ldots, b_{k_i}^i\}$ for $i = 1, \ldots, l$. It follows from properties of the equivalence classes that $|B_1| + \cdots + |B_l| = |B|$, $N_G(b_1^i) = N_G(x)$ for every $x \in B_i$, and $N_G(b_1^i) \neq N_G(b_1^j)$ if $i, j \in \{1, \ldots, l\}$ and $i \neq j$.

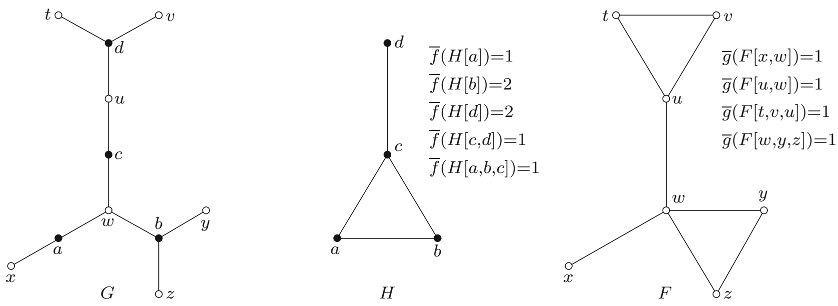

Fig. 2 Graph G is the bipartization of the two non-isomorphic graphs H and F

Now, let $H = (V_H, E_H)$ be a graph in which $V_H = A$ and two vertices x and y are adjacent in H if and only if they are at distance two apart from each other in G. Let $\mathcal{K}_H$ be the set of all complete subgraphs of H, and let $f: \mathcal{K}_H \to \mathbb{N}$ be a function such that $f(K) = |\{b \in B: N_G(b) = V_K\}|$ for $K \in \mathcal{K}_H$. Next, let K_i be the induced subgraph $H[N_G(b_1^i)]$ of H. It follows from the definition of H that K_i is a complete subgraph of H. In addition, from the definition of f and from properties of the classes $B_1, \ldots, B_l$, it follows that $f(K_i) = |B_i| > 0$ ($i = 1, \ldots, l$), and $f(K) = 0$ if $K \in \mathcal{K}_H - \{K_1, \ldots, K_l\}$. Consequently, $\mathcal{K}_H^f = \{K_1, \ldots, K_l\}$.

Finally, consider the bipartite graph $B_f(H) = ((X, Y), E_{B_f(H)})$ in which $X = V_H = A$, $Y = \bigcup_{K \in \mathcal{K}_H} \mathcal{F}_K = \bigcup_{K \in \mathcal{K}_H^f} \mathcal{F}_K = \bigcup_{i=1}^{l} \{(K_i, 1), \ldots, (K_i, k_i)\}$, and where $N_{B_f(H)}((K_i, j)) = V_{K_i} = N_G(b_1^i)$ for every $(K_i, j) \in Y$. Now, one can observe that the function $\varphi: A \cup B \to X \cup Y$, where $\varphi(x) = x$ if $x \in A$, and $\varphi(b_j^i) = (K_i, j)$ if $b_j^i \in B$, is an isomorphism between graphs G and $B_f(H)$. □

We have proved that a bipartite graph $G = ((A, B), E_G)$ is the bipartization $B_f(H)$ of a graph $H = (V_H, E_H)$ (in which $V_H = A$ and $E_H = \{xy: x, y \in A \text{ and } d_G(x, y) = 2\}$) with respect to a function $f: \mathcal{K}_H \to \mathbb{N}$, where $f(K) = |\{b \in B: N_G(b) = V_K\}|$ for $K \in \mathcal{K}_H$. The same graph G is also the bipartization $B_g(F)$ of a graph $F = (V_F, E_F)$ (in which $V_F = B$ and $E_F = \{xy: x, y \in B \text{ and } d_G(x, y) = 2\}$) with respect to a function $g: \mathcal{K}_F \to \mathbb{N}$, where $g(K) = |\{a \in A: N_G(a) = V_K\}|$ for $K \in \mathcal{K}_F$. Consequently, every bipartite graph may be the bipartization of two non-isomorphic graphs.

Example 2 Figure 2 depicts the bipartite graph G which is the bipartization of the non-isomorphic graphs H and F with respect to functions $\overline{f}: \mathcal{K}_H \to \mathbb{N}$ and $\overline{g}: \mathcal{K}_F \to \mathbb{N}$, respectively, which non-zero values are displayed in the figure.

It is obvious from Theorem 1 that every tree is a bipartization. We are now interested in providing a simple characterization of graphs H and functions $f: \mathcal{K}_H \to \mathbb{N}$ for which the bipartization $B_f(H)$ is a tree. We begin with the following notation: An alternating sequence of vertices and complete graphs $(v_0, F_1, v_1, \ldots, v_{k-1}, F_k, v_k)$ is said to be a *positively f-valued complete* $v_0 - v_k$ *path* if $v_{i-1}v_i$ is an edge in the complete graph F_i for $i = 1, \ldots, k$. We now have the following two useful lemmas.

Lemma 1 *Let H be a connected graph, and let $f : \mathcal{K}_H \to \mathbb{N}$ be a function. If there are two vertices u and v and two distinct internally vertex-disjoint positively f-valued complete $u - v$ paths in H, then the bipartization $B_f(H)$ contains a cycle.*

Proof If $(v_0 = u, F_1, v_1, \ldots, v_{m-1}, F_m, v_m = v)$ and $(v'_0 = u, F'_1, v'_1, \ldots, v'_{n-1}, F'_n, v'_n = v)$ are distinct internally vertex-disjoint positively f-valued complete $u - v$ paths in H, then $(v_0, (F_1, 1), v_1, \ldots, v_{m-1}, (F_m, 1), v_m)$ and $(v'_0, (F'_1, 1), v'_1, \ldots, v'_{n-1}, (F'_n, 1), v'_n)$ are distinct $u - v$ paths in $B_f(H)$, and so they generate at least one cycle in $B_f(H)$. □

Let us recall first that a maximal connected subgraph without a cutvertex is called a *block*. A graph H is said to be a *block graph* if each block of H is a complete graph. The next lemma is probably known, therefore we omit its easy inductive proof.

Lemma 2 *If $\mathcal{S}$ is the set of all blocks of a graph H, then $\sum_{B\in\mathcal{S}} (|V_B| - 1) = |V_H| - 1$.*

Now we are ready for a characterization of graphs which bipartizations (with respect to some functions) are trees.

Theorem 2 *Let H be a connected graph, and let $f : \mathcal{K}_H \to \mathbb{N}$ be a function such that every edge of H belongs to some positively f-valued complete subgraph of H. Then the bipartization $B_f(H)$ is a tree if and only if the following conditions hold:*

(1) *$f(K) \le 1$ for every non-trivial complete subgraph K of H.*
(2) *H is a block graph.*
(3) *For a non-trivial complete subgraph K of H is $f(K) = 1$ if and only if K is a block of H.*

Proof Assume that $B_f(H)$ is a tree. The statement (1) is obvious, for if there were a non-trivial complete subgraph K of H for which $f(K) \ge 2$, then for any two vertices u and v belonging to K, the sequence $(u, (K, 1), v, (K, 2), u)$ would be a cycle in $B_f(H)$.

Suppose now that H is not a block graph. Then there exists a block in H, say B, which is not a complete graph. Thus in B there exists a cycle such that not all its chords belong to B. Let $C = (v_0, v_1, \ldots, v_l, v_0)$ be a shortest such cycle in B. Then $l \ge 3$ and we distinguish two cases. If C is chordless, then, by Lemma 1, $B_f(H)$ contains a cycle. Thus assume that C has a chord. We may assume that v_0 is an end-vertex of a chord of C, and then let k be the smallest integer such that $v_0 v_k$ is a chord of C. Now the choice of C implies that the vertices $v_0, v_1, \ldots, v_k$ are mutually adjacent, and therefore, $k = 2$. Similarly, $v_0, v_k, \ldots, v_l$ are mutually adjacent, and so we must have $l = 3$. Consequently, $C = (v_0, v_1, v_2, v_3, v_0)$ and $v_0 v_2$ is the only chord of C. Now it is obvious that there are at least two $v_0 - v_2$ positively f-valued complete paths in H. From this and from Lemma 1 it follows that the bipartition $B_f(H)$ contains a cycle. This contradiction completes the proof of the statement (2).

Let B be a block of H. We have already proved that B is a complete graph. Let B' be a proper non-trivial complete subgraph of B. To prove (3), it suffices to observe that $f(B') = 0$. On the contrary, suppose that $f(B') \ne 0$. We now choose two distinct vertices v and u belonging to B', and a vertex w belonging to B but not to B'. This clearly forces that there are at least two $v-u$ positively f-valued complete paths in H. Consequently, by Lemma 1, $B_f(H)$ contains a cycle, and this contradiction completes the proof of the statement (3).

Assume now that the conditions (1)–(3) are satisfied for H and f. Since end-vertices of $B_f(H)$, corresponding to positively f-valued one-vertex complete subgraphs of H, are not important to our study of tree-like structure of $B_f(H)$, we can assume without loss of generality that $f(H[v]) = 0$ for every vertex $v \in V_H$. Consequently, H is a block graph and $f(K) = 1$ for every block K of H, while $f(K') = 0$ for every other complete subgraph K' of H. It remains to prove that $B_f(H)$ is a tree. Since $B_f(H)$ is a connected graph, it suffices to show that $|E_{B_f(H)}| = |V_{B_f(H)}| - 1$. Let $\mathcal{S}$ be the set of all blocks of H. Then $\mathcal{K}_H^f = \mathcal{S}$, $|V_{B_f(H)}| = |V_H| + \sum_{K\in\mathcal{K}_H^f} f(K) = |V_H| + |\mathcal{S}|$, and $|E_{B_f(H)}| = \sum_{K\in\mathcal{K}_H^f} f(K)|V_K| = \sum_{K\in\mathcal{S}} |V_K| = \sum_{K\in\mathcal{S}}(|V_K| - 1) + |\mathcal{S}|$. Now, since $\sum_{K\in\mathcal{S}}(|V_K| - 1) = |V_H| - 1$ (by Lemma 2), we finally have $|E_{B_f(H)}| = (|V_H| - 1) + |\mathcal{S}| = (|V_H| + |\mathcal{S}|) - 1 = |V_{B_f(H)}| - 1$. □

Corollary 1 *For every connected graph H, there exists a function $f : \mathcal{K}_H \to \mathbb{N}$ such that the bipartization $B_f(H)$ is a tree.*

Proof Let F be a spanning block graph of H and let $f : \mathcal{K}_F \to \{0, 1\}$ be a function such that $f(K) = 1$ if and only if K is a block of F. Clearly, f satisfies the conditions (1)–(3) of Theorem 2, and so the bipartization $B_f(H)$ is a tree. □

Example 3 Figure 2 shows the tree G which is the bipartization of two block graphs H and F with respect to functions $\overline{f}$ and $\overline{g}$, respectively, which non-zero values are listed in the same figure.

4 Graphs Belonging to the Family $\mathcal{B}$

In this section, we provide an alternative characterization of all bipartite graphs whose domination number is equal to the cardinality of its smaller partite set, that is, we prove that a connected graph G belongs to the class $\mathcal{B}$ if and only if G is some bipartization of a graph. For that purpose, we need the following lemma.

Lemma 3 [4] *Let $G = ((A, B), E_G)$ be a connected bipartite graph with $1 \le |A| \le |B|$. Then the following statements are equivalent:*

(1) $\gamma(G) = |A|$.
(2) $\gamma(G) = \beta(G) = |A|$.
(3) *G has the following two properties:*
   (a) *Each support vertex of G belonging to B is a weak support and each of its non-leaf neighbors is a support.*
   (b) *If x and y are vertices belonging to $A-(L_G \cup S_G)$ and $d_G(x, y) = 2$, then there are at least two vertices $\overline{x}$ and $\overline{y}$ in B such that $N_G(\overline{x}) = N_G(\overline{y}) = \{x, y\}$.*

We are ready to establish our main theorem that provides an alternative characterization of the graphs belonging to $\mathcal{B}$ in terms of the bipartization of a graph.

Theorem 3 *Let $G = ((A, B), E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. Then $\gamma(G) = |A|$ if and only if G is the bipartization $B_f(H)$ of a connected graph H with respect to a non-zero function $f: \mathcal{K}_H \to \mathbb{N}$ and f has the following two properties:*

(1) *If $uv \in E_H$ and $f(H[u, v]) = 0$, then $f(H') > 0$ for some complete subgraph H' of H containing the edge uv.*
(2) *If $uv \in E_H$ and $f(H[u]) = f(H[v]) = 0$, then $f(H[u, v]) \geq 2$.*

Proof Assume first that $\gamma(G) = |A|$. Then G has the properties (3a) and (3b) of Lemma 3. Let $H = (V_H, E_H)$ be a graph in which $V_H = A$ and $E_H = \{xy: x, y \in A \text{ and } d_G(x, y) = 2\}$, and let $f: \mathcal{K}_H \to \mathbb{N}$ be a function such that $f(K) = |\{x \in B: N_G(x) = V_K\}|$ for each $K \in \mathcal{K}_H$. Then G is the bipartization $B_f(H)$ of H with respect to f, as we have shown in the proof of Theorem 1. It is obvious that if $H = K_1$, then $\mathcal{K}_H = \{H\}$ and it must be $f(H) \geq 1$ (as otherwise $G = B_f(H)$ would be a graph of order one). Thus assume that H is non-trivial. Now it remains to prove that f has the properties (1) and (2).

Let uv be an edge of H such that $f(H[u, v]) = 0$. Suppose on the contrary that $f(H') = 0$ for every complete subgraph H' containing the edge uv. Then the vertices u and v do not share a neighbor in $B_f(H) = G$, so $d_G(u, v) > 2$ and uv is not an edge in H, a contradiction. This proves the property (1).

Now let uv be an edge of H such that $f(H[u]) = f(H[v]) = 0$. From these assumptions it follows that $d_G(u, v) = 2$ and neither u nor v is a support vertex in $G = B_f(H)$. Now we shall prove that none of the vertices u and v is a leaf in G. First, because $u, v \in A$ and they have a common neighbor, it follows from the first part of the property (3a) of Lemma 3 that at least one of the vertices u and v is not a leaf in G. Suppose now that exactly one of the vertices u and v is a leaf in G, say u is a leaf. Then it follows from the second part of the property (3a) of Lemma 3 that v is a support vertex in $G = B_f(H)$ and, therefore, $f(H[v]) > 0$, a contradiction. Consequently, both u and v are elements of $A - N_G[L_G]$. Thus, since $d_G(u, v) = 2$, the property (3b) of Lemma 3 implies that there are at least two vertices $\bar{u}, \bar{v} \in B$ such that $N_G(\bar{u}) = N_G(\bar{u}) = \{u, v\}$. Therefore $f(H[u, v]) = |\{x \in B: N_G(x) = \{u, v\}\}| \geq |\{\bar{u}, \bar{v}\}| = 2$ and this proves the property (2).

Assume now that H is a connected graph, and $f: \mathcal{K}_H \to \mathbb{N}$ is a non-zero function having the properties (1) and (2). We shall prove that in the bipartization $B_f(H) = ((A, B), E_{B_f(H)})$, where $A = V_H$ and $B = \bigcup_{K \in \mathcal{K}_H} \mathcal{F}_K$, is $|A| \leq |B|$ and $\gamma(B_f(H)) = |A|$. This is obvious if H is a graph of order 1. Thus assume that H is a graph of order at least 2. From the property (1) it follows that $B_f(H)$ is a connected graph. We first prove the inequality $|A| \leq |B|$. To prove this, it suffices to show that $B_f(H)$ has an A-saturating matching. We begin by dividing $A = V_H$ into two subsets $V_H^1 = \{v \in V_H: f(H[v]) \geq 1\}$ and $V_H^0 = \{v \in V_H: f(H[v]) = 0\}$. It is obvious that the edge-set $M^1 = \{v(H[v], 1): v \in V_H^1\}$ is a V_H^1-saturating matching in $B_f(H)$. Next, we order the set V_H^0 in an arbitrary way, say $V_H^0 = \{v_1, \ldots, v_n\}$. Now, depending on this order, we consecutively choose edges $e_1, \ldots, e_n$ in such a way that $M^1 \cup \{e_1, \ldots, e_i\}$ is a $(V_H^1 \cup \{v_1, \ldots, v_i\})$-saturating matching in $B_f(H)$.

Assume that we have already chosen a $(V_H^1 \cup \{v_1, \ldots, v_{i-1}\})$-saturating matching $M^1 \cup \{e_1, \ldots, e_{i-1}\}$ in $B_f(H)$, and consider the next vertex $v_i \in V_H^0$. If $N_H(v_i) \cap V_H^0 \neq \emptyset$, say $v_j \in N_H(v_i) \cap V_H^0$, then $f(H[v_j]) = 0$ and therefore $f(H[v_i, v_j]) \geq 2$ (by the property (2)) and the edge $e_i = v_i(H[v_i, v_j], 1)$ if $j > i$ ($e_i = v_i(H[v_i, v_j], 2)$ if $j < i$) together with $M^1 \cup \{e_1, \ldots, e_{i-1}\}$ form a $(V_H^1 \cup \{v_1, \ldots, v_i\})$-saturating matching in $B_f(H)$. Thus assume that $N_H(v_i) \subseteq V_H^1$. Let v be a neighbor of v_i in H. If $f(H[v_i, v]) \geq 1$, then the edge $e_i = v_i(H[v_i, v], 1)$ has the desired property. Finally, if $f(H[v_i, v]) = 0$, then $f(H') > 0$ for some complete subgraph H' of H containing the edge $v_i v$ (by the property (1)) and in this case the edge $e_i = v_i(H', 1)$ has the desired property (as $N_H(v_i) \subseteq V_H^1$). Repeating this procedure as many times as needed, an A-saturating matching in $B_f(H)$ can be obtained.

To complete the proof, it remains to show that $\gamma(B_f(H)) = |A|$. In a standard way, suppose to the contrary that $\gamma(B_f(H)) < |A|$. Let D be a minimum dominating set of $B_f(H)$ with $|D \cap A|$ as large as possible. Since $\gamma(B_f(H)) = |D|$, the inequality $\gamma(B_f(H)) < |A|$ implies that $|A - D| > |D \cap B| \geq 1$. In addition, since $|D \cap A|$ is as large as possible, the set V_H^1 $(= \{v \in V_H : f(H[v]) \geq 1\})$ is a subset of $D \cap A$, while $A - D$ is a subset of V_H^0 $(= \{v \in V_H : f(H[v]) = 0\})$. Now, because $|A - D| > |D \cap B|$ and each vertex of $A - D$ has a neighbor in $D \cap B$, the pigeonhole principle implies that there are two vertices x and y in $A - D$ which are adjacent to the same vertex in $D \cap B$. Hence, x and y are adjacent in H (by the definition of $B_f(H)$). Now, since $f(H[x]) = f(H[y]) = 0$, the property (2) implies that $f(H[x, y]) \geq 2$. Next, since $N_{B_f(H)}((H[x, y], 1)) = N_{B_f(H)}((H[x, y], 2)) = \{x, y\}$ and $\{x, y\} \cap D = \emptyset$, the vertices $(H[x, y], 1)$ and $(H[x, y], 2)$ belong to $D \cap B$. Consequently, it is easy to observe that the set $D' = (D - \{(H[x, y], 1), (H[x, y], 2)\}) \cup \{x, y\}$ is a dominating set of $B_f(H)$, which is impossible as $|D'| = |D|$ and $|D' \cap A| > |D \cap A|$. This completes the proof. □

Example 4 The graph H and the function $f : \mathcal{K}_H \to \mathbb{N}$ given in Example 1 have the properties (1) and (2) of Theorem 3 and therefore the bipartization $B_f(H)$ belongs to the family $\mathcal{B}$, that is, $\gamma(B_f(H)) = |A|$, where A is the smaller of two partite sets of $B_f(H)$ shown in Fig. 1.

The graph F and the function $\overline{g}$ given in Fig. 2 do not satisfy the condition (2) of Theorem 3. However, the bipartization $G = B_{\overline{g}}(F)$ is a graph belonging to the family $\mathcal{B}$ since G is also the bipartization $B_{\overline{f}}(H)$, with H and $\overline{f}$ given in Fig. 2 and possessing properties (1) and (2) of Theorem 3.

It is obvious that the complete bipartite graph $K_{m,n}$ is the bipartization of the complete graph K_m (resp. K_n) with respect to the function $f : \mathcal{K}_{K_m} \to \{0, n\}$, where $f(K) = 0$ if and only if $K \in \mathcal{K}_{K_m} - \{K_m\}$ (resp. $g : \mathcal{K}_{K_n} \to \{0, m\}$, where $g(K) = 0$ if and only if $K \in \mathcal{K}_{K_n} - \{K_n\}$). It is also evident that if $\min\{m, n\} \geq 3$, then $K_{m,n}$ does not belong to the family $\mathcal{B}$ (as $\gamma(K_{m,n}) = 2 < \min\{m, n\}$), and neither K_m and f nor K_n and g possess the property (2) of Theorem 3.

Finally, as an immediate consequence of Theorems 2 and 3 we have the following simple characterization of trees in which the domination number is equal to the size of a smaller of its partite sets. All such trees are bipartizations of block graphs.

Corollary 2 *Let $T = ((A, B), E_T)$ be a tree in which $1 \le |A| \le |B|$. Then $\gamma(T) = |A|$ if and only if T is the bipartization $B_f(H)$ of a block graph H with respect to a non-zero function $f : \mathcal{K}_H \to \mathbb{N}$ and f has the following two properties:*

(1) *$f(K) = 1$ if K is a block of H, and $f(K') = 0$ if K' is a non-trivial complete subgraph of H which is not a block of H.*
(2) *$\max\{f(H[u]), f(H[v])\} \ge 1$ for every edge uv of H (or, equivalently, the set $\{v \in V_H : f(H[v]) \ge 1\}$ is a covering set of H).*

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