Graphs and Combinatorics (2013) 29:617–635
DOI 10.1007/s00373-011-1126-y

ORIGINAL PAPER

Ehrhart Series for Connected Simple Graphs

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Received: 26 May 2011 / Revised: 13 December 2011 / Published online: 11 January 2012

Abstract The Ehrhart ring of the edge polytope $\mathcal{P}_G$ for a connected simple graph G is known to coincide with the edge ring of the same graph if G satisfies the odd cycle condition. This paper gives for a graph which does not satisfy the condition, a generating set of the defining ideal of the Ehrhart ring of the edge polytope, described by combinatorial information of the graph. From this result, two factoring properties of the Ehrhart series are obtained; the first one factors out bipartite biconnected components, and the second one factors out a even cycle which shares only one edge with other part of the graph. As an application of the factoring properties, the root distribution of Ehrhart polynomials for bipartite polygon trees is determined.

Keywords Ehrhart series · Ehrhart polynomial · Hilbert series · Edge polytope · Non-edge-normal graph · Polygon tree

Mathematics Subject Classification (2010) Primary 52C07; Secondary 05A15 · 05C25 · 13F20

1 Introduction

1.1 Background

This paper studies explicit construction and factoring properties of Ehrhart series of edge polytope for connected simple graphs. It is motivated by the root distribution of

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Ehrhart polynomials, which is one of the current topics on computational commutative algebra. In particular, the conjecture of Beck et al. [1] attracts much attention.

Conjecture 1 [1] *All roots* α *of Ehrhart polynomials of lattice D-polytopes satisfy* $-D \leq \Re(\alpha) \leq D - 1$.

The author contributed to a recent paper [6] providing some computational evidence of the conjecture. However, in the paper, rigorous proofs are shown only in a few cases, because the Ehrhart polynomials are known only for a few families: such as complete graphs or complete multipartite graphs.[1]

The Ehrhart polynomial is always related to the Hilbert series of a certain graded K-algebra, called the Ehrhart ring. We call the Hilbert series of the Ehrhart ring the *Ehrhart series*. The subject of this paper is those for the edge polytopes. An edge polytope $\mathcal{P}_G$ is an integral convex polytope defined for a graph G (see Sect. 1.2). Associated with the graph G, there is a graded K-algebra $K[G]$, called an edge ring, which gives the Ehrhart series for G if the algebra is normal; the normality of $K[G]$ is equivalent to the "odd cycle condition" on the graph G [7,11]. The definition of the odd cycle condition is as follows [2]:

Definition 1 The *odd cycle condition* is a condition for a graph G whereby any two odd cycles in G share a vertex or they are connected by a path whose length is one.

In such cases, the Ehrhart series is explicitly computable from the Gröbner basis of a toric ideal I_G, called an edge ideal. However, if $K[G]$ is not normal, there have been no direct construction for the Ehrhart ring for the graph G; one has had to go through the edge polytope and use other techniques. In such a way of construction, it is hard to see the relationship between a graph and the corresponding Ehrhart ring. This paper describes the Ehrhart ring directly from combinatorial information of the graph.

1.2 Graphs, Edge Rings and Edge Polytopes

In this and the next subsections, we recall some known facts and prepare notations.

A graph is a triple (V, E, ϕ) of a finite set V, another finite set E disjoint with V, and a map ϕ from E to the power set of V. An element of V is called a vertex and an element of E is called an edge. The map ϕ sends an edge $e \in E$ to a two-element subset of V. As a consequence of this definition, any graphs have no loops. Besides, throughout this paper, we assume that any graphs have no multiple edges, i.e., we only consider simple graphs.

In order to handle graphs in a ring theoretical setting, it is convenient to interpret a graph map as an algebra homomorphism between polynomial rings. Let $K[\mathbf{V}]$ (respectively $K[\mathbf{E}]$) be the polynomial ring over a field K of characteristic zero, whose variables are vertices (respectively edges). Then, the map ϕ is linearly extended to a

[1] During the submission and the revision of this paper, two preprints have appeared [4,9], which construct counterexamples of Conjecture 1, though these counterexamples come from different polytopes than edge polytopes.

K-algebra homomorphism $\phi^* : K[\mathbf{E}] \longrightarrow K[\mathbf{V}]$, where a subset of V is interpreted as a product of them. The notation $\mathbf{V}$ (respectively $\mathbf{E}$) is solely used to name the free commutative monoid of monomials generated by V (respectively E) written multiplicatively.

Now, let G be a connected simple graph. The edge ideal $I_G \subset K[\mathbf{E}]$ is defined as:

$$I_G = \left(t - u \mid t, u \in \mathbf{E} \,\text{such that}\, \phi^*(t) = \phi^*(u)\right).$$

It is a homogeneous binomial ideal. The edge ring $K[G]$ of G is the image of ϕ^*:

$$K[G] = \phi^*(K[\mathbf{E}]) \cong K[\mathbf{E}]/\ker\phi^*,$$

and $I_G = \ker\phi^*$. The generators of I_G correspond to a certain class of even closed walks on G. For example, the Graver basis of I_G, which consists of the primitive (Definition 4) even closed walks, is given in the following theorem of Tatakis and Thoma [13].

Theorem 1 [13] *Let G be a graph and w an even closed walk of G. The binomial B_w is primitive if and only if*

1. *every block of w is a cycle or a cut edge,*
2. *every multiple edge of the walk w is a double edge of the walk and a cut edge of w,*
3. *every cut vertex of w belongs to exactly two blocks and it is a sink of both.*

As already mentioned in Sect. 1.1, the edge ring $K[G]$ gives the Ehrhart series if and only if G is an edge-normal graph; here, we mean by edge-normal graph, a graph G which satisfies the odd cycle condition (Definition 1).

The edge polytope $\mathcal{P}_G$ of a simple graph $G = (V, E, \phi)$ is defined as follows. Let the vertex set V be $\{v_1, \ldots, v_n\}$, and let a monoid homomorphism ϵ from $\mathbf{V}$ to $\mathbb{Z}^n$ be defined by

$$\epsilon : \prod v_i^{m_i} \longmapsto \sum m_i \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the ith fundamental unit vector. Then, a map ρ from the edge set E to $\mathbb{Z}^n$ is defined as the restriction of the composition $\epsilon \circ \phi^*$ to E. The edge polytope $\mathcal{P}_G \subset \mathbb{R}^n$ is the convex hull of the image of ρ:

$$\begin{aligned}\mathcal{P}_G &= \text{CONV}\,\rho(E)\\ &= \left\{ \sum_{e_i \in E} \lambda_i \rho(e_i) \,\middle|\, 0 \le \lambda_i \le 1,\ \sum_{e_i \in E} \lambda_i = 1,\ \lambda_i \in \mathbb{R} \right\}.\end{aligned}$$

The Ehrhart polynomial $i_G = i_{\mathcal{P}_G}$ of the edge polytope $\mathcal{P}_G$ is the counting function of the integral points in dilated polytopes, that is, $i_G(m) = |m\mathcal{P}_G \cap \mathbb{Z}^n|$. For convenience, we define $i_G(0) = 1$, and we call the generating function

$$\sum_{m=0}^{\infty} i_G(m)t^m$$

the Ehrhart series $H_G(t) = H_{\mathcal{P}_G}(t)$ for $\mathcal{P}_G$.

1.3 Hypergraphs and Hyperedge Rings

A hypergraph is, as a generalization of a graph, a triple (V, E, ϕ) of a finite set V, another finite set E disjoint with V, and a map ϕ from E to the power set of V. An element of V is called a vertex and an element of E is called an edge or a hyperedge. The map ϕ of a hypergraph sends an edge $e \in E$ to a non-empty subset of V. The same algebraic interpretation with graphs in sect. 1.2 is applied.

We construct a hypergraph for each connected non-edge-normal graph. Let $G = (V, E, \phi)$ be a connected non-edge-normal graph with fixed numbering of its odd cycles; let C_i denote the ith odd cycle. We say a pair of odd cycles in a graph is an exceptional pair if any connecting path of the cycles are of length at least two. A set Θ consists of symbols θ_{ij} each corresponding to an exceptional pair (C_i, C_j) with $i < j$. Let F denote the union $E \cup \Theta$, and ψ the map extending ϕ, which sends θ_{ij} in F to the subset of vertices on C_i and C_j in V. Then, $\Gamma = (V, F, \psi)$ is the hypergraph we need.

The K-algebra homomorphism ψ^* is defined from ψ similarly to ϕ^* from ϕ. Accordingly, we define the hyperedge ideal $I_\Gamma \subset K[\mathbf{F}]$:

$$I_\Gamma = \left(t - u \mid t, u \in \mathbf{F} \text{ such that } \psi^*(t) = \psi^*(u)\right).$$

We need a degree function on $K[\mathbf{F}]$, that is not a standard one.

Definition 2 For any monomial T in $K[\mathbf{F}]$, ψ^*-*degree* of T (denote $\deg_{\psi^*} T$) is half the number of vertices multiplied in the image $\psi^*(T)$. Moreover, any non-zero element $f \in K[\mathbf{F}]$ is a sum $f = \sum_{i=1}^{k} c_i T_i$ with $c_i \neq 0$, and the degree of f is $\max_{i=1,\ldots,k} \deg_{\psi^*} T_i$.

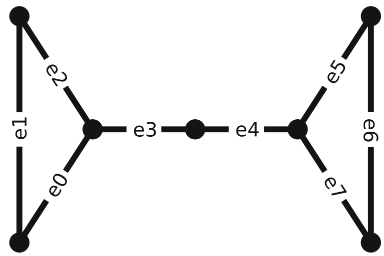
More precisely, each edge $e \in E$ has ψ^*-degree one; each $\theta_{ij} \in \Theta$ has ψ^*-degree $\frac{1}{2}(n_i + n_j)$ where n_i (respectively n_j) is the number of vertices in the odd cycle C_i (respectively C_j). Then, the binomial ideal I_Γ is homogeneous with respect to ψ^*-degree. The hyperedge ring $K[\Gamma]$ of Γ is the image of ψ^*:

$$K[\Gamma] = \psi^*(K[\mathbf{F}]) \cong K[\mathbf{F}]/\ker \psi^*,$$

and $I_\Gamma = \ker \psi^*$.

To describe generating sets of I_Γ, we use the following notations of edge sets. Suppose C_i and C_j are an exceptional pair; then there are paths connecting the cycles, all of which have lengths at least two. Let $N_{ij}^{(p)}$ denote the pth such path connecting C_i and C_j. Moreover, if $N_{ij}^{(p)}$ is $(e_{k_0}, e_{k_1}, \ldots, e_{k_r})$, let $N_{ij}^{(p)+} = \prod_{l:\text{even}} e_{k_l}$ and $N_{ij}^{(p)-} = \prod_{l:\text{odd}} e_{k_l}$. Similarly, C_i^+ and C_i^- denote the alternating products of edges on the cycles. The choice of the sign, C_i^+ or C_i^-, depends on the sign of $N_{ij}^{(p)\pm}$; that

Fig. 1 "bow-tie"



is, the shared vertex of C_i and $N_{ij}^{(p)}$ is incident to either edges of C_i^+ and an edge of $N_{ij}^{(p)-}$ or edges of C_i^- and an edge of $N_{ij}^{(p)+}$. Finally, by abuse of notations, we let C_i denote also the product of edges in C_i.

1.4 Example

The example below illustrates how to use the results of this paper, Theorem 2 in particular, to obtain the Ehrhart series and the Ehrhart polynomial from a given connected simple graph.

Example 1 Let G be the "bow-tie" graph of Fig. 1. We take C_0 as the left triangle consisting of edges $\{e_0, e_1, e_2\}$ and C_1 the right triangle. Since the length of the path connecting triangles is two, (C_0, C_1) is an exceptional pair; we let $\theta = \theta_{01}$ be the corresponding hyperedge. Let $N = N_{01}^{(0)}$ denote the path (e_3, e_4), then we have $N^+ = e_3$ and $N^- = e_4$. The sign of edges on the triangles are determined accordingly as $C_0^+ = e_1$, $C_0^- = e_0e_2$, $C_1^+ = e_5e_7$ and $C_1^- = e_6$.

Then, by Theorem 2, we pick up four generators in the ideal I_Γ: $(\mathcal{G}1)$ a type (1): $e_0e_2e_4^2e_6 - e_1e_3^2e_5e_7$, $(\mathcal{G}2)$ a type (2): $\theta^2 - e_0e_1e_2e_5e_6e_7$, $(\mathcal{G}3)$ a type (3): $\theta e_3 - e_0e_2e_4e_6$, and $(\mathcal{G}4)$ another type (3): $\theta e_4 - e_1e_3e_5e_7$. Here, "type (i)" means that it is a binomial of ith form described in Theorem 2. For example, $(\mathcal{G}3)$ is of 3rd form, as it can be expressed as $\theta N^+ - C_0^- C_1^- N^-$ using the notations introduced in the previous paragraph.

It is easy to see that these binomials immediately correspond to a Gröbner basis of the ideal I_Γ for, say, a lexicographic order $\theta > e_0 > \cdots > e_7$. Then, the Ehrhart series is obtained through a multivariate generating function as explained in Sect. 3.1. Let $\hat{H}_G$ denote the generating function obtained:

$$\begin{aligned}
&\hat{H}_G(e_0, \ldots, e_7, \theta) \\
&\quad = \frac{1 - e_0e_2e_4^2e_6 - \theta^2 - \theta e_3 - \theta e_4 + \theta e_0e_2e_4^2e_6 + \theta e_3e_4 + \theta^2 e_3 + \theta^2 e_4 - \theta^2 e_3e_4}{(1-\theta)\prod_{i=0}^{7}(1-e_i)} \\
&\quad = \frac{1 - e_0e_2e_4^2e_6 + \theta(1-e_3)(1-e_4)}{\prod_{i=0}^{7}(1-e_i)}.
\end{aligned}$$

Since the ψ^*-degree (Definition 2) of each e_i is one and of θ is three, substituting t for each e_i and t^3 for θ gives the Ehrhart series H_G.

$$H_G(t) = \hat{H}_G(t, \dots, t, t^3) = \frac{1 - t^5 + t^3(1-t)^2}{(1-t)^8} = \frac{1 + t + t^2 + 2t^3}{(1-t)^7}.$$

(Notice the difference from the Hilbert series for $K[G] : \frac{1+t+t^2+t^3+t^4}{(1-t)^7}$).
Then, the Ehrhart polynomial $i_G(m)$ is

$$\begin{aligned} i_G(m) &= \binom{m+6}{6} + \binom{m+5}{6} + \binom{m+4}{6} + 2\binom{m+3}{6} \\ &= \frac{1}{720}(m+3)(m+2)(m+1)(5m^3 + 21m^2 + 94m + 120). \end{aligned}$$

1.5 Structure of the Paper

In Sect. 2, motivated by the fact that the hyperedge ring $K[\Gamma]$ is the Ehrhart ring for a non-edge-normal graph G (Proposition 1), we prove the following main theorem which gives a generating set of the hyperedge ideal I_Γ consisting of the crude elements (Definition 3).

Theorem 2 *The following elements form a generating set of* I_Γ*:*

1. *a set of crude generators of* I_G*;*
2. $\theta_{ij}^2 - C_i C_j$ *for any* $\theta_{ij} \in \Theta$*;*
3. $\theta_{ij} N_{ij}^{(p)\pm} - C_i^\mp C_j^\mp N_{ij}^{(p)\mp}$ *for any* $\theta_{ij} \in \Theta$ *and* $N_{ij}^{(p)\pm}$ *without* $N_{ij}^{(q)\pm}$ *which properly divides* $N_{ij}^{(p)\pm}$*;*
4. $\theta_{ij} N_{jk}^{(p)\pm} C_k^\pm - \theta_{ik} N_{jk}^{(p)\mp} C_j^\mp$ *for any* $\theta_{ij} \in \Theta$ *and* $N_{ij}^{(p)\pm}$ *without* $N_{ij}^{(q)\pm}$ *which properly divides* $N_{ij}^{(p)\pm}$*;*
5. $\theta_{ij}\theta_{kl} - \tilde{\theta}_{ik}\tilde{\theta}_{jl}$ *for any* $\theta_{ij}, \theta_{kl} \in \Theta$ *with* i, j, k, l *are different each other; and*
6. $\theta_{ij}\theta_{ik} - \tilde{\theta}_{jk} C_i$ *and* $\theta_{ij}\theta_{lj} - \tilde{\theta}_{il} C_j$ *for any* $\theta_{ij}, \theta_{ik}, \theta_{lj} \in \Theta$.

Here, $\tilde{\theta}_{ij}$ *means either* θ_{ij} *if* C_i *and* C_j *are an exceptional pair or* $C_i^\pm C_j^\pm$ *otherwise.*

Though Theorem 2 determines the Ehrhart series for any connected simple graphs, computation becomes easier if the series is expressible with those of subgraphs. In Sect. 3, we present two factoring properties of the Ehrhart series corresponding to a decomposition into subgraphs; both properties are derived from properties of Möbius sums on lcm-lattices.

Theorem 3 (First factoring property) *The Ehrhart series* H_G *of a graph* G *has a factorization*

$$H_G(t) = H_{G_0}(t) \prod_{i=1}^{r'} H_{B_i}(t),$$

where $G_0, B_1, \dots, B_{r'}$ *are the biconnected decomposition of* G *with oddments.*

Theorem 4 (Second factoring property) *Let G be a connected graph and $G^{(1)}$ and $G^{(2)}$ be its subgraphs. Assume* (1) *each edge of G belongs either $G^{(1)}$ or $G^{(2)}$, except exactly one edge e which is shared by both;* (2) *$G^{(2)}$ is a bipartite graph; and* (3) *e is a part of a cycle in $G^{(2)}$. Then, the Ehrhart series $H_G(t)$ can be factored as*

$$H_G(t) = H_{G^{(1)}}(t)(H_{G^{(2)}}(t)(1-t)).$$

Finally, Sect. 4 applies the lemma of Rodriguez-Villegas [10] to obtain the root distribution of the Ehrhart polynomials for bipartite polygon trees (Proposition 4), whose Ehrhart series are determined by using the second factoring property.

2 Non-Edge-Normal Graphs

2.1 Ehrhart Series

The Hilbert series H_A of a graded K-algebra A is:

$$H_A(t) = \sum_{n=0}^{\infty} (\dim_K A_n) t^n.$$

The Hilbert series for the K-algebra $K[G]$ is the Ehrhart series for $\mathcal{P}_G$ if G is edge-normal. Unfortunately, it differs from the Ehrhart series for $\mathcal{P}_G$ if G is a non-edge-normal graph. However, we can overcome the gap. This is the motivation to consider the hyperedge ring $K[\Gamma]$.

Proposition 1 *The hyperedge ring $K[\Gamma]$ is a graded K-algebra with respect to ψ^*-degree, whose Hilbert series is the Ehrhart series $H_G(t)$ for edge polytope $\mathcal{P}_G$.*

Proof Let $K[\Gamma]_m$ denote the K-vector space generated by ψ^*-degree m elements in $K[\Gamma]$. Then, $K[\Gamma]_i K[\Gamma]_j \subset K[\Gamma]_{i+j}$ since the ideal I_Γ equates only elements of the same ψ^*-degree.

It is shown in [7] that normalization of $K[G]$ can be obtained with the exceptional pairs of odd cycles.[2] Thus, F contains all necessary elements, i.e., all the integer points in $m\mathcal{P}_G$ for any m are in $\epsilon \circ \psi^*(\mathbf{F})$. There are integer points counted multiple times in the image, but it is possible to count each of them only once by identifying the preimage of each point. Thus, the monomials of $K[\Gamma]$, which is isomorphic to $K[\mathbf{F}]/I_\Gamma$, have one-to-one correspondence with integer points in $m\mathcal{P}_G$ for some m. Because all integer points of $m\mathcal{P}_G$ correspond to degree m elements of $K[\Gamma]$, the Ehrhart polynomial $i_G(m) = \dim_K K[\Gamma]_m$.

This means that the Ehrhart ring for a non-edge-normal graph G is given as a hyperedge ring $K[\Gamma]$ of extended hypergraph Γ.

[2] In [7], it is claimed that the only exceptional pairs that have no vertex in common should be considered. However, this is too restrictive; in fact, two exceptional pairs that, for example, have a cycle in common correspond to independent integral points in $\mathcal{P}_G$.

We have the Ehrhart series H_G as

$$H_G(t) = \sum_{n=0}^{\infty} (\dim_K K[\Gamma]_n) t^n = \sum_{n=0}^{\infty} i_G(n) t^n = \frac{i_G^*(t)}{(1-t)^{D+1}},$$

where $D = \dim \mathcal{P}_G$ and $i_G^*(t) \in \mathbb{Z}[t]$ with $\deg i_G^* \leq D$.

2.2 Crude Elements

A binomial $\theta_{ij}^2 - C_i C_j$ is in the hyperedge ideal I_Γ, because the product of edges of C_i and C_j in $K[\mathbf{E}]$ is sent by ϕ^* (and ψ^*) to the square of $\psi^*(\theta_{ij})$. If $N_{ij}^{(p)}$ is a path between an exceptional pair C_i and C_j, then, $\theta_{ij} N_{ij}^{(p)+} - C_i^- C_j^- N_{ij}^{(p)-}$ is in I_Γ. Moreover, if there are plenty of exceptional pairs, one recognizes that $\theta_{ij}\theta_{kl} - \theta_{ik}\theta_{jl}$, $\theta_{ij}\theta_{kl}\theta_{mn} - \theta_{il}\theta_{kn}\theta_{mj}$, etc. are all in I_Γ. How many exceptional pairs should we consider at once as elements in a generating set of the hyperedge ideal? To answer the question, this section introduces the notion of crude elements in a slightly generalized ground. They form a special generating set of an ideal in a finitely generated K-algebra, shown after the definitions.

Definition 3 For given a graded K-algebra R and a homogeneous binomial ideal I, an element $T - U$ of the ideal is *crude* if and only if $T \neq U$ and there are no $T_i - U_i$ $(i = 1, \ldots, k)$ in I satisfying all of the following conditions:

1. $\forall i \deg(T_i) < \deg(T)$,
2. T_1 and U_k are proper divisors of T and U, respectively,
3. $\exists V_i \in R (i = 2, \ldots, k)$ such that $V_i U_i = V_{i+1} T_{i+1}$ for $i = 1, \ldots, k-1$, with $V_1 = T/T_1$.

The crudeness above is a tightening of the following primitiveness on graph walks in [8], rephrased in terms of ideal:

Definition 4 An element $T - U$ of an ideal I in a graded K-algebra R is *primitive* if and only if $T \neq U$ and there is no $T_1 - U_1$ in I satisfying that T_1 and U_1 are proper divisors of T and U, respectively.

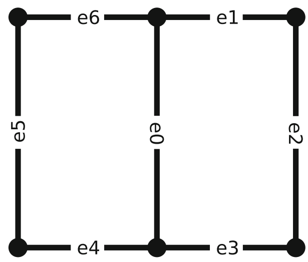
The condition of primitiveness uses only 1 in place of k in the conditions of crudeness. Hence, if an element is crude then it is primitive.

Example 2 Consider the graph of Fig. 2. Let $f_0 = e_1e_3e_5 - e_2e_4e_6$, $f_1 = e_1e_3 - e_0e_2$ and $f_2 = e_0e_5 - e_4e_6$. Each of them corresponds to an even closed walk in the graph and is primitive. However, f_0 is not crude:

1. $\deg f_1 = \deg f_2 = 2 < 3 = \deg f_0$.
2. $e_1e_3 | e_1e_3e_5$ and $e_4e_6 | e_2e_4e_6$.
3. Let $V_2 = e_2$, then with $V_1 = (e_1e_3e_5)/(e_1e_3) = e_5$, we have $V_1e_0e_2 = V_2e_0e_5$.

In fact, f_0 can be written as a sum $e_5 f_1 + e_2 f_2$.

Fig. 2 Graph for Example 2



It is crucial from Proposition 1 to know a generating system of I_Γ. The following proposition is essential for the purpose of this section.

Proposition 2 *A homogeneous binomial ideal I of a finitely generated graded K-algebra R can be generated by crude elements.*

Proof Assume $X - Y$ is not a crude element, but is in a generating set $S \subset I$. Then, there exist $X_i - Y_i$ $(i = 1, \ldots, k)$ in I and V_i $(i = 2, \ldots, k)$ in R satisfying the conditions of Definition 3. Let $I' = (X_i - Y_i \mid i = 1, \ldots, k)$. Then,

$$\begin{aligned} X - Y &= (X/X_1)X_1 - Y \\ &\equiv (X/X_1)Y_1 - Y \pmod{I'} \\ &= V_2X_2 - Y \\ &\equiv V_2Y_2 - Y \pmod{I'} \\ &\cdots \\ &\equiv V_kY_k - Y \pmod{I'} \\ &= (V_k - Y/Y_k)Y_k. \end{aligned}$$

Thus, $X - Y$ is in $I' + (V_k - Y/Y_k)$. Hence, I is generated by

$$S \cup \{X_i - Y_i \mid i = 1, \ldots, k\} \cup \{V_k - Y/Y_k\} \setminus \{X - Y\}.$$

Since degrees strictly decrease on every replacement and R is Noetherian, this process will eventually stop. The resulting generating set is a finite one consisting of crude elements.

As a consequence of this proposition, it is sufficient to consider the crude elements in I_Γ for giving a generating set. In order to determine whether an element in I_Γ is crude or not, we prepare the following lemma.

Lemma 1 *In the same situation with Proposition 2, assume for $T - U$ in an ideal I that there exist $T_1 - U_1$ and $T_3 - U_3$ in I such that T_1 and U_3 properly divides T and U, respectively, and there exists a non-trivial common divisor for U_1 and T_3. Then, $T - U$ is not crude.*

Proof Let X be a non-trivial common divisor for U_1 and T_3. Then, $V_1 = T/T_1$ by definition leads $V_1U_1 = XV_1(U_1/X)$, and $X(U/U_3)(T_3/X) = (U/U_3)T_3$ is obvious. Now, since $T - U \in I$, $XV_1(U_1/X) - X(U/U_3)(T_3/X)$ is also in the ideal. However, because X is a monomial, it is not an element of the binomial ideal. Then, $V_1(U_1/X) - (U/U_3)(T_3/X) \in I$. Let $T_2 = V_1(U_1/X)$, $U_2 = (U/U_3)(T_3/X)$, $V_2 = X$ and $V_3 = U/U_3$. Verifying that $\deg(T_2) < \deg(T)$ and other conditions is easy.

The last lemma means that if part of T and part of U are transformed by the ideal to elements having a non-trivial common divisor, then $T - U$ is ignorable.

2.3 Proof of the Main Theorem

Going back to $K[\Gamma]$, we prove the main theorem by a series of lemmata.

Lemma 2 *Let $N^{(1)}$ and $N^{(2)}$ are two connecting path between an exceptional pair of odd cycles C_1 and C_2. If $N^{(1)+}$ properly divides $N^{(2)+}$, then $\theta_{12}N^{(2)+} - C_1^-C_2^-N^{(2)-}$ is not crude.*

Proof Obviously, both $\theta_{12}N^{(1)+} - C_1^-C_2^-N^{(1)-}$ and $\theta_{12}N^{(2)+} - C_1^-C_2^-N^{(2)-}$ are in I_Γ. Since $N^{(1)+}$ divides $N^{(2)+}$, path $N^{(2)}$ branches at some vertex u from path $N^{(1)}$ but joins again at the vertex v just an edge apart from u along with $N^{(1)}$. Moreover, since the next edge is shared by both half paths, the number of edges on the subpath P of $N^{(2)}$ from u to v is odd. Then, the edge e on $N^{(1)}$ connecting u and v forms an even cycle with the subpath P. The even cycle corresponds to an element in I_G: $P^+e - P^-$, where $P^\pm$ are restrictions of $N^{(2)\pm}$ on P. By Lemma 1, the existence of e as a common divisor of $C_1^-C_2^-N^{(1)-}$ and P^+e is sufficient to conclude that $\theta_{12}N^{(2)+} - C_1^-C_2^-N^{(2)-}$ is not crude.

In Sect. 3.3, the lemma above will be generalized, but we continue the proof of the theorem for now.

Lemma 3 *The following elements in the ideal I_Γ are crude.*

1. $\theta_{ij}^2 - C_iC_j$ *for any* $\theta_{ij} \in \Theta$;
2. $\theta_{ij}N_{ij}^{(p)\pm} - C_i^\mp C_j^\mp N_{ij}^{(p)\mp}$ *for any* $\theta_{ij} \in \Theta$ *and* $N_{ij}^{(p)\pm}$ *without* $N_{ij}^{(q)\pm}$ *which properly divides* $N_{ij}^{(p)\pm}$;
3. $\theta_{ij}N_{jk}^{(p)\pm}C_k^\pm - \theta_{ik}N_{jk}^{(p)\mp}C_j^\mp$ *for any* $\theta_{ij} \in \Theta$ *and* $N_{ij}^{(p)\pm}$ *without* $N_{ij}^{(q)\pm}$ *which properly divides* $N_{ij}^{(p)\pm}$;
4. $\theta_{ij}\theta_{kl} - \tilde{\theta}_{ik}\tilde{\theta}_{jl}$ *for any* $\theta_{ij}, \theta_{kl} \in \Theta$ *with* i, j, k, l *are different each other; and*
5. $\theta_{ij}\theta_{ik} - \tilde{\theta}_{jk}C_i$ *and* $\theta_{ij}\theta_{lj} - \tilde{\theta}_{il}C_j$ *for any* $\theta_{ij}, \theta_{ik}, \theta_{lj} \in \Theta$.

Here, $\tilde{\theta}_{ij}$ means either θ_{ij} if C_i and C_j are an exceptional pair or $C_i^\pm C_j^\pm$ otherwise.

Proof (1) Since θ_{ij} is an irreducible element, there is no monomial T in $K[\mathbf{F}]$ other than itself that $\theta_{ij} \equiv T \pmod{I_\Gamma}$. The proper divisor of θ_{ij}^2 is only θ_{ij}; thus, $\theta_{ij}^2 - C_iC_j$ is crude.

(2) Assume the contrary that $\theta_{ij}N_{ij}^{(p)+} - C_i^- C_j^- N_{ij}^{(p)-}$ is not crude. Then, there exists a proper divisor $T \in K[\mathbf{F}]$ of $\theta_{ij}N_{ij}^{(p)+}$, which is congruent to some U. As in the argument of (1), T cannot be θ_{ij}. Thus, there is a divisor of T which divides $N_{ij}^{(p)+}$. Let D be the greatest common divisor of T and $N_{ij}^{(p)+}$. The degree of D is in a range 1 to $\deg(N_{ij}^{(p)+}) - 1$. Thus, the number of edges in $N_{ij}^{(p)+}$ is more than one. Hence, there are edges $e_{2k}^{(p)}$ in $N_{ij}^{(p)+}$ and $e_{2k+1}^{(p)}$ in $N_{ij}^{(p)-}$, where $\psi(e_\kappa^{(p)}) = v_\kappa^{(p)} v_{\kappa+1}^{(p)}$. Suppose $e_{2k}^{(p)}$ does not divide D. Then, $v_{2k+1}^{(p)}$ does not divide $\psi^*(T)$. As assumed, $T \equiv U \pmod{I_\Gamma}$, neither $e_{2k}^{(p)}$ nor $e_{2k+1}^{(p)}$ divides U, and $v_{2k+2}^{(p)}$ does not divide $\psi^*(U)$. This argument continues until all edges in $N_{ij}^{(p)}$ are excluded, or we find a shortcut path directly connecting $v_{2k}^{(p)}$ to $v_{2k+2l+1}^{(p)}$. The former contradicts with the existence of the divisor T, and the latter contradicts with the assumption that there is no dividing path from Lemma 2. Therefore, $\theta_{ij}N_{ij}^{(p)\pm} - C_i^{\mp} C_j^{\mp} N_{ij}^{(p)\mp}$ is crude.

We omit the rest of the cases; the proof of (3) is similar to that of (2), while the proofs of (4) and (5) are similar to (1).

Before proving the next lemma, we should introduce some terminology. A cycle C_i (and C_j) *semi-supports* θ_{ij}. We define a *T-induced subgraph G'* of G for T, a monomial of $K[\mathbf{F}]$ as a subgraph G' of G consisting of every edge dividing T and every edge of cycle C_i semi-supporting θ_{ij} dividing T. Moreover, if a subgraph G'' of G is either a connected component with non-zero even semi-supporting cycles or a pair of connected components both with odd semi-supporting cycles, we call the subgraph G'' an *even component*; it corresponds to a connected subhypergraph of the hypergraph Γ.

Lemma 4 *If $T - U$ is in the ideal I_Γ but not in I_G and an $N_{ij}^{(p)\pm}$ divides T, then $T - U$ is not a crude element unless itself is one of (2) and (3) of Lemma 3.*

Proof Assume that a crude binomial $T - U$ is not one of (2) and (3) of Lemma 3, and that $N_{12}^{(1)+}$ divides T. Then, $N_{12}^{(1)+}$ does not divide U.

If both C_1 and C_2 semi-support θ's, then there exists $X \neq 1$ such that $T \equiv \theta_{12}N_{12}^{(1)+}X \pmod{I_\Gamma}$, by using (4) of Lemma 3 if necessary. Then, by (2) of Lemma 3, we have $T \equiv C_1^- C_2^- N_{12}^{(1)-}X \pmod{I_\Gamma}$. If there is a vertex on N_{12} that does not divide $\psi^*(X)$, then since $N_{12}^{(1)+}$ does not divide U, $N_{12}^{(1)-}$ must divide U. Thus, Lemma 1 implies that $T - U$ is not crude, contradicting the assumption. Otherwise, all the vertices divide $\psi^*(X)$. However, edges of X are impossible to be shortcut or detour of N_{12} by Lemma 2. Then, T is divisible by $\theta_{12}N_{12}^{(1)+}$ and U by $C_1^- C_2^- N_{12}^{(1)-}$ regardless of X. Thus, $T - U$ is not even primitive, and contradicting the assumption.

The case when both C_1 and C_2 do not semi-support θ's is just a reverse course of the above.

We, then assume that C_1 semi-supports a θ but C_2 does not. Then, there exists $X \neq 1$ such that $T \equiv \theta_{13}C_2^+ N_{12}^{(1)+}X \pmod{I_\Gamma}$. Then, by (3) of Lemma 3, we have $T \equiv C_1^- \theta_{23}N_{12}^{(1)-}X \pmod{I_\Gamma}$. Almost same argument applies to U to be divided by $N_{12}^{(1)-}$, and it contradicts the assumption that $T - U$ is crude.

Lemma 5 *If $T - U$ is a crude element in the ideal I_Γ, then TU-induced subgraph of G has at most two disjoint even components.*

Proof Without loss of generality, we can assume TU is not divisible by any product of all edges in an even closed walk. Let G' denote the TU-induced subgraph of G.

Assume that G' has three disjoint even components G_0, G_1 and G_2. Then, by arranging θ in T and U with (4) or (5) of Lemma 3, we have $T' \equiv T$ and $U' \equiv U$ (mod I_Γ) satisfying the following condition: if an odd cycle in an even component G_i semi-supports a θ, the other cycle semi-supporting the same θ is also in G_i for both T' and U'. Then, each of T' and U' is decomposed into G_i parts T'_i and U'_i, respectively, for $i = 0, 1, 2$ and possibly a $G' \setminus (\bigcup G_i)$ part.

The decomposition implies that $T' - U'$ is not primitive. Therefore, $T - U$ is not crude: this is a contradiction.

The above lemmata lead us to the main theorem.

Theorem 2 *The following elements form a generating set of I_Γ:*

1. *a set of crude generators of I_G;*
2. *$\theta_{ij}^2 - C_i C_j$ for any $\theta_{ij} \in \Theta$;*
3. *$\theta_{ij} N_{ij}^{(p)\pm} - C_i^\mp C_j^\mp N_{ij}^{(p)\mp}$ for any $\theta_{ij} \in \Theta$ and $N_{ij}^{(p)\pm}$ without $N_{ij}^{(q)\pm}$ which properly divides $N_{ij}^{(p)\pm}$;*
4. *$\theta_{ij} N_{jk}^{(p)\pm} C_k^\pm - \theta_{ik} N_{jk}^{(p)\mp} C_j^\mp$ for any $\theta_{ij} \in \Theta$ and $N_{ij}^{(p)\pm}$ without $N_{ij}^{(q)\pm}$ which properly divides $N_{ij}^{(p)\pm}$;*
5. *$\theta_{ij}\theta_{kl} - \tilde{\theta}_{ik}\tilde{\theta}_{jl}$ for any $\theta_{ij}, \theta_{kl} \in \Theta$ with i, j, k, l are different each other; and*
6. *$\theta_{ij}\theta_{ik} - \tilde{\theta}_{jk} C_i$ and $\theta_{ij}\theta_{lj} - \tilde{\theta}_{il} C_j$ for any $\theta_{ij}, \theta_{ik}, \theta_{lj} \in \Theta$.*

Here, $\tilde{\theta}_{ij}$ means either θ_{ij} if C_i and C_j are an exceptional pair or $C_i^\pm C_j^\pm$ otherwise.

Proof Lemmata 3 through 5 determine the crude elements in the ideal I_Γ. That is, besides elements from I_G of (1), if an $N_{ij}^{(p)\pm}$ appears in a crude element, that element is of (3) or (4) by Lemma 4; otherwise there are only pure odd cycles with at most four cycles by Lemma 5. Thus, the element is of (2), (5) or (6). From Proposition 2, these crude elements generate the ideal I_Γ. All monomials appearing in (2) through (6) are not divisible by any monomial appearing in (1).

3 Factoring Properties

3.1 Möbius Sum on lcm-Lattice

There may be various methods to compute Ehrhart series from a Gröbner basis, but we use multivariate series as a convenient tool. By Macaulay's theorem, the dimension of $K[\Gamma]_n$ can be computed by counting the monomials of degree n outside the initial ideal. The main part of the computation is the Möbius sum on lcm-lattice, which is a lattice on all least common multiples of monomials ordered by divisibility [3]. The lcm-lattice of our case is defined on initial monomials $\{f_i = \mathrm{in}_<(g_i) \mid g_i \in \mathcal{G}\}$ of a

Gröbner basis $\mathcal{G}$ with respect to a term order $<$; the elements of the lattice are least common multiples of initial monomials with 1 as the bottom element (the least common multiple of empty set). Let $L(X)$ denote the lcm-lattice on atoms $X = \{\xi_1, \ldots, \xi_s\}$. Moreover, let $M(L(X))$ denote the Möbius sum on $L(X)$

$$M(L(X)) = \sum_{x \in L(X)} \mu(x)x,$$

where $\mu(x) = \mu(1, x)$ is the Möbius function on $L(X)$ of interval $[1, x]$ (see [12], for example). It is used to obtain a multivariate generating function $\hat{H}_G$:

$$\hat{H}_G(\tau_1, \ldots, \tau_s) = \frac{M(L(\mathrm{in}_<(\mathcal{G})))}{\prod_{i=1}^{s}(1 - \tau_i)},$$

where τ_i denote each elements of F. Finally, substituting $t^{\deg \tau_i}$ to each $\tau_i \in F$ gives the Ehrhart series H_G.

In the following sections, the factoring properties of the Ehrhart series are discussed based on the factorization of the Möbius sum.

Lemma 6 *Let X and Y be two finite sets of monomials such that any pair $x \in X$ and $y \in Y$ are coprime. Then, the Möbius sum $M(L(X \cup Y))$ can be factored as $M(L(X \cup Y)) = M(L(X))M(L(Y))$.*

Proof We claim that the Möbius function on a lcm-lattice is multiplicative; that is, $\mu(1) = 1$ and $\mu(xy) = \mu(x)\mu(y)$ if $x \in X$ and $y \in Y$. If this claim is valid, the lemma follows:

$$\begin{aligned} M(L(X \cup Y)) &= \sum_{x \in X \wedge y \in Y} \mu(xy)xy = \sum_{x \in X \wedge y \in Y} \mu(x)\mu(y)xy \\ &= \sum_{x \in X} \mu(x)x \sum_{y \in Y} \mu(y)y = M(L(X))M(L(Y)). \end{aligned}$$

Thus, we prove the claim. First, by definition, $\mu(1) = 1$. Second, assume that for any $x'y' < xy$ the claim is correct. Then,

$$\begin{aligned} \mu(xy) &= -\sum_{x'y' < xy} \mu(x'y') = -\sum_{x'y' < xy} \mu(x')\mu(y') \\ &= \mu(x)\mu(y) - \left(\sum_{x' \le x} \mu(x')\right)\left(\sum_{y' \le y} \mu(y')\right) = \mu(x)\mu(y). \end{aligned}$$

Finally, by induction on the lattice order, the claim holds.

3.2 First Factoring Property

The first factoring property of the Ehrhart series corresponds, roughly, to biconnected decomposition of a graph. The main discrepancy presents with odd cycles, which are always the most complicated part of the discussion of Ehrhart series of edge polytopes. We avoid digging deeper into the complications, parenthesize the hard part as a whole. Let $B_1, \ldots, B_r$ be the biconnected decomposition of a graph G. If there are odd cycle subgraphs in G, let G_0 be the minimum connected subgraph containing all biconnected components with odd cycle subgraphs of G. By renumbering, if necessary, we have a decomposition of G as $G_0, B_1, \ldots, B_{r'}$. We call such decomposition the biconnected decomposition of G *with oddments*.

We apply Lemma 6 to obtain the first factoring property.

Theorem 3 *The Ehrhart series H_G of a graph G has a factorization*

$$H_G(t) = H_{G_0}(t) \prod_{i=1}^{r'} H_{B_i}(t),$$

where $G_0, B_1, \ldots, B_{r'}$ are the biconnected decomposition of G with oddments.

Proof From Theorem 2, the only patterns that odd cycles affect the Ehrhart series are in the oddments subgraph G_0.

Let $\mathrm{in}_<(\mathcal{G})$ be the initial monomials of Gröbner basis $\mathcal{G}$ of the ideal I_Γ with respect to a term order $<$. The Ehrhart series is obtained through the multivariate generating function: $\hat{H}_G(\tau_1, \ldots, \tau_s) = \frac{M(L(\mathrm{in}_<(\mathcal{G})))}{\prod_{\tau_i \in F}(1-\tau_i)}$, as in Sect. 3.1.

We know a generating set of I_Γ from Theorem 2, but do not know a Gröbner basis, explicitly. If an initial monomial of the generating set is in a decomposed component, then it is coprime to those in other components, since the non-initial monomial also in the same decomposed component with the initial monomial. Then, the monomial remains coprime to those from other components after the Buchberger algorithm by Buchberger's criterion. Therefore, we have a Gröbner basis whose initial monomials are classified into each decomposed component.

By Lemma 6, the numerator of $\hat{H}_G$ is factored along with the biconnected decomposition with oddments. The denominator is also factored, because each edge is classified into a decomposed component.

The Ehrhart series is obtained from $\hat{H}_G$ by substituting $t^{\deg \tau_i}$ to each $\tau_i \in F$.

3.3 Second Factoring Property

The second factoring property focuses on an edge. As we have seen in Lemma 2, a chordal path can be separated into shortcut path and an even cycle, if parity permits. We generalize the property not only on a path but also on an even cycle. The key lemma is Lemma 7 that explains phenomena like Example 2 that a binomial corresponding to an even cycle with a chord can be a $K[\mathbf{E}]$-linear combination of two binomials corresponding to smaller even cycles.

A separating pair of vertices of a graph G is a pair of vertices v_1, v_2 of G that the number of connected components of $G - \{v_1, v_2\}$ is greater than that of G. Let e be an edge of G, and v_1 and v_2 be the end vertices of the edge e. Then, $G - \tilde{e}$ denotes $G - \{v_1, v_2\}$. We call an edge with its end vertices a *separating face*, if the number of connected components of $G - \tilde{e}$ is greater than that of G.

Lemma 7 *Let G be a biconnected graph. If a separating face (e with u and v) decomposes G into at least two components one of which is bipartite, then, there is a generating set of the ideal I_Γ having no cycles stretching over the bipartite component and another component.*

Proof By assumption, we have two decomposed components $G^{(1)}$ and $G^{(2)}$, one of which, say $G^{(2)}$, is a bipartite subgraph of G. In subgraphs $G^{(i)}$ for both $i = 1, 2$, the vertices u and v are degree at least 2; one of the adjacent edges is e. Let A_u and A_v denote ones of the other edges adjacent to u in $G^{(1)}$ and to v, respectively, and similarly B_u and B_v in $G^{(2)}$.

Consider a big even cycle in G passing A_u, B_u and B_v, A_v. By assumption of bipartiteness of $G^{(2)}$, if we numbers A_u the first and B_u the second on the cycle, then the numbering of B_v is even and that of A_v is odd. Hence we can name the other edges on the cycle:

$$A_1 = A_u, B_2 = B_u, A_3, B_4 \ldots, B_{2k} = B_v, A_{2k+1} = A_v, B_{2k+2}, \ldots, A_{2m-1}, B_{2m}.$$

Then,

$$\prod_{i=1}^{m} A_{2i-1} - \prod_{i=1}^{m} B_{2i}$$

is in I_Γ. We claim that this is redundant in a generating set of the ideal. If the claim is valid, since the choice of even cycle is arbitrary, there is no need to include cycles stretching over both $G^{(1)}$ and $G^{(2)}$ in the generating set of the ideal.

Now we prove the claim. Let $A^{(i)} = \prod_{A_j \in E(G^{(i)})} A_j$ and $B^{(i)} = \prod_{B_j \in E(G^{(i)})} B_j$ for $i = 1, 2$. Then

$$\prod_{i=1}^{m} A_{2i-1} - \prod_{i=1}^{m} B_{2i} = A^{(1)}A^{(2)} - B^{(1)}B^{(2)}.$$

There are cycles in $G^{(1)}$ and $G^{(2)}$, each corresponds to $A^{(1)} - B^{(1)}e$ and $A^{(2)}e - B^{(2)}$, respectively: in other words, each half the big cycle with e. Then,

$$A^{(1)}A^{(2)} - B^{(1)}B^{(2)} = A^{(2)}(A^{(1)} - B^{(1)}e) + B^{(1)}(A^{(2)}e - B^{(2)}).$$

Thus the binomial is generated by the small cycles, one of which is in $G^{(1)}$ and another in $G^{(2)}$.

Theorem 3 *Let G be a connected graph and $G^{(1)}$ and $G^{(2)}$ be its subgraphs. Assume* (1) *each edge of G belongs either $G^{(1)}$ or $G^{(2)}$, except exactly one edge e which is shared by both;* (2) *$G^{(2)}$ is a bipartite graph; and* (3) *e is a part of a cycle in $G^{(2)}$. Then, the Ehrhart series $H_G(t)$ can be factored as*

$$H_G(t) = H_{G^{(1)}}(t)(H_{G^{(2)}}(t)(1-t)).$$

Proof By Theorem 3, we can assume that $G^{(2)}$ is a biconnected graph. If $G^{(1)}$ also is a biconnected graph, by Lemma 7, there is a generating set consisting of binomials from each subgraph. Moreover, even if $G^{(1)}$ is not a biconnected graph, the same argument applies on any cycles stretching over both subgraphs. Hence, the remaining concerns are connecting paths passing through $G^{(2)}$ between odd cycles, both of which are in $G^{(1)}$. However, the condition of Theorem 2(3) based on Lemma 2 have already excluded such paths.

Because $G^{(2)}$ is bipartite, one can chose a term order that the shared edge e does not appear in the initial terms. Thus, the same argument with in the Theorem 3 applies, i.e., initial monomials from different components are coprime, then the Möbius sum is factored along with the decomposition.

Finally, since the shared edge e is counted in both $H_{G^{(1)}}(t)$ and $H_{G^{(2)}}(t)$, we should cancel a $(1-t)$ from the denominator of $H_{G^{(2)}}(t)$.

Note that both factoring properties are also applicable to the Hilbert series of edge rings.

4 Bipartite Polygon Trees

4.1 Explicit Series

We apply the factoring properties to a few families of graphs to obtain explicit Ehrhart series for them.

Recall a *polygon tree* is a connected simple graph defined recursively as follows (see [5], for example). A polygon, or a cycle, is a polygon tree. If G is a polygon tree, then picking an edge of it and make a new cycle graph G' share the edge with G, then resulting graph is a polygon tree. We call a polygon tree a bipartite polygon tree, if all involving cycles are even cycles.

Before proving the result of polygon trees, let us recall the basic examples of the Ehrhart series.

Fact 1 *The followings are well-known Ehrhart series of a few biconnected graphs.*

1. $\frac{1}{1-t}$ *if G is an edge;*
2. $\frac{1+t+\cdots+t^{n-1}}{(1-t)^{2n-1}}$ *if G is an even cycle with $2n$ edges;*
3. $\frac{1}{(1-t)^{2n-1}}$ *if G is an odd cycle with $2n-1$ edges.*

Proposition 3 *The Ehrhart series $H_G(t)$ for a bipartite polygon tree graph G with e edges and f_{2n} cycles with $2n$ edges for $n \geq 2$ is:*

$$H_G(t) = \frac{\prod(1 + t + \cdots + t^{n-1})^{f_{2n}}}{(1-t)^{e-f}} \tag{*}$$

with $f = \sum f_{2n}$.

Proof We show the proposition by induction on the number of even cycles. If the number of cycles is one, the graph is an even cycle with $2n$ edges, then, from Fact 1(2), the Ehrhart polynomial is $\frac{1+t+\cdots+t^{n-1}}{(1-t)^{2n-1}}$. It coincides with $e = 2n$ and $f = 1$ case of (*), as desired. Assume that (*) is valid for bipartite polygon trees with $f - 1$ cycles. Then, a polygon tree G consisting of e edges and f even cycles is considered as an even cycle C' of $2n$ edges and a polygon tree G' of $e - (2n - 1)$ edges and $f - 1$ even cycles sharing an edge. Since the sharing edge is a separating face, by Theorem 3, the Ehrhart series can be factored as

$$H_G(t) = H_{G'}(t)(H_{C'}(t)(1 - t)).$$

With the induction hypothesis and Fact 1(2), the degree of denominator in total is

$$(e - (2n - 1)) - (f - 1) + (2n - 1) - 1 = e - f,$$

and the numerator is in the form of (*).

Since the graph G of Proposition 3 is bipartite, the dimension D of the edge polytope is $v - 2$, as shown in [7]. The degree of denominator is equals to $D + 1$ for any polytopes, thus it should be $v - 1$ in the current case. Since the polygon trees are planar, the Euler characteristic of the graph gives the equation $v - e + f = 1$, i.e., $v - 1 = e - f$, which is equal to the degree of our formula.

Note that the formula (*) is valid by Theorem 3 for bipartite graphs whose biconnected components are all polygon trees, including bipartite cacti. Moreover, the formula (*) is also valid if a single odd cycle is in a polygon tree; since we can start the induction from the odd cycle, whose Ehrhart series is known as Fact 1(3). Note also that since the outerplanar graphs are subfamily of the polygon tree, if it is bipartite or with a single odd cycle as above, the formula applies to these cases as well.

Example 3 Ladders L_k are Cartesian products $K_2 \times P_k$, where K_2 is the complete graph of order two and P_k is the path graph of order k. It is an even outerplanar graph and thus a bipartite polygon tree graph, consisting of $k - 1$ squares. Thus, the Ehrhart series $H_{L_k}(t)$ can be deduced from Proposition 3:

$$H_{L_k}(t) = \frac{(1+t)^{k-1}}{(1-t)^{2k-1}}.$$

Example 4 We know the Ehrhart series of the "bow-tie" (Example 1) and the ladders (Example 3). Then, for any combined graphs of a bow-tie and a ladder L_k, sharing a vertex or an edge, we know their Ehrhart series. In case sharing a vertex, it is given by the first factoring property (Theorem 3) as

$$H_G(t) = \frac{(1+t+t^2+2t^3)(1+t)^{k-1}}{(1-t)^{2k+6}}.$$

In case sharing an edge, it is given by the second factoring property (Theorem 3) as

$$H_G(t) = \frac{(1+t+t^2+2t^3)(1+t)^{k-1}}{(1-t)^{2k+5}}.$$

4.2 Root Distribution

The root distribution of the Ehrhart polynomials can be obtained from the Ehrhart series without explicit computation of the polynomials themselves in some cases. We use the results of Rodriguez-Villegas [10]. For an integer a, let S_a be a set of non-zero polynomials $p(x)$ such that

$$p(x) = v(x)\prod_{i=1}^{a}(x+i)$$

where all roots of $v(x)$ lie on $\Re(x) = -(a+1)/2$. Then the following lemma holds.

Lemma 8 [10] *Let α be a root of unity and $f \in S_a$ for some $a \in \mathbb{Z}$. Then*

$$f(x-1) - \alpha f(x) \in S_{a-1}.$$

Proposition 4 *The Ehrhart polynomial $i_G(m)$ for a bipartite polygon tree G with e edges and f_{2n} cycles with $2n$ edges for $n \geq 2$ is in $S_{e-1-\sum nf_{2n}}$. In other words, the roots of $i_G(m)$ are negative integers or on $\Re(x) = -(e - \sum nf_{2n})/2$.*

Proof Let E^- denote the negative shift operator. Then $f(x-1) - \alpha f(x)$ can be rewritten as $(E^- - \alpha)f(x)$. The Ehrhart polynomial $i_G(m)$ is related to $i_G^*(t)$, the numerator of the Ehrhart series, as

$$\begin{aligned} i_G(m) &= i_G^*(E^-)\binom{m+D}{D} \\ &= c_h \prod_{j=1}^{h}(E^- - \alpha_j)\binom{m+D}{D}, \end{aligned}$$

where h is the degree, α_j are the roots and c_h is the initial coefficient of $i_G^*(t)$. From Proposition 3, all roots of $i_G^*(t)$ are roots of unity, and the initial coefficient is 1. Moreover, notice that $\binom{m+D}{D}$ is in S_D. When applying each factor $E^- - \alpha_j$, we track the roots using Lemma 8. Then, the intermediate polynomials are in S_{D-1}, S_{D-2}, and so on, and finally the Ehrhart polynomial is in S_{D-h}. As noted after the proof of Proposition 3, $D = e - f - 1$. On the other hand, the degree of $i^*_G(t)$ is $h = \sum(n-1)f_{2n}$. Since $f = \sum f_{2n}$,

$$D - h = e - f - 1 - \sum(n-1)f_{2n} = e - 1 - \sum n f_{2n}$$

as required.

Remark that the roots are on the strip of Conjecture 1, and in fact on the left half-plane part of the region.

Acknowledgments I would like to thank Hidefumi Ohsugi for kind and helpful comments, on an early stage in particular. Takayuki Hibi gave me several invaluable and encouraging comments on the first draft. Discussions with Akihiro Higashitani greatly motivated me to think factorization properties. Suggestions from Takeaki Uno about choice of graphs were very useful for Sect. 4.

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