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ORIGINAL PAPER

Minimum Number of Palettes in Edge Colorings

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Abstract A proper edge-coloring of a graph defines at each vertex the set of colors of its incident edges. This set is called the palette of the vertex. In this paper we are interested in the minimum number of palettes taken over all possible proper colorings of a graph.

Keywords Edge colouring · Palette of a vertex

Mathematics Subject Classification 05C15

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1 Introduction

Let G be a finite simple graph, let C be a set of *colors* and let $f : E(G) \to C$ be an edge-coloring of G. We shall always assume that f is *proper*, i.e., any two adjacent edges get distinct colors. The *palette* of a vertex $v \in V(G)$ with respect to f is the set $S_f(v)$ of colors of edges incident to v.

Two vertices of G are distinguished by a coloring f if their palettes are distinct. There are many papers discussing colorings distinguishing vertices in a graph, that is, colorings with maximum possible number of pallettes (see, e.g., [1–3], and references given there). To our knowledge, this paper is the first one dealing with an opposite problem. Namely, we are interested in the minimum number of palettes taken over all possible proper (edge-) colorings of a graph. For a given graph G, we denote this number by $\check{s}(G)$ and call the *palette index* of G.

The minimum number of colors required in a proper coloring of a graph G is called the *chromatic index* of G and is denoted by $\chi'(G)$. Recall that, by Vizing's theorem, the chromatic index of G equals either $\Delta(G)$ or $\Delta(G) + 1$. A graph with $\chi'(G) = \Delta(G)$ is called class 1, while a graph with $\chi'(G) = \Delta(G) + 1$ is called class 2. Our first result on the palette index is (almost) obvious.

Proposition 1 *The palette index of a graph G is* 1 *if and only if G is regular and class* 1. □

A proper coloring of G using $\chi'(G)$ colors is called *minimum*. In general, however, minimum colorings do not provide the minimum number of palettes. In our analysis the following lemma will be useful.

Lemma 2 *If a graph G is regular, then $\check{s}(G) \neq 2$.*

Proof Suppose that $G = (V, E)$ is r-regular with $\check{s}(G) = 2$, and let f be the corresponding coloring of G. Denote by P_1 and P_2 the palettes induced by f and let $V_i = \{x \in V : S_f(x) = P_i\}$, $i = 1, 2$. Without loss of generality, we may suppose that f is chosen in such a way that the set $P_1 \setminus P_2$ is as small as possible. Since $P_1 \neq P_2$ and $|P_1| = |P_2|$, there exists a color $\alpha \in P_1 \setminus P_2$, as well as a color $\beta \in P_2 \setminus P_1$. Clearly, the edges colored with α form a perfect matching of the subgraph $G[V_1]$ (induced in G by the set V_1) and the same is true for the edges colored with β and the subgraph $G[V_2]$. Since $\{V_1, V_2\}$ is a partition of V, by replacing α with β we get a coloring that induces also two palettes P'_1, P'_2, but with $|P'_1 \setminus P'_2| = |P_1 \setminus P_2| - 1$, a contradiction. □

A set of edges E' of a graph G with a coloring $f : E(G) \to C$ is called *f-rainbow* if $|f(E')| = |E'|$, that is, if each edge of E' has different color. The following lemma deals with class 1 graphs K_{2k} and $K_{2k+1,2k+1}$.

Lemma 3 (a) *For every minimum coloring f of a complete graph K_{2k} with $k \neq 2$, there exists an f-rainbow perfect matching in K_{2k}.*

(b) *There exists a minimum coloring f of $K_{2k+1,2k+1}$ such that there is an f-rainbow perfect matching in $K_{2k+1,2k+1}$.*

Proof The part (a) follows, for instance, from results given in [4].

To prove the part (b), assume that $V(K_{2k+1,2k+1}) = X \cup Y$, where $X = \{x_0, x_1, \ldots x_{2k}\}$ and $Y = \{y_0, y_1, \ldots y_{2k}\}$. In order to construct an appropriate coloring f, color the edges $x_{i+j}y_{i+2j}$, $j = 0, 1, \ldots 2k$ (where indices are taken modulo $2k+1$), with the color i. An f-rainbow perfect matching consists of the edges $x_i y_i$, $i = 0, 1, \ldots 2k$. □

2 Complete Graphs

Since the graphs K_1 and K_{2k} are class 1, we have $\check{s}(K_1) = \check{s}(K_{2k}) = 1$. On the other hand, it is easy to see that the minimum coloring of K_n for odd n induces n distinct palettes. Indeed, each palette has $n-1$ colors. That means that at each vertex exactly one color is missing. Further, since n is odd, each color misses at least one vertex. Consequently, each color misses exactly one vertex and missing colors are distinct for distinct vertices. However, by increasing the number of colors we can reach the number of palettes 3 or 4.

The aim of this section is to determine palette indices of complete graphs:

Theorem 4

$$\check{s}(K_n) = \begin{cases} 1, \text{ if } & n \equiv 0 \pmod 2 \quad \text{or } n = 1 \\ 3, \text{ if } & n \equiv 3 \pmod 4 \\ 4, \text{ if } & n \equiv 1 \pmod 4 \end{cases}$$

The proof follows from five partial results, namely Proposition 1, Proposition 5, Theorem 6, Theorem 7 and Proposition 8.

2.1 Complete Graphs with Palette Index 3

First, we deal with the case of orders in congruence class 3 modulo 4.

Proposition 5 *If* $k \geq 0$, *then* $\check{s}(K_{4k+3}) = 3$.

Proof We begin with showing the existence of a coloring f inducing three palettes. Partition the vertex set V of the graph K_{4k+3} into three sets: two sets X and Y, each having $2k+1$ elements, and one one-element set, say $\{u\}$.

We first color the edges of the complete graph induced by X using $2k+1$ colors from the set $A = \{a_1, a_2, \ldots, a_{2k+1}\}$ to obtain a minimum coloring of K_{2k+1}. Thus, there is exactly one color of A missing at each vertex of X.

Next, we color the edges of the complete graph induced by Y using $2k+1$ colors from the set $B = \{b_1, b_2, \ldots, b_{2k+1}\}$ to obtain a minimum coloring of K_{2k+1}. Thus, there is exactly one color of B missing at each vertex of Y. We assume that A and B are disjoint.

Now, we color each edge ux joining the vertex u to a vertex $x \in X$ in such a way that ux gets the color from the set A missing at x. Analogously, we color each edge uy joining u to a vertex $y \in Y$ in such a way that uy gets the color from the set B missing at y. Therefore, in this moment, the (partial) palettes of vertices of X coincide with the set A and those of vertices of Y coincide with B.

Finally, we color the edges between X and Y using the colors from a set C with $|C| = 2k+1$ and $C \cap (A \cup B) = \emptyset$, in order to get a minimum coloring of the bipartite graph $K_{2k+1,2k+1}$.

Then all vertices of X have the palette $A \cup C$, all vertices of Y have the palette $B \cup C$ and the vertex u has the palette $A \cup B$.

The above defined coloring shows that $\check{s}(K_{4k+3}) \leq 3$. However, by Lemma 2, $\check{s}(K_{4k+3}) \neq 2$, and, by Proposition 1, $\check{s}(K_{4k+3}) \neq 1$. □

The next theorem provides a full characterization of complete graphs with the palette index 3.

Theorem 6 *The palette index of the complete graph K_n equals 3 if and only if $n \equiv 3 \pmod 4$.*

Proof Suppose that $\check{s}(K_n) = 3$. By Proposition 1 then K_n is class 2, so that n is odd and $n \geq 3$. There is a coloring of K_n inducing three distinct palettes P_i, $i = 1, 2, 3$. Let V_i be the set of all vertices of K_n with palette P_i and let $n_i = |V_i|$, $i = 1, 2, 3$. Clearly, we have $|P_i| = n - 1$ and $|P_i \setminus P_j| = |P_j \setminus P_i|$, $i, j = 1, 2, 3$.

Observe first, that there is no color belonging to all three palettes. Indeed, otherwise this color would induce a perfect matching of G, which is impossible (n is odd). Consequently, a color of any edge joining V_i and V_j, $i \neq j$, induces a perfect matching of the graph $G[V_i \cup V_j]$. Therefore, $n_i + n_j$ with $i \neq j$ is always even. So, all n_i's are of the same parity, and, since $n = n_1 + n_2 + n_3$ is odd, all n_i's are odd. This implies in particular, that there is no color belonging to exactly one palette. Hence, each color belongs to exactly two palettes.

Set $\xi = |P_1 \setminus P_2|$ and $\eta = |P_1 \cap P_2|$. Thus, $\xi + \eta = n - 1$. Moreover, since $P_3 = (P_1 \setminus P_2) \cup (P_2 \setminus P_1)$ and $|P_2 \setminus P_1| = |P_1 \setminus P_2| = \xi$, we have $2\xi = n - 1$ and, as a consequence, we obtain $\xi = \eta = \frac{n-1}{2}$. Of course, by symmetry it holds

$$|P_i \setminus P_j| = |P_i \cap P_j| = \frac{n-1}{2} \qquad \text{for} \quad i, j = 1, 2, 3,\ i \neq j.$$

Let $E_{i,j}$ denote the set of all edges joining V_i and V_j. Since all n_i's are odd, $|E_{i,j}|$ is odd, too. Moreover, for every color α from $P_i \cap P_j$, the number e_α of edges in $E_{i,j}$ colored with α is odd (otherwise, $n_i - e_\alpha$ is odd and then color α is missing at some vertex in V_i). Thus the total number of colors in $P_i \cap P_j$ is odd, what implies $\frac{n-1}{2} \equiv 1 \pmod 2$, and finally $n \equiv 3 \pmod 4$.

The reverse implication has been proved as Proposition 5. □

2.2 Complete Graphs K_{4k+5} with $k \neq 1$

Theorem 7 *If $n = 4k + 5$, $k \neq 1$, then $\check{s}(K_n) = 4$.*

Proof By Proposition 1, Lemma 2 and Theorem 6 we see that $\check{s}(K_n) \geq 4$. Therefore, it will be sufficient to find a coloring of K_n inducing four palettes.

Partition the vertex set V of the graph K_n with $n = 4k+5$ into four sets: two sets X and Y, each having $2k+1$ elements, one one-element set, say $\{u\}$ and one two-element set, say $\{v, w\}$. Let $X = \{x_0, x_1, \ldots, x_{2k}\}$ and $Y = \{y_0, y_1, \ldots, y_{2k}\}$.

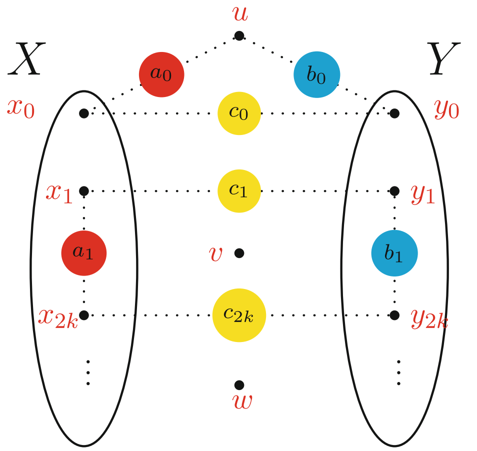

Fig. 1 After step 1

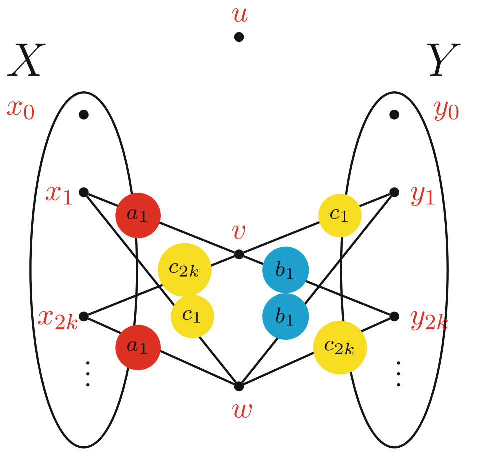

Fig. 2 After step 2

Step 1. Let $A = \{a_0, a_1, \ldots, a_{2k}\}$, $B = \{b_0, b_1, \ldots, b_{2k}\}$ and $C = \{c_0, c_1, \ldots, c_{2k}\}$ be three pairwise disjoint sets of $2k + 1$ colors. Consider (arbitrary) minimum colorings $f_A : E(K_n[X \cup \{u\}]) \to A$ and $f_B : E(K_n[Y \cup \{u\}]) \to B$. By Lemma 3 a) there exists an f_A-rainbow perfect matching M_A in the graph induced by $X \cup \{u\}$ and an f_B-rainbow perfect matching M_B in the graph induced by $Y \cup \{u\}$. If H is the subgraph of K_n isomorphic to $K_{2k+1,2k+1}$ with bipartition $\{X, Y\}$, by Lemma 3 b) there is a minimum coloring $f_C : E(H) \to C$ such that H has an f_C-rainbow perfect matching M_C.

Without loss of generality we may assume that M_A consists of $k + 1$ edges ux_0, $x_1x_{2k}, \ldots, x_kx_{k+1}$ colored with colors $a_0, a_1, \ldots, a_k$, respectively; M_B consists of $k+1$ edges uy_0, $y_1y_{2k}, \ldots, y_ky_{k+1}$ colored with colors $b_0, b_1, \ldots, b_k$, respectively; and M_C consists of $2k + 1$ edges x_iy_i, $i = 0, 1, \ldots, 2k$ colored with colors $c_0, c_1, \ldots, c_k$, respectively.

We uncolor the edges belonging to $M_A \cup M_B \cup M_C$ (see Fig. 1).

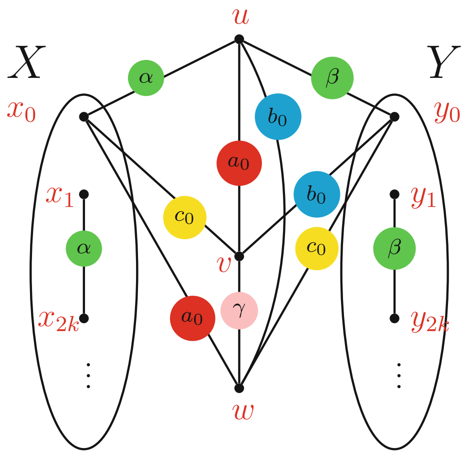

Fig. 3 After step 3. For the sake of clarity of the drawing edges $x_i y_i$, colored with γ, are not presented

Step 2. For each i, $1 \leq i \leq k$ we color the four edges joining the vertices v, w to the vertices x_i, x_{2k-i+1}, as well the four edges joining the vertices v, w to the vertices y_i, y_{2k-i+1} in the following way (see Fig. 2):

- the edges vx_i and wx_{2k-i+1} are colored with a_i, the color missing at both vertices x_i and x_{2k-i+1};
- the edges wy_i and vy_{2k-i+1} are colored with b_i, the color missing at both vertices y_i and y_{2k-i+1};
- the edges wx_i and vy_i are colored with c_i, the color missing at both vertices x_i and y_i;
- the edges vx_{2k-i+1} and wy_{2k-i+1} are colored with c_{2k-i+1}, the color missing at both vertices x_{2k-i+1} and y_{2k-i+1};

Observe that after this step all vertices of X except for x_0 have the (partial) palette $A \cup C$, while all vertices of Y except for y_0 have the palette $B \cup C$.

Step 3. In this step we color all the edges containing the vertices v and w which were not colored yet, as well as the edges uncolored in Step 1.

First, the edges vu and wx_0 are colored with a_0, the edges wu and vy_0 are colored with b_0, and the edges vx_0 and wy_0 are colored with c_0.

Next, we pick three distinct colors α, β, γ so that $\{\alpha, \beta, \gamma\} \cap (A \cup B \cup C) = \emptyset$ and color the edges of M_A with α, the edges of M_B with β and the edges of M_C with γ. Finally, we color the last edge not colored yet, vw, with γ.

Observe that the vertices of X have the palette $A \cup C \cup \{\alpha, \gamma\}$, the vertices of Y have the palette $B \cup C \cup \{\beta, \gamma\}$, and both vertices v and w have the palette $\{a_i, b_i : i = 0, 1, \ldots, k\} \cup C \cup \{\gamma\}$. The fourth palette, that of the vertex u, is $A \cup B \cup \{\alpha, \beta\}$.

This finishes the proof of the theorem. □

2.3 The Complete Graph K_9

To complete the proof of Theorem 4, it is enough to settle the palette index of K_9.

Proposition 8 $\check{s}(K_9) = 4$.

Proof For the same reason as in the Sect. 2.2, it suffices to define a coloring f of edges of K_9 inducing four palettes.

Suppose that $V(K_9) = \{x_i : i = 1, 2, \ldots, 9\}$ and let f be a coloring of K_9 with 13 colors, described by the following symmetric 9×9 matrix $A = (a_{ij})$, namely by setting $f(x_i x_j) = a_{ij}$, $i, j = 1, 2, \ldots, 9$, $i \neq j$ (the empty diagonal corresponds to the fact that in K_9 there are no loops):

$$A = \begin{pmatrix} & 1 & 2 & 3 & 4 & \bar{0} & 6 & \bar{1} & 5 \\ 1 & & 3 & 2 & 6 & 4 & \bar{0} & 5 & \bar{1} \\ 2 & 3 & & 1 & 5 & \bar{1} & 4 & 6 & \bar{0} \\ 3 & 2 & 1 & & \bar{2} & 9 & 7 & \bar{0} & 8 \\ 4 & 6 & 5 & \bar{2} & & 8 & \bar{1} & 9 & 7 \\ \bar{0} & 4 & \bar{1} & 9 & 8 & & 5 & 7 & 6 \\ 6 & \bar{0} & 4 & 7 & \bar{1} & 5 & & 8 & 9 \\ \bar{1} & 5 & 6 & \bar{0} & 9 & 7 & 8 & & 4 \\ 5 & \bar{1} & \bar{0} & 8 & 7 & 6 & 9 & 4 & \end{pmatrix}$$

Clearly, the elements of the i-th row (as well as those of the i-th column) of the matrix A form the palette of the vertex x_i with respect to f. Hence, the palettes are: $\{1, 2, 3, 4, 5, 6, \bar{0}, \bar{1}\}$ for the vertices x_1, x_2, x_3, $\{1, 2, 3, 7, 8, 9, \bar{0}, \bar{2}\}$ for the vertex x_4, $\{4, 5, 6, 7, 8, 9, \bar{1}, \bar{2}\}$ for the vertex x_5, and $\{4, 5, 6, 7, 8, 9, \bar{0}, \bar{1}\}$ for the vertices x_6, x_7, x_8, x_9. □

3 Cubic Graphs

Theorem 9 *Let G be a connected cubic graph.*
If G is class 1, then $\check{s}(G) = 1$.
If G is class 2 and has a perfect matching, then $\check{s}(G) = 3$.
If G is class 2 without a perfect matching, then $\check{s}(G) = 4$.

Proof The first statement follows from Proposition 1.

Suppose now that G is class 2 and has a perfect matching M. Then the graph $G - M$ is the union of disjoint cycles with at least one of an odd length. Each minimum coloring of this graph induces exactly three palettes (partial in G). Observe that by coloring the edges of M with a new color each palette acquires the new color, hence the number of palettes is three as before and $\check{s}(G) \leq 3$. On the other hand, $\check{s}(G) \geq 3$ by Proposition 1 and Lemma 2, so we are done.

Suppose finally that G is class 2, but does not contain a perfect matching. Observe that, since all palettes have three elements, each minimum coloring of G (which uses four colors) induces at most four palettes. For the same reason as above we have $\check{s}(G) \geq 3$. Thus, it suffices to show that it is impossible to have three palettes.

So, assume there exists a coloring f of G inducing three palettes P_1, P_2 and P_3. Denote by V_1, V_2 and V_3 the corresponding sets of vertices of G. Observe first that there is no color belonging to all three palettes (otherwise G would have a perfect matching).

Since G is connected, we may suppose without loss of generality that there are edges joining V_1 to V_2, and, consequently, $P_1 \cap P_2 \neq \emptyset$. For any color $a \in P_1 \cap P_2$ the edges colored with a form a perfect matching M_a of the graph $G[V_1 \cup V_2]$. This implies, in particular, that there is no color belonging just to P_3. Indeed, suppose that $b \in P_3 \setminus (P_1 \cup P_2)$. Then the edges colored with b form a perfect matching M_b of the graph $G[V_3]$ and $M_a \cup M_b$ is a perfect matching of G, a contradiction. Thus, $P_3 = (P_1 \setminus P_2) \cup (P_2 \setminus P_1)$.

Recall that in a cubic graph all palettes are of size three. Therefore, $|P_1 \setminus P_2| = |P_2 \setminus P_1|$ and $|P_3| = |(P_1 \setminus P_2) \cup (P_2 \setminus P_1)| \equiv 0 \pmod 2$, a contradiction. □

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