Graphs and Combinatorics (2014) 30:957–962
DOI 10.1007/s00373-013-1314-z

ORIGINAL PAPER

Closure and Spanning k-Trees

Ryota Matsubara · Masao Tsugaki · Tomoki Yamashita

Received: 15 April 2011 / Revised: 26 November 2012 / Published online: 19 April 2013

Abstract In this paper, we propose a new closure concept for spanning k-trees. A k-tree is a tree with maximum degree at most k. We prove that: Let G be a connected graph and let u and v be nonadjacent vertices of G. Suppose that $\sum_{w\in S} d_G(w) \geq |V(G)|-1$ for every independent set S in G of order k with $u, v \in S$. Then G has a spanning k-tree if and only if $G + uv$ has a spanning k-tree. This result implies Win's result (Abh Math Sem Univ Hamburg, 43:263–267, 1975) and Kano and Kishimoto's result (Graph Comb, 2013) as corollaries.

Keywords Spanning tree · k-tree · Closure

1 Introduction

All graphs considered in this paper are only simple and finite. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [1].

Bondy and Chvátal [2] introduced the closure concept, and showed that it plays an important role for the existence of cycles, paths, and other subgraphs in graphs. In this

R. Matsubara
College of Engineering, Shibaura Institute of Technology, 307 Fukasaku,
Minuma-ku, Saitama-shi, Saitama, 337-8570, Japan
e-mail: ryota@sic.shibaura-it.ac.jp

M. Tsugaki (✉)
Department of Mathematical Information Science, Tokyo University of Science,
1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162-8601, Japan
e-mail: tsugaki@hotmail.com

T. Yamashita
Department of Mathematics, Kinki University, Kowakae 3-4-1, Higashi-Osaka,
Osaka, 577-8502, Japan
e-mail: yamashita@math.kindai.ac.jp

paper, we consider a closure concept for spanning k-trees, and refer the reader to the survey [3] on closure concept. A k-tree is a tree with maximum degree at most k. Win [6] obtained a degree sum condition for the existence of spanning k-trees.

Theorem 1 *(*Win [6]*) Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sum_{v \in S} d_G(v) \geq |V(G)| - 1$ for every independent set S in G of order k, then G has a spanning k-tree.*

Recently, Kano and Kishimoto [4] considered a closure concept for spanning k-trees, and proved the following theorem.

Theorem 2 *(*Kano and Kishimoto [4]*) Let $k \geq 2$ be an integer, and let G be an m-connected graph. Let u and v be two nonadjacent vertices of G. Suppose that $d_G(u) + d_G(v) \geq |V(G)| - m(k-2) - 1$. Then G has a spanning k-tree if and only if $G + uv$ has a spanning k-tree.*

In this paper, we give a closure result which implies the above theorems as corollaries.

Theorem 3 *Let $k \geq 2$ be an integer, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. Suppose that $\sum_{w \in S} d_G(w) \geq |V(G)| - 1$ for every independent set S in G of order k such that $u, v \in S$. Then G has a spanning k-tree if and only if $G + uv$ has a spanning k-tree.*

We now show that a graph satisfying the condition of Theorem 2 also satisfies that of Theorem 3.

Proof of Theorem 2 Assume that G is an m-connected graph and satisfies $d_G(u) + d_G(v) \geq |V(G)| - m(k-2) - 1$ for some $u, v \in V(G)$ with $uv \notin E(G)$. Since $|V(G)| - m(k-2) - 1 \geq |V(G)| - \delta(G)(k-2) - 1 \geq |V(G)| - \sum_{w \in T} d_G(w) - 1$ for every independent set $T \subseteq V(G) \backslash \{u, v\}$ of order $k-2$, we have $\sum_{w \in S} d_G(w) \geq |V(G)| - 1$ for every independent set $S \subseteq V(G)$ of order k such that $u, v \in S$. Hence G satisfies the condition of Theorem 3. □

2 Proof of Theorem 3

We prove a slightly stronger theorem than Theorem 3. For a graph G and $S \subseteq V(G)$ with $|S| \geq k$, let $\Delta_k(S; G) := \max \left\{\sum_{x \in X} d_G(x) \colon X \text{ is a subset of } S \text{ of order } k\right\}$. If there is no confusion, then we abbreviate $\Delta_k(S; G)$ to $\Delta_k(S)$.

Theorem 4 *Let $k \geq 2$ be an integer, and let G be a connected graph. Let u and v be two nonadjacent vertices of G. Suppose that there exists no independent set of order $k+1$ containing both u and v, or $\Delta_k(S) \geq |V(G)| - 1$ for every independent set S in G of order $k+1$ such that $u, v \in S$. Then G has a spanning k-tree if and only if $G + uv$ has a spanning k-tree.*

1. The degree condition of Theorem 4 is best possible in the following sense. Let G be a complete bipartite graph $K_{n,n(k-1)+2}$ with partite sets X and Y such that $|X| = n$ and $|Y| = n(k-1) + 2$, where $n \geq 1$ and $k \geq 2$. Let u and v be

two vertices of Y. Then $\Delta_k(S) = nk = |V(G)| - 2$ for every independent set S of order $k+1$ such that $u, v \in S$, and $G + uv$ has a spanning$k$-tree. But G has no spanning k-tree, because if G has a spanning k-tree T, then $|V(G)| - 1 = |V(T)| - 1 = |E(T)| \le k|X| = kn = |V(G)| - 2$, a contradiction.

2. The closure $\mathrm{cl}^{\Delta}(G)$ obtained from Theorem 4 is well-defined.
   Let G_1 and G_2 be graphs obtained from G by recursively joining pairs of nonadjacent vertices which satisfy the condition of Theorem 4 until there exists no such a pair. Let $e_1, e_2, \ldots, e_m$ and $f_1, f_2, \ldots, f_n$ be the sequences of edges added to G in obtaining G_1 and G_2, respectively. Suppose that $e_1, e_2, \ldots, e_l \in E(G_2)$ and $e_{l+1} \notin E(G_2)$. Let $e_{l+1} := uv$ and $H := G + e_1 + \cdots + e_l$. Then, by the definition of G_2, there exists an independent set S in G_2 of order $k+1$ such that $u, v \in S$ and $\Delta_k(S; G_2) \le |V(G_2)| - 2 = |V(G)| - 2$. Since H is a subgraph of G_2, S is an independent set in H and $\Delta_k(S; G_2) \ge \Delta_k(S; H)$. By the choice of e_{l+1}, we have $\Delta_k(S; H) \ge |V(H)| - 1 = |V(G)| - 1$. Hence $|V(G)| - 2 \ge \Delta_k(S; G_2) \ge \Delta_k(S; H) \ge |V(G)| - 1$, a contradiction. Hence $e_1, e_2, \ldots, e_m \in E(G_2)$. Similarly, we can obtain $f_1, f_2, \ldots, f_n \in E(G_1)$. This implies that $G_1 = G_2$, and so $\mathrm{cl}^{\Delta}(G)$ is well-defined.
3. Theorem 4 implies a result due to Neumann-Lara and Rivera-Campo.
   Neumann-Lara and Rivera-Campo [5] obtained an independence number condition for the existence of spanning k-trees. (In fact, they proved a stronger result as we mention in Sect. 3.)

Theorem 5 *(*Neumann-Lara and Rivera-Campo [5]*) Let $k \ge 2$ be an integer, and let G be a connected graph. If there exists no independent set of order $k+1$, then G has a spanning k-tree.*

If a graph G satisfies the hypothesis of Theorem 5, then $\mathrm{cl}^{\Delta}(G)$ is complete, and hence Theorem 4 implies Theorem 5.

Proof of Theorem 4 For a subgraph H of a graph G and a vertex $v \in V(H)$, we denote the set of neighbors of v in H by $N_H(v)$, and let $d_H(v) := |N_H(v)|$.

If G has a spanning k-tree, then trivially also $G + uv$ has a spanning k-tree. Hence we prove the converse.

Suppose that $G + uv$ has a spanning k-tree T and G does not have a spanning k-tree. Then $T - uv$ consists of two trees T_1 and T_2 such that $u \in V(T_1)$ and $v \in V(T_2)$. Note that for $i = 1, 2$, T_i is a k-tree in G, and $d_{T_i}(w) = d_T(w)$ for $w \in V(T_i) \setminus \{u, v\}$, $d_{T_1}(u) \le k - 1$ and $d_{T_2}(v) \le k - 1$. Since G is a connected graph, there exist $w_1 \in V(T_1)$ and $w_2 \in V(T_2)$ with $w_1 w_2 \in E(G)$. Choose w_1 and w_2 such that $d_{T_1}(w_1) + d_{T_2}(w_2)$ is as small as possible. Since G does not have a spanning k-tree, it follows that for some $i = 1, 2$, there exists no k-tree S_i such that $V(S_i) = V(T_i)$ and $d_{S_i}(w_i) \le k - 1$. Without loss of generality, we may assume that

$$\text{there exists no } k\text{-tree } S_1 \text{ such that } V(S_1) = V(T_1) \ \text{ and } \ d_{S_1}(w_1) \le k - 1. \quad (1)$$

Hence we have $d_{T_1}(w_1) = k$. Then $w_1 \ne u$ because $d_{T_1}(u) \le k - 1$.

Let $T_3 := T_1 \cup T_2 + w_1 w_2$ and let $F_0, \ldots, F_k$ be $k+1$ components of $T_3 - w_1$. Since F_i is a tree, there exists a vertex x_i of F_i with $d_{T_1 \cup T_2}(x_i) \le k - 1$ for $0 \le i \le k$. Let $X := \{x_0, x_1, \ldots, x_k\}$. We can choose X so that $u, v \in X$, because $d_{T_1}(u) \le k - 1$ and

$d_{T_2}(v) \le k-1$. Without loss of generality, we may assume that $d_G(x_0) = \min\{d_G(x_i) : 0 \le i \le k\}$. Let $\{z_i\} := N_{T_3}(w_1) \cap V(F_i)$ for each $0 \le i \le k$. We regard F_0 as a rooted tree with root z_0 and F_i as a rooted tree with root x_i for $1 \le i \le k$.

Claim 1 *Let i, j be integers with $0 \le i \ne j \le k$. Then $d_{T_1 \cup T_2}(y) = k$ for all $y \in N_G(x_i) \cap V(F_j)$.*

Proof Suppose that $d_{T_1 \cup T_2}(y) \le k-1$ for some $y \in N_G(x_p) \cap V(F_q)$, where p, q are integers with $0 \le p \ne q \le k$. If $v \in \{x_p, x_q\}$, then $T' := T_1 \cup T_2 + x_p y$ is a spanning k-tree in G, a contradiction. Hence $v \notin \{x_p, x_q\}$. Then $S_1 := T_1 - w_1 z_q + x_p y$ is a k-tree with $V(S_1) = V(T_1)$ and $d_{S_1}(w_1) = k-1$. This contradicts (1). □

By Claim 1 and the choice of x_0, we obtain the following.

Claim 2 *X is an independent set in G, and $\Delta_k(X) = \sum_{i=1}^{k} d_G(x_i)$.*

We define

$$Y_j := \bigcup_{1 \le i \ne j \le k} \Big(N_G(x_i) \cap V(F_j)\Big) \qquad \text{for } 1 \le j \le k$$

and

$$Y_0 := \bigcup_{1 \le i \le k-1} \Big(N_G(x_i) \cap V(F_0)\Big).$$

For $0 \le i \le k$ and $z \in V(F_i)$, we denote the parent and the children of z in F_i by z^- and $ch(z)$, respectively and we let $Y_i^+ := \bigcup_{y \in Y_i} ch(y)$.

Claim 3 $Y_i^+ \cap N_G(x_i) = \emptyset$ *for each* $1 \le i \le k$, *and* $Y_0^+ \cap N_G(x_k) = \emptyset$.

Proof First, suppose that there exists $y \in Y_p^+ \cap N_G(x_p)$ for some $1 \le p \le k$. Then $y^- \in N_G(x_q)$ for some $1 \le q \ne p \le k$. If $v \in \{x_p, x_q\}$, then $T_1 \cup T_2 - yy^- + x_p y + x_q y^-$ is a spanning k-tree in G, a contradiction.Otherwise, $S_1 := T_1 - yy^- - w_1 z_p + x_p y + x_q y^-$ is a k-tree and $d_{S_1}(w_1) = k-1$. This contradicts (1). Next, suppose that there exists $y \in Y_0^+ \cap N_G(x_k)$. Then $y^- \in N_G(x_r)$ for some $1 \le r \le k-1$. If $v \in \{x_0, x_r\}$, then $T_1 \cup T_2 - yy^- + x_k y + x_r y^-$ is a spanning k-tree in G, a contradiction. Assume that $x_k = v$. Then $x_k \in V(T_2)$ and $y \in V(T_1)$, and the minimality of $d_{T_1}(w_1) + d_{T_2}(w_2)$ and $d_{T_1}(y) + d_{T_2}(x_k) \le k + k - 1$ yields that $d_{T_2}(w_2) \le k-1$. Therefore $T_3 - w_1 z_0 - yy^- + x_k y + x_r y^-$ is a spanning k-tree in G, a contradiction. If $v \notin \{x_0, x_r, x_k\}$, then $S_1' := T_1 - w_1 z_0 - yy^- + x_k y + x_r y^-$ is a k-tree with $V(S_1') = V(T_1)$ and $d_{S_1'}(w_1) = k-1$. This contradicts (1).

Claim 4 $z_i \notin N_G(x_j)$ *for each* $0 \le i \ne j \le k$.

Proof Suppose that $z_p \in N_G(x_q)$ for some $0 \le p \ne q \le k$. Assume that $x_p = v$. Then $z_p = w_2$ and the minimality of $d_{T_1}(w_1) + d_{T_2}(w_2)$ yields that $k + d_{T_2}(w_2) = d_{T_1}(w_1) + d_{T_2}(w_2) \le d_{T_1}(x_q) + d_{T_2}(z_p) \le k - 1 + d_{T_2}(w_2)$, a contradiction. Assume that $x_q = v$. Then note that $d_G(w_2) \le k-1$ by the choice of w_1 and w_2. Thus, $T_3 - z_p w_1 + x_q z_p$ is a spanning k-tree in G, a contradiction. If $v \notin \{x_p, x_q\}$, then $S_1 := T_1 - z_p w_1 + x_q z_p$ is a k-tree with $V(S_1) = V(T_1)$ and $d_{S_1}(w_1) = k-1$, which contradicts (1).

Claim 5 $|Y_i^+| = (k-1)|Y_i|$ *for each* $0 \le i \le k$.

Proof By Claim 4, $z_i \notin Y_i$ for all $0 \le i \le k$, and hence $d_{F_i}(y) = d_{T_1 \cup T_2}(y)$ for all $y \in Y_i$. It follows from Claim 1 that $|ch(y)| = d_{F_i}(y) - 1 = k-1$ for all $y \in Y_i$. Since F_i is a tree, $ch(y_1) \cap ch(y_2) = \emptyset$ for every $y_1, y_2 \in Y_i$ with $y_1 \ne y_2$. Therefore we obtain $|Y_i^+| = \sum_{y \in Y_i} |ch(y)| = (k-1)|Y_i|$ for each $0 \le i \le k$.

By Claims 3–5, for $1 \le h \le k$, we obtain

$$\begin{aligned} |N_G(x_h) \cap V(F_h)| &\le |V(F_h)| - |\{x_h\}| - |Y_h^+| \\ &= |V(F_h)| - 1 - (k-1)|Y_h| \\ &\le |V(F_h)| - 1 - \sum_{1 \le i \le k, i \ne h} |N_G(x_i) \cap V(F_h)| \end{aligned}$$

and

$$\begin{aligned} |N_G(x_k) \cap V(F_0)| &\le |V(F_0)| - |\{z_0\}| - |Y_0^+| \\ &= |V(F_0)| - 1 - (k-1)|Y_0| \\ &\le |V(F_0)| - 1 - \sum_{1 \le i \le k-1} |N_G(x_i) \cap V(F_0)|. \end{aligned}$$

Therefore we deduce that

$$\sum_{i=1}^{k} |N_G(x_i) \cap V(F_j)| \le |V(F_j)| - 1 \text{ for each } 0 \le j \le k. \tag{2}$$

Since $d_G(x_i) \le |\{w_1\}| + \sum_{j=0}^{k} |N_G(x_i) \cap V(F_j)|$ for each $1 \le i \le k$, it follows from the inequality (2) that

$$\begin{aligned} \Delta_k(X) &= \sum_{i=1}^{k} d_G(x_i) \\ &\le \sum_{i=1}^{k} \left(|\{w_1\}| + \sum_{j=0}^{k} |N_G(x_i) \cap V(F_j)| \right) \\ &\le k + \sum_{j=0}^{k} (|V(F_j)| - 1) \\ &\le |V(G)| - 2, \end{aligned}$$

a contradiction.

3 Problem

In this section, we propose a problem concerning a closure involving the independence number and the connectivity. Let $\alpha(G)$ and $\kappa(G)$ be the independence number and the connectivity of G, respectively. Neumann-Lara and Rivera-Campo [5] obtained the following result.

Theorem 6 *(*Neumann-Lara and Rivera-Campo [5]*) Let $k \geq 2$ be an integer, and let G be a graph. If $\alpha(G) \leq (k-1)\kappa(G) + 1$, then G has a spanning k-tree.*

We can consider the following problem as a closure result for Theorem 6. For a graph G and $u, v \in V(G)$ with $uv \notin E(G)$, let $\alpha(u, v; G)$ be the cardinality of a maximum independent set containing u and v. For a graph G and $u, v \in V(G)$, the local connectivity $\kappa(u, v; G)$ is defined to be the maximum number of internally-disjoint paths connecting u and v in G.

Problem 7 *Let $k \geq 2$ be an integer, and let G be a graph. Let u and v be two nonadjacent vertices of G. Assume that $\alpha(u, v; G) \leq (k-1)\kappa(u, v; G) + 1$. Then G has a spanning k-tree if and only if $G + uv$ has a spanning k-tree.*

References

1. Bondy, J.A.: Basic Graph Theory—Paths and Circuits. Handbook of Combinatorics, vol. I, pp. 5–110. Elsevier, Amsterdam (1995)
2. Bondy, J.A., Chvátal, V.: A method in graph theory. Discret. Math **15**, 111–135 (1976)
3. Broersma, H.J., Ryjáček, Z., Schiermeyer, I.: Closure concepts: a survey. Graph. Combin. **16**, 17–48 (2000)
4. Kano, M., Kishimoto, H.: Spanning k-tree of n-connected graphs. Graph. Combin (2013, to appear)
5. Neumann-Lara, V., Rivera-Campo, E.: Spanning trees with bounded degrees. Combinatorica **11**, 55–61 (1991)
6. Win, S.: Existenz von Gerüsten mit vorgeschriebenem Maximalgrad in Graphen (German). Abh. Math. Sem. Univ. Hamburg **43**, 263–267 (1975)