Graphs and Combinatorics (2014) 30:573–579
DOI 10.1007/s00373-013-1293-0

ORIGINAL PAPER

On Some Ramsey Numbers for Quadrilaterals Versus Wheels

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Received: 24 January 2012 / Revised: 30 January 2013 / Published online: 28 February 2013

Abstract For given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the least integer n such that every 2-coloring of the edges of K_n contains a subgraph isomorphic to G_1 in the first color or a subgraph isomorphic to G_2 in the second color. Surahmat et al. proved that the Ramsey number $R(C_4, W_n) \leq n+\lceil(n-1)/3\rceil$. By using asymptotic methods one can obtain the following property: $R(C_4, W_n) \leq n + \sqrt{n} + o(1)$. In this paper we show that in fact $R(C_4, W_n) \leq n+\sqrt{n-2}+1$ for $n \geq 11$. Moreover, by modification of the Erdős-Rényi graph we obtain an exact value $R(C_4, W_{q^2+1}) = q^2+q+1$ with $q \geq 4$ being a prime power. In addition, we provide exact values for Ramsey numbers $R(C_4, W_n)$ for $14 \leq n \leq 17$.

Keywords Ramsey numbers · Quadrilateral · Wheels

Mathematics Subject Classification (2000) 05C55 · 05C15

1 Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. Let $d(v)$ be the

This research was funded by the Polish National Science Centre (contract number DEC-2012/05/N/ST6/03063).

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degree of vertex v, and let $d_1(v)$ and $d_2(v)$ denote the number of the edges incident to v colored with the first and the second color, respectively. By $\delta_i(G)$ we denote the minimum degree of G in color i. The open neighborhood in color i of vertex v in graph G is $N_i(v) = \{u \in V(G) | \{u, v\} \in E(G) \text{and} \{u, v\} \text{ is colored with color} i\}$. Define $G[S]$ to be the subgraph of G induced by the set of vertices $S \subset V(G)$. Let P_n (resp. C_n) be the path (resp. cycle) on n vertices. A *wheel* W_n is a graph on n vertices obtained from a C_{n-1} by adding one vertex w and making w adjacent to all vertices of the C_{n-1}.

For given graphs G_1, G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with 2 colors, then it always contains a monochromatic copy of G_1 colored with the first color or a monochromatic copy of G_2 colored with the second color. A coloring of the edges of n-vertex complete graph with 2 colors is called a $(G_1, G_2; n)$-coloring if it does not contain a subgraph isomorphic to G_1 colored with the first color nor a subgraph isomorphic to G_2 colored with the second color.

The *Turán number* $t(n, G)$ is the maximum number of edges in any n-vertex graph which does not contain a subgraph isomorphic to G. A graph on n vertices is said to be *extremal with respect to* G if it does not contain a subgraph isomorphic to G and has exactly $t(n, G)$ edges.

Some well known theorems will be used to prove the main result of this paper.

Theorem 1 *(Ore [3]) Let G be a graph on n ($n \geq 3$) vertices. If $d(v) + d(w) \geq n$ for every pair of non-adjacent vertices v and w of G, then G is Hamiltonian.*

Theorem 2 *(Rosta [7], Faudree and Schelp [2]) For all integers $n \geq 5$*

$$R(C_4, C_n) = max\{n + 1, 7\}.$$

Theorem 3 *(Reiman [6]) For all integers $n \geq 4$*

$$t(n, C_4) < \frac{1}{4}n(1 + \sqrt{4n - 3}).$$

Several results have been obtained for wheels and quadrilaterals. Surahmat et al. [8] showed that $R(C_4, W_m) = 9, 10$ and 9 for $m = 4, 5$ and 6 respectively. Independently, Kung-Kuen Tse [10] showed that $R(C_4, W_m) = 10, 9, 10, 9, 11, 12, 13, 14, 16$ and 17 for $m = 4, 5, 6, 7, 8, 9, 10, 11, 12$ and 13, respectively. In 2005, Surahmat et al. [9] obtained property that $R(C_4, W_n) \leq n + \lceil (n - 1)/3 \rceil$. Suppose that we have an admissible coloring of K_m without C_4 in color 1 and without W_n in color 2. Asymptotically we have a well-known property that $t(n, C_4) \approx \frac{1}{2}n^{\frac{3}{2}}$. Since $R(C_4, C_{n-1}) = n$ for $n \geq 7$, we obtain $\frac{1}{2}m(m - n) \approx \frac{1}{2}m^{\frac{3}{2}}$, which implies that $m - n \approx \sqrt{m}$ and $R(C_4, W_n) = n + \sqrt{n} + o(1)$. The main result of this work is the following.

Theorem 4 *For all integers $n \geq 11$*

$$R(C_4, W_n) \leq n + \left\lfloor \sqrt{n - 2} \right\rfloor + 1.$$

2 Main Theorem

Proof (Theorem 4) For simplicity of notation, we set $k = \lfloor\sqrt{n-2}\rfloor$. Let us consider a graph $G = K_{n+k+1}$ and its decomposition $G = G_1 \cup G_2$, where $V(G) = V(G_1) = V(G_2)$ and $E(G_i)$ consists of all edges of G in ith color. Suppose that for graph G there is a $(C_4, W_n; n+k+1)$-coloring and let us consider such coloring.

First let us assume that there is a vertex $v \in V(G)$ such that $d_1(v) \leq k$. Then $d_2(v) \geq n$ and by $R(C_4, C_{n-1}) = n$ we immediately obtain a W_n in the second color.

Now, suppose that $\delta_1(G) \geq k+2$. Let us consider integer p such that $n \in \{(p-1)^2+2, \cdots, p^2+1\}$. Then $k = p-1$. Let $s = n-(p-1)^2$, one can see that $2 \leq s \leq 2p$. In this case the minimum possible number of edges in color 1 in G is

$$\lceil\frac{1}{2}(n+k+1)\delta_1(G)\rceil \geq \frac{1}{4}(n+k+1)(2p+2) \geq$$

$$\geq \frac{1}{4}(n+k+1)\left(1+\sqrt{4(p^2+p+1)-3}\right) \geq$$

$$\geq \frac{1}{4}\left(n+k+1)(1+\sqrt{4(p^2-p+1+s)-3}\right) \geq$$

$$\geq \frac{1}{4}\left(n+k+1)(1+\sqrt{4(n+k+1)-3}\right) > t(n+k+1, C_4),$$

a contradiction.

The last case to consider is $\delta_1(G) = k+1$. In this case G_1 has at most $t(n+k+1, C_4) = \lceil\frac{(n+k+1)\delta_1(G)}{2}\rceil + A$ edges. Similarly to the previous case let us consider integer p such that $n \in \{(p-1)^2+2, \cdots, p^2+1\}$. Then $k+1 = p$. Let us take vertex $v \in V(G)$ such that $d_1(v) = k+1$, subgraph $G' = G_2[N_2(v)]$ and two vertices $v_1, v_2 \in V(G')$, where the edge $\{v_1, v_2\} \in E(G_1)$. Then $|V(G')| = n-1$ and in subgraph G' we have $d_2(v_1)+d_2(v_2) = 2(n-2)-(d_1(v_1)+d_1(v_2))$. We have the following

Claim $d_1(v_1)+d_1(v_2) \leq 2\delta_1(G)+A$ or $d_1(v_1)+d_1(v_2) \leq 2\delta_1(G)+A+1$ depending on the parity of $\delta_1(G)$ and $(n+k+1)$.

Proof If $\delta_1(G)$ and $|V(G)| = (n+k+1)$ are odd, then it is impossible that for all vertices $w \in V(G)$ we have $d_1(w) = \delta_1(G)$. In the worst situation, when all A edges are adjacent to v_1 or v_2, we have that $d_1(v_1)+d_1(v_2) \leq 2\delta_1(G)+A+1$. □

We will prove that $d_2(v_1)+d_2(v_2) \geq n-1$ for all vertices $v_1, v_2 \in V(G')$ such that $\{v_1, v_2\} \in E(G_1)$. In this case we obtain a contradiction because by Ore's Theorem subgraph G' contains a C_{n-1} and G contains a W_n in the second color.

The remaining part of the proof is divided into three parts.

Table 1 Values needed to prove that $d_2(v_1) + d_2(v_2) \geq n - 1$ for $11 \leq n \leq 17$

n	11	12	13	14	15	16	17
$\lvert V(G)\rvert = n + k + 1$	15	16	17	18	19	20	21
$n - 1$	10	11	12	13	14	15	16
$t(\lvert V(G)\rvert, C_4)$	30	33	36	39	42	46	50
A	0	1	2	3	4	6	8
$d_2(v_1) + d_2(v_2) \geq$	10	11	12	13	14	14	14

Table 2 Values needed to prove that $d_2(v_1) + d_2(v_2) \geq n - 1$ for $18 \leq n \leq 26$

n	18	19	20	21	22	23	24	25	26
$\lvert V(G)\rvert = n + k + 1$	23	24	25	26	27	28	29	30	31
$n - 1$	17	18	19	20	21	22	23	24	25
$t(\lvert V(G)\rvert, C_4)$	56	59	63	67	71	76	80	85	90
A	–	–	0	2	3	6	7	10	12
$d_2(v_1) + d_2(v_2) \geq$	21	24	25	26	26	26	26	26	25

1. $11 \leq n \leq 17$
   In this case $\delta_1(G) = p = 4$. The exact values of $t(n, C_4)$ are known for all $n \leq 21$, see [1]. In addition, this paper covers all extremal graphs. Table 1 contains all values needed to prove the inequality $d_2(v_1) + d_2(v_2) \geq n - 1$.
   One can see that for all $11 \leq n \leq 15$ the proof is complete. For case $n = 16$ let us consider the graph G_1. If it is the only extremal graph for $t(20, C_4)$ [1] then its maximum degree is 5, so by Ore's Theorem G' contains a C_{15} and G contains a W_{16} in the second color. If $|E(G_1)| \leq 45$, then $A \leq 5$ and $d_2(v_1) + d_2(v_2) \geq 15$. By similar considerations in case $n = 17$, if G_1 is the only extremal graph for $t(21, C_4)$ [1] then G' contains a C_{16} and G contains a W_{17}. If $|E(G_1)| = 49$ and there exists a vertex $w \in V(G)$ such that $d_1(w) = 8$, then we obtain a C_4 in color 1 in G (consider $\delta_1(G) = 4$ and all possible edges in color 1 from $N_1(w)$ to the remaining vertices of G). If $d_1(w) \leq 7$ for all vertices $w \in V(G)$, then by Ore's Theorem G' contains a C_{16} and G contains a W_{17}. Then $A \leq 6$ and $d_2(v_1) + d_2(v_2) \geq 16$ and we are done.
2. $18 \leq n \leq 26$ In this case $\delta_1(G) = p = 5$. The exact values and extremal graphs for $t(n, C_4)$ are known for all $22 \leq n \leq 31$, see [11]. Table 2 presents all values needed to finish the checking the inequality $d_2(v_1) + d_2(v_2) \geq n - 1$ for $18 \leq n \leq 26$. We will mark with $'-'$ the case when $A < 0$.
3. $n \geq 27$
   In this case $p \geq 6$. We have that in $G' d_1(v_1) + d_1(v_2) \leq 2\delta_1(G) + 1 + A$, then in $G' d_2(v_1) + d_2(v_2) \geq 2(n - 2) - (2\delta_1(G) + 1 + A) = 2n - 2p - 5 - A$. In order to finish the proof we have to show that $2n - 2p - 5 - A \geq n - 1$, i.e. $A \leq n - 2p - 4$. Observe that $w(n, p) = t(n + p, C_4) - \lceil \frac{(n+p)p}{2} \rceil \leq \frac{1}{4}(n + p)(1 + \sqrt{4(n + p) - 3}) - \lceil \frac{(n+p)p}{2} \rceil$ is an increasing function of n, i.e. $w(n_1, p) > w(n_2, p)$ if $n_1 > n_2$. Then, the maximal possible value of A holds for $n = p^2 + 1$. For even p we have that $t(n + p, C_4) \leq \frac{(p^2+p+1)(p+1)}{2} - \frac{1}{2}$ and

$\lceil \frac{(n+p)p}{2} \rceil = \frac{(p^2+p+1)p}{2}$. For odd p we have that $t(n+p, C_4) \leq \frac{(p^2+p+1)(p+1)}{2}$ and $\lceil \frac{(n+p)p}{2} \rceil = \frac{(p^2+p+1)p}{2} + \frac{1}{2}$. In both situations we obtain that $A \leq \frac{p^2+p}{2}$ and for all $p \geq 6$, $A \leq p^2 - 2p - 3$.

□

Taking $n = q^2 + 1$ in Theorem 4, we have

Corollary 5 *For all integers* $q, q \geq 4$

$$R(C_4, W_{q^2+1}) \leq q^2 + q + 1.$$

3 Erdős-Rényi Graph

Let q be a prime power. The famous Erdős-Rényi graph $ER(q)$, first constructed by Erdős and Rényi in 1962, was studied in detail by Parsons in [4]. We know the following properties of $ER(q)$:

- $ER(q)$ has $q^2 + q + 1$ vertices, $q + 1$ vertices with degree q and q^2 vertices with degree $q + 1$
- $ER(q)$ does not contain a subgraph C_4
- in $ER(q)$ there are no two adjacent vertices of degree q
- in $ER(q)$ no vertex of degree q belongs to a subgraph K_3

Let $H(q)$ denote the subgraph of $ER(q)$ obtained by deleting one vertex of degree q. By the third property of $ER(q)$, the subgraph $H(q)$ contains $2q$ vertices with degree q and $q^2 - q$ vertices with degree $q + 1$. One can observe that for all vertices w, the degree $d(w)$ in the complement of $H(q)$ is at most $q^2 - 1$. By this fact, the complement of $H(q)$ does not contain a W_{q^2+1}, so there exists a $(C_4, W_{q^2+1}; q^2 + q)$-coloring. By this fact and by Corollary 5 we have the following

Theorem 6 *For* $q \geq 4$ *being a prime power*

$$R(C_4, W_{q^2+1}) = q^2 + q + 1.$$

4 Exact Values for Small Wheels

Up to date values for $R(C_4, W_n)$ are known only for $n \leq 13$. We determined the next four values as follows:

Theorem 7 *1.* $R(C_4, W_{14}) = 18$,
2. $R(C_4, W_{15}) = 19$,
3. $R(C_4, W_{16}) = 20$,
4. $R(C_4, W_{17}) = 21$.

Proof By Theorem 6 we immediately obtain $R(C_4, W_{17}) = 21$. In order to determine an upper bound for all remaining cases we use Theorem 4. For a lower bound we present appropriate matrix of critical coloring (see Fig. 1). These matrices were obtained by using simulated annealing to find C_4-free graphs with a minimum degree 4. □

```
X1111100000000000
1X100011000000000
11X00000110000000
100X1000001100000
1001X000000011000
10000X00000000111
010000X1001000100
0100001X000100010
00100000X01010010
001000000X0101001
0001001010X000100
00010001010X00010
000010001000X1010
0000100001001X001
00000110001000X01
000001011001100X0
0000010001000110X
```

$(C_4, W_{14}; 17)$-coloring

```
X11111000000000000
1X1000110000000000
11X000001100000000
100X10000011000000
1001X0000000110000
10000X000000001110
010000X10010001000
0100001X0000100001
00100000X010010101
001000000X01100010
0001001010X0000100
00010000010X000011
000010010100X00100
0000100010000X1001
00000110000001X010
000001001010100X00
0000010001010010X0
00000001100101000X
```

$(C_4, W_{15}; 18)$-coloring

```
X111110000000000000
1X10001110000000000
11X0000001100000000
100X100000011000000
1001X00000000110000
10000X0000000001110
010000X100010001000
0100001X00000000101
01000000X0001100010
001000000X010100001
0010000000X01010100
00010010010X1001000
000100001011X000000
0000100011000X00100
00001000001000X0011
000001100001000X010
0000010100100100X00
00000100100000110X1
000000010100001001X
```

$(C_4, W_{16}; 19)$-coloring

Fig. 1 Lower bound for $R(C_4, W_n),\ 14 \leq n \leq 16$

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