On the existence and stability of fuzzy CF variable fractional differential equation for COVID-19 epidemic

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Abstract

In this paper, we convert the recent COVID-19 model with the use of the most influential theories, such as variable fractional calculus and fuzzy theory. We propose the fuzzy variable fractional differential equation for the COVID-19 model in which the variable fractional-order derivative is described using the Caputo-Fabrizio in the Caputo sense. Furthermore, we provide the results on the existence and uniqueness using Lipschitz conditions. Also, discuss the stability analysis of the present new COVID-19 model by employing Hyers-Ulam stability.

Keywords Novel coronavirus · Variable Caputo-Fabrizio fractional derivative · Fixed point theorems · Existence and uniqueness · Hyers-Ulam stability

Mathematics Subject Classification 47H10 · 58K25 · 34A08

1 Introduction

The World Health Organization (WHO) declared that the new coronavirus infection (COVID-19) was a Public Health Emergency of International Concern on January 30th 2020. It has exploded to 64.2 million cases worldwide and caused 1.49 million deaths by December 2nd, 2020 [1, 2]. The symptoms of the COVID-19 include coughing, breathing difficulties, and fever etc. The transmission of the infection from human to human is known, but there are still some specific cases where this virus has also been recognized in animals. The infection can cause pneumonia, severe acute respiratory syndrome, kidney failure, and even death in the next steps.

Many researchers are working to analyse the effects of COVID-19 via mathematical modeling. The fractional-order derivative is the extended version of the integer-order derivative. In recent years, the fractional-order

derivative has produced efficient results in modeling the real problem phenomena [3, 4]. Due to the advantages of fractional-order derivatives, the current studies mainly focused on the fractional-order COVID-19 model. There are two types of definitions developed in the literature to introduce fractional operators (see [5–10]) where they involved singular or non-singular kernel. Most studies for COVID-19 are based on a non-singular kernel present in the definition of the fractional operator. The fractional operator definition has recently developed, including non-singular kernels such as Caputo-Fabrizio (CF) fractional derivative and Atangana-Baleanu (AB) fractional derivative. The Caputo-Fabrizio operator with a regular kernel is the direct consequence of the Caputo derivative operator. The only difference between these two definitions is that the new definition does not have any singular point. Via using a non-singular kernel, the full effect on memory accurately describe, and many more advantages developed with the new definition called CF fractional derivative.

This study's main objective is to present the new COVID-19 model and discuss the common but essential aspects to analyze the new COVID-19 model. First, we select the recently developed COVID-19 model and convert it into a more effective and fruitful COVID-19 model with the fuzzy theory and variable fractional calculus. Here, the derivative

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describes by variable fractional derivative and follows CF fractional derivative definition. This analysis includes the existence, uniqueness, and Hyers-Ulam stability of the new COVID-19 model using fixed point theory. In [11], authors first time developed the theory on COVID-19 system using variable fractional derivative using fixed point theory. Obtain the new results for existence, uniqueness and stability with some more new and exciting developments for the proposed COVID-19 system.

The outline of the paper is as follows: Sect. 2, we present some fundamental results about variable Caputo-Fabrizio fractional and fuzzy theory definitions. In Sect. 3, introduces the classical model of the COVID-19 model. We describe the proposed COVID-19 model with variable Caputo-Fabrizio fractional and fuzzy concept in Sect. 4. We obtain the new results existence, uniqueness, and stability for the generalized variable fractional COVID-19 model in Sect. 5, the article is finished by concluding remarks in Sect. 6.

2 Background material and preliminaries

In this section, we recall some preliminaries needed for this study in this paper [11–14].

Definition 1 ([12, 13]). Let $\Lambda : \mathbb{R} \rightarrow [0, 1]$ be a fuzzy set of the real line satisfies the following conditions:

- (i) Λ is normal (for any $k_0 \in \mathbb{R}; \Lambda(k_0) = 1$),
- (ii) Λ is upper semi-continuous on $\mathbb{R}$ that is forever $\epsilon > 0$ there exist $\delta > 0, |\Lambda(k) - \Lambda(k_0)| \leq \epsilon, |k - k_0| \leq \delta$,
- (iii) Λ is convex
($\Lambda(\rho k + (1 - \rho)p) \geq (\Lambda(k) \wedge \Lambda(p)) \forall \rho \in [0, 1], k, p \in \mathbb{R}$)
- (iv) $cl\{k \in \mathbb{R}, \Lambda(k) > 0\}$ is compact.

Then, it is called a fuzzy number.

Definition 2 ([12, 13]). On a fuzzy number Λ , the λ -level set is defined by

$$[\Lambda]^\lambda = \{h \in \mathbb{R}; \Lambda(h) \geq \lambda\},$$

where $\lambda \in (0, 1]$ and $h \in \mathbb{R}$.

Definition 3 ([12, 13]). Let $[\underline{\Lambda}(\xi), \bar{\Lambda}(\xi)]$ be the parametric form of a fuzzy number Λ , where $0 \leq \xi \leq 1$, which satisfies the following properties:

- (i) $\underline{\Lambda}(\xi)$ is left continuous, bounded, and increasing function over $(0, 1]$, and right continuous at 0.
- (ii) $\bar{\Lambda}(\xi)$ is right continuous, bounded, and decreasing over $[0, 1]$, and right continuous at 0.
- (iii) $\underline{\Lambda}(\xi) \leq \bar{\Lambda}(\xi)$.

Also, if $\underline{\Lambda}(\xi) = \bar{\Lambda}(\xi) = 0$, then ξ is called a crisp number.

Let $\mathcal{E}$ denote the set of upper semi-continuous, convex and normal fuzzy numbers with bounded λ -level interval which yields that $g \in \mathcal{E}$, then λ -level set

$$[g]^\lambda = \{\xi : g(\xi) \geq \lambda\}, \lambda \in (0, 1],$$

where is bounded and closed interval represented by $g = [\underline{g}(\xi), \bar{g}(\xi)]$, $g' = [\underline{g}'(\xi), \bar{g}'(\xi)]$, and for $\gamma \geq 0$, various operator are defined as follows:

- (i) **Addition**:
 $[\underline{g}(\xi) + \underline{g}'(\xi), \bar{g}(\xi) + \bar{g}'(\xi)] = [\underline{g}(\xi) + \underline{g}'(\xi), \bar{g}(\xi) + \bar{g}'(\xi)]$
- (ii) **Subtraction**:
 $[\underline{g}(\xi) - \underline{g}'(\xi), \bar{g}(\xi) - \bar{g}'(\xi)] = [\underline{g}(\xi) - \underline{g}'(\xi), \bar{g}(\xi) - \bar{g}'(\xi)]$
- (iii) **Scalar multiplication**:

$$\gamma \cdot g(\xi) = \begin{cases} \gamma \underline{g}(\xi) \cdot \gamma \bar{g}(\xi), \gamma > 0, \\ \gamma \bar{g}(\xi) \cdot \gamma \underline{g}(\xi), \gamma < 0. \end{cases} \quad (1)$$

Definition 4 ([12, 13]). Consider a mapping $\mathcal{X} : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}$ and let $g = [\underline{g}(\xi), \bar{g}(\xi)]$, $g' = [\underline{g}'(\xi), \bar{g}'(\xi)]$ be two fuzzy numbers in their parametric form. The Hausdorff distance between g and g' is defined by

$$\mathcal{X}(g, g') = \sup_{\xi \in [0, 1]} [\max\{|\underline{g}(\xi) - \underline{g}'(\xi)|, |\bar{g}(\xi) - \bar{g}'(\xi)|\}]$$

In $\mathcal{E}$, the metric $\mathcal{X}$ has the following conditions:

- (i) $\mathcal{X}(g + \omega, g' + \omega) = \mathcal{X}(g, g')$ for all $g, g', \omega \in \mathcal{E}$,
- (ii) $\mathcal{X}(g\tau, g'\tau) = |\tau| \mathcal{X}(g, g')$ for all $g, g' \in \mathcal{E}, \tau \in \mathbb{R}$,
- (iii) $\mathcal{X}(g + \tau_1, g' + \tau_2) \leq \mathcal{X}(g, g') + \mathcal{X}(\tau_1, \tau_2)$ for all $g, g', \tau_1, \tau_2 \in \mathcal{E}$
- (iv) $(\mathcal{E}, \mathcal{X})$ is a complete metric space.

Definition 5 ([12, 13]). Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{E}$. If there exist $\mathcal{L}_3 \in \mathcal{E}$ such that $\mathcal{L}_1 = \mathcal{L}_2 + \mathcal{L}_3$ then $\mathcal{L}_3$ is said to be the H -difference of $\mathcal{L}_1$ and $\mathcal{L}_2$, denoted by $\mathcal{L}_1 \ominus \mathcal{L}_2$.

Definition 6 ([12, 13]). Let $\mathcal{K} : \mathbb{R} \rightarrow \mathcal{E}$ be a fuzzy mapping. Then, $\mathcal{K}$ is called continuous if for any $\epsilon > 0 \exists \delta > 0$ and a fixed value of $C_0 \in [p_1, p_2]$, we have $\mathcal{X}(\mathcal{K}(C), \mathcal{K}(C_0)) < \epsilon$ whenever $|C - C_0| < \delta$.

Definition 7 ([14]). Let two fuzzy numbers $\mathcal{P}, \mathcal{Q}$, the arithmetic operations in the parametric form are given as

$$\begin{aligned}
(\mathcal{P} \oplus \mathcal{Q})[\lambda] &= [\underline{\mathcal{P}}(\lambda) + \underline{\mathcal{Q}}(\lambda), \overline{\mathcal{P}}(\lambda) + \overline{\mathcal{Q}}(\lambda)], (\rho \odot \mathcal{P})[\lambda] \\
&= \begin{cases} [\rho \underline{\mathcal{P}}(\lambda) + \rho \underline{\mathcal{Q}}(\lambda)], \rho \geq 0, \\ [\rho \overline{\mathcal{P}}(\lambda) + \rho \overline{\mathcal{Q}}(\lambda)], \rho < 0, \end{cases} \quad (2)
\end{aligned}$$

where $\lambda \in [0, 1]$.

The λ -level set of a fuzzy number, we imply a closed and bounded interval $[\underline{\mathcal{P}}(\lambda), \overline{\mathcal{P}}(\lambda)]$. Where $\underline{\mathcal{P}}(\lambda), \overline{\mathcal{P}}(\lambda)$ are left and right hand end points of the $[\mathcal{P}]_\lambda$.

Now, we introduce the helpful notations for this work [14]:

1. The set of measurable FV function $\mathbf{G}^*$ is denoted by $\mathbf{N}^E(s_1, s_2)$ on $[s_1, s_2]$.
2. The set of continuous FV function $\mathbf{G}^*$ is denoted by $\mathbf{M}^E(s_1, s_2)$ on $[s_1, s_2]$.
3. The set of all fuzzy numbers is denoted by $\mathbb{R}_F^\lambda$.
4. $C([0, 1], \mathbb{R}_F^\lambda)(\rho) = \{\mathcal{P}(\theta) \in C([0, 1], \mathbb{R}_F^\lambda) : \|\mathcal{P}\| \leq \rho\}$.

The Hausdorff distance is $\mathfrak{N} : \mathbb{R}_F^\lambda \times \mathbb{R}_F^\lambda \rightarrow \mathbb{R} \cup \{0\}$ with $\mathfrak{N}(\mathcal{P}, \mathcal{Q}) = \sup_{\theta \in [0, 1]} \max\{|\underline{\mathcal{P}}(\theta) - \underline{\mathcal{Q}}(\theta)|, |\overline{\mathcal{P}}(\theta) - \overline{\mathcal{Q}}(\theta)|\}$. The $(\mathbb{R}_F, \mathfrak{N})$ is a complete metric space satisfying the following:

1. $\mathfrak{N}(\mathcal{P} \oplus \mathcal{F}, \mathcal{Q} \oplus \mathcal{F}) = \mathfrak{N}(\mathcal{P}, \mathcal{Q}), \forall \mathcal{P}, \mathcal{Q}, \mathcal{F} \in \mathbb{R}_F^\lambda$.
2. $\mathfrak{N}(\mathcal{P} \oplus \mathcal{F}, 0) = \mathfrak{N}(\mathcal{P}, 0) + \mathfrak{N}(\mathcal{F}, 0), \forall \mathcal{P}, \mathcal{Q}, \mathcal{F} \in \mathbb{R}_F^\lambda$.
3. $\mathfrak{N}(\mathcal{P} \oplus \mathcal{Q}, \mathcal{P} \oplus \mathcal{F}) = \mathfrak{N}(\mathcal{Q}, \mathcal{F}), \forall \mathcal{P}, \mathcal{Q}, \mathcal{F} \in \mathbb{R}_F^\lambda$.
4. $\mathfrak{N}(\mathcal{P} \oplus \mathcal{Q}, \mathcal{V} \oplus \mathcal{F}) \leq \mathfrak{N}(\mathcal{P}, \mathcal{V}) + \mathfrak{N}(\mathcal{Q}, \mathcal{F}), \forall \mathcal{P}, \mathcal{Q}, \mathcal{F} \in \mathbb{R}_F^\lambda$.
5. $\mathfrak{N}(\mathcal{P} \oplus \mathcal{Q}, \mathcal{V} \oplus \mathcal{F}) \leq \mathfrak{N}(\mathcal{P}, \mathcal{V}) + \mathfrak{N}(\mathcal{Q}, \mathcal{F}), \forall \mathcal{P}, \mathcal{Q}, \mathcal{F} \in \mathbb{R}_F^\lambda$, $\mathcal{P} \oplus \mathcal{Q}, \mathcal{V} \oplus \mathcal{F}$ exist.
6. $\mathfrak{N}(\mu \odot \mathcal{P}, \mu \odot \mathcal{Q}) \leq |\mu| \mathfrak{N}(\mathcal{P}, \mathcal{Q}), \forall \mathcal{P}, \mathcal{Q} \in \mathbb{R}_F, \mu \in \mathbb{R}_F^\lambda$.

Definition 8 ([11]). The function $\rho(\theta)$ is differentiable, the variable Caputo derivative of order $\vartheta(\theta) \in [0, 1]$ is given as

$${}_0^C D_\theta^{\vartheta(\theta)} \rho(\theta) = \frac{1}{\Gamma(1 - \vartheta(\theta))} \int_0^\theta \frac{1}{(\theta - a)^{\vartheta(\theta)}} \rho'(a) da. \quad (3)$$

Note: The VO fractional operators (derivatives/integral) of $\ell - 1 < \vartheta(\theta) \leq \ell$ and $\rho(\theta)$ is bounded in interval $\theta \in [0, \nu]$.

Definition 9 ([11]). Let $\rho(\theta) \in H^1([0, T])$, and $\vartheta(\theta) \in [0, 1]$, then the $\vartheta(\theta)$ th-order variable Caputo-Fabrizio derivative of $\rho(\theta)$ in the Caputo sence is defined as

$${}_0^{CF} D_\theta^{\vartheta(\theta)} \rho(\theta) = \frac{\mathbb{E}(\vartheta(\theta))}{1 - \vartheta(\theta)} \int_0^\theta \rho'(a) \exp\left(\frac{-\vartheta(\theta)(\theta - a)}{(1 - \vartheta(\theta))}\right) da, \quad (4)$$

Here, $H^1([0, T])$ is a Hilbert space. The integral (4) on right converges and the function $\mathbb{E}(\vartheta(\theta))$ is a normalizing function depending on $\vartheta(\theta)$ such that $\mathbb{E}(0) = \mathbb{E}(1) = 1$, and

$${}_0^{CF} D_\theta^{\vartheta(\theta)} A = 0, \quad (5)$$

where A is a constant.

Definition 10 ([11]). The nonsingular kernel type variable fractional integral is defined by

$$\begin{aligned}
{}_0^{CF} J_\theta^{\vartheta(\theta)} \rho(\theta) &= \frac{(1 - \vartheta(\theta))}{\mathbb{E}(\vartheta(\theta))} \rho(\theta) \\
&+ \frac{\vartheta(\theta)}{\mathbb{E}(\vartheta(\theta))} \int_0^\theta \rho(a) da, 0 < \vartheta(\theta) \leq 1. \quad (6)
\end{aligned}$$

Lemma 1 ([11]) Let $\rho(\theta) \in C([0, T])$, then the solution of the following variable Caputo-Fabrizio fractional differential equation:

$$\begin{cases} {}_0^{CF} D_\theta^{\vartheta(\theta)} \rho(\theta) = \omega(\theta), \theta \in [0, T], 0 < \vartheta(\theta) \leq 1, \\ \rho(0) = \rho_0, \omega_0 \in \mathbb{R}, \end{cases} \quad (7)$$

is given by

$$\rho(\theta) = \omega_0 + \frac{(1 - \vartheta(\theta))}{\mathbb{E}(\vartheta(\theta))} \omega(\theta) + \frac{\vartheta(\theta)}{\mathbb{E}(\vartheta(\theta))} \int_0^\theta \omega(a) da. \quad (8)$$

Theorem 1 ([14]) The fuzzy variable fractional differential equation (FVFDE) in Caputo-Fabrizio sense of variable fractional derivative is given as

$$\begin{cases} {}_0^{CF} D_\theta^{\vartheta(\theta)} g(\theta) = \mathbf{G}^*(\theta, g), 0 < \vartheta(\theta) \leq 1, \\ g(0) = g_0, \omega_0 \in \mathbb{R}, \end{cases} \quad (9)$$

for $\mathbf{G}^* \in \mathbf{N}^F(J) \cap \mathbf{M}^C(J), J = [s_1, s_2]$. Then, consider two cases for the gH -differentiability.

Case 1 (i-differentiability).

$$i-gH[g']_r = [g'(\theta; r_0), \overline{g'(\theta; r_0)}], r_0 \in [0, 1],$$

Case 2 (ii-differentiability).

$$ii-gH[g']_r = [\overline{g'(\theta; r_0)}, g'(\theta; r_0)], r_0 \in [0, 1].$$

Then, the FVFDE (9) has a solution of the following form as

$$\begin{aligned}
&(g(\theta) \ominus_{gH} g(0))_{E_{\vartheta(\theta)} \geq 0} \oplus (g(\theta) \ominus_{gH} g(0))_{E_{\vartheta(\theta)} < 0} \\
&= \frac{(1 - \vartheta(\theta))}{\mathbb{E}(\vartheta(\theta))} \mathbf{G}^*(\theta, g(\theta)) + \frac{\vartheta(\theta)}{\mathbb{E}(\vartheta(\theta))} \int_0^\theta \mathbf{G}^*(a, g(a)) da. \quad (10)
\end{aligned}$$

Corollary 1 Consider non-negative kernel for all $\theta, \vartheta(\theta)$, then we have the following solution given by

$$g(\theta) = g(0) \oplus \frac{(1 - \vartheta(\theta))}{\mathbb{E}(\vartheta(\theta))} \odot \mathbf{G}^*(\theta, g(\theta)) \oplus \frac{\vartheta(\theta)}{\mathbb{E}(\vartheta(\theta))} \odot \int_0^\theta \mathbf{G}^*(a, g(a)) da. \quad (11)$$

while in case 2

$$g(\theta) = g(0) \ominus (-1) \frac{(1 - \vartheta(\theta))}{\mathbb{E}(\vartheta(\theta))} \odot \mathbf{G}^*(\theta, g(\theta)) \ominus \frac{\vartheta(\theta)}{\mathbb{E}(\vartheta(\theta))} \odot \int_0^\theta \mathbf{G}^*(a, g(a)) da. \quad (12)$$

Definition 11 ([14]). The variable Caputo fractional derivative in parametric form of order $\vartheta(\theta) \in (0, 1]$ is given by

Case 1. ${}_0^C D_\theta^{i, \vartheta(\theta)} g(\theta)_r = [{}_0^C D_\theta^{i, \vartheta(\theta)} \underline{g}(\theta; r), {}_0^C D_\theta^{i, \vartheta(\theta)} \bar{g}(\theta; r)]$,

Case 2. ${}_0^C D_\theta^{ii, \vartheta(\theta)} g(\theta)_r = [{}_0^C D_\theta^{ii, \vartheta(\theta)} \bar{g}(\theta; r), {}_0^C D_\theta^{ii, \vartheta(\theta)} \underline{g}(\theta; r)]$, where

$${}_0^C D_\theta^{i, \vartheta(\theta)} \varphi(\theta) = \frac{1}{\Gamma(1 - \vartheta(\theta))} \int_0^\theta \frac{1}{(\theta - a)^{\vartheta(\theta)}} \left({}_{i-} \varphi'(a) \right) da, \quad (13)$$

$${}_0^C D_\theta^{ii, \vartheta(\theta)} \varphi(\theta) = \frac{1}{\Gamma(1 - \vartheta(\theta))} \int_0^\theta \frac{1}{(\theta - a)^{\vartheta(\theta)}} \left({}_{ii-} \varphi'(a) \right) da. \quad (14)$$

3 Classical COVID-19 model

In this section, we describe the chosen recent COVID-19 model [15] and give the information regarding the parameters and used variable in the model, described as

$$\left. \begin{aligned} \mathcal{S}(\varpi_0) &= \mathcal{S}_0, \\ \Omega(\varpi_0) &= \Omega_0, \\ \mathcal{I}(\varpi_0) &= \mathcal{I}_0, \\ \mathcal{I}_v(\varpi_0) &= \mathcal{I}_{v_0}, \\ \mathcal{H}_r(\varpi_0) &= \mathcal{H}_{r_0}, \\ \mathcal{H}_d(\varpi_0) &= \mathcal{H}_{d_0}, \\ r_d(\varpi_0) &= r_{d_0}, \\ r_v(\varpi_0) &= r_{v_0}, \\ d(\varpi_0) &= d_{d_0}. \end{aligned} \right\} \quad (16)$$

The model (15), monitors the dynamics of nine classes (compartments), namely denoted as: susceptible ($\mathcal{S}$), exposed (Ω), infectious ($\mathcal{I}$), infectious but undetected ($\mathcal{I}_v$), recover ($\mathcal{H}_r$), hospitalized that will die ($\mathcal{H}_d$), dead by COVID-19 (d), recovered after being previously detected as infectious (r_d), recovered after being previously infectious but undetected (r_v), the full informations of the model (15) given in the reference [15]. The following table present the details for the used parameters in (15) (Table 1):

In [15], authors have developed the new mathematical model for coronavirus (COVID-19) infection based on the China case. Also, check the validation of the presented COVID-19 model via reported data on China. Furthermore, estimate the errors when identifying parameters at initial stages of the virus and obtain the impact of unknown cases for various stages on the COVID-19.

In the next section, using the model (15), we introduce the new COVID-19 model via the variable fractional derivative and fuzzy theory. The presented fuzzy variable fractional differential (FVFDE) COVID-19 has many advantages due to variable fractional order. Using variable fractional deriva-

$$\left. \begin{aligned} D_\varpi(\mathcal{S}(\varpi)) &= -\frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_\Omega(\varpi) \kappa_\Omega \Omega(\varpi) + n_{\mathcal{I}}(\varpi) \kappa_{\mathcal{I}} \mathcal{I}(\varpi) + n_{\mathcal{I}_v}(\varpi) \kappa_{\mathcal{I}_v} (\rho(\varpi)) \mathcal{I}_v(\varpi) \right) \\ &\quad - \frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\mathcal{H}_r}(\varpi) \kappa_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi) + n_{\mathcal{H}_d}(\varpi) \kappa_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi) \right), \\ D_\varpi(\Omega(\varpi)) &= \frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_\Omega(\varpi) \kappa_\Omega(\varpi) \Omega(\varpi) + n_{\mathcal{I}}(\varpi) \kappa_{\mathcal{I}} \mathcal{I}(\varpi) + n_{\mathcal{I}_v}(\varpi) \kappa_{\mathcal{I}_v} (\rho(\varpi)) \mathcal{I}_v(\varpi) \right) + \\ &\quad \frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\mathcal{H}_r}(\varpi) \kappa_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi) + n_{\mathcal{H}_d}(\varpi) \kappa_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi) \right) - \Gamma_\Omega \Omega(\varpi) + \Lambda_1(\varpi) - \Lambda_2(\varpi), \\ D_\varpi(\mathcal{I}(\varpi)) &= \Gamma_\Omega \Omega(\varpi) - \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi), \\ D_\varpi(\mathcal{I}_v(\varpi)) &= (1 - \rho(\varpi)) \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) - \Gamma_{\mathcal{I}_v}(\varpi) \mathcal{I}_v(\varpi), \\ D_\varpi(\mathcal{H}_r(\varpi)) &= \rho(\varpi) \left(1 - \frac{\omega(\varpi)}{\rho(\varpi)} \right) \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) - \Gamma_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi), \\ D_\varpi(\mathcal{H}_d(\varpi)) &= \omega(\varpi) \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) - \Gamma_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi), \\ D_\varpi(r_d(\varpi)) &= \Gamma_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi), \\ D_\varpi(r_v(\varpi)) &= \Gamma_{\mathcal{I}_v}(\varpi) \mathcal{I}_v(\varpi), \\ D_\varpi(d(\varpi)) &= \Gamma_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi), \end{aligned} \right\} \quad (15)$$

associated initial conditions

tive in (15), we can found some interesting and important

Table 1 Parameters of the model (15) and their biological interpretations

Symbol	Description	Value/Source
Γ_{Ω}	Transition rate of a person in compartment Ω (day^{-1})	$\mathbb{R}^+$ [15]
$\Gamma_{\mathcal{I}}(\varpi)$	Transition rate of a person in compartment $\mathcal{I}$ (day^{-1}) at time t	$\mathbb{R}^+$ [15]
$\Gamma_{\mathcal{H}_r}(\varpi)$	Transition rate of a person in compartment $\mathcal{H}_r$ To compartment r_d (day^{-1}) at time t	$\mathbb{R}^+$ [15]
$\Gamma_{\mathcal{I}_v}(\varpi)$	Transition rate of a person in compartment $\mathcal{I}_v$ To compartment r_v (day^{-1}) at time t	$\mathbb{R}^+$ [15]
κ_{Ω}	The disease contact rates (day^{-1}) of a person In the corresponding compartments Ω	$\mathbb{R}^+$ [15]
$\kappa_{\mathcal{I}}, \kappa_{\mathcal{I}_v}$	The disease contact rates (day^{-1}) of a person In the corresponding compartments $\mathcal{I}, \mathcal{I}_v$	$\mathbb{R}^+$ [15]
$\kappa_{\mathcal{H}_r}, \kappa_{\mathcal{H}_d}$	The disease contact rates (day^{-1}) of a person In the corresponding compartments $\mathcal{H}_r, \mathcal{H}_d$	$\mathbb{R}^+$ [15]
n_{Ω}	The functions representing the efficiency of the control measures applied To the corresponding compartments Ω	$[0, 1]$ [15]
$n_{\mathcal{I}}, n_{\mathcal{I}_v}$	The functions representing the efficiency of the control measures applied To the corresponding compartments $\mathcal{I}, \mathcal{I}_v$	$[0, 1]$ [15]
$n_{\mathcal{H}_r}, n_{\mathcal{H}_d}$	The (%) functions representing the efficiency of the control measures applied To the corresponding compartments $\mathcal{H}_r, \mathcal{H}_d$	$[0, 1]$ [15]
$\Lambda_1(\varpi)$	The people infected arrives (day^{-1})	[15]
$\Lambda_2(\varpi)$	The people infected that leaves (day^{-1})	[15]

results via simulation methods, and theoretical aspect have also needed before proceeding further, such as existence and uniqueness of the solution for (15) and stability. These are the most focused points in the theory of the differential equations before solving it with the numerical methods. To take more benefit of the proposed FVFDE COVID-19 model, we use the fuzzy theory concept.

4 The FVFDE COVID-19 model

In this section, we describe the COVID-19 model (15) with the help of the variable Caputo-Fabrizio (CF) fractional derivative in the Caputo sense and fuzzy concept in the following way:

$$\left. \begin{aligned}
 {}_0^{\text{CF}} D_{\varpi}^{\psi(\varpi)}(\mathcal{S}(\varpi)) &= -\frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\Omega}(\varpi) \kappa_{\Omega}(\varpi) \Omega(\varpi) + n_{\Omega}(\varpi) \kappa_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) + n_{\mathcal{I}_v} \kappa_{\mathcal{I}_v}(\rho(\varpi)) \mathcal{I}_v(\varpi) \right) \\
 &\quad - \frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\mathcal{H}_r}(\varpi) \kappa_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi) + n_{\mathcal{H}_d}(\varpi) \kappa_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi) \right), \\
 {}_0^{\text{CF}} D_{\varpi}^{\psi(\varpi)}(\Omega(\varpi)) &= \frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\Omega}(\varpi) \kappa_{\Omega}(\varpi) \Omega(\varpi) + n_{\mathcal{I}}(\varpi) \kappa_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) + n_{\mathcal{I}_v} \kappa_{\mathcal{I}_v}(\rho(\varpi)) \mathcal{I}_v(\varpi) \right) + \\
 &\quad \frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\mathcal{H}_r}(\varpi) \kappa_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi) + n_{\mathcal{H}_d}(\varpi) \kappa_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi) \right) - \Gamma_{\Omega} \Omega(\varpi) + \Lambda_1(\varpi) - \Lambda_2(\varpi), \\
 {}_0^{\text{CF}} D_{\varpi}^{\psi(\varpi)}(\mathcal{I}(\varpi)) &= \Gamma_{\Omega} \Omega(\varpi) - \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi), \\
 {}_0^{\text{CF}} D_{\varpi}^{\psi(\varpi)}(\mathcal{I}_v(\varpi)) &= (1 - \rho(\varpi)) \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) - \Gamma_{\mathcal{I}_v}(\varpi) \mathcal{I}_v(\varpi), \\
 {}_0^{\text{CF}} D_{\varpi}^{\psi(\varpi)}(\mathcal{H}_r(\varpi)) &= \rho(\varpi) \left(1 - \frac{\omega(\varpi)}{\rho(\varpi)} \right) \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) - \Gamma_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi), \\
 {}_0^{\text{CF}} D_{\varpi}^{\psi(\varpi)}(\mathcal{H}_d(\varpi)) &= \omega(\varpi) \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) - \Gamma_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi), \\
 {}_0^{\text{CF}} D_{\varpi}^{\psi(\varpi)}(r_d(\varpi)) &= \Gamma_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi), \\
 {}_0^{\text{CF}} D_{\varpi}^{\psi(\varpi)}(r_v(\varpi)) &= \Gamma_{\mathcal{I}_v}(\varpi) \mathcal{I}_v(\varpi), \\
 {}_0^{\text{CF}} D_{\varpi}^{\psi(\varpi)}(d(\varpi)) &= \Gamma_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi),
 \end{aligned} \right\} \quad (17)$$

where $\psi(\varpi)$ denotes the variable CF fractional order and the associated initial conditions:

$$\left. \begin{aligned}
 \mathcal{S}(\varpi_0) &= \mathcal{S}_0 \geq 0, \\
 \Omega(\varpi_0) &= \Omega_0 \geq 0, \\
 \mathcal{I}(\varpi_0) &= \mathcal{I}_0 \geq 0, \\
 \mathcal{I}_v(\varpi_0) &= \mathcal{I}_{v_0} \geq 0, \\
 \mathcal{H}_r(\varpi_0) &= \mathcal{H}_{r_0} \geq 0, \\
 \mathcal{H}_d(\varpi_0) &= \mathcal{H}_{d_0} \geq 0, \\
 r_d(\varpi_0) &= r_{d_0} \geq 0, \\
 r_v(\varpi_0) &= r_{v_0} \geq 0, \\
 d(\varpi_0) &= d_0 \geq 0.
 \end{aligned} \right\} \quad (18)$$

$$\left. \begin{aligned}
 \mathcal{S}(\varpi) &= \mathcal{S}_0 + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon_1(\cdot)), \\
 \Omega(\varpi) &= \Omega_0 + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon_2(\cdot)), \\
 \mathcal{I}(\varpi) &= \mathcal{I}_0 + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon_3(\cdot)), \\
 \mathcal{I}_v(\varpi) &= \mathcal{I}_{v_0} + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon_4(\cdot)), \\
 \mathcal{H}_r(\varpi) &= \mathcal{H}_{r_0} + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon_5(\cdot)), \\
 \mathcal{H}_d(\varpi) &= \mathcal{H}_{d_0} + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon_6(\cdot)), \\
 r_d(\varpi) &= r_{d_0} + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon_7(\cdot)), \\
 r_v(\varpi) &= r_{v_0} + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon_8(\cdot)), \\
 d(\varpi) &= d_0 + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon_9(\cdot)).
 \end{aligned} \right\} \quad (21)$$

Consider the right-hand side of the propped model (17):

Let us introduce the notation $\Upsilon_i(\varpi, \mathcal{S}(\varpi), \Omega(\varpi), \mathcal{I}(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)) = \Upsilon_i(\cdot), i = 1, 2, \dots, 9$. Re-write the equation (21) as

$$\left. \begin{aligned}
 \Upsilon_1(\cdot) &= -\frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\Omega}(\varpi) \kappa_{\Omega}(\varpi) + n_{\Omega}(\varpi) \kappa_{\mathcal{I}}(\varpi) + n_{\mathcal{I}_v} \kappa_{\mathcal{I}_v}(\rho(\varpi)) \mathcal{I}_v(\varpi) \right) \\
 &\quad - \frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\mathcal{H}_r}(\varpi) \kappa_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi) + n_{\mathcal{H}_d}(\varpi) \kappa_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi) \right), \\
 \Upsilon_2(\cdot) &= \frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\Omega}(\varpi) \kappa_{\Omega}(\varpi) \Omega(\varpi) + n_{\mathcal{I}}(\varpi) \kappa_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) + n_{\mathcal{I}_v} \kappa_{\mathcal{I}_v}(\rho(\varpi)) \mathcal{I}_v(\varpi) \right) + \\
 &\quad \frac{\mathcal{S}(\varpi)}{\mathbf{M}} \left(n_{\mathcal{H}_r}(\varpi) \kappa_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi) + n_{\mathcal{H}_d}(\varpi) \kappa_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi) \right) - \Gamma_{\Omega} \Omega(\varpi) + \Lambda_1(\varpi) - \Lambda_2(\varpi), \\
 \Upsilon_3(\cdot) &= \Gamma_{\Omega} \Omega(\varpi) - \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi), \\
 \Upsilon_4(\cdot) &= (1 - \rho(\varpi)) \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) - \Gamma_{\mathcal{I}_v}(\varpi) \mathcal{I}_v(\varpi), \\
 \Upsilon_5(\cdot) &= \rho(\varpi) \left(1 - \frac{\omega(\varpi)}{\rho(\varpi)} \right) \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) - \Gamma_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi), \\
 \Upsilon_6(\cdot) &= \omega(\varpi) \Gamma_{\mathcal{I}}(\varpi) \mathcal{I}(\varpi) - \Gamma_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi), \\
 \Upsilon_7(\cdot) &= \Gamma_{\mathcal{H}_r}(\varpi) \mathcal{H}_r(\varpi), \\
 \Upsilon_8(\cdot) &= \Gamma_{\mathcal{I}_v}(\varpi) \mathcal{I}_v(\varpi), \\
 \Upsilon_9(\cdot) &= \Gamma_{\mathcal{H}_d}(\varpi) \mathcal{H}_d(\varpi),
 \end{aligned} \right\} \quad (19)$$

where $\Upsilon_i(\cdot), i = 1, 2, \dots, 9$ are fuzzy function. Hence, for $\psi(\varpi) \in (0, 1]$, then the model (17) can be written as

$$\left. \begin{aligned}
 {}_0^{CF}D_{\varpi}^{\psi(\varpi)}(\mathcal{S}(\varpi)) &= \Upsilon_1(\cdot), \\
 {}_0^{CF}D_{\varpi}^{\psi(\varpi)}(\Omega(\varpi)) &= \Upsilon_2(\cdot), \\
 {}_0^{CF}D_{\varpi}^{\psi(\varpi)}(\mathcal{I}(\varpi)) &= \Upsilon_3(\cdot), \\
 {}_0^{CF}D_{\varpi}^{\psi(\varpi)}(\mathcal{I}_v(\varpi)) &= \Upsilon_4(\cdot), \\
 {}_0^{CF}D_{\varpi}^{\psi(\varpi)}(\mathcal{H}_r(\varpi)) &= \Upsilon_5(\cdot), \\
 {}_0^{CF}D_{\varpi}^{\psi(\varpi)}(\mathcal{H}_d(\varpi)) &= \Upsilon_6(\cdot), \\
 {}_0^{CF}D_{\varpi}^{\psi(\varpi)}(r_d(\varpi)) &= \Upsilon_7(\cdot), \\
 {}_0^{CF}D_{\varpi}^{\psi(\varpi)}(r_v(\varpi)) &= \Upsilon_8(\cdot), \\
 {}_0^{CF}D_{\varpi}^{\psi(\varpi)}(d(\varpi)) &= \Upsilon_9(\cdot),
 \end{aligned} \right\} \quad (20)$$

with non-negative initial conditions (18).

With the help of the initial conditions (18) and variable fractional integral operator ${}_0^{CF}J_{\varpi}^{\psi(\varpi)}$, we transform the proposed model (17) into the following integral equations:

$$\Theta(\varpi) = \Theta(0) + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(\Upsilon(\varpi, \Theta(\varpi))), \quad (22)$$

where

$$\Theta(\varpi) = \begin{cases} \mathcal{S}(\varpi), \\ \Omega(\varpi), \\ \mathcal{I}(\varpi), \\ \mathcal{I}_v(\varpi), \\ \mathcal{H}_r(\varpi), \\ \mathcal{H}_d(\varpi), \\ r_d(\varpi), \\ r_v(\varpi), \\ d(\varpi) \end{cases}, \quad \Theta(0) = \begin{cases} \mathcal{S}_0, \\ \Omega_0, \\ \mathcal{I}_0, \\ \mathcal{I}_{v_0}, \\ \mathcal{H}_{r_0}, \\ \mathcal{H}_{d_0}, \\ r_{d_0}, \\ r_{v_0}, \\ d_0. \end{cases}$$

and

$$Y(\varpi, \Theta(\varpi)) = \begin{cases} Y_1(\varpi, S(\varpi), \Omega(\varpi), I(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)), \\ Y_2(\varpi, S(\varpi), \Omega(\varpi), I(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)), \\ Y_3(\varpi, S(\varpi), \Omega(\varpi), I(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)), \\ Y_4(\varpi, S(\varpi), \Omega(\varpi), I(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)), \\ Y_5(\varpi, S(\varpi), \Omega(\varpi), I(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)), \\ Y_6(\varpi, S(\varpi), \Omega(\varpi), I(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)), \\ Y_7(\varpi, S(\varpi), \Omega(\varpi), I(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)), \\ Y_8(\varpi, S(\varpi), \Omega(\varpi), I(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)), \\ Y_9(\varpi, S(\varpi), \Omega(\varpi), I(\varpi), \mathcal{I}_v(\varpi), \mathcal{H}_r(\varpi), \mathcal{H}_d(\varpi), r_d(\varpi), r_v(\varpi), d(\varpi)). \end{cases}$$

In the next section, we prove our main results and present some known definitions and theorems.

5 Main results for the proposed FVFDE COVID-19 model

This section defines some definitions and theorems that helps us to prove the results for existence and uniqueness conditions for variable CF fractional integral operator [11].

Definition 12 If $\Theta \in \mathbb{R}$ is said to be a solution of the equation (21) with non-negative initial condition, if $\Theta(0) = \Theta_0$ and there exists $Y(\varpi, \Theta(\varpi)) \in \mathbb{N}^F([0, \mu] \times \mathbb{R})$ such that $\varpi \in [0, \mu]$ and the integral equation

$$\Delta(\varpi, \Theta(\varpi)) = \frac{(1 - \psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot Y(\varpi, \Theta(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi Y(\zeta, \Theta(\zeta)) d\zeta. \quad (27)$$

$$\Theta(\varpi) = \Theta_0 + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(Y(\varpi, \Theta(\varpi))) \quad (23)$$

is satisfied.

Theorem 2 Let $\Theta \in \mathbb{R}$, then the solution of the equation (21)

$${}_0^{CF}D_{\varpi}^{\psi(\varpi)}(\Theta(\varpi)) = Y(\zeta, \Theta(\zeta)), \varpi \in [0, \mu], 0 < \psi(\varpi) \leq 1, \quad (24)$$

$$\Theta(0) = \Theta_0, \Theta_0 \in \mathbb{R} \quad (25)$$

is given as

$$\Theta(\varpi) = \Theta(0) + {}_0^{CF}J_{\varpi}^{\psi(\varpi)}(Y(\varpi, \Theta(\varpi))), \quad (26)$$

Proof Using initial conditions (18), we can easily prove the Theorem 2 (see [5–11]).

5.1 Existence and uniqueness results

This section devoted to develop the new conditions for existence and uniqueness of the solutions for the Eq. (21) under some assumptions. We know that the conditions for the Eq. (21) is similar to the proposed FVFDE COVID-19 model (17)–(18).

Let $\psi^* = \min_{\varpi \in [0, \mu]} \{\psi(\varpi), \varpi \in [0, \mu]\}$ and $\psi^{**} = \max_{\varpi \in [0, \mu]} \{\psi(\varpi), \varpi \in [0, \mu]\}$ be the minimum and maximum value of the variable fractional order $\psi(\varpi)$ on $[0, \mu]$.

First, we present the following operator, defined as:

The following hypothesis are helpful to the proof of our existence and uniqueness results:

[$\mathcal{O}_1$] Assume the continuous Θ_1, Θ_2 , there exists a constant Ξ , with

$$\aleph(Y(\varpi, \Theta_1(\varpi)), Y(\varpi, \Theta_2(\varpi))) \leq \Xi \aleph(\Theta_1, \Theta_2).$$

Theorem 3 Under the hypothesis of [$\mathcal{O}_1$], the operator (27) satisfies the Lipschitz condition provided that

$$\Xi \times \left[\frac{(1 - \psi^*)}{\mathbb{E}(\psi^*)} + \frac{\psi^* \times \mu}{\mathbb{E}(\psi^*)} \right] < 1. \quad (28)$$

Proof To claim that the operator (27) that is Δ satisfies Lipschitz condition. Let $\Theta_1, \Theta_2 \in \mathbb{R}$, we have

$$\begin{aligned}
& \aleph(\Delta(\varpi, \Theta_1(\varpi)), \Delta(\varpi, \Theta_2(\varpi))) \\
&= \aleph\left(\frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta_1(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta_1(\zeta)) d\zeta, \right. \\
&\quad \left. \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta_2(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta_2(\zeta)) d\zeta\right) \\
&\leq \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \left[\aleph\left(\Upsilon(\varpi, \Theta_1(\varpi)), \Upsilon(\varpi, \Theta_2(\varpi))\right)\right] \\
&\quad \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \left[\int_0^\varpi \aleph\left(\Upsilon(\varpi, \Theta_1(\varpi)), \Upsilon(\varpi, \Theta_2(\varpi))\right) d\zeta\right] \\
&\leq \frac{\Xi(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \left[\aleph\left(\Theta_1(\varpi), \Theta_2(\varpi)\right)\right] \\
&\quad \oplus \frac{\Xi\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \left[\int_0^\varpi \aleph\left(\Theta_1(\varpi), \Theta_2(\varpi)\right) d\zeta\right] \\
&\leq \left[\frac{\Xi(1-\psi^*)}{\mathbb{E}(\psi^*)} + \frac{\Xi\psi^* \times \mu}{\mathbb{E}(\psi^*)}\right] \aleph\left(\Theta_1(\varpi), \Theta_2(\varpi)\right) \\
&\leq \Xi \times \left[\frac{(1-\psi^*)}{\mathbb{E}(\psi^*)} + \frac{\psi^* \times \mu}{\mathbb{E}(\psi^*)}\right] \aleph\left(\Theta_1(\varpi), \Theta_2(\varpi)\right).
\end{aligned} \tag{29}$$

From the inequality (28), the operator Δ in the Eq. (27), satisfies the Lipschitz condition. Hence the proof is completed.

Using the solution of the Eq. (22), given by the Eq. (26) in the Theorem 2, we assume the following recursive formula describe as

Theorem 4 Under the hypothesis of $[\mathcal{O}_1]$, the fractional integral Eq. (22) has a solution provided that the following restriction holds true:

$$F = \left[\frac{\Xi(1-\psi^*)}{\mathbb{E}(\psi^*)} + \frac{\Xi\psi^* \times \mu}{\mathbb{E}(\psi^*)}\right] < 1. \tag{31}$$

$$\Theta_q(\varpi) = \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta_q(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta_q(\zeta)) d\zeta. \tag{30}$$

Proof We define the sequence $\aleph\left(\Theta_q(\varpi), \Theta(\varpi)\right)$. Then, by using the Eq. (30), we have the following calculation:

$$\begin{aligned}
& \aleph \left(\Theta_q(\varpi), \Theta(\varpi) \right) \\
&= \aleph \left(\frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta_q(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta_q(\zeta)) d\zeta, \right. \\
&\quad \left. \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta(\zeta)) d\zeta \right) \\
&\leq \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \left[\aleph \left(\Upsilon(\varpi, \Theta_q(\varpi)), \Upsilon(\varpi, \Theta(\varpi)) \right) \right] \\
&\quad \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \left[\int_0^\varpi \aleph \left(\Upsilon(\varpi, \Theta_q(\varpi)), \Upsilon(\varpi, \Theta(\varpi)) \right) \right] \\
&\leq \frac{\Xi(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \left[\aleph \left(\Theta_q(\varpi), \Theta(\varpi) \right) \right] \\
&\quad \oplus \frac{\Xi\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \left[\int_0^\varpi \aleph \left(\Theta_q(\varpi), \Theta(\varpi) \right) \right] \\
&\leq \left[\frac{\Xi(1-\psi^*)}{\mathbb{E}(\psi^*)} + \frac{\Xi\psi^* \times \mu}{\mathbb{E}(\psi^*)} \right] \aleph \left(\Theta_q(\varpi), \Theta(\varpi) \right) \\
&\leq_F \aleph \left(\Theta_q(\varpi), \Theta(\varpi) \right),
\end{aligned} \tag{32}$$

where $F = \Xi \times \left[\frac{(1-\psi^*)}{\mathbb{E}(\psi^*)} + \frac{\psi^* \times \mu}{\mathbb{E}(\psi^*)} \right] < 1$.

Hence, the sequence $\aleph \left(\Theta_q(\varpi), \Theta(\varpi) \right) \rightarrow 0$, as $q \rightarrow \infty$.

Now, we prove the uniqueness of the solution for the Eq. (22).

Theorem 5 The CF-FVFIE (22)–(18) has a unique solution provided that

$$\left[\frac{\Xi(1-\psi^*)}{\mathbb{E}(\psi^*)} + \frac{\Xi\psi^* \times \mu}{\mathbb{E}(\psi^*)} \right] < 1 \tag{33}$$

holds true.

Proof Consider there exists another solution that is $\Theta^*(\varphi)$ for the CF-FVFIE (22)–(18), satisfying

$$\Theta^*(\varpi) = \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta^*(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta^*(\zeta)) d\zeta. \tag{34}$$

Now, let

$$\begin{aligned}
& \aleph \left(\Theta^*(\varpi), \Theta(\varpi) \right) \\
&= \aleph \left(\frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta^*(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta^*(\zeta)) d\zeta, \right. \\
&\quad \left. \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta(\zeta)) d\zeta \right) \\
&\leq \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \left[\aleph \left(\Upsilon(\varpi, \Theta^*(\varpi)), \Upsilon(\varpi, \Theta(\varpi)) \right) \right] \\
&\quad \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \left[\int_0^\varpi \aleph \left(\Upsilon(\varpi, \Theta^*(\varpi)), \Upsilon(\varpi, \Theta(\varpi)) \right) \right] \\
&\leq \frac{\Xi(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \left[\aleph \left(\Theta^*(\varpi), \Theta(\varpi) \right) \right] \\
&\quad \oplus \frac{\Xi\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \left[\int_0^\varpi \aleph \left(\Theta^*(\varpi), \Theta(\varpi) \right) \right] \\
&\leq \Xi \times \left[\frac{(1-\psi^*)}{\mathbb{E}(\psi^*)} + \frac{\psi^* \times \mu}{\mathbb{E}(\psi^*)} \right] \times \aleph \left(\Theta^*(\varpi), \Theta(\varpi) \right).
\end{aligned} \tag{35}$$

Further, implies that

$$\Xi \times \left[\frac{(1-\psi^*)}{\mathbb{E}(\psi^*)} + \frac{\psi^* \times \mu}{\mathbb{E}(\psi^*)} \right] \times \aleph \left(\Theta^*(\varpi), \Theta(\varpi) \right) \geq 0. \tag{36}$$

From the condition (33), the relation in the Eq. (36) is true and provided that $\aleph \left(\Theta^*(\varpi), \Theta(\varpi) \right) = 0$, which next

5.2 Stability analysis

This section is presents the stability analysis for the CF-FVFIE (22)–(18) using the famous Hyers-Ulam type stability.

Definition 13 The CF-FVFIE (22) is known as Hyers-Ulam stable if there exists constant $\mathcal{A} > 0$ satisfying.

For every $\mathcal{B}$, for

$$\aleph \left(\Theta(\varpi), \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta(\zeta)) d\zeta \right), \tag{37}$$

there exists $\Theta^{**}(\varpi)$, satisfying

$$\Theta^{**}(\varpi) = \frac{(1-\psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \Upsilon(\varpi, \Theta^{**}(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^\varpi \Upsilon(\zeta, \Theta^{**}(\zeta)) d\zeta, \tag{38}$$

implies that $\Theta^*(\varpi) = \Theta(\varpi)$. Overall, the CF-FVFIE (22)–(18) has a unique solution. This implies that the proposed FVFDE COVID-19 model has unique solution. Hence proof the Theorem 5.

such that

$$\aleph(\Theta^{**}(\varpi), \Theta(\varpi)) \leq \mathcal{A}\mathcal{B}. \tag{39}$$

Theorem 6 Using the Theorem 5, the considered CF-FVFIE (22)–(18) is Hyers-Ulam stable.

Proof From the Theorem 5, the considered CF-FVFIE (22)–(18) has a unique solutions called $\Theta(\varpi)$. Consider an approximate solution say $\Theta^{**}(\varpi)$ for the CF-FVFIE (22)–(18) satisfying the Eq. (38). Then, we have

$$\Theta^{**}(\varpi) = \frac{(1 - \psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot Y(\varpi, \Theta^{**}(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^{\varpi} Y(\zeta, \Theta^{**}(\zeta)) d\zeta, \quad (40)$$

Now, let us consider

$$\begin{aligned} & \aleph \left(\Theta^{**}(\varpi), \Theta(\varpi) \right) \\ &= \aleph \left(\frac{(1 - \psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot Y(\varpi, \Theta^{**}(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^{\varpi} Y(\zeta, \Theta^{**}(\zeta)) d\zeta, \right. \\ & \quad \left. \frac{(1 - \psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot Y(\varpi, \Theta(\varpi)) \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \int_0^{\varpi} Y(\zeta, \Theta(\zeta)) d\zeta \right) \\ &\leq \frac{(1 - \psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \left[\aleph \left(Y(\varpi, \Theta^{**}(\varpi)), Y(\varpi, \Theta(\varpi)) \right) \right] \\ &\quad \oplus \frac{\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \left[\int_0^{\varpi} \aleph \left(Y(\zeta, \Theta^{**}(\zeta)), Y(\zeta, \Theta(\zeta)) \right) d\zeta \right] \\ &\leq \frac{\Xi(1 - \psi(\varpi))}{\mathbb{E}(\psi(\varpi))} \odot \left[\aleph \left(\Theta^{**}(\varpi), \Theta(\varpi) \right) \right] \\ &\quad \oplus \frac{\Xi\psi(\varpi)}{\mathbb{E}(\psi(\varpi))} \left[\int_0^{\varpi} \aleph \left(\Theta^{**}(\zeta), \Theta(\zeta) \right) d\zeta \right] \\ &\leq \Xi \times \left[\frac{(1 - \psi^*)}{\mathbb{E}(\psi^*)} + \frac{\psi^* \times \mu}{\mathbb{E}(\psi^*)} \right] \times \aleph \left(\Theta^{**}(\varpi), \Theta(\varpi) \right). \end{aligned} \quad (41)$$

This implies that

$$|\Theta^{**}(\varpi) - \Theta(\varpi)| \leq \mathcal{AB}, \quad (42)$$

where $\mathcal{A} = \Xi \times \left[\frac{(1 - \psi^*)}{\mathbb{E}(\psi^*)} + \frac{\psi^* \times \mu}{\mathbb{E}(\psi^*)} \right]$ and $\mathcal{B} = \aleph \left(\Theta^{**}(\varpi), \Theta(\varpi) \right)$. From then Eq. (42), the CF-FVFIE (22)–(18) is Hyers-Ulam stable. This implies that the proposed FVFDE COVID-19 model is Hyers-Ulam stable.

6 Conclusion

This study is devoted to a novel coronavirus infection (COVID-19) for its developed model. Here, we choose the recent COVID-19 model to discuss the important aspects of the considered model using interesting and important theories such as fuzzy theory and variable fractional calculus. In the literature, there are no study developed on the fuzzy variable fractional equation for the COVID-19 model.

Keeping in mind the importance of this situation. The following points have given ideals of the study on the significance and how it is helpful for researchers on the COVID-19 community:

1. First, we transform the proposed COVID-19 model into a fuzzy variable fractional differential equation, and the

variable fractional order introduce by the Caputo-Fabrizio derivative in Caputo sense.

2. Lipschitz condition is used to prove the main results: existence and the uniqueness for the solution of the fuzzy variable fractional differential equation (FVFDE) COVID-19 model via $\aleph$ operator as per supremum norm definition.
3. To prove the FVFDE COVID-19 model's stability, we used the most used the famous definitions, namely, the stability of the Hyers-Ulam type.

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