Constructive Approximation
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CONSTRUCTIVE APPROXIMATION

On the Characteristic Polynomial of the Eigenvalue Moduli of Random Normal Matrices

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Abstract

We study the characteristic polynomial $p_n(x) = \prod_{j=1}^{n}(|z_j| - x)$ where the z_j are drawn from the Mittag–Leffler ensemble, i.e. a two-dimensional determinantal point process which generalizes the Ginibre point process. We obtain precise large n asymptotics for the moment generating function $\mathbb{E}[e^{\frac{u}{\pi}\,\mathrm{Im}\,\ln p_n(r)}e^{a\,\mathrm{Re}\,\ln p_n(r)}]$, in the case where r is in the bulk, $u \in \mathbb{R}$ and $a \in \mathbb{N}$. This expectation involves an $n \times n$ determinant whose weight is supported on the whole complex plane, is rotation-invariant, and has both jump- and root-type singularities along the circle centered at 0 of radius r. This “circular" root-type singularity differs from earlier works on Fisher–Hartwig singularities, and surprisingly yields a new kind of ingredient in the asymptotics, the so-called *associated Hermite polynomials*.

Keywords Jump- and root-type singularities along circles · Moment generating functions · Random matrix theory · Asymptotic analysis

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1 Introduction and Statement of Results

The Mittag–Leffler ensemble with parameters $b > 0$ and $\alpha > -1$ is the joint probability distribution

$$\frac{1}{n!Z_n}\prod_{1\leq j<k\leq n}|z_k-z_j|^2\prod_{j=1}^{n}|z_j|^{2\alpha}e^{-n|z_j|^{2b}}d^2z_j, \qquad z_1,\ldots,z_n\in\mathbb{C}, \tag{1.1}$$

where Z_n is the normalization constant. This determinantal point process can be realized as the eigenvalues of a random normal matrix M with distribution proportional to $|\det(M)|^{2\alpha}e^{-n\,\mathrm{tr}((MM^*)^b)}dM$ [54]. The special case $(b,\alpha)=(1,0)$ also corresponds to the eigenvalue distribution of a Ginibre matrix [15, 43], i.e. an $n\times n$ matrix with independent complex Gaussian entries with mean 0 and variance $\frac{1}{n}$.

Consider the characteristic polynomial $p_n(x)=\prod_{j=1}^{n}(|z_j|-x)$ of the process of the moduli $\{|z_j|\}_{j=1}^{n}$. The main result of this work is a precise asymptotic formula as $n\to+\infty$ for

$$\mathbb{E}\Big[e^{\frac{u}{\pi}\,\mathrm{Im}\,\ln p_n(r)}e^{a\,\mathrm{Re}\,\ln p_n(r)}\Big], \tag{1.2}$$

where $u\in\mathbb{R}$, $a\in\mathbb{N}:=\{0,1,\ldots\}$, $r\in(0,b^{-\frac{1}{2b}})$ are fixed, and $\ln p_n(r):=\ln|p_n(r)|+\pi i\,\#\{z_j\,:\,|z_j|<r\}$. The macroscopic large n behavior of the $|z_j|$ is described by the probability measure $d\mu(y)=2b^2y^{2b-1}dy$, whose support is $[0,b^{-\frac{1}{2b}}]$ [57]; thus the condition that $r\in(0,b^{-\frac{1}{2b}})$ is fixed means that we focus on "the bulk regime". By definition, the expectation (1.2) is equal to D_n/Z_n, where

$$D_n:=\frac{1}{n!}\int_{\mathbb{C}}\cdots\int_{\mathbb{C}}\prod_{1\leq j<k\leq n}|z_k-z_j|^2\prod_{j=1}^{n}w(z_j)d^2z_j=\det\left(\int_{\mathbb{C}}z^j\overline{z}^k w(z)d^2z\right)_{j,k=0}^{n-1}$$

and the weight w is given by

$$w(z):=|z|^{2\alpha}e^{-n|z|^{2b}}\omega(|z|), \qquad \omega(x):=|x-r|^a\begin{cases}e^u, & \text{if } x<r,\\ 1, & \text{if } x\geq r.\end{cases}$$

Hence our results can also be seen as large n asymptotics for $n\times n$ determinants whose weight is supported on $\mathbb{C}$, rotation-invariant, has both jump- and root-type singularities along the circle centered at 0 of radius r (which we will call "circular" jump- and root-type singularities), and a "pointwise" root-type singularity at 0.

Since w is rotation-invariant, $\int_{\mathbb{C}}z^j\overline{z}^k w(z)d^2z=0$ for $j\neq k$, and therefore

$$D_n=\prod_{j=0}^{n-1}\int_{\mathbb{C}}|z|^{2j}w(z)d^2z=(2\pi)^n\prod_{j=1}^{n}\int_0^{+\infty}v^{2j-1}w(v)dv. \tag{1.3}$$

Over the past 50 years or so a lot of works have been done on structured determinants with singularities, and we briefly pause here to review the literature. In their pioneering work [38], Fisher and Hartwig made a conjecture for the asymptotics of large Toeplitz determinants when the weight is supported on the unit circle and has root- and jump-type singularities—such singularities are now called Fisher–Hartwig singularities. Many authors have contributed in proving this conjecture for certain parameter ranges, among which Lenard [53], Widom [65], Basor [7], Böttcher and Silbermann [13], and Ehrhardt [33]. A counterexample to the Fisher–Hartwig conjecture was found by Basor and Tracy in [10], and the corrected conjecture was solved for general values of the parameters by Deift, Its and Krasovsky [31]. The study of these singular determinants was motivated mainly from questions arising in the Ising model and impenetrable bosons, see [9, 32] for more historical background. In recent years, these determinants have also attracted considerable attention in the random matrix community. One reason for that is the well-known work [45] of Keating and Snaith, where numerical evidences were found of links between the characteristic polynomials of unitary and Hermitian random matrices and the zeros of the Riemann zeta function on the critical line. Expectations of powers of the absolute value of the characteristic polynomial of the Gaussian Unitary Ensemble, which are Hankel determinants with a Gaussian weight on $\mathbb{R}$ and root-type singularities, were investigated in [14] and their asymptotics were obtained by Garoni [42] for integer values of the parameters and by Krasovsky [46] for the general case. This result was then generalized by Berestycki, Webb and Wong [11] for one-cut regular ensembles. In a different direction, Its and Krasovsky in [44] obtained asymptotics of Hankel determinants with a jump-type singularity and a Gaussian weight. Such determinants provide information about the imaginary part of the log-characteristic polynomial of the Gaussian Unitary Ensemble. The results [11, 44] have been generalized in [19] for general Fisher–Hartwig singularities and one-cut regular ensembles. The case of one-cut regular ensembles with hard edges was then treated in [23], and the multi-cut case in [24]. Strong results on Toeplitz determinants with merging Fisher–Hartwig singularities are also available in the literature [27, 35]; these results have been useful to prove a conjecture of [41] on “the moments of the moments" of the characteristic polynomial of random unitary matrices, and [27] has also been used by Webb in [63] to establish a connection between random matrix theory and Gaussian multiplicative chaos. There exists also a vast literature on other structured determinants with Fisher–Hartwig singularities, see e.g. [28, 29] for Fredholm determinants, [8, 26, 39] for Toeplitz+Hankel determinants, and [22] for a biorthogonal generalization of Hankel determinants.

The literature on determinants associated with a singular weight supported on $\mathbb{C}$ is more limited. For $a = 0$, (1.2) is the moment generating function of the disk counting function

$$\mathbb{E}\Big[e^{\frac{u}{\pi}\,\mathrm{Im}\ln p_n(r)}\Big] = \mathbb{E}\Big[e^{u\,\#\{z_j : |z_j| < r\}}\Big], \tag{1.4}$$

and in this case w is discontinuous along a circle but has no circular root-type singularity. Counting statistics of two-dimensional point processes have attracted a lot of interest in recent years [1, 18, 20, 25, 34, 37, 47–49, 59, 60]. The first two terms in the large n asymptotics of (1.4) were derived in [18, 48][1]. More precise asymptotics, including the third term of order 1, were obtained in [37] for the Ginibre ensemble (i.e. $(b,\alpha)=(1,0)$),[2] and the asymptotics of (1.4) including the fourth term of order $n^{-\frac{1}{2}}$ were then obtained in [20] for general $b>0$ and $\alpha>-1$. Several works on determinants with pointwise root-type singularities in dimension two are also available in the literature. In [5, 6, 12, 17, 50–52], the orthogonal polynomials for the planar Gaussian weight perturbed with a finite number of pointwise root-type singularities have been studied, see also [3] where microscopic properties of the associated point process have been analyzed. Building on [5, 50], Webb and Wong in [64] obtained a precise asymptotic formula for $\mathbb{E}[e^{a\,\mathrm{Re}\ln q_n(r)}]$ where $a\in\mathbb{C}$ is fixed, $\mathrm{Re}\, a>-2$, $r<1$ and q_n is the characteristic polynomial of a Ginibre matrix, i.e. $q_n(x)=\prod_{j=1}^{n}(z_j-x)$ and the z_j are drawn from (1.1) with $(b,\alpha)=(1,0)$. This expectation involves a determinant with a single pointwise root-type singularity in the bulk. The case where r is close to 1, which corresponds to the edge regime, was then investigated by Deaño and Simm in [30]. Determinants with two merging planar pointwise root-type singularities were also considered in [30], and the asymptotics were found to involve some Painlevé transcendents. The regime where a is proportional to n reveals a topological phase transition and was studied in [17]. (We also mention that determinants with regular weights supported on $\mathbb{C}$ have recently been studied in [2, 16].)

Determinants with circular root-type singularities have not been considered before to our knowledge. A main difficulty in the analysis of pointwise root-type singularities in dimension two stems from the fact that they break the rotation-invariance of the weight (unless of course if they are located at 0). The circular root-type singularities preserve the rotation-invariance of the weight, which makes them simpler to analyze in this respect, but they also pose a series of new challenges, which we discuss at the end of this section, and which we have been able to overcome only for integer values of a. Interestingly, these circular root-type singularities also produce some *associated Hermite polynomials* (see below for the definition) in the asymptotics. This came as a surprise to us and appears to be completely new. There are of course many *exact* formulas in random matrix theory which involve the Hermite polynomials, but we are not aware of an earlier work where these polynomials show up explicitly in the *asymptotics* of large determinants, let alone the associated Hermite polynomials (see however [36] where Legendre polynomials appear in the asymptotics of the sine-kernel determinant in a transition regime). For comparison, pointwise root-type singularities typically produce other kinds of ingredients in the asymptotics, such as Barnes' G-function (as was discovered by Basor [7] in dimension one and by Webb and Wong [64] in dimension two), and circular jump-type singularities involve the error function. We also mention that ensembles with circular root-type singularities have been studied in [58, 67], and ensembles with "elliptic" root-type singularities in [55]. In [55, 58, 67], the singularities are located at the hard edge and the focus was on the leading

[1] [18] considers counting statistics on more general domains (not only centered disks) for a class of determinantal processes on a Kähler manifold (which includes Ginibre), and [48] considers general "one-cut" rotation-invariant potentials (including Mittag–Leffler).

[2] Some generalizations of the Ginibre point process (different from the Mittag–Leffler ensemble) and some hyperbolic models have also been considered in [37].

order behavior of the kernel; in particular, the (associated) Hermite polynomials do not show up in these works.

The ν-th *associated* Hermite polynomials $\{\mathrm{He}_k^{(\nu)} : k = 0, 1, \dots\}$ are defined recursively by

$$\begin{cases} \mathrm{He}_{k+1}^{(\nu)}(x) = x\,\mathrm{He}_k^{(\nu)}(x) - (k+\nu)\mathrm{He}_{k-1}^{(\nu)}(x), & k \geq 1, \\ \mathrm{He}_0^{(\nu)}(x) = 1, \quad \mathrm{He}_1^{(\nu)}(x) = x, \end{cases} \tag{1.5}$$

and satisfy the orthogonality relations [4]

$$\int_{-\infty}^{+\infty} \mathrm{He}_k^{(\nu)}(x)\mathrm{He}_\ell^{(\nu)}(x)\frac{dx}{|D_{-\nu}(ix)|^2} = \sqrt{2\pi}\,(k+\nu)!\,\delta_{k\ell}, \qquad k, \ell = 0, 1, \dots$$

where $D_{-\nu}$ is the parabolic cylinder function (see e.g. [56, Chapter 12]). These polynomials are explicitly given by

$$\mathrm{He}_k^{(\nu)}(x) = \begin{cases} k! \sum_{\ell=0}^{\lfloor k/2 \rfloor} \frac{(-1)^\ell}{\ell!(k-2\ell)!}\frac{x^{k-2\ell}}{2^\ell} = \left[\frac{d}{dt}\right]^k \left[e^{xt-\frac{t^2}{2}}\right]\Big|_{t=0}, & \text{if } \nu = 0, \\ \sum_{\ell=0}^{\lfloor k/2 \rfloor} \frac{(-1)^\ell}{(k-2\ell)!}\left(\sum_{j=0}^{\ell} \frac{(k-j)!(\nu-1+j)!}{j!(\ell-j)!(\nu-1)!2^{\ell-j}}\right)x^{k-2\ell}, & \text{if } \nu \geq 1, \end{cases} \tag{1.6}$$

see [66, eq. (4.12)]. For $\nu = 0$, they reduce to the standard Hermite polynomials[3], i.e. $\mathrm{He}_k^{(0)}(x) = \mathrm{He}_k(x)$ for all $k \in \mathbb{N}$. We refer to [62] for basic properties of general associated polynomials, and to [66] for a focus on the Hermite case.

Only the polynomials $\{\mathrm{He}_k, \mathrm{He}_k^{(1)} : k = 0, 1, \dots\}$, corresponding to $\nu = 0$ and $\nu = 1$, will appear in our asymptotic formula. Our results can be presented in a unified way if we formally define $\mathrm{He}_k, \mathrm{He}_k^{(1)}$ for the first few negative k as follows:

$$\begin{aligned} &\mathrm{He}_{-1}(x) := 0, && \mathrm{He}_{-2}(x) := 1, && \mathrm{He}_{-3}(x) := -\frac{x}{2}, \\ &\mathrm{He}_{-1}^{(1)}(x) := 0, && \left[k\mathrm{He}_{k-2}^{(1)}(x)\right]_{k=0} := -1, && \left[k\mathrm{He}_{k-3}^{(1)}(x)\right]_{k=0} := x, \\ &\left[k\mathrm{He}_{k-4}^{(1)}(x)\right]_{k=0} := -\frac{x^2+1}{2}. \end{aligned} \tag{1.7}$$

These definitions are consistent with the recurrence (1.5). For general $a \in \mathbb{N}$, we define

$$p_{0,a}(x) := \frac{1}{i^a}\mathrm{He}_a(ix) = \sum_{s=0}^{\lfloor a/2 \rfloor} \frac{a!}{s!(a-2s)!}\frac{x^{a-2s}}{2^s}, \tag{1.8}$$

[3] There are two commonly used Hermite polynomials in the literature, denoted He_k and H_k, and which are related by $H_k(x) = 2^{\frac{k}{2}}\mathrm{He}_n(\sqrt{2}\,x)$. For us it is more convenient to work with He_k.

$$q_{0,a}(x) := \frac{1}{i^{a-1}}\mathrm{He}^{(1)}_{a-1}(ix) = \sum_{s=0}^{\lfloor (a-1)/2\rfloor}\left(\sum_{j=0}^{s}\frac{(a-1-j)!\,2^j}{(s-j)!}\right)\frac{1}{(a-1-2s)!}\frac{x^{a-1-2s}}{2^s}, \quad (1.9)$$

$$p_{1,a}(x) := -\frac{a}{2}p_{0,a+1}(x) - ab\Big(p_{0,a+1}(x) - (3a-1)\,p_{0,a-1}(x) + \frac{5}{3}(a-1)(a-2)\,p_{0,a-3}(x)\Big), \quad (1.10)$$

$$q_{1,a}(x) := -\frac{a}{2}q_{0,a+1}(x) - b\Big(aq_{0,a+1}(x) - (3a-1)[a\,q_{0,a-1}(x)] + \frac{5}{3}[a(a-1)(a-2)\,q_{0,a-3}(x)]\Big), \quad (1.11)$$

where the brackets in (1.11) emphasize that for the first values of a, one needs to use (1.7), namely

$$[aq_{0,a-1}(x)] := \begin{cases} 1, & \text{if } a=0,\\ aq_{0,a-1}(x), & \text{if } a\geq 1,\end{cases}$$
$$[a(a-1)(a-2)q_{0,a-3}(x)] := \begin{cases} x^2-1, & \text{if } a=0,\\ -x, & \text{if } a=1,\\ 2, & \text{if } a=2,\\ a(a-1)(a-2)q_{0,a-3}(x), & \text{if } a\geq 3.\end{cases} \quad (1.12)$$

To be concrete, the first polynomials are given by

$$\begin{aligned}
\big(p_{0,a}(x)\big)_{a=0}^4 &= \big(p_{0,0}(x), p_{0,1}(x), p_{0,2}(x), p_{0,3}(x), p_{0,4}(x)\big)\\
&= \Big(1,\ x,\ x^2+1,\ x^3+3x,\ x^4+6x^2+3\Big),\\
\big(q_{0,a}(x)\big)_{a=0}^4 &= \Big(0,\ 1,\ x,\ x^2+2,\ x^3+5x\Big),\\
\big(p_{1,a}(x)\big)_{a=0}^4 &= -\big(\frac{a}{2}p_{0,a+1}(x)\big)_{a=0}^4\\
&\quad + b\,\Big(0, -x^2+1, -2x^3+4x, -3x^4+6x^2+5, -4x^5+4x^3+32x\Big),\\
\big(q_{1,a}(x)\big)_{a=0}^4 &= -\big(\frac{a}{2}q_{0,a+1}(x)\big)_{a=0}^4 + b\,\Big(-\frac{5}{3}x^2+\frac{2}{3}, \frac{2}{3}x, -2x^2+\frac{8}{3},\\
&\quad -3x^3+9x, -4x^4+8x^2+16\Big).
\end{aligned}$$

The large n asymptotics of $\mathbb{E}\big[e^{\frac{u}{\pi}\,\mathrm{Im}\,\ln p_n(r)}e^{a\,\mathrm{Re}\,\ln p_n(r)}\big]$ are naturally described in terms of the two functions

$$\mathcal{G}_0(y;u,a) := p_{0,a}(-\sqrt{2}y)\Big((-1)^a + \frac{e^u-(-1)^a}{2}\mathrm{erfc}(y)\Big) + q_{0,a}(-\sqrt{2}y)(e^u-(-1)^a)\frac{e^{-y^2}}{\sqrt{2\pi}}, \quad (1.13)$$

$$\begin{aligned}\mathcal{G}_1(y;u,a) :=& p_{1,a}(-\sqrt{2}y)\bigg((-1)^a + \frac{e^u-(-1)^a}{2}\text{erfc}(y)\bigg)\\ &+ q_{1,a}(-\sqrt{2}y)(e^u-(-1)^a)\frac{e^{-y^2}}{\sqrt{2\pi}},\end{aligned} \tag{1.14}$$

where $y \in \mathbb{R}$, $u \in \mathbb{R}$, $a \in \mathbb{N}$, and erfc is the complementary error function

$$\text{erfc}(y) = \frac{2}{\sqrt{\pi}}\int_y^{\infty} e^{-x^2}dx. \tag{1.15}$$

In the statement of our theorem, $\mathcal{G}_0$ appears inside a logarithm and in a denominator. It turns out that $\mathcal{G}_0(y;u,a) > 0$ for all $y \in \mathbb{R}$, $u \in \mathbb{R}$ and $a \in \mathbb{N}$. This fact is not obvious so we defer the proof to Sect. 4, see Lemma 4.9.

Theorem 1.1 *Let $b > 0$, $\alpha > -1$, $r \in (0, b^{-\frac{1}{2b}})$, $u \in \mathbb{R}$ and $a \in \mathbb{N}$ be fixed parameters. As $n \to +\infty$,*

$$\mathbb{E}\Big[e^{\frac{u}{\pi}\text{Im}\ln p_n(r)}e^{a\,\text{Re}\ln p_n(r)}\Big] = \exp\bigg(C_1 n + C_2\sqrt{n} + C_3 + \mathcal{O}\Big(n^{-\frac{1}{2b}} + (\ln n)^{\frac{5}{8}}n^{-\frac{1}{8}}\Big)\bigg), \tag{1.16}$$

where

$$\begin{aligned}C_1 &= \int_0^r \Big(u + a\ln(r-y)\Big)d\mu(y) + \int_r^{b^{-\frac{1}{2b}}} a\ln(y-r)\,d\mu(y),\\ C_2 &= \sqrt{2}\,br^b\int_{-\infty}^{+\infty}\Big(\ln\mathcal{G}_0(y;u,a) - a\ln(\sqrt{2}|y|) - u\chi_{(-\infty,0)}(y)\Big)\,dy,\\ C_3 &= -\Big(\frac{1}{2}+\alpha\Big)u + \frac{a(1-a)}{4(1-(br^{2b})^{\frac{1}{2b}})} + \frac{a}{4}(2+a-2b+4\alpha)\ln\Big((br^{2b})^{-\frac{1}{2b}}-1\Big)\\ &\quad+\int_{-\infty}^{+\infty}\bigg\{\frac{1}{\sqrt{2}}\frac{\mathcal{G}_1(y;u,a)}{\mathcal{G}_0(y;u,a)} + 4by\Big(\ln(\mathcal{G}_0(y;u,a)) - u\chi_{(-\infty,0)}(y)\Big)\\ &\quad-\frac{a}{2}y\Big(1+2b+8b\ln(\sqrt{2}|y|)\Big) + \frac{(2ab-a^2)y}{4(1+y^2)}\bigg\}\,dy,\end{aligned}$$

$d\mu(y) = 2b^2y^{2b-1}dy$, and $\chi_{(-\infty,0)}(y) = 0$ for $y \geq 0$ and $\chi_{(-\infty,0)}(y) = 1$ for $y < 0$.

Remark 1.2 For $a = 0$, the next term in (1.16) is of order $n^{-\frac{1}{2}}$ and was obtained explicitly in [20]. For $a \geq 1$, numerical simulations suggest that the $\mathcal{O}$-term is in fact of order $n^{-\frac{1}{2}} + n^{-\frac{1}{2b}}$. This also suggests that our estimate in (1.16) for the $\mathcal{O}$-term is optimal for $b > 4$ and $a \geq 1$.

Outline of the proof of Theorem 1.1

Let $\mathcal{E}_n := \mathbb{E}\big[e^{\frac{u}{\pi}\operatorname{Im}\ln p_n(r)}e^{a\operatorname{Re}\ln p_n(r)}\big]$. Our starting point is the following exact formula

$$\ln \mathcal{E}_n = \sum_{j=1}^{n} \ln\Bigg(\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \left[1+((-1)^a e^u-1)\frac{\gamma(\frac{2j+2\alpha+k}{2b},nr^{2b})}{\Gamma(\frac{2j+2\alpha+k}{2b})}\right]\Bigg), \tag{1.17}$$

where $\gamma(\tilde{a},z)$ is the incomplete gamma function

$$\gamma(\tilde{a},z)=\int_0^z t^{\tilde{a}-1}e^{-t}dt.$$

Formula (1.17) can be derived using the facts that (1.1) is determinantal and that w is rotation-invariant, see Lemma 2.1 below. For fixed j and $a \geq 1$, it is easy to see that the summand in (1.17) contains a term proportional to $n^{-\frac{1}{2b}}$ in its large n asymptotics. This already explains why our estimate for the error term in (1.16) contains $n^{-\frac{1}{2b}}$.

To obtain precise asymptotics for $\mathcal{E}_n$, up to and including the term C_3 of order 1, we must take into account *each* of the n terms in the sum (1.17) (they all contribute). As can be seen from (1.17), this means that we need precise uniform asymptotics for $\gamma(\tilde{a},z)$ as both $z \to +\infty$ and $\tilde{a} \to +\infty$ at various different relative speeds. Fortunately, these asymptotics are available in the literature [61]. Following the approach of [40] (which was further developed in [20, 21]), we will split the sum (1.17) in several parts,

$$\ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3,$$

where S_ℓ, $\ell = 0,1,2,3$ are given in (2.5–(2.8). There is a critical transition in the large $\tilde{a}$ asymptotics of $\gamma(\tilde{a},z)$ when $z \to +\infty$ is such that $\lambda = \frac{z}{\tilde{a}} \approx 1$. The sum S_2 is the hardest one and precisely corresponds to this critical transition; it requires a "local analysis" involving the j-terms in (1.17) for which $\frac{bnr^{2b}}{j} \approx 1$.

We found that, quite surprisingly, circular root-type singularities are significantly more involved to analyze than circular jump-type singularities. Let us highlight some of the reasons for that:

- The "global analysis" needed for S_0, S_1 and S_3 requires some precise Riemann sum approximations for functions with singularities. For comparison, the analogue of S_0, S_1 and S_3 in [20] in the case of pure circular jump-type singularities are straightforward to analyze, because the corresponding Riemann sum approximations only involve constant functions.
- Huge cancellations occur in the "local analysis" of S_2. In fact, to obtain C_3, we need to expand up to the $(a+2)$-th order the summand of the k-sum in (1.17). This is because, curiously, the first a terms in the expansion cancel perfectly after summing over k. To treat the general case $a \in \mathbb{N}$, this means that we need to

expand various quantities to all orders. An important technical obstacle is that the coefficients in these various expansions are not always readily available in an explicit form; sometimes they can only be found recursively and involve heavy combinatorics, see e.g. Lemma 4.5 and the all-order expansion of γ in Lemma A.2. The analysis of S_2 is in fact the only part in the proof where solving the problem for general $a \in \mathbb{N}$ is clearly harder than solving the problem for a finite number of values of a, say $a \in \{0, 1, 2, 3, 4\}$. This is also the only place in the proof where the (associated) Hermite polynomials arise, see Lemmas 4.6 and 4.7.

Remark 1.3 For non-integer values of a, formula (1.17) does not hold and the connection with the incomplete gamma function is lost (and therefore the strong results from [61] cannot be used anymore). This is the main reason as to why we decided to restrict ourselves to $a \in \mathbb{N}$ in this work. For $a \notin \mathbb{N}$, the exact expression for $\mathcal{E}_n$ involves hypergeometric functions that generalize the incomplete gamma function. Also, because of the well-known relation $D_k(z) = e^{-\frac{z^2}{4}}\mathrm{He}_k(z)$, $k \in \mathbb{N}$ (see [56, eq 12.7.2]), it is tempting to conjecture that for the general case $a \in (-1, +\infty)$ the large n asymptotics of $\mathcal{E}_n$ involve the parabolic cylinder function. It would be very interesting to figure that out in detail.

Outline of the paper. In Sect. 2, we prove (1.17), define the sums S_j, $j = 0, 1, 2, 3$, and establish many useful lemmas. In Sect. 3, we obtain the large n asymptotics of S_0, S_3 and S_1. The large n asymptotics of S_2 are then obtained in Sect. 4. We finish the proof of Theorem 1.1 in Sect. 5.

2 Preliminaries

This section contains the proof of (1.17) and the definitions of $S_0, \ldots, S_3$. We also establish here various preliminary lemmas that will be used in Sects. 3 and 4.

Lemma 2.1 *Formula (1.17) holds for all $n \in \mathbb{N}_{>0} := \{1, 2, \ldots\}$.*

Proof The partition function Z_n of the Mittag–Leffler ensemble is known to be

$$Z_n = n^{-\frac{n^2}{2b}} n^{-\frac{1+2\alpha}{2b}n} \frac{\pi^n}{b^n} \prod_{j=1}^{n} \Gamma(\tfrac{j+\alpha}{b}), \tag{2.1}$$

see e.g. [20, eq. (1.23)]. Since $\mathcal{E}_n = D_n/Z_n$, it only remains to find a simplified exact expression for D_n. Since $a \in \mathbb{N}$,

$$w(v) = v^{2\alpha}e^{-nv^{2b}} \begin{cases} e^u \displaystyle\sum_{k=0}^{a} \binom{a}{k}(-1)^k v^k r^{a-k}, & \text{if } v < r, \\ \displaystyle\sum_{k=0}^{a} \binom{a}{k}(-1)^{a-k} v^k r^{a-k}, & \text{if} v \geq r, \end{cases}$$

and thus, by (1.3),

$$D_n = n^{-\frac{n^2}{2b}} n^{-\frac{1+2\alpha}{2b}n} \frac{\pi^n}{b^n} \prod_{j=1}^{n} \sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \Big(\Gamma(\tfrac{2j+2\alpha+k}{2b}) + ((-1)^a e^u - 1) \gamma(\tfrac{2j+2\alpha+k}{2b}, nr^{2b}) \Big). \tag{2.2}$$

The claim now follows directly from (2.1), (2.2) and $\mathcal{E}_n = D_n/Z_n$. □

Throughout the paper, c and C denote positive constants which may change within a computation, and ln always denotes the principal branch of the logarithm.

Let M' be a large integer independent of n, let $\epsilon > 0$ be a small constant independent of n, and let $M := n^{\frac{1}{8}}(\ln n)^{-\frac{1}{8}}$. Define

$$j_- := \lceil \tfrac{bnr^{2b}}{1+\epsilon} - \alpha \rceil, \qquad j_+ := \lfloor \tfrac{bnr^{2b}}{1-\epsilon} - \alpha \rfloor. \tag{2.3}$$

We choose ϵ small enough so that

$$\frac{br^{2b}}{1-\epsilon} < \frac{1}{1+\epsilon}.$$

Using (1.17), we divide $\ln \mathcal{E}_n$ into 4 parts

$$\ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3, \tag{2.4}$$

with

$$S_0 = \sum_{j=1}^{M'} \ln \left(\sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \left[1 + ((-1)^a e^u - 1) \frac{\gamma(\frac{2j+2\alpha+k}{2b}, nr^{2b})}{\Gamma(\frac{2j+2\alpha+k}{2b})} \right] \right), \tag{2.5}$$

$$S_1 = \sum_{j=M'+1}^{j_- -1} \ln \left(\sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \left[1 + ((-1)^a e^u - 1) \frac{\gamma(\frac{2j+2\alpha+k}{2b}, nr^{2b})}{\Gamma(\frac{2j+2\alpha+k}{2b})} \right] \right), \tag{2.6}$$

$$S_2 = \sum_{j=j_-}^{j_+} \ln \left(\sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \left[1 + ((-1)^a e^u - 1) \frac{\gamma(\frac{2j+2\alpha+k}{2b}, nr^{2b})}{\Gamma(\frac{2j+2\alpha+k}{2b})} \right] \right), \tag{2.7}$$

$$S_3 = \sum_{j=j_++1}^{n} \ln \left(\sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \left[1 + ((-1)^a e^u - 1) \frac{\gamma(\frac{2j+2\alpha+k}{2b}, nr^{2b})}{\Gamma(\frac{2j+2\alpha+k}{2b})} \right] \right). \tag{2.8}$$

In Sects. 3 and 4, we will analyze these sums in order of increasing difficulty: first S_0, then S_3, then S_1, and finally S_2. The sum S_0 is straightforward to analyze, but S_1, S_2 and S_3 are more involved and require some preparation. This preparation is carried out in the next subsection.

2.1 Useful Lemmas

For $\ell \in \mathbb{N} := \{0, 1, \ldots\}$, let

$$\mathfrak{g}_\ell(x) := \sum_{k=0}^{a} \binom{a}{k} x^k k^\ell. \tag{2.9}$$

If $k = \ell = 0$ in (2.9), then $k^\ell := 1$ so that $\mathfrak{g}_0(x) = (1+x)^a$.

Remark 2.2 The sequence $\{\mathfrak{g}_\ell\}_{\ell=0}^{+\infty}$ satisfies $\mathfrak{g}_{\ell+1}(x) = x\mathfrak{g}_\ell'(x)$, $\ell \in \mathbb{N}$. Solving this recurrence relation using the initial value $\mathfrak{g}_0(x) = (1+x)^a$ yields

$$\mathfrak{g}_\ell(x) = \left[\frac{d}{dt}\right]^\ell \left[(1+e^t)^a\right]\Big|_{t=\ln x},$$

which is an interesting alternative representation of $\mathfrak{g}_\ell$.

The next lemma establishes yet another representation of $\mathfrak{g}_\ell$.

Lemma 2.3 *For $\ell \in \mathbb{N}_{>0}$, we have*

$$\mathfrak{g}_\ell(x) = x(x+1)^{a-\min(\ell,a)} \sum_{j=1}^{\min(\ell,a)} S(\ell, j) \frac{a!}{(a-j)!} x^{j-1}(x+1)^{\min(\ell,a)-j}, \tag{2.10}$$

where $S(\ell, j)$ is the Stirling number of the second kind, i.e. the number of partitions of $\{1, \ldots, \ell\}$ into exactly j nonempty subsets. Furthermore,

$$\mathfrak{g}_\ell(x) = \mathcal{O}(x), \qquad \text{as } x \to 0, \quad \ell \in \mathbb{N}_{>0}, \tag{2.11}$$

$$\mathfrak{g}_\ell(x) = \mathcal{O}(\min\{1, |x+1|^{a-\ell}\}), \qquad \text{as } x \to -1, \quad \ell \in \mathbb{N}. \tag{2.12}$$

Remark 2.4 Since $\mathfrak{g}_0(x) = (1+x)^a$, (2.10) and (2.11) do not hold for $\ell = 0$.

Proof Let $\ell \in \mathbb{N}_{>0}$ be fixed. By [56, eq. 26.8.10], we have

$$k^\ell = \sum_{j=1}^{\ell} (k)_j S(\ell, j), \qquad \text{for all } k \in \mathbb{C}, \tag{2.13}$$

where $(k)_j := k(k-1)(k-2)\ldots(k-j+1)$ is the descending factorial. Substituting (2.13) in (2.9), we obtain

$$\begin{aligned}\mathfrak{g}_\ell(x) &= \sum_{k=0}^{a} \sum_{j=1}^{\min(\ell,k)} \frac{a! S(\ell, j) x^k}{(a-k)!(k-j)!} = \sum_{j=1}^{\min(\ell,a)} \sum_{k=j}^{a} \frac{a! S(\ell, j) x^k}{(a-k)!(k-j)!} \\ &= \sum_{j=1}^{\min(\ell,a)} S(\ell, j) \frac{a!}{(a-j)!} \sum_{k=j}^{a} \binom{a-j}{k-j} x^k\end{aligned}$$

$$= \sum_{j=1}^{\min(\ell,a)} S(\ell,j) \frac{a!}{(a-j)!} x^j (x+1)^{a-j},$$

which is (2.10). The expansions (2.11) and (2.12) for $\ell \geq 1$ directly follows from (2.10), and (2.12) for $\ell = 0$ follows from $\mathfrak{g}_0(x) = (1+x)^a$. □

The sums S_1, S_2 and S_3 naturally involve the functions

$$\gamma_\ell(x) := \begin{cases} \sum_{k=0}^{a} \binom{a}{k} (-r)^{a-k} \left(\frac{x}{b}\right)^{\frac{k}{2b}} k^\ell, & x > br^{2b}, \\ \sum_{k=0}^{a} \binom{a}{k} (-1)^k r^{a-k} \left(\frac{x}{b}\right)^{\frac{k}{2b}} k^\ell, & x \in (0, br^{2b}). \end{cases} \tag{2.14}$$

The next lemma collects some properties of γ_ℓ.

Lemma 2.5 *Let $\ell \in \mathbb{N}$. The function γ_ℓ can be written as*

$$\gamma_\ell(x) = \begin{cases} \left| r - \left(\frac{x}{b}\right)^{\frac{1}{2b}} \right|^a, & \text{if } \ell = 0, \\ \left(\frac{x}{b}\right)^{\frac{1}{2b}} \left| r - \left(\frac{x}{b}\right)^{\frac{1}{2b}} \right|^a \sum_{j=1}^{\min(\ell,a)} \frac{a! S(\ell,j)}{(a-j)!} \left(\frac{x}{b}\right)^{\frac{j-1}{2b}} \left(\left(\frac{x}{b}\right)^{\frac{1}{2b}} - r \right)^{-j}, & \text{if } \ell \geq 1. \end{cases} \tag{2.15}$$

In particular,

$$\gamma_\ell(x) = \mathcal{O}(x^{\frac{1}{2b}}), \qquad \text{as } x \to 0, \quad \ell \in \mathbb{N}_{>0}, \tag{2.16}$$

$$\gamma_\ell(x) = \mathcal{O}(\min\{1, |x - br^{2b}|^{a-\ell}\}), \qquad \text{as } x \to br^{2b}, \quad \ell \in \mathbb{N}. \tag{2.17}$$

Proof It is easily checked that

$$\gamma_\ell(x) = \begin{cases} (-r)^a \mathfrak{g}_\ell\Big(- \Big(\frac{x}{br^{2b}}\Big)^{\frac{1}{2b}} \Big), & x > br^{2b}, \\ r^a \mathfrak{g}_\ell\Big(- \Big(\frac{x}{br^{2b}}\Big)^{\frac{1}{2b}} \Big), & x \in (0, br^{2b}). \end{cases}$$

The claim is now a straightforward consequence of Lemma 2.3. □

Lemma 2.6 *Let $k \in \mathbb{N}$ be fixed. As $j \to +\infty$,*

$$\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \sim \left(\frac{j}{b}\right)^{\frac{k}{2b}} \left(1 + \sum_{\ell=1}^{+\infty} \frac{\mathfrak{p}_{2\ell}(k)}{j^\ell}\right), \tag{2.18}$$

where

$$\mathfrak{p}_{2\ell}(k) := b^{\ell}\binom{\frac{k}{2b}}{\ell}B_{\ell}^{(1+\frac{k}{2b})}\Big(\frac{2\alpha+k}{2b}\Big) =: \sum_{m=1}^{2\ell}\mathfrak{p}_{2\ell,m}\,k^{m}, \tag{2.19}$$

$\binom{\frac{k}{2b}}{\ell} := \frac{\frac{k}{2b}(\frac{k}{2b}-1)\ldots(\frac{k}{2b}-\ell+1)}{\ell!}$, *and* $B_{\ell}^{(k)}(x)$ *is the generalized Bernoulli*[4] *polynomial of degree* ℓ *defined through the generating function*

$$\Big(\frac{t}{e^{t}-1}\Big)^{k}e^{xt} = \sum_{\ell=0}^{+\infty}B_{\ell}^{(k)}(x)\frac{t^{n}}{n!}, \qquad |t|<2\pi. \tag{2.20}$$

Remark 2.7 The degree 2ℓ polynomial $\mathfrak{p}_{2\ell}$ satisfies $\mathfrak{p}_{2\ell}(0)=0$. This is consistent with the fact that for $k=0$ the left-hand side of (2.18) is 1.

Remark 2.8 The notation "$\sim$" in (2.18) means that for any $N\in\mathbb{N}$, $N\geq 1$, we have

$$\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} = \Big(\frac{j}{b}\Big)^{\frac{k}{2b}}\Big(1+\sum_{\ell=1}^{N}\frac{\mathfrak{p}_{2\ell}(k)}{j^{\ell}}+\mathcal{O}(j^{-N-1})\Big), \qquad \text{as } j\to+\infty.$$

We will use this notation repetitively in the paper.

Proof The claim directly follows from [56, eq. 5.11.13]

$$\frac{\Gamma(v+p_2+p_1)}{\Gamma(v+p_2)} \sim v^{p_1}\sum_{\ell=0}^{+\infty}\binom{p_1}{\ell}\frac{B_{\ell}^{(p_1+1)}(p_1+p_2)}{v^{\ell}} \qquad \text{as } v\to+\infty,\ p_1,p_2 \text{ fixed.}$$

□

Lemma 2.9 *As* $n\to+\infty$,

$$\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \sim \gamma_0(j/n)+\sum_{\ell=1}^{+\infty}\frac{1}{n^{\ell}}\frac{\sum_{m=1}^{2\ell}\mathfrak{p}_{2\ell,m}\gamma_m(j/n)}{(j/n)^{\ell}}, \tag{2.21}$$

uniformly for $j\in\{j_+ +1,\ldots,n\}$. *Furthermore,* M' *can be chosen sufficiently large (but fixed) such that the following holds: there exists* $C>0$ *such that*

$$\Big|\sum_{k=0}^{a}\binom{a}{k}\frac{r^{a-k}(-1)^{k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} - \Big\{\gamma_0(j/n)+\frac{\gamma_2(j/n)+(4\alpha-2b)\gamma_1(j/n)}{8bj}\Big\}\Big| \leq C\frac{(j/n)^{\frac{1}{2b}-2}}{n^{2}}, \tag{2.22}$$

[4] $B_{\ell}^{(1)}(x)$ is the classical Bernoulli polynomial of degree ℓ, and $B_{\ell}^{(0)}(x)=x^{\ell}$.

for all sufficiently large n and all $j \in \{M'+1,\dots,j_- - 1\}$.

Proof Since a is fixed, Lemma 2.6 implies that

$$\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \sim \sum_{k=0}^{a}\binom{a}{k}(-r)^{a-k}\left(\frac{j/n}{b}\right)^{\frac{k}{2b}}\left(1+\sum_{\ell=1}^{+\infty}\frac{\mathfrak{p}_{2\ell}(k)}{j^{\ell}}\right), \tag{2.23}$$

as $j \to +\infty$. Since $j/n \in (br^{2b}, 1]$ for all $j \in \{j_+ + 1,\dots,n\}$, the expansion (2.21) directly follows from (2.23) and the definition (2.14) of γ_ℓ. In the same way, but using now the definition (2.14) of $\gamma_\ell(x)$ for $x \in (0, br^{2b})$, we infer that

$$\sum_{k=0}^{a}\binom{a}{k}\frac{r^{a-k}(-1)^{k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \sim \gamma_0(j/n)+\sum_{\ell=1}^{+\infty}\frac{\sum_{m=1}^{2\ell}\mathfrak{p}_{2\ell,m}\gamma_m(j/n)}{j^{\ell}}, \tag{2.24}$$

as $j \to +\infty$ uniformly for n such that $j/n \in (0, br^{2b})$. The estimate (2.22) then follows from (2.24) and the behavior (2.16). □

For the large n analysis of S_3, S_1, S_2, we will need to approximate various large sums of the form $\sum_j f(\frac{j}{n})$ involving some functions f with singularities and some j's for which $\frac{j}{n}$ is close to these singularities; for this we will also rely on the following lemma from [21].

Lemma 2.10 ([21, Lemma 3.4]) *Let* A, a_0, B, b_0 *be bounded function of* $n \in \{1,2,\dots\}$*, such that*

$$a_n := An + a_0 \qquad \textit{and} \qquad b_n := Bn + b_0$$

are integers. Assume also that $B - A$ *is positive and remains bounded away from 0. Let* f *be a function independent of* n*, and which is* $C^4([\min\{\frac{a_n}{n}, A\}, \max\{\frac{b_n}{n}, B\}])$ *for all* $n \in \{1,2,\dots\}$*. Then as* $n \to +\infty$*, we have*

$$\begin{aligned}\sum_{j=a_n}^{b_n} f(\tfrac{j}{n}) &= n\int_A^B f(x)dx + \frac{(1-2a_0)f(A)+(1+2b_0)f(B)}{2}\\ &+\frac{(-1+6a_0-6a_0^2)f'(A)+(1+6b_0+6b_0^2)f'(B)}{12n}\\ &+\frac{(-a_0+3a_0^2-2a_0^3)f''(A)+(b_0+3b_0^2+2b_0^3)f''(B)}{12n^2}\\ &+\mathcal{O}\left(\frac{\mathfrak{m}_A(f''')+\mathfrak{m}_B(f''')}{n^3}+\sum_{j=a_n}^{b_n-1}\frac{\mathfrak{m}_{j,n}(f'''')}{n^4}\right),\end{aligned}$$

where, for a given function g continuous on $[\min\{\frac{a_n}{n}, A\}, \max\{\frac{b_n}{n}, B\}]$,

$$\mathfrak{m}_A(g) := \max_{x\in[\min\{\frac{a_n}{n},A\},\max\{\frac{a_n}{n},A\}]} |g(x)|, \qquad \mathfrak{m}_B(g) := \max_{x\in[\min\{\frac{b_n}{n},B\},\max\{\frac{b_n}{n},B\}]} |g(x)|,$$

and for $j \in \{a_n, \dots, b_n - 1\}$, $\mathfrak{m}_{j,n}(g) := \max_{x\in[\frac{j}{n},\frac{j+1}{n}]} |g(x)|$.

3 Global Analysis: Large *n* Asymptotics of S_0, S_3 and S_1

As mentioned earlier, we will analyze the sums (2.5) –(2.8) in order of increasing difficulty: first S_0, then S_3, then S_1, and finally S_2. In this section we focus on S_0, S_3 and S_1. We defer the analysis of S_2 to the next section.

Lemma 3.1 *As* $n \to +\infty$,

$$S_0 = M' \ln(r^a e^u) + \mathcal{O}(n^{-\frac{1}{2b}}).$$

Proof Using (2.5) and Lemma A.1, we obtain

$$\begin{aligned} S_0 &= \sum_{j=1}^{M'} \ln\left(\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\left[1+((-1)^a e^u - 1)[1+\mathcal{O}(e^{-cn})]\right]\right), \\ &= \sum_{j=1}^{M'} \ln\left(\sum_{k=0}^{a}\binom{a}{k}\frac{(-1)^k r^{a-k} e^u}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\right) + \mathcal{O}(e^{-cn}), \qquad \text{as } n \to +\infty. \end{aligned}$$

Since M' is fixed, only the $(k = 0)$-terms contribute to order 1 in the large n asymptotics of S_0; the other terms are $\mathcal{O}(n^{-\frac{1}{2b}})$. ☐

Recall that S_1, S_3 are given by (2.6) and (2.8). Following the approach of [20, 21], we define

$$\theta_+^{(n,\epsilon)} = \left(\frac{bnr^{2b}}{1-\epsilon} - \alpha\right) - \left\lfloor \frac{bnr^{2b}}{1-\epsilon} - \alpha \right\rfloor, \qquad \theta_-^{(n,\epsilon)} = \left\lceil \frac{bnr^{2b}}{1+\epsilon} - \alpha \right\rceil - \left(\frac{bnr^{2b}}{1+\epsilon} - \alpha\right),$$

and for $j = 1, \dots, n$ and $k = 0, 1, \dots, a$, we also define

$$a_j := \frac{j+\alpha}{b}, \lambda_j := \frac{bnr^{2b}}{j+\alpha}, \eta_j := (\lambda_j - 1)\sqrt{\frac{2(\lambda_j - 1 - \ln \lambda_j)}{(\lambda_j - 1)^2}}, \tag{3.1}$$

$$a_{j,k} := \frac{2j+2\alpha+k}{2b}, \lambda_{j,k} := \frac{bnr^{2b}}{j+\alpha+\frac{k}{2}},$$

$$\eta_{j,k} := (\lambda_{j,k} - 1)\sqrt{\frac{2(\lambda_{j,k} - 1 - \ln \lambda_{j,k})}{(\lambda_{j,k} - 1)^2}}. \tag{3.2}$$

Lemma 3.2 *As $n \to +\infty$,*

$$\begin{aligned} S_3 = {} & n \int_{\frac{br^{2b}}{1-\epsilon}}^{1} a \ln\left(\left(\frac{x}{b}\right)^{\frac{1}{2b}} - r\right) dx \\ & + a\frac{(2\alpha + 2\theta_{+}^{(n,\epsilon)} - 1)\ln(r(1-\epsilon)^{-\frac{1}{2b}} - r) + \ln(b^{-\frac{1}{2b}} - r)}{2} \\ & + \frac{a}{4}\left(\frac{1-a}{(br^{2b})^{-\frac{1}{2b}} - 1} - \frac{1-a}{(1-\epsilon)^{-\frac{1}{2b}} - 1} + (a - 2b + 4\alpha)\ln\left(\frac{(br^{2b})^{-\frac{1}{2b}} - 1}{(1-\epsilon)^{-\frac{1}{2b}} - 1}\right)\right) \\ & + \mathcal{O}(n^{-1}). \end{aligned}$$

Proof Using (2.8) and Lemma A.2 (ii) with a and λ replaced by $a_{j,k}$ and $\lambda_{j,k}$ respectively, where $j \in \{j_+ + 1, \ldots, n\}$ and $k \in \{0, \ldots, a\}$, we obtain

$$S_3 = \sum_{j=j_+ + 1}^{n} \ln\left(\sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\left[1 + ((-1)^a e^u - 1)\mathcal{O}(e^{-\frac{a_{j,k}\eta_{j,k}^2}{2}})\right]\right),$$

as $n \to +\infty$. It is easy to check from (3.2) that $c_1 n \le a_{j,k} \le c_1' n$, $c_2 \le |\lambda_{j,k} - 1| \le c_2'$ and $c_3 \le \eta_{j,k}^2 \le c_3'$ hold for some positive constants $\{c_j, c_j'\}_{j=1}^3$, for all n sufficiently large, for all $j \in \{j_+ + 1, \ldots, n\}$ and for all $k \in \{0, \ldots, a\}$. Thus

$$S_3 = \sum_{j=j_+ + 1}^{n} \ln\left(\sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\right) + \mathcal{O}(e^{-cn}), \qquad \text{as } n \to +\infty. \tag{3.3}$$

To complete the proof of this lemma, we need the following weaker version of (2.21):

$$\begin{aligned} \sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} = {} & \gamma_0(j/n) + \frac{1}{n}\frac{\gamma_2(j/n) + (4\alpha - 2b)\gamma_1(j/n)}{8bj/n} \\ & + \mathcal{O}(n^{-2}), \end{aligned}$$

as $n \to +\infty$ and simultaneously $j \in \{j_+ + 1, \ldots, n\}$. Note from (2.3) that j/n lies in $(br^{2b}, 1]$ and remains bounded away from br^{2b} as $n \to +\infty$ and simultaneously $j \in \{j_+ + 1, \ldots, n\}$; in particular $\gamma_0(j/n)$ remains bounded away from 0. Hence, by substituting the above expansion in (3.3) and using (2.15) with $\ell = 0, 1, 2$, we obtain after a computation that

$$S_3 = \Sigma_0 + \frac{1}{n}\Sigma_1 + \mathcal{O}(n^{-1}), \qquad \Sigma_\ell := \sum_{j=j_+ + 1}^{n} f_\ell(j/n), \ \ \ell = 0, 1,$$

$$f_0(x) := \ln \gamma_0(x) = a \ln \left(\left(\frac{x}{b} \right)^{\frac{1}{2b}} - r \right),$$

$$f_1(x) := \frac{\gamma_2(x) + (4\alpha - 2b)\gamma_1(x)}{8bx\,\gamma_0(x)} = \frac{a(\frac{x}{b})^{\frac{1}{2b}}}{8bx((\frac{x}{b})^{\frac{1}{2b}} - r)^2} \left((a + 4\alpha - 2b) \left(\frac{x}{b} \right)^{\frac{1}{2b}} - (1 + 4\alpha - 2b) r \right), \tag{3.4}$$

where we have also used that $\ln(1+x) = x + \mathcal{O}(x^2)$ as $x \to 0$. From Lemma 2.10 (with $A = \frac{br^{2b}}{1-\epsilon}$, $a_0 = 1 - \alpha - \theta_+^{(n,\epsilon)}$, $B = 1$ and $b_0 = 0$), we infer that

$$\Sigma_\ell = n \int_{\frac{br^{2b}}{1-\epsilon}}^1 f_\ell(x) dx + \frac{(2\alpha + 2\theta_+^{(n,\epsilon)} - 1) f_\ell(\frac{br^{2b}}{1-\epsilon}) + f_\ell(1)}{2} + \mathcal{O}(n^{-1}), \qquad \ell = 0, 1,$$

as $n \to +\infty$. We then obtain the claim for S_3 after a computation using the simplification

$$\int_{\frac{br^{2b}}{1-\epsilon}}^1 f_1(x) dx = \frac{a}{4} \left(\frac{1-a}{(br^{2b})^{-\frac{1}{2b}} - 1} - \frac{1-a}{(1-\epsilon)^{-\frac{1}{2b}} - 1} + (a - 2b + 4\alpha) \ln \left(\frac{(br^{2b})^{-\frac{1}{2b}} - 1}{(1-\epsilon)^{-\frac{1}{2b}} - 1} \right) \right).$$

□

The large n asymptotics of S_1 are harder to obtain than those of S_3. The main reason for it is that S_1 involves small j's, and for such j's the quantities $\gamma_\ell(j/n)$ have a singular behavior, see (2.16). This is also the reason why the error term in Lemma 3.3 is more complicated than in Lemma 3.2.

Lemma 3.3 *As $n \to +\infty$,*

$$S_1 = n \int_0^{\frac{br^{2b}}{1+\epsilon}} \left(u + a \ln \left(r - \left(\frac{x}{b} \right)^{\frac{1}{2b}} \right) \right) dx - M' \ln(r^a e^u) + \frac{-(u + a \ln r) + (2\theta_-^{(n,\epsilon)} - 1 - 2\alpha)\big(u + a \ln(r - r(1+\epsilon)^{-\frac{1}{2b}}\big)}{2} + \frac{a}{4} \left(\frac{a-1}{(1+\epsilon)^{\frac{1}{2b}} - 1} + (a - 2b + 4\alpha) \ln \big(1 - (1+\epsilon)^{-\frac{1}{2b}}\big) \right) + \mathcal{O}\big(n^{-\frac{1}{2b}} + n^{-1}\big).$$

Proof Lemma A.2 (i) implies that for any $\epsilon' > 0$ there exist $A = A(\epsilon')$, $C = C(\epsilon') > 0$ such that $|\frac{\gamma(a,z)}{\Gamma(a)} - 1| \le Ce^{-\frac{a\eta^2}{2}}$ for all $a \ge A$, for all $\lambda = \frac{z}{a} \in [1 + \epsilon', +\infty]$, and where η is given by (A.1). Let us take $\epsilon' = \frac{\epsilon}{2}$ and choose M' so large that

$a_j = \frac{j+\alpha}{b} \geq A(\frac{\epsilon}{2})$ for all $j \in \{M'+1, \ldots, j_{1,-}-1\}$. Thus

$$S_1 = \sum_{j=M'+1}^{j_--1} \ln\Bigg(\sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \\ \Big[1 + ((-1)^a e^u - 1)(1 + \mathcal{O}(e^{-\frac{a_{j,k}\eta_{j,k}^2}{2}}))\Big]\Bigg),$$

as $n \to +\infty$. From a direct analysis of (3.2), we infer that $a_{j,k}\eta_{j,k}^2$ decreases as j increases from $M'+1$ to $j_- - 1$, and $a_{j,k}\eta_{j,k}^2$ decreases also as k increases from 0 to a. Therefore

$$\frac{a_{j,k}\eta_{j,k}^2}{2} \geq \frac{a_{j_--1}\eta_{j_--1}^2}{2} \geq cn, \qquad \text{for all } j \in \{M'+1, \ldots, j_- - 1\},$$

for a small enough $c > 0$. Thus

$$S_1 = \sum_{j=M'+1}^{j_--1} \ln\Bigg(\sum_{k=0}^{a} \binom{a}{k} \frac{r^{a-k}(-1)^k e^u}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\Bigg) + \mathcal{O}(e^{-cn}), \qquad \text{as } n \to +\infty. \tag{3.5}$$

Substituting (2.22) in (3.5) and using (2.15) with $\ell = 0, 1, 2$, we obtain

$$S_1 = \widetilde{\Sigma}_0 + \frac{1}{n}\widetilde{\Sigma}_1 + \mathcal{O}\Bigg(n^{-2} \sum_{j=M'+1}^{j_--1} (j/n)^{\frac{1}{2b}-2}\Bigg) = \widetilde{\Sigma}_0 + \frac{1}{n}\widetilde{\Sigma}_1 + \mathcal{O}\big(n^{-\frac{1}{2b}} + n^{-1}\big),$$

$$\widetilde{\Sigma}_0 := \sum_{j=M'+1}^{j_--1} \widetilde{f}_0(j/n), \quad \widetilde{\Sigma}_1 := \sum_{j=M'+1}^{j_--1} f_1(j/n), \qquad \widetilde{f}_0(x) := u + a\ln\Bigg(r - \Big(\frac{x}{b}\Big)^{\frac{1}{2b}}\Bigg), \tag{3.6}$$

as $n \to +\infty$, where f_1 is given by (3.4). Note that $f_1(x) \sim cx^{\frac{1}{2b}-1}$ as $x \to 0$; thus f_1 blows up at 0 if $b > \frac{1}{2}$. Using now Lemma 2.10 (with $A = \frac{M'}{n}$, $a_0 = 1$, $B = \frac{br^{2b}}{1+\epsilon}$ and $b_0 = \theta_-^{(n,\epsilon)} - 1 - \alpha$), we get

$$\widetilde{\Sigma}_0 = n\int_{\frac{M'}{n}}^{\frac{br^{2b}}{1+\epsilon}} \widetilde{f}_0(x)dx + \frac{-\widetilde{f}_0(\frac{M'}{n}) + (2\theta_-^{(n,\epsilon)} - 1 - 2\alpha)\widetilde{f}_0(\frac{br^{2b}}{1+\epsilon})}{2} + \mathcal{O}\big(n^{-\frac{1}{2b}} + n^{-1}\big),$$

$$\frac{1}{n}\widetilde{\Sigma}_1 = \int_{\frac{M'}{n}}^{\frac{br^{2b}}{1+\epsilon}} f_1(x)dx + \mathcal{O}\big(n^{-\frac{1}{2b}} + n^{-1}\big).$$

as $n \to +\infty$. Furthermore, by a direct analysis of $\widetilde{f}_0$ and f_1,

$$\widetilde{f}_0\Big(\frac{M'}{n}\Big) = u + a\ln r + \mathcal{O}\big(n^{-\frac{1}{2b}}\big),$$

$$\int_{\frac{M'}{n}}^{\frac{br^{2b}}{1+\epsilon}} f_1(x)\,dx = \frac{a}{4}\left(\frac{a-1}{(1+\epsilon)^{\frac{1}{2b}}-1} + (a-2b+4\alpha)\ln\left(1-(1+\epsilon)^{-\frac{1}{2b}}\right)\right) + \mathcal{O}\big(n^{-\frac{1}{2b}}\big),$$

$$n\int_{\frac{M'}{n}}^{\frac{br^{2b}}{1+\epsilon}} \widetilde{f}_0(x)\,dx = n\int_0^{\frac{br^{2b}}{1+\epsilon}} \widetilde{f}_0(x)\,dx - (u+a\ln r)M' + \mathcal{O}\big(n^{-\frac{1}{2b}}\big),$$

as $n \to +\infty$. The claim follows after substituting the above expansions in (3.6). □

4 Large *n* Asymptotics of S_2

It remains to obtain the large n asymptotics of S_2, which was defined in (2.7). For this, let us split S_2 in three pieces,

$$S_2 = S_2^{(1)} + S_2^{(2)} + S_2^{(3)},$$

where

$$S_2^{(v)} := \sum_{j:\lambda_j\in I_v} \ln\left(\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\left[1+((-1)^a e^u-1)\frac{\gamma(\frac{2j+2\alpha+k}{2b},nr^{2b})}{\Gamma(\frac{2j+2\alpha+k}{2b})}\right]\right) \tag{4.1}$$

for $v = 1, 2, 3$, λ_j is given by (3.1), and

$$I_1 := [1-\epsilon, 1-\tfrac{M}{\sqrt{n}}), \qquad I_2 := [1-\tfrac{M}{\sqrt{n}}, 1+\tfrac{M}{\sqrt{n}}], \qquad I_3 := (1+\tfrac{M}{\sqrt{n}}, 1+\epsilon].$$

Equivalently, the above sums can be rewritten using

$$\sum_{j:\lambda_j\in I_3} = \sum_{j=j_-}^{g_--1}, \qquad \sum_{j:\lambda_j\in I_2} = \sum_{j=g_-}^{g_+}, \qquad \sum_{j:\lambda_j\in I_1} = \sum_{j=g_++1}^{j_+}, \tag{4.2}$$

where $g_- := \lceil \frac{bnr^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha \rceil$, $g_+ := \lfloor \frac{bnr^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha \rfloor$. We also define $\theta_-^{(n,M)}, \theta_+^{(n,M)} \in [0,1)$ by

$$\theta_-^{(n,M)} := g_- - \left(\frac{bnr^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha\right) = \left\lceil \frac{bnr^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha \right\rceil - \left(\frac{bnr^{2b}}{1+\frac{M}{\sqrt{n}}} - \alpha\right),$$

$$\theta_+^{(n,M)} := \left(\frac{bnr^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha\right) - g_+ = \left(\frac{bnr^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha\right) - \left\lfloor \frac{bnr^{2b}}{1-\frac{M}{\sqrt{n}}} - \alpha \right\rfloor.$$

Note that the sums $S_2^{(1)}$ and $S_2^{(3)}$ each contain a number of elements proportional to n, while $S_2^{(2)}$ contains roughly $M\sqrt{n}$ elements.

4.1 Global Analysis: Large *n* Asymptotics of $S_2^{(1)}$ and $S_2^{(3)}$

We first treat $S_2^{(1)}$, $S_2^{(3)}$. These sums are delicate to analyze because they involve the asymptotics of $\gamma(a,z)$ in the regime $a \to +\infty$, $z \to +\infty$, when $\lambda = \frac{z}{a}$ is close to 1 but not very close (more precisely, $\lambda \in [1-\epsilon, 1-\frac{M}{\sqrt{n}}) \cup (1+\frac{M}{\sqrt{n}}, 1+\epsilon])$.

Lemma 4.1 *As $n \to +\infty$,*

$$
\begin{aligned}
S_2^{(1)} &= n\int_{br^{2b}}^{\frac{br^{2b}}{1-\epsilon}} a\ln\left(\left(\frac{x}{b}\right)^{\frac{1}{2b}} - r\right)dx + \left\{abr^{2b}M\left(\ln\left(\frac{2b\sqrt{n}}{rM}\right)+1\right) + \frac{a(a-1)b}{2M}\right. \\
&\quad \left. - \frac{ab(3-5a+2a^2)}{12r^{2b}M^3}\right\}\sqrt{n} + \left(-abr^{2b}M^2 + a\frac{2\alpha+2\theta_+^{(n,M)}-1}{2}\right. \\
&\quad \left. - \frac{a}{4}(a-2b+4\alpha)\right)\ln\left(\frac{M}{\sqrt{n}}\right) \\
&\quad + \frac{ar^{2b}(2b-1+8b\ln(\frac{2b}{r}))}{8}M^2 + \frac{2\alpha+2\theta_+^{(n,M)}-1}{2}a\ln\left(\frac{r}{2b}\right) \\
&\quad + \frac{1-2\alpha-2\theta_+^{(n,\epsilon)}}{2}a\ln\left(r(1-\epsilon)^{-\frac{1}{2b}} - r\right) \\
&\quad + \frac{a}{4}\left\{\frac{(1-a)(1+2b)}{2} + (a-2b+4\alpha)\ln(2b)\right. \\
&\quad \left. + \frac{1-a}{(1-\epsilon)^{-\frac{1}{2b}}-1} + (a-2b+4\alpha)\ln\left((1-\epsilon)^{-\frac{1}{2b}}-1\right)\right\} + \mathcal{O}\left(\frac{\sqrt{n}}{M^5} + \frac{M^3\ln n}{\sqrt{n}}\right).
\end{aligned}
$$

Remark 4.2 Since $M = n^{\frac{1}{8}}(\ln n)^{-\frac{1}{8}}$, the $\mathcal{O}$-term above is small as $n \to +\infty$.

Remark 4.3 The above asymptotics are of the form

$$S_2^{(1)} = E_1^{(\epsilon)}n + \tilde{E}_2^{(M)}\sqrt{n}\ln n + E_2^{(M)}\sqrt{n} + \tilde{E}_3^{(n,M)}\ln n + E_3^{(n,\epsilon,M)} + o(1).$$

This may seem a bit counter intuitive as the asymptotics of Theorem 1.1 only contain terms proportional to n, $\sqrt{n}$ and 1. In fact, remarkable cancellations will occur in the asymptotics of $S_2^{(1)} + S_2^{(2)} + S_2^{(3)}$; in particular $\tilde{E}_2^{(M)}$ and $\tilde{E}_3^{(n,M)}$ will get perfectly canceled by other terms in the large n asymptotics of $S_2^{(2)}$ and $S_2^{(3)}$, and we will show in Lemma 4.12 below that the large n asymptotics of $S_2 = S_2^{(1)} + S_2^{(2)} + S_2^{(3)}$ are of the form $S_2 = \widehat{C}_1^{(\epsilon)}n + \widehat{C}_2\sqrt{n} + \widehat{C}_3^{(n,\epsilon)} + o(1)$.

Proof By (4.1) and Lemma A.2, $S_2^{(1)}$ admits the following exact formula

$$S_2^{(1)} = \sum_{j:\lambda_j \in I_1} \ln\left(\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\right.$$

$$\times \left[1+((-1)^a e^u - 1)\left(\frac{1}{2}\text{erfc}\left(-\eta_{j,k}\sqrt{\tfrac{a_{j,k}}{2}}\right)\right.\right.$$
$$\left.\left.\times -R_{a_{j,k}}(\eta_{j,k})\right)\right]\Bigg),$$

where $\eta_{j,k}$ and $a_{j,k}$ are given by (3.2). Since $I_1 = [1-\epsilon, 1-\frac{M}{\sqrt{n}})$,

$$\eta_{j,k} = \lambda_{j,k} - 1 + \mathcal{O}((\lambda_{j,k}-1)^2) \leq -\frac{M}{\sqrt{n}} + \mathcal{O}(\tfrac{M^2}{n}), \qquad \text{as } n \to +\infty,$$
$$-\eta_{j,k}\sqrt{a_{j,k}/2} = -\eta_{j,k}\sqrt{\frac{nr^{2b}}{2\lambda_{j,k}}} \geq \frac{Mr^b}{\sqrt{2}} + \mathcal{O}(\tfrac{M^2}{\sqrt{n}}), \qquad \text{as } n \to +\infty,$$

uniformly for $j \in \{j : \lambda_j \in I_1\}$ and $k \in \{0, 1, \ldots, a\}$. Also, $M = n^{\frac{1}{8}}(\ln n)^{-\frac{1}{8}}$, and thus, by (A.2),

$$R_{a_{j,k}}(\eta_{j,k}) = \mathcal{O}(e^{-\frac{r^{2b}M^2}{4}}) = \mathcal{O}(e^{-n^c}), \qquad \frac{1}{2}\text{erfc}\left(-\eta_{j,k}\sqrt{\tfrac{a_{j,k}}{2}}\right) = \mathcal{O}(e^{-n^c}),$$

as $n \to +\infty$ uniformly for $j \in \{j : \lambda_j \in I_1\}$ and $k \in \{0, 1, \ldots, a\}$, and (using also (4.2))

$$S_2^{(1)} = \sum_{j=g_+ +1}^{j_+} \ln\left(\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\right) + \mathcal{O}(e^{-n^c}), \qquad \text{as } n \to +\infty. \tag{4.3}$$

In the same way as for (2.21), we obtain

$$\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \sim \gamma_0(j/n) + \sum_{\ell=1}^{+\infty}\frac{\sum_{m=1}^{2\ell}\mathfrak{p}_{2\ell,m}\gamma_m(j/n)}{j^\ell} \tag{4.4}$$

as $n \to +\infty$ uniformly for $j \in \{g_+ + 1, \ldots, j_+\}$. However, unlike for S_3, the first subleading term (corresponding to $\ell = 1$) is not sufficient for our purpose; we also need the coefficients of $\mathfrak{p}_4$ which are given by (see (2.19))

$$\begin{aligned}\mathfrak{p}_4(k) &= \frac{k(k-2b)}{8}B_2^{(1+\frac{k}{2b})}\left(\frac{2\alpha+k}{2b}\right)\\ &= \frac{k(k-2b)(8b^2+3(k+4\alpha)^2-2b(7k+24\alpha))}{384b^2}.\end{aligned} \tag{4.5}$$

The reason as to why we need $\mathfrak{p}_4$ can be seen as follows. The sum on the left-hand side of (4.4) appears inside ln in (4.3). Since $\ln(\gamma_0 + B) = \ln\gamma_0 + \ln(1+B/\gamma_0)$, what is relevant is to estimate the asymptotics series of (4.4) divided by $\gamma_0(j/n)$. Lemma

2.5 implies

$$\frac{\gamma_\ell(x)}{\gamma_0(x)} = \mathcal{O}(\min\{|x-br^{2b}|^{-a}, |x-br^{2b}|^{-\ell}\}) = \mathcal{O}(|x-br^{2b}|^{-\min\{a,\ell\}}), \quad \text{as } x \to br^{2b}. \tag{4.6}$$

Using the definitions of j_+ and g_+ (see 2.3 and (4.2)), we see that for $j \in \{g_+ + 1, \ldots, j_+\}$, j/n lies in $(br^{2b}, 1]$ and $g_+/n - br^{2b}$ is of order $\frac{M}{\sqrt{n}}$ as $n \to +\infty$. Therefore, for $\ell = 1, 2, \ldots$,

$$\sum_{j=g_++1}^{j_+} \frac{\sum_{m=1}^{2\ell} \mathfrak{p}_{2\ell,m}\gamma_m(j/n)}{j^\ell \gamma_0(j/n)} = \mathcal{O}\bigg(\sum_{j=g_++1}^{j_+} \frac{|j/n - br^{2b}|^{-\min\{a,2\ell\}}}{n^\ell}\bigg) = \mathcal{O}\bigg(\frac{\sqrt{n}}{M^{2\ell}}\bigg), \tag{4.7}$$

as $n \to +\infty$ uniformly for $j \in \{g_+ + 1, \ldots, j_+\}$. Since $M = n^{\frac{1}{8}}(\ln n)^{-\frac{1}{8}}$, only the terms corresponding to $\ell \geq 3$ in (4.4) (i.e. the terms associated with $\mathfrak{p}_6, \mathfrak{p}_8, \ldots$) will give an error in the asymptotics of $S_2^{(1)}$. Substituting (4.4)–(4.5) in (4.3) and using (2.15) (with $\ell = 0, 1, 2, 3, 4$), (2.17) and (4.7), we obtain

$$S_2^{(1)} = \Sigma_0^{(1)} + \frac{1}{n}\Sigma_1^{(1)} + \frac{1}{n^2}\Sigma_2^{(1)} + \mathcal{O}\bigg(\frac{\sqrt{n}}{M^5}\bigg) \tag{4.8}$$

as $n \to +\infty$, where

$$\Sigma_\ell^{(1)} := \sum_{j=g_++1}^{j_+} f_\ell(j/n), \qquad \ell = 0, 1, 2,$$

f_0 and f_1 are as in (3.4), and f_2 is defined by

$$\begin{aligned} f_2(x) := \frac{1}{384b^2x^2}\bigg\{ &-16b(b^2-6b\alpha+6\alpha^2)\frac{a\left(\frac{x}{b}\right)^{\frac{1}{2b}}}{\left(\frac{x}{b}\right)^{\frac{1}{2b}}-r} + 12(3b^2-8b\alpha+4\alpha^2) \\ &\frac{a\left(\frac{x}{b}\right)^{\frac{1}{2b}}\left(a\left(\frac{x}{b}\right)^{\frac{1}{2b}}-r\right)}{\left(a\left(\frac{x}{b}\right)^{\frac{1}{2b}}-r\right)^2} \\ &+4(6\alpha-5b)\frac{a\left(\frac{x}{b}\right)^{\frac{1}{2b}}\left(a^2\left(\frac{x}{b}\right)^{\frac{1}{b}}+(1-3a)r\left(\frac{x}{b}\right)^{\frac{1}{2b}}+r^2\right)}{\left(a\left(\frac{x}{b}\right)^{\frac{1}{2b}}-r\right)^3} \\ &+\frac{3a\left(\frac{x}{b}\right)^{\frac{1}{2b}}\left[a^3\left(\frac{x}{b}\right)^{\frac{3}{2b}}+(4a-6a^2-1)r\left(\frac{x}{b}\right)^{\frac{1}{b}}+(7a-4)r^2\left(\frac{x}{b}\right)^{\frac{1}{2b}}-r^3\right]}{\left(a\left(\frac{x}{b}\right)^{\frac{1}{2b}}-r\right)^4}\bigg\} \\ &-\frac{f_1(x)^2}{2}. \end{aligned} \tag{4.9}$$

Using Lemma 2.10 (with $A = \frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}$, $a_0 = 1-\alpha-\theta_+^{(n,M)}$, $B = \frac{br^{2b}}{1-\epsilon}$ and $b_0 = -\alpha-\theta_+^{(n,\epsilon)}$), we get

$$\Sigma_0^{(1)} = n\int_{\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{br^{2b}}{1-\epsilon}} f_0(x)dx + \frac{(2\alpha+2\theta_+^{(n,M)}-1)f_0(\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}) + (1-2\alpha-2\theta_+^{(n,\epsilon)})f_0(\frac{br^{2b}}{1-\epsilon})}{2} + \mathcal{O}\Big(\frac{1}{M\sqrt{n}}\Big),$$

$$\frac{1}{n}\Sigma_1^{(1)} = \int_{\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{br^{2b}}{1-\epsilon}} f_1(x)dx + \mathcal{O}\Big(\frac{1}{M^2}\Big), \qquad \frac{1}{n^2}\Sigma_2^{(1)} = \frac{1}{n}\int_{\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{br^{2b}}{1-\epsilon}} f_2(x)dx + \mathcal{O}\Big(\frac{1}{M^4}\Big), \tag{4.10}$$

as $n \to +\infty$. To obtain the above error terms, we also used (4.6), (2.15), and in particular that

$$f_0'\Big(\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}\Big) = \mathcal{O}\Big(\frac{\sqrt{n}}{M}\Big), \qquad f_1\Big(\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}\Big) = \mathcal{O}\Big(\frac{n}{M^2}\Big), \qquad f_2\Big(\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}\Big) = \mathcal{O}\Big(\frac{n^2}{M^4}\Big)$$

as $n \to +\infty$. Using the asymptotics of f_0, f_1 and f_2 near $x = br^{2b}$ with $x > br^{2b}$, namely

$$f_0(x) = a\log(x-br^{2b}) + a\log(\tfrac{r^{1-2b}}{2b^2}) + a\frac{(1-2b)(x-br^{2b})}{4b^2r^{2b}} + \mathcal{O}\big((x-br^{2b})^2\big),$$

$$f_1(x) = \frac{(a-1)ab^2r^{2b}}{2(x-br^{2b})^2} + \frac{a(a-2b+4\alpha)}{4(x-br^{2b})} + \mathcal{O}(1),$$

$$f_2(x) = -\frac{a(3-5a+2a^2)b^4r^{4b}}{4(x-br^{2b})^4} + \mathcal{O}\big((x-br^{2b})^{-3}\big),$$

we obtain

$$f_0\Big(\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}\Big) = a\ln\Big(\frac{Mr}{2b\sqrt{n}}\Big) + \mathcal{O}\Big(\frac{M}{\sqrt{n}}\Big), \tag{4.11}$$

$$\begin{aligned} n\int_{\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{br^{2b}}{1-\epsilon}} f_0(x)dx &= n\int_{br^{2b}}^{\frac{br^{2b}}{1-\epsilon}} f_0(x)dx + \sqrt{n}\,abr^{2b}M\Big(1+\ln\Big(\frac{2b\sqrt{n}}{rM}\Big)\Big) \\ &\quad + abr^{2b}M^2\ln\Big(\frac{\sqrt{n}}{M}\Big) \\ &\quad + \frac{ar^{2b}M^2(-1+2b+8b\ln(\frac{2b}{r}))}{8} + \mathcal{O}\Big(\frac{M^3\ln n}{\sqrt{n}}\Big), \end{aligned} \tag{4.12}$$

$$\int_{\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{br^{2b}}{1-\epsilon}} f_1(x)dx = \frac{a(a-1)b\sqrt{n}}{2M} + \frac{a}{4}\bigg\{\frac{(1-a)(1+2b)}{2} + \frac{1-a}{(\frac{1}{1-\epsilon})^{\frac{1}{2b}}-1} - (a-2b+4\alpha)\ln\left(\frac{M}{2b\sqrt{n}}\right) + (a-2b+4\alpha)\ln\left((1-\epsilon)^{-\frac{1}{2b}}-1\right)\bigg\} + \mathcal{O}\left(\frac{M}{\sqrt{n}}\right), \quad (4.13)$$

$$\frac{1}{n}\int_{\frac{br^{2b}}{1-\frac{M}{\sqrt{n}}}}^{\frac{br^{2b}}{1-\epsilon}} f_2(x)dx = -\frac{ab(3-5a+2a^2)\sqrt{n}}{12r^{2b}M^3} + \mathcal{O}\left(\frac{1}{M^2}\right), \quad (4.14)$$

as $n \to +\infty$. Substituting (4.11)–(4.14) in (4.10) and then in (4.8), we obtain the claim after another long but direct computation. □

Lemma 4.4 *As* $n \to +\infty$,

$$\begin{aligned}
S_2^{(3)} &= n\int_{\frac{br^{2b}}{1+\epsilon}}^{br^{2b}} \left[u + a\ln\left(r - \left(\frac{x}{b}\right)^{\frac{1}{2b}}\right)\right]dx + \bigg\{br^{2b}M\left(a\ln\left(\frac{2b\sqrt{n}}{rM}\right) + a - u\right) \\
&\quad + \frac{a(a-1)b}{2M} - \frac{ab(3-5a+2a^2)}{12r^{2b}M^3}\bigg\}\sqrt{n} + \bigg(abr^{2b}M^2 + a\frac{2\theta_-^{(n,M)}-1-2\alpha}{2} \\
&\quad + \frac{a}{4}(a-2b+4\alpha)\bigg)\ln\left(\frac{M}{\sqrt{n}}\right) + \frac{r^{2b}(a-2ab+8bu+8ab\ln(\frac{r}{2b}))}{8}M^2 \\
&\quad + \frac{2\theta_-^{(n,M)}-1-2\alpha}{2}\left(u + a\ln\left(\frac{r}{2b}\right)\right) + \frac{1+2\alpha-2\theta_-^{(n,\epsilon)}}{2}\left(u + a\ln\left(r - r(1+\epsilon)^{-\frac{1}{2b}}\right)\right) \\
&\quad + \frac{a}{4}\bigg\{\frac{(a-1)(1+2b)}{2} - (a-2b+4\alpha)\ln(2b) \\
&\quad + \frac{1-a}{1-(1+\epsilon)^{-\frac{1}{2b}}} - (a-2b+4\alpha)\ln\left(1-(1+\epsilon)^{-\frac{1}{2b}}\right)\bigg\} \\
&\quad + \mathcal{O}\left(\frac{\sqrt{n}}{M^5} + \frac{M^3\ln n}{\sqrt{n}}\right).
\end{aligned}$$

Proof By (4.1) and Lemma A.2,

$$\begin{aligned}
S_2^{(3)} &= \sum_{j:\lambda_j\in I_3} \ln\left(\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\right. \\
&\quad \left.\times\left[1 + ((-1)^a e^u - 1)\left(\frac{1}{2}\mathrm{erfc}\left(-\eta_{j,k}\sqrt{\frac{a_{j,k}}{2}}\right) - R_{a_{j,k}}(\eta_{j,k})\right)\right]\right).
\end{aligned}$$

Because $I_3 = (1 + \frac{M}{\sqrt{n}}, 1+\epsilon]$, we have

$$\eta_{j,k} = \lambda_{j,k} - 1 + \mathcal{O}((\lambda_{j,k}-1)^2) \geq \frac{M}{\sqrt{n}} + \mathcal{O}(\frac{M^2}{n}), \qquad \text{as } n \to \infty,$$
$$-\eta_{j,k}\sqrt{a_{j,k}/2} \leq -\frac{Mr^b}{\sqrt{2}} + \mathcal{O}(\frac{M^2}{\sqrt{n}}), \qquad \text{as } n \to \infty,$$

uniformly for $j \in \{j : \lambda_j \in I_3\}$ and $k \in \{1, \dots, a\}$. Since $M = n^{\frac{1}{8}}(\ln n)^{-\frac{1}{8}}$, we get

$$R_{a_{j,k}}(\eta_{j,k}) = \mathcal{O}(e^{-\frac{r^{2b}M^2}{4}}) = \mathcal{O}(e^{-n^c}), \quad \frac{1}{2}\mathrm{erfc}\Big(-\eta_{j,k}\sqrt{\tfrac{a_{j,k}}{2}}\Big) = 1 - \mathcal{O}(e^{-\frac{r^{2b}M^2}{4}})$$
$$= 1 - \mathcal{O}(e^{-n^c}),$$

as $n \to +\infty$ uniformly for $j \in \{j : \lambda_j \in I_3\}$ and $k \in \{1, \dots, a\}$, and thus

$$S_2^{(3)} = \sum_{j=j_-}^{g_--1} \ln\Bigg(\sum_{k=0}^{a}\binom{a}{k}\frac{r^{a-k}(-1)^k e^u}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\Bigg) + \mathcal{O}(e^{-n^c}), \tag{4.15}$$

where we have also used (4.2). By (2.24),

$$\sum_{k=0}^{a}\binom{a}{k}\frac{r^{a-k}(-1)^k e^u}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \sim e^u\Bigg\{\gamma_0(j/n) + \sum_{\ell=1}^{+\infty}\frac{\sum_{m=1}^{2\ell}\mathfrak{p}_{2\ell,m}\gamma_m(j/n)}{j^\ell}\Bigg\}, \tag{4.16}$$

as $n \to +\infty$ uniformly for $j \in \{j_-, \dots, g_- - 1\}$. Using Lemma 2.5, (4.6) and the definitions (2.3), (4.2) of j_- and g_-, we infer that (4.7) holds also as $n \to +\infty$ uniformly for $j \in \{j_-, \dots, g_- - 1\}$. Hence, substituting (4.16) in (4.15) and using (2.15) (with $\ell = 0, 1, 2, 3, 4$), (2.17) and (4.7), we get

$$S_2^{(3)} = \Sigma_0^{(3)} + \frac{1}{n}\Sigma_1^{(3)} + \frac{1}{n^2}\Sigma_2^{(3)} + \mathcal{O}\Bigg(\frac{\sqrt{n}}{M^5}\Bigg) \tag{4.17}$$

as $n \to +\infty$, where

$$\Sigma_0^{(3)} := \sum_{j=j_-}^{g_--1}\widetilde{f}_0(j/n), \qquad \Sigma_\ell^{(3)} := \sum_{j=j_-}^{g_--1} f_\ell(j/n), \qquad \ell = 1, 2,$$

$\widetilde{f}_0$ is given by (3.6), f_1 is given by (3.4), and f_2 is given by (4.9).

Using Lemma 2.10 (with $A = \frac{br^{2b}}{1+\epsilon}$, $a_0 = \theta_-^{(n,\epsilon)} - \alpha$, $B = \frac{br^{2b}}{1+\frac{M}{\sqrt{n}}}$ and $b_0 = \theta_-^{(n,M)} - 1 - \alpha$), we get

$$\Sigma_0^{(3)} = n\int_{\frac{br^{2b}}{1+\epsilon}}^{\frac{br^{2b}}{1+\frac{M}{\sqrt{n}}}}\widetilde{f}_0(x)dx + \frac{(1+2\alpha-2\theta_-^{(n,\epsilon)})\widetilde{f}_0(\frac{br^{2b}}{1+\epsilon}) + (2\theta_-^{(n,M)}-1-2\alpha)\widetilde{f}_0(\frac{br^{2b}}{1+\frac{M}{\sqrt{n}}})}{2}$$
$$+ \mathcal{O}\Bigg(\frac{1}{M\sqrt{n}}\Bigg),$$

$$\frac{1}{n}\Sigma_1^{(3)} = \int_{\frac{br^{2b}}{1+\epsilon}}^{\frac{br^{2b}}{1+\frac{M}{\sqrt{n}}}} f_1(x)dx + \mathcal{O}\bigg(\frac{1}{M^2}\bigg), \qquad \frac{1}{n^2}\Sigma_2^{(3)} = \frac{1}{n}\int_{\frac{br^{2b}}{1+\epsilon}}^{\frac{br^{2b}}{1+\frac{M}{\sqrt{n}}}} f_2(x)dx + \mathcal{O}\bigg(\frac{1}{M^4}\bigg), \tag{4.18}$$

as $n \to +\infty$. Furthermore, a long but straightforward analysis of $\widetilde{f}_0$, f_1 and f_2 shows that

$$\widetilde{f}_0\bigg(\frac{br^{2b}}{1+\frac{M}{\sqrt{n}}}\bigg) = u + a\ln\bigg(\frac{Mr}{2b\sqrt{n}}\bigg) + \mathcal{O}\bigg(\frac{M}{\sqrt{n}}\bigg), \tag{4.19}$$

$$\begin{aligned} n\int_{\frac{br^{2b}}{1+\epsilon}}^{\frac{br^{2b}}{1+\frac{M}{\sqrt{n}}}} \widetilde{f}_0(x)dx &= n\int_{\frac{br^{2b}}{1+\epsilon}}^{br^{2b}} \widetilde{f}_0(x)dx + \sqrt{n}\, br^{2b} \\ &\quad M\bigg(a - u + a\ln\bigg(\frac{2b\sqrt{n}}{rM}\bigg)\bigg) + abr^{2b}M^2\ln\bigg(\frac{M}{\sqrt{n}}\bigg) \\ &\quad + \frac{r^{2b}M^2(a - 2ab + 8bu + 8ab\ln(\frac{r}{2b}))}{8} + \mathcal{O}\bigg(\frac{M^3\ln n}{\sqrt{n}}\bigg), \end{aligned} \tag{4.20}$$

$$\begin{aligned} \int_{\frac{br^{2b}}{1+\epsilon}}^{\frac{br^{2b}}{1+\frac{M}{\sqrt{n}}}} f_1(x)dx &= \frac{a(a-1)b\sqrt{n}}{2M} + \frac{a}{4}\bigg\{\frac{(a-1)(1+2b)}{2} + \frac{1-a}{1-(\frac{1}{1+\epsilon})^{\frac{1}{2b}}} \\ &\quad + (a - 2b + 4\alpha)\ln\bigg(\frac{M}{2b\sqrt{n}}\bigg) \\ &\quad - (a - 2b + 4\alpha)\ln\Big(1 - (1+\epsilon)^{-\frac{1}{2b}}\Big)\bigg\} + \mathcal{O}\bigg(\frac{M}{\sqrt{n}}\bigg), \end{aligned} \tag{4.21}$$

$$\frac{1}{n}\int_{\frac{br^{2b}}{1+\epsilon}}^{\frac{br^{2b}}{1+\frac{M}{\sqrt{n}}}} f_2(x)dx = -\frac{ab(3 - 5a + 2a^2)\sqrt{n}}{12r^{2b}M^3} + \mathcal{O}\bigg(\frac{1}{M^2}\bigg). \tag{4.22}$$

Substituting (4.19)–(4.22) in (4.18) and then in (4.17), we obtain the claim after another long but direct computation. □

4.2 Local Analysis: Large n Asymptotics of $S_2^{(2)}$

Our next goal is to obtain the large n asymptotics of $S_2^{(2)}$. This is the most technical part of the proof of Theorem 1.1. As mentioned in the introduction, a major obstacle in the asymptotic analysis of $S_2^{(2)}$ is that, in order to treat the general case $a \in \mathbb{N}$, we need to expand various quantities to all orders. Lemma 4.5 below provides a general scheme to compute the coefficients appearing in these expansions in a recursive way. These coefficients are not all readily available in explicit forms, however only a few of those will really matter for us. Lemmas 4.6 and 4.7 establish some non-trivial identities between those relevant coefficients and the (associated) Hermite polynomials. The large n asymptotics of $S_2^{(2)}$ are then obtained in Lemma 4.11.

Let us define

$$M_{j,k} := \sqrt{n}(\lambda_{j,k}-1), \quad k \in \{1,\dots,m\},\ j \in \{j : \lambda_j \in I_2\} = \{g_-,\dots,g_+\}, \tag{4.23}$$

$$M_j := \sqrt{n}(\lambda_j - 1), \qquad j \in \{j : \lambda_j \in I_2\} = \{g_-,\dots,g_+\}. \tag{4.24}$$

Since $I_2 = [1-\frac{M}{\sqrt{n}}, 1+\frac{M}{\sqrt{n}}]$ and λ_j is decreasing in j, we have $-M \le M_{g_+} < \cdots < M_{g_-} \le M$. Note from (3.1)–(3.2) that $\lambda_{j,k}$ is close to λ_j for large n, and from (4.23)–(4.24) that $M_{j,k}$ is close to M_j for large n. For convenience, for $j \in \{g_-,\dots,g_+\}$, we also define

$$\Xi_{j,k} := \frac{M_j r^b}{\sqrt{2}} - \frac{\sqrt{a_{j,k}}}{\sqrt{2}}\eta_{j,k}, \tag{4.25}$$

where $a_{j,k}$ and $\eta_{j,k}$ are given in (3.2). Also, by (4.1) and Lemma A.2, we have

$$\begin{aligned} S_2^{(2)} = \sum_{j:\lambda_j \in I_2} \ln \Bigg(\sum_{k=0}^{a} \binom{a}{k} \frac{(-r)^{a-k}}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \\ \times \Big[1 + ((-1)^a e^u - 1)\Big(\mathrm{erfc}\Big(-\eta_{j,k}\sqrt{\tfrac{a_{j,k}}{2}}\Big) - R_{a_{j,k}}(\eta_{j,k})\Big)\Big]\Bigg). \end{aligned} \tag{4.26}$$

The next lemma provides the asymptotics of $\Xi_{j,k}$ and of the summand of the k-sum in (4.26) as $n \to +\infty$ uniformly for $j \in \{j : \lambda_j \in I_2\} = \{g_-,\dots,g_+\}$.

Lemma 4.5 *Let $k \in \{0,1,\dots,a\}$ be fixed. As $n \to +\infty$ and uniformly for $j \in \{g_-,\dots,g_+\}$, we have*

$$\frac{1}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \sim r^k \sum_{\ell=0}^{+\infty} \frac{\sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \mathfrak{q}_{\ell,m,p}^{(1)} k^m M_j^p}{n^{\frac{\ell}{2}}}, \tag{4.27}$$

$$\Xi_{j,k} \sim \Xi_{j,k}^{\mathrm{formal}} := \sum_{\ell=1}^{+\infty} \frac{\sum_{m=0}^{\lfloor(\ell+1)/2\rfloor} d_{\ell,m}\, k^m M_j^{\ell+1-2m}}{n^{\ell/2}}, \tag{4.28}$$

$$\begin{aligned} \frac{1}{2}\mathrm{erfc}\Big(-\eta_{j,k}\sqrt{\tfrac{a_{j,k}}{2}}\Big) \sim \frac{1}{2}\mathrm{erfc}\Big(-\frac{M_j r^b}{\sqrt{2}}\Big) \\ - \frac{e^{-\frac{M_j^2 r^{2b}}{2}}}{\sqrt{2\pi}} \sum_{\ell=1}^{+\infty} \frac{\sum_{m=0}^{\ell} \sum_{p=0}^{3\ell-1-2m} \mathfrak{q}_{\ell,m,p}^{(2)} k^m M_j^p}{n^{\frac{\ell}{2}}}, \end{aligned} \tag{4.29}$$

$$R_{a_{j,k}}(\eta_{j,k}) \sim -\frac{e^{-\frac{M_j^2 r^{2b}}{2}}}{3\sqrt{2\pi}\, r^b \sqrt{n}} \sum_{\ell=0}^{+\infty} \frac{\sum_{m=0}^{\ell} \sum_{p=0}^{3\ell-2m} \mathfrak{q}_{\ell,m,p}^{(3)} k^m M_j^p}{n^{\frac{\ell}{2}}}, \tag{4.30}$$

$$\frac{1}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\bigg[1+((-1)^a e^u-1)\bigg(\frac{1}{2}\mathrm{erfc}\Big(-\eta_{j,k}\sqrt{\tfrac{a_{j,k}}{2}}\Big)-R_{a_{j,k}}(\eta_{j,k})\bigg)\bigg]$$
$$\sim r^k\sum_{\ell=0}^{+\infty}\frac{\mathcal{A}_\ell(M_j;k)}{\sqrt{n^\ell}}, \tag{4.31}$$

for some $\mathfrak{q}^{(1)}_{\ell,m,p}, \mathfrak{q}^{(2)}_{\ell,m,p}, \mathfrak{q}^{(3)}_{\ell,m,p}, \mathfrak{q}^{(4)}_{\ell,m,p}, d_{\ell,m}\in\mathbb{C}$, $\mathfrak{q}^{(1)}_{0,0,0}:=1$, $\mathfrak{q}^{(3)}_{0,0,0}:=1$, *where* $a_{j,k}$, $\eta_{j,k}$ *are given in (3.2), and the* $\mathcal{A}_\ell$*'s are defined by*

$$\mathcal{A}_0(x;k)=1+\frac{(-1)^a e^u-1}{2}\mathrm{erfc}\Big(-\frac{r^b x}{\sqrt{2}}\Big), \tag{4.32}$$
$$\mathcal{A}_\ell(x;k)=\bigg(1+\frac{(-1)^a e^u-1}{2}\mathrm{erfc}\Big(-\frac{r^b x}{\sqrt{2}}\Big)\bigg)\sum_{m=1}^{\ell}\sum_{p=0}^{\ell}\mathfrak{q}^{(1)}_{\ell,m,p}k^m x^p$$
$$+((-1)^a e^u-1)\frac{e^{-\frac{r^{2b}x^2}{2}}}{\sqrt{2\pi}}\sum_{m=0}^{\ell}\sum_{p=0}^{3\ell-1-2m}\mathfrak{q}^{(4)}_{\ell,m,p}k^m x^p,\quad \ell\geq 1. \tag{4.33}$$

For $\ell\geq 1$, $m=0$ *and* $p=0,\ldots,\ell$, $\mathfrak{q}^{(1)}_{\ell,m,p}:=0$. *The other coefficients* $\{\mathfrak{q}^{(1)}_{\ell,m,p}\}$ *are given by*

$$\sum_{m=1}^{\ell}\sum_{p=0}^{\ell}\mathfrak{q}^{(1)}_{\ell,m,p}k^m x^p=\sum_{s=0}^{\lfloor \ell/2\rfloor}\frac{1}{(r^{2b})^s}\binom{\frac{k}{2b}}{s}B_s^{(1+\frac{k}{2b})}\Big(\frac{k}{2b}\Big)\binom{s-\frac{k}{2b}}{\ell-2s}x^{\ell-2s}, \tag{4.34}$$

and the coefficients $\{\mathfrak{q}^{(j)}_{\ell,m,p}\}_{j=2}^4$ *can be found by equaling the terms of the same order in n in the following formal power series*

$$\sum_{\ell=1}^{+\infty}\frac{\sum_{m=0}^{\ell}\sum_{p=0}^{3\ell-1-2m}\mathfrak{q}^{(2)}_{\ell,m,p}k^m M_j^p}{n^{\frac{\ell}{2}}}=\sum_{\ell=1}^{+\infty}\frac{2^{\ell/2}}{\ell!}\mathrm{He}_{\ell-1}(M_j r^b)(\Xi^{\mathrm{formal}}_{j,k})^\ell, \tag{4.35}$$
$$\sum_{\ell=0}^{+\infty}\frac{\sum_{m=0}^{\ell}\sum_{p=0}^{3\ell-2m}\mathfrak{q}^{(3)}_{\ell,m,p}k^m M_j^p}{n^{\frac{\ell}{2}}}$$
$$=-3r^b\sqrt{n}\bigg(\sum_{\ell=0}^{+\infty}\frac{2^{\ell/2}}{\ell!}\mathrm{He}_\ell(M_j r^b)(\Xi^{\mathrm{formal}}_{j,k})^\ell\bigg)\bigg(\sum_{\ell=0}^{+\infty}\frac{c_\ell(\eta_{j,k})}{a_{j,k}^{\ell+\frac{1}{2}}}\bigg), \tag{4.36}$$
$$\sum_{\ell=1}^{+\infty}\frac{1}{n^{\frac{\ell}{2}}}\sum_{m=0}^{\ell}\sum_{p=0}^{3\ell-1-2m}\mathfrak{q}^{(4)}_{\ell,m,p}k^m M_j^p=-\bigg(1+\sum_{\ell=1}^{+\infty}\frac{\sum_{m=1}^{\ell}\sum_{p=0}^{\ell}\mathfrak{q}^{(1)}_{\ell,m,p}k^m M_j^p}{n^{\frac{\ell}{2}}}\bigg)$$
$$\times\bigg\{\sum_{\ell=1}^{+\infty}\frac{\sum_{m=0}^{\ell}\sum_{p=0}^{3\ell-1-2m}\mathfrak{q}^{(2)}_{\ell,m,p}k^m M_j^p}{n^{\frac{\ell}{2}}}$$

$$- \frac{1}{3r^b\sqrt{n}} \sum_{\ell=0}^{+\infty} \frac{\sum_{m=0}^{\ell} \sum_{p=0}^{3\ell-2m} \mathfrak{q}_{\ell,m,p}^{(3)} k^m M_j^p}{n^{\frac{\ell}{2}}} \Bigg\}, \tag{4.37}$$

where the c_ℓ's are defined recursively as in (A.3).

Proof By (3.1) and (4.24), we have $j+\alpha = \frac{bnr^{2b}}{1+\frac{M_j}{\sqrt{n}}}$. We conclude that the left-hand side of (4.27) is independent of α, and thus the coefficients $\mathfrak{q}_{\ell,m,p}^{(1)}$ do not depend on α. For $\alpha = 0$, using (2.18), we get

$$\begin{aligned}
\frac{1}{n^{\frac{k}{2b}}} \frac{\Gamma(\frac{2j+k}{2b})}{\Gamma(\frac{2j}{2b})} &\sim r^k \sum_{\ell=0}^{+\infty} \frac{1}{(r^{2b})^\ell} \binom{\frac{k}{2b}}{\ell} B_\ell^{(1+\frac{k}{2b})}\Big(\frac{k}{2b}\Big) \frac{1}{n^\ell} \Big(1+\frac{M_j}{\sqrt{n}}\Big)^{\ell-\frac{k}{2b}} \\
&= r^k \sum_{\ell=0}^{+\infty} \frac{1}{(r^{2b})^\ell} \binom{\frac{k}{2b}}{\ell} B_\ell^{(1+\frac{k}{2b})}\Big(\frac{k}{2b}\Big) \frac{1}{n^\ell} \sum_{s=0}^{+\infty} \binom{\ell-\frac{k}{2b}}{s} \frac{M_j^s}{n^{\frac{s}{2}}} \\
&= r^k \sum_{\ell=0}^{+\infty} \Bigg[\sum_{s=0}^{\lfloor \ell/2 \rfloor} \frac{1}{(r^{2b})^s} \binom{\frac{k}{2b}}{s} B_s^{(1+\frac{k}{2b})}\Big(\frac{k}{2b}\Big) \binom{s-\frac{k}{2b}}{\ell-2s} M_j^{\ell-2s} \Bigg] \frac{1}{n^{\frac{\ell}{2}}},
\end{aligned}$$

and (4.27), (4.34) follow. By (4.25),

$$\begin{aligned}
\Xi_{j,k} - \frac{M_j r^b}{\sqrt{2}} &= -\sqrt{nr^{2b}}(\lambda_{j,k}-1)\sqrt{\frac{\lambda_{j,k}-1-\ln\lambda_{j,k}}{\lambda_{j,k}(\lambda_{j,k}-1)^2}} \\
&\sim -\sqrt{nr^{2b}} \sum_{s=1}^{+\infty} \mathfrak{a}_s^{(1)} (\lambda_{j,k}-1)^s
\end{aligned}$$

as $n \to +\infty$ uniformly for $j \in \{g_-, \ldots, g_+\}$, for some $\{\mathfrak{a}_s^{(1)}\}_{s=1}^{+\infty} \subset \mathbb{C}$ that are independent of j and k. The all-order expansion (4.28) now follows from

$$\begin{aligned}
\lambda_{j,k} &= \Big(1+\frac{M_j}{\sqrt{n}}\Big)\Big(1+\frac{k}{2bnr^{2b}}\Big(1+\frac{M_j}{\sqrt{n}}\Big)\Big)^{-1} \sim \sum_{\ell=0}^{+\infty} \Big(\frac{-k}{2bnr^{2b}}\Big)^\ell \Big(1+\frac{M_j}{\sqrt{n}}\Big)^{\ell+1} \\
&= \sum_{\ell=0}^{+\infty} \frac{[\lambda_{j,k}]_\ell}{n^{\ell/2}},
\end{aligned}$$

where $[\lambda_{j,k}]_\ell$ is given by

$$[\lambda_{j,k}]_\ell = \sum_{s=\lceil(\ell-1)/3\rceil}^{\lfloor \ell/2 \rfloor} \Big(\frac{-1}{2br^{2b}}\Big)^s \binom{s+1}{\ell-2s} k^s M_j^{\ell-2s}.$$

Using that

$$\frac{1}{2}\frac{d^\ell}{dz^\ell}\mathrm{erfc}(z) = -\frac{1}{\sqrt{\pi}}\frac{d^{\ell-1}}{dz^{\ell-1}}e^{-z^2} = \frac{(-1)^\ell}{\sqrt{2\pi}}\,2^{\frac{\ell}{2}}\mathrm{He}_{\ell-1}(\sqrt{2}z)\,e^{-z^2}, \quad \ell \in \mathbb{N}, \tag{4.38}$$

and $\mathrm{He}_\ell(z) = (-1)^\ell \mathrm{He}_\ell(-z)$, we infer that

$$\frac{1}{2}\mathrm{erfc}\Big(-\frac{M_j r^b}{\sqrt{2}} + \Xi_{j,k}\Big) \sim \frac{1}{2}\mathrm{erfc}\Big(-\frac{M_j r^b}{\sqrt{2}}\Big) - \frac{e^{-\frac{M_j^2 r^{2b}}{2}}}{\sqrt{2\pi}}\sum_{\ell=1}^{+\infty}\frac{2^{\ell/2}}{\ell!}\mathrm{He}_{\ell-1}(M_j r^b)\,\Xi_{j,k}^\ell$$

as $n \to +\infty$ uniformly for $j \in \{g_-, \dots, g_+\}$. To verify (4.29), it is thus enough to show (4.35). For this purpose, note that by (4.28), we have

$$\begin{aligned}(\Xi_{j,k}^{\mathrm{formal}})^\ell &= \left(\sum_{s=1}^{+\infty}\frac{\sum_{m=0}^{\lfloor (s+1)/2\rfloor} d_{s,m}\,k^m M_j^{s+1-2m}}{n^{s/2}}\right)^\ell \\ &= \sum_{s=\ell}^{+\infty}\frac{\sum_{m=0}^{\lfloor (s+\ell)/2\rfloor} d_{s,m}^{(\ell)}\,k^m M_j^{s+\ell-2m}}{n^{s/2}}\end{aligned} \tag{4.39}$$

for some coefficients $\{d_{s,m}^{(\ell)}\} \subset \mathbb{C}$. Inserting this expansion and (1.6) in the right-hand side of (4.35), we obtain

$$\begin{aligned}\sum_{\ell=1}^{+\infty}\frac{2^{\ell/2}}{\ell!}\mathrm{He}_{\ell-1}(M_j r^b)(\Xi_{j,k}^{\mathrm{formal}})^\ell &\sim \sum_{\ell=1}^{+\infty}\sum_{s=\ell}^{+\infty}\frac{2^{\ell/2}}{\ell!}\mathrm{He}_{\ell-1}(M_j r^b) \\ &\quad \frac{\sum_{m=0}^{\lfloor (s+\ell)/2\rfloor} d_{s,m}^{(\ell)}\,k^m M_j^{s+\ell-2m}}{n^{s/2}} \\ &\sim \sum_{\ell=1}^{+\infty}\sum_{s=\ell}^{+\infty}\frac{2^{\ell/2}}{\ell!}\left((\ell-1)!\sum_{q=0}^{\lfloor (\ell-1)/2\rfloor}\frac{(-1)^q}{q!(\ell-1-2q)!}\frac{(M_j r^b)^{\ell-1-2q}}{2^q}\right) \\ &\quad \frac{\sum_{m=0}^{\lfloor (s+\ell)/2\rfloor} d_{s,m}^{(\ell)}\,k^m M_j^{s+\ell-2m}}{n^{s/2}}.\end{aligned}$$

Rearranging the summations in the right-hand side of this equation, we obtain (4.35).

We now turn to the proof of (4.30) and (4.36). By (A.2), we have

$$R_{a_{j,k}}(\eta_{j,k}) \sim \frac{e^{-\frac{1}{2}a_{j,k}\eta_{j,k}^2}}{\sqrt{2\pi a_{j,k}}}\sum_{\ell=0}^{+\infty}\frac{c_\ell(\eta_{j,k})}{a_{j,k}^\ell}, \qquad \text{as } n \to +\infty,$$

uniformly for $j \in \{g_-, \dots, g_+\}$. Using again (4.38), we get

$$e^{-\frac{1}{2}a_{j,k}\eta_{j,k}^2} \sim e^{-\frac{M_j^2 r^{2b}}{2}}\sum_{\ell=0}^{+\infty}2^{\ell/2}\,\mathrm{He}_\ell(M_j r^b)\frac{\Xi_{j,k}^\ell}{\ell!}.$$

Combining these expansions yields

$$R_{a_{j,k}}(\eta_{j,k}) \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{M_j^2 r^{2b}}{2}} \Bigg(\sum_{\ell=0}^{+\infty} \frac{2^{\ell/2}}{\ell!} \mathrm{He}_\ell(M_j r^b) \Xi_{j,k}^{\ell} \Bigg) \Bigg(\sum_{\ell=0}^{+\infty} \frac{c_\ell(\eta_{j,k})}{a_{j,k}^{\ell+\frac{1}{2}}} \Bigg).$$

Therefore, by (4.28), to prove (4.30) it is enough to show (4.36). The first term (inside the left parenthesis) on the right-hand side of (4.36) can be expanded in a similar way as (4.35). On the other hand, the second term (inside the right parenthesis) can be expanded using that the c_ℓ's (which are defined below (A.2)) are smooth, and the expansions

$$a_{j,k} = \frac{nr^{2b}}{1+\frac{M_j}{\sqrt{n}}} + \frac{k}{2b} \sim \frac{k}{2b} + nr^{2b} \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell M_j^\ell}{n^{\ell/2}}, \tag{4.40}$$

$$\eta_{j,k} \sim \sum_{s=1}^{+\infty} \mathfrak{b}_s (\lambda_{j,k}-1)^s = \sum_{s=1}^{+\infty} \mathfrak{b}_s \Bigg(\sum_{\ell=1}^{+\infty} \frac{[\lambda_{j,k}]_\ell}{n^{\ell/2}} \Bigg)^s, \tag{4.41}$$

where the coefficients $\{\mathfrak{b}_s\}_{s=1}^{+\infty} \subset \mathbb{C}$ are independent of j and k. Combining all of the above, we obtain the desired asymptotic expansion (4.36).

Finally, (4.31), (4.32), (4.33) and (4.37) are direct consequences of (4.27)–(4.30). □

Lemma 4.6 *The following relations hold*

$$(2b)^a r^{ab} (-1)^a a! \sum_{p=0}^{a} \mathfrak{q}_{a,a,p}^{(1)} \Big(\frac{x}{r^b} \Big)^p = p_{0,a}(x), \tag{4.42}$$

$$\begin{aligned} &(2b)^{a+1} r^{(a+1)b} (-1)^a a! \sum_{p=0}^{a+1} \mathfrak{q}_{a+1,a,p}^{(1)} \Big(\frac{x}{r^b} \Big)^p \\ &= -ab \Big(p_{0,a+1}(x) + (1-3a) p_{0,a-1}(x) \\ &\quad + \frac{5}{3}(a-1)(a-2) p_{0,a-3}(x) \Big), \end{aligned} \tag{4.43}$$

where $p_{0,a}(x)$ is given by (1.8).

Proof It follows from (4.34) that

$$\begin{aligned} &(2b)^a r^{ab} (-1)^a a! \sum_{p=0}^{a} \mathfrak{q}_{a,a,p}^{(1)} \Big(\frac{x}{r^b} \Big)^p \\ &= (2b)^a (-1)^a \sum_{s=0}^{\lfloor a/2 \rfloor} \Big[\frac{d}{dk} \Big]^a \Big[B_s^{(1+\frac{k}{2b})} \Big(\frac{k}{2b} \Big) \binom{\frac{k}{2b}}{s} \binom{s-\frac{k}{2b}}{a-2s} \Big] \Big|_{k=0} x^{a-2s}. \end{aligned}$$

For each $s \in \{0, \ldots, \lfloor \frac{a}{2} \rfloor\}$, the polynomial $k \mapsto B_s^{(1+\frac{k}{2b})}(\frac{k}{2b})\binom{\frac{k}{2b}}{s}\binom{s-\frac{k}{2b}}{a-2s}$ is of the form $\sum_{\ell=0}^{a} \tilde{c}_\ell (k/b)^a$, where $\tilde{c}_0, \ldots, \tilde{c}_a \in \mathbb{C}$ are independent of b. Thanks to the prefactor $(2b)^a$, the above expression is thus independent of b. Replacing b by $\frac{1}{2}$ yields

$$(2b)^a r^{ab}(-1)^a a! \sum_{p=0}^{a} \mathfrak{q}_{a,a,p}^{(1)} \left(\frac{x}{r^b}\right)^p = (-1)^a \sum_{s=0}^{\lfloor a/2 \rfloor} \left[\frac{d}{dk}\right]^a \left[B_s^{(1+k)}(k)\binom{k}{s}\binom{s-k}{a-2s}\right]\Big|_{k=0} x^{a-2s}. \tag{4.44}$$

By (2.20),

$$B_s^{(1+k)}(k) = \left(\frac{d}{dt}\right)^s \left[\left(\frac{t}{e^t-1}\right)^{k+1} e^{kt}\right]\Big|_{t=0} = \frac{1}{2^s}\left(k^s - \frac{s(s+5)}{6}k^{s-1} + \cdots\right) \tag{4.45}$$

is a polynomial in k of degree s, and

$$\binom{k}{s}\binom{s-k}{a-2s} = \frac{(-1)^a}{s!(a-2s)!}\left(k^{a-s} + \frac{7s^2-(6a-3)s+a^2-a}{2}k^{a-1-s} + \cdots\right) \tag{4.46}$$

is a polynomial in k of degree $a-s$. Combining (4.45) and (4.46), we obtain

$$\left[\frac{d}{dk}\right]^a \left[B_s^{(1+k)}(k)\binom{k}{s}\binom{s-k}{a-2s}\right]\Big|_{k=0} = \frac{(-1)^a a!}{s!(a-2s)!2^s},$$

and now (4.42) follows directly from the right-most expression of $p_{0,a}$ in (1.8). Now we turn to the proof of (4.43). In a similar way as (4.44), using (4.34),

$$\begin{aligned}&(2b)^{a+1} r^{(a+1)b}(-1)^a a! \sum_{p=0}^{a+1} \mathfrak{q}_{a+1,a,p}^{(1)} \left(\frac{x}{r^b}\right)^p \\ &= 2b\,(-1)^a \sum_{s=0}^{\lfloor (a+1)/2 \rfloor} \left[\frac{d}{dk}\right]^a \left[B_s^{(1+k)}(k)\binom{k}{s}\binom{s-k}{a+1-2s}\right]\Big|_{k=0} x^{a+1-2s}.\end{aligned}$$

From (4.45) and (4.46) with a replaced by $a+1$, we get

$$\begin{aligned}&\left[\frac{d}{dk}\right]^a \left[B_s^{(1+k)}(k)\binom{k}{s}\binom{s-k}{a+1-2s}\right]\Big|_{k=0} \\ &= \frac{(-1)^{a+1} a!}{s!(a+1-2s)!2^s}\,\frac{20s^2-2(9a+7)s+3a(a+1)}{6},\end{aligned}$$

and thus

$$
\begin{aligned}
&(2b)^{a+1} r^{(a+1)b} (-1)^a a! \sum_{p=0}^{a+1} \mathfrak{q}^{(1)}_{a+1,a,p} \left(\frac{x}{r^b}\right)^p \\
&= -\frac{b}{3} \sum_{s=0}^{\lfloor (a+1)/2 \rfloor} \frac{a!\,(20s^2 - 2(9a+7)s + 3a(a+1))}{s!(a+1-2s)!} \frac{x^{a+1-2s}}{2^s}.
\end{aligned}
$$

Finally, the expression (4.43) follows from the manipulation

$$
\begin{aligned}
&\sum_{s=0}^{\lfloor (a+1)/2 \rfloor} \frac{a!\,(20s^2 - 2(9a+7)s + 3a(a+1))}{s!(a+1-2s)!} \frac{x^{a+1-2s}}{2^s} \\
&= 3a\, p_{0,a+1}(x) + a! \sum_{s=0}^{\lfloor (a-1)/2 \rfloor} \frac{10s + 3(1-3a)}{s!(a-1-2s)!} \frac{x^{a-1-2s}}{2^s} \\
&= 3a\, p_{0,a+1}(x) + 3a(1-3a)\, p_{0,a-1}(x) \\
&\quad + a! \sum_{s=1}^{\lfloor (a-1)/2 \rfloor} \frac{10}{(s-1)!(a-1-2s)!} \frac{x^{a-1-2s}}{2^s} \\
&= 3a\, p_{0,a+1}(x) + 3a(1-3a)\, p_{0,a-1}(x) + 5a(a-1)(a-2)\, p_{0,a-3}(x).
\end{aligned}
$$

□

Lemma 4.7 *The following relations hold*

$$
(2br^b)^{a+1} (-1)^{a+1} (a+1)! \sum_{p=0}^{a} \mathfrak{q}^{(4)}_{a+1,a+1,p} \left(\frac{x}{r^b}\right)^p = q_{0,a+1}(x), \tag{4.47}
$$

$$
\begin{aligned}
&(2br^b)^{a+1} (-1)^a a! \sum_{p=0}^{a+2} \mathfrak{q}^{(4)}_{a+1,a,p} \left(\frac{x}{r^b}\right)^p \\
&\quad = -b\Big(a\, q_{0,a+1}(x) + (1-3a)[a q_{0,a-1}(x)] \\
&\qquad + \frac{5}{3}[a(a-1)(a-2) q_{0,a-3}(x)]\Big),
\end{aligned} \tag{4.48}
$$

where $q_{0,a}(x)$, $[aq_{0,a-1}(x)]$ and $[a(a-1)(a-2)q_{0,a-3}(x)]$ are given by (1.9) and (1.12).

Remark 4.8 The degree of the polynomial in the right-hand side of (4.48) is given by

$$
\begin{cases} 2 & \text{if } a = 0, \\ a & \text{if } a \geq 1. \end{cases}
$$

In particular, $\mathfrak{q}^{(4)}_{a+1,a,a+1} = \mathfrak{q}^{(4)}_{a+1,a,a+2} = 0$.

Proof Let us first rewrite the sums on the left-hand sides of (4.47) and (4.48) in terms of the coefficients $\{\mathfrak{q}^{(1)}_{\ell,m,p}, \mathfrak{q}^{(2)}_{\ell,m,p}, \mathfrak{q}^{(3)}_{\ell,m,p}\}$. By (4.37),

$$\sum_{m=0}^{a+1}\sum_{p=0}^{3a+2-2m}\mathfrak{q}^{(4)}_{a+1,m,p}k^m x^p = -\sum_{s=0}^{a}\left\{\sum_{m=0}^{s}\sum_{p=0}^{s}\mathfrak{q}^{(1)}_{s,m,p}k^m x^p\right.$$
$$\left.\times\left\{\sum_{m=0}^{a+1-s}\sum_{p=0}^{3(a+1-s)-1-2m}\mathfrak{q}^{(2)}_{a+1-s,m,p}k^m x^p - \frac{1}{3r^b}\sum_{m=0}^{a-s}\sum_{p=0}^{3(a-s)-2m}\mathfrak{q}^{(3)}_{a-s,m,p}k^m x^p\right\}\right\}.$$

Equaling the coefficients of k^{a+1} and of k^a gives the identities

$$\sum_{p=0}^{a}\mathfrak{q}^{(4)}_{a+1,a+1,p}x^p = -\sum_{s=0}^{a}\left\{\sum_{p=0}^{s}\mathfrak{q}^{(1)}_{s,s,p}x^p \times \left\{\sum_{p=0}^{a-s}\mathfrak{q}^{(2)}_{a+1-s,a+1-s,p}x^p\right\}\right\} \tag{4.49}$$

and

$$\sum_{p=0}^{a+2}\mathfrak{q}^{(4)}_{a+1,a,p}x^p = -\sum_{s=1}^{a}\left\{\sum_{p=0}^{s}\mathfrak{q}^{(1)}_{s,s-1,p}x^p \times \left\{\sum_{p=0}^{a-s}\mathfrak{q}^{(2)}_{a+1-s,a+1-s,p}x^p\right\}\right\}$$
$$-\sum_{s=0}^{a}\left\{\sum_{p=0}^{s}\mathfrak{q}^{(1)}_{s,s,p}x^p \times \left\{\sum_{p=0}^{a-s+2}\mathfrak{q}^{(2)}_{a+1-s,a-s,p}x^p - \frac{1}{3r^b}\sum_{p=0}^{a-s}\mathfrak{q}^{(3)}_{a-s,a-s,p}x^p\right\}\right\}. \tag{4.50}$$

It remains to simplify the right-hand sides of (4.49) and (4.50). Sums of the form $\sum_{p=0}^{s}\mathfrak{q}^{(1)}_{s,s,p}x^p$ and $\sum_{p=0}^{s}\mathfrak{q}^{(1)}_{s,s-1,p}x^p$ were already simplified in (4.42) and (4.43). Also, by (4.35),

$$\sum_{p=0}^{\ell-1}\mathfrak{q}^{(2)}_{\ell,\ell,p}\left(\frac{x}{r^b}\right)^p = \sum_{s=1}^{\ell}\frac{2^{s/2}}{s!}\mathrm{He}_{s-1}(x)\frac{1}{\ell!}\left[\frac{d}{dk}\right]^{\ell}\left[[(\Xi^{\mathrm{formal}}_{j,k})^s]_\ell\right]\Big|_{k=0,M_j\to\frac{x}{r^b}} \tag{4.51}$$

where $[(\Xi^{\mathrm{formal}}_{j,k})^s]_\ell$ is the coefficient of the term of order $n^{-\frac{\ell}{2}}$ in the asymptotic series $(\Xi^{\mathrm{formal}}_{j,k})^s$. Using (4.28), we infer that

$$\frac{1}{\ell!}\left[\frac{d}{dk}\right]^{\ell}\left[[(\Xi^{\mathrm{formal}}_{j,k})^s]_\ell\right]\Big|_{k=0} = \delta_{s,\ell}d^{\ell}_{1,1}, \qquad s=1,\ldots,\ell. \tag{4.52}$$

Also, a direct computation using (4.25) shows that

$$\Xi_{j,k} = \frac{1}{2\sqrt{2}br^b\sqrt{n}}\left\{k + \frac{5(r^bM_j)^2}{3}b + \frac{1}{\sqrt{n}}\left(\frac{kM_j}{3} - \frac{53}{36}bM_j^3r^{2b}\right) + \mathcal{O}(n^{-1})\right\} \tag{4.53}$$

as $n \to +\infty$ uniformly for $j \in \{g_-, \ldots, g_+\}$. In particular, $d_{1,1} = \frac{1}{2\sqrt{2}br^b}$ and thus, by (4.51)-(4.52),

$$\sum_{p=0}^{\ell-1} \mathfrak{q}^{(2)}_{\ell,\ell,p} \left(\frac{x}{r^b}\right)^p = \frac{1}{(2br^b)^\ell} \frac{1}{\ell!} \mathrm{He}_{\ell-1}(x), \qquad \ell \geq 1. \tag{4.54}$$

Combining (4.49) with (1.8), (4.42) and (4.54) yields

$$(2br^b)^{a+1}(-1)^{a+1}(a+1)! \sum_{p=0}^{a} \mathfrak{q}^{(4)}_{a+1,a+1,p} \left(\frac{x}{r^b}\right)^p = (-1)^a \sum_{s=0}^{a} \binom{a+1}{s} i^s \mathrm{He}_s(ix)\mathrm{He}_{a-s}(x).$$

Using the functional equation (see e.g. [66, eq.(9.6)])

$$\sum_{s=0}^{a} \binom{a+1}{s} i^s \mathrm{He}_s(ix)\mathrm{He}_{a-s}(x) = i^a\, \mathrm{He}^{(1)}_a(ix), \qquad a \in \mathbb{N}, \tag{4.55}$$

we obtain the desired identity (4.47). It remains to simplify the right-hand side of (4.50) (with x replaced by $\frac{x}{r^b}$) and to prove (4.48). In view of (4.42), (4.43) and (4.54), it only remains to evaluate explicitly sums of the forms

$$\sum_{p=0}^{\ell+2} \mathfrak{q}^{(2)}_{\ell+1,\ell,p} \left(\frac{x}{r^b}\right)^p \quad \text{and} \quad \frac{1}{3r^b} \sum_{p=0}^{\ell} \mathfrak{q}^{(3)}_{\ell,\ell,p} \left(\frac{x}{r^b}\right)^p, \qquad \ell \geq 0. \tag{4.56}$$

For the first sum, we use (4.35) to get

$$\sum_{p=0}^{\ell+2} \mathfrak{q}^{(2)}_{\ell+1,\ell,p} \left(\frac{x}{r^b}\right)^p = \sum_{s=1}^{\ell+1} \frac{2^{s/2}}{s!} \mathrm{He}_{s-1}(x) \frac{1}{\ell!} \left[\frac{d}{dk}\right]^\ell \left[[(\Xi^{\mathrm{formal}}_{j,k})^s]_{\ell+1} \right]\Big|_{k=0, M_j \to \frac{x}{r^b}}, \quad \ell \geq 0. \tag{4.57}$$

A direct computation using (4.28) shows that

$$\frac{1}{\ell!}\left[\frac{d}{dk}\right]^\ell \left[[(\Xi^{\mathrm{formal}}_{j,k})^s]_{\ell+1} \right]\Big|_{k=0} = \begin{cases} 0, & \text{if } 1 \leq s \leq \ell-2, \\ (\ell-1)d_{1,1}^{\ell-2}d_{3,2}, & \text{if } s = \ell-1, \\ \ell \frac{x}{r^b} d_{1,1}^{\ell-1} d_{2,1}, & \text{if } s = \ell, \\ (\ell+1)\frac{x^2}{r^{2b}} d_{1,1}^{\ell} d_{1,0}, & \text{if } s = \ell+1, \end{cases}$$

and by (4.25) and (4.53),

$$d_{1,1} = \frac{1}{2\sqrt{2}br^b}, \qquad d_{1,0} = \frac{5r^b}{6\sqrt{2}}, \qquad d_{2,1} = \frac{1}{6\sqrt{2}br^b}, \qquad d_{3,2} = \frac{-1}{24\sqrt{2}b^2r^{3b}}.$$

Substituting the above in (4.57), for $\ell \geq 0$ we get

$$\sum_{p=0}^{\ell+2} \mathfrak{q}_{\ell+1,\ell,p}^{(2)} \left(\frac{x}{r^b}\right)^p = \frac{1}{(2b)^\ell r^{b(\ell+1)}} \frac{1}{\ell!} \Big(\frac{5}{6} x^2 \,\mathrm{He}_\ell(x) + \frac{\ell}{3} x \,\mathrm{He}_{\ell-1}(x) - \frac{\ell(\ell-1)}{6} \mathrm{He}_{\ell-2}(x)\Big),$$
$$= \frac{1}{(2b)^\ell r^{b(\ell+1)}} \frac{1}{\ell!} \Big[\frac{5x^2+2\ell}{6} \mathrm{He}_\ell(x) + \frac{\ell(\ell-1)}{6} \mathrm{He}_{\ell-2}(x)\Big], \tag{4.58}$$

where $\mathrm{He}_{-2}(x) \equiv \mathrm{He}_{-1}(x) \equiv 0$, and for the second line we have used the three-term recurrence relation (1.5) of He_ℓ.

Now we turn to the problem of simplifying the second sum in (4.56). Let us write

$$\sqrt{n}\Big(\sum_{\ell=0}^{+\infty} \frac{c_\ell(\eta_{j,k})}{a_{j,k}^{\ell+\frac{1}{2}}}\Big) \sim \sum_{\ell=0}^{+\infty} \frac{e_\ell}{n^{\frac{\ell}{2}}}, \qquad \text{as } n \to +\infty$$

for some $\{e_\ell\}_{\ell=0}^{+\infty} \subset \mathbb{C}$. By (4.36),

$$\frac{1}{3r^b} \sum_{p=0}^{\ell} \mathfrak{q}_{\ell,\ell,p}^{(3)} = -\frac{1}{\ell!}\Big[\frac{d}{dk}\Big]^\ell \Big\{e_\ell + \sum_{s=1}^{\ell} e_{\ell-s} \sum_{m=1}^{s} \frac{2^{m/2}}{m!} \mathrm{He}_m(x) \Big[[(\Xi_{j,k}^{\mathrm{formal}})^m]_s\Big]\Big\}\Big|_{k=0, M_j \to \frac{x}{r^b}}$$
$$= -\frac{1}{\ell!}\Big\{\Big[\frac{d}{dk}\Big]^\ell e_\ell + \sum_{s=1}^{\ell}\sum_{m=1}^{s} \frac{2^{m/2}}{m!} \mathrm{He}_m(x) \sum_{q=0}^{\ell} \binom{\ell}{q}$$
$$\Big[\frac{d}{dk}\Big]^q e_{\ell-s} \Big[\frac{d}{dk}\Big]^{\ell-q} \Big[[(\Xi_{j,k}^{\mathrm{formal}})^m]_s\Big]\Big\}\Big|_{k=0, M_j \to \frac{x}{r^b}}. \tag{4.59}$$

Long but direct calculations using (4.39), (4.40) and (4.41) show that

$$\Big[\frac{d}{dk}\Big]^q e_{\ell-s}\Big|_{k=0} = 0, \qquad \text{for } \ell \geq 0,\ 0 \leq s \leq \ell,\ q \geq 1 + \Big\lfloor \frac{\ell-s}{2} \Big\rfloor,$$
$$\Big[\frac{d}{dk}\Big]^{\ell-q} \Big[[(\Xi_{j,k}^{\mathrm{formal}})^m]_s\Big]\Big|_{k=0} = 0, \quad \text{for } \ell \geq 1,\ 1 \leq s \leq \ell,\ 1 \leq m \leq s,\ q$$
$$\leq \Big\lfloor \ell - \frac{s+m+1}{2} \Big\rfloor.$$

This means that for $\ell \geq 1$, the only term that contributes in (4.59) corresponds to $m = s = \ell$ and $q = 0$. Thus, for any $\ell \geq 0$, we have

$$\frac{1}{3r^b} \sum_{p=0}^{\ell} \mathfrak{q}_{\ell,\ell,p}^{(3)} = -\Big\{e_0 \frac{2^{\ell/2}}{\ell!} \mathrm{He}_\ell(x)\ \frac{1}{\ell!}\Big[\frac{d}{dk}\Big]^\ell \Big[[(\Xi_{j,k}^{\mathrm{formal}})^\ell]_\ell\Big]\Big\}\Big|_{k=0, M_j \to \frac{x}{r^b}}.$$

A direct computation shows that $e_0 = -\frac{1}{3r^b}$. Using also (4.52), we get

$$\frac{1}{3r^b} \sum_{p=0}^{\ell} \mathfrak{q}_{\ell,\ell,p}^{(3)} \left(\frac{x}{r^b}\right)^p = \frac{1}{3} \frac{1}{(2b)^\ell r^{b(\ell+1)}} \frac{1}{\ell!} \mathrm{He}_\ell(x), \qquad \ell \geq 0. \tag{4.60}$$

Using (4.58) and (4.60), for $\ell \geq 0$ we obtain

$$\sum_{p=0}^{\ell+2} \mathfrak{q}^{(2)}_{\ell+1,\ell,p} \left(\frac{x}{r^b}\right)^p - \frac{1}{3r^b} \sum_{p=0}^{\ell} \mathfrak{q}^{(3)}_{\ell,\ell,p} \left(\frac{x}{r^b}\right)^p$$
$$= \frac{1}{(2b)^\ell r^{b(\ell+1)}} \frac{1}{\ell!} \left[\frac{5x^2 + 2(\ell-1)}{6} \mathrm{He}_\ell(x) + \frac{\ell(\ell-1)}{6} \mathrm{He}_{\ell-2}(x) \right]. \tag{4.61}$$

Combining (4.50) with (4.42), (4.43) and (4.61), we obtain after some calculations that

$$(2br^b)^{a+1}(-1)^a a! \sum_{p=0}^{a+2} \mathfrak{q}^{(4)}_{a+1,a,p} \left(\frac{x}{r^b}\right)^p$$
$$= -\frac{b}{3} \left\{ \sum_{s=1}^{a} (-1)^{a-s} \binom{a}{s-1} \widehat{p}_{0,s}(x) \mathrm{He}_{a-s}(x) + \mathfrak{s}_a(x) \right\}, \tag{4.62}$$

where

$$\widehat{p}_{0,a+1}(x) := 3a\, p_{0,a+1}(x) + 3a(1-3a)\, p_{0,a-1}(x) + 5a(a-1)(a-2)\, p_{0,a-3}(x), \tag{4.63}$$

$$\mathfrak{s}_a(x) := (-1)^a \sum_{s=0}^{a} \binom{a}{s} i^s \mathrm{He}_s(ix) \Big[(5x^2 + 2(a-s-1))\, \mathrm{He}_{a-s}(x)$$
$$+ (a-s)(a-s-1)\mathrm{He}_{a-s-2}(x) \Big]. \tag{4.64}$$

The polynomial $\mathfrak{s}_a$ can actually be considerably simplified. Indeed, since

$$\mathrm{He}_s(x) = \left[\frac{d}{dt}\right]^s \left[e^{xt - \frac{t^2}{2}} \right]\Big|_{t=0}, \qquad i^s \mathrm{He}_s(ix) = \left[\frac{d}{dt}\right]^s \left[e^{-xt + \frac{t^2}{2}} \right]\Big|_{t=0},$$

we have

$$\sum_{s=0}^{a} \binom{a}{s} i^s \mathrm{He}_s(ix) \mathrm{He}_{a-s}(x) = \left[\frac{d}{dt}\right]^a \left[1\right]\Big|_{t=0} = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{if } a \geq 1. \end{cases} \tag{4.65}$$

Hence,

$$\sum_{s=0}^{a} \binom{a}{s} (a-s)(a-s-1) i^s \mathrm{He}_s(ix) \mathrm{He}_{a-s-2}(x)$$
$$= a(a-1) \sum_{s=0}^{a-2} \binom{a-2}{s} i^s \mathrm{He}_s(ix) \mathrm{He}_{a-s-2}(x) = \begin{cases} 2 & \text{if } a = 2, \\ 0 & \text{if } a \neq 2, \end{cases} \tag{4.66}$$

and using also the recurrence relation (1.5) we get

$$\begin{aligned}
&\sum_{s=0}^{a}\binom{a}{s}(a-s)i^s\mathrm{He}_s(ix)\mathrm{He}_{a-s}(x)\\
&\quad= a\sum_{s=0}^{a-1}i^s\binom{a-1}{s}\mathrm{He}_s(ix)\Big(x\mathrm{He}_{a-s-1}(x)-(a-s-1)\mathrm{He}_{a-s-2}(x)\Big)\\
&\quad=\begin{cases} x & \text{if } a=1,\\ -2 & \text{if } a=2,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned} \tag{4.67}$$

Using (4.65), (4.66) and (4.67) to simplify $\mathfrak{s}_a$, we finally obtain

$$\mathfrak{s}_a(x)=\begin{cases} 5x^2-2 & \text{if } a=0,\\ -2x & \text{if } a=1,\\ -2 & \text{if } a=2,\\ 0 & \text{if } a\geq 3.\end{cases}$$

Let us now simplify the sum in (4.62). First, substituting the definitions (4.63) and (1.8), we rewrite it as

$$\sum_{s=1}^{a}(-1)^{a-s}\binom{a}{s-1}\widehat{p}_{0,s}(x)\mathrm{He}_{a-s}(x)=A_1+A_2+A_3, \tag{4.68}$$

where

$$\begin{aligned}
A_1 &:= 3(-1)^a\sum_{s=2}^{a}\binom{a}{s-1}(s-1)i^s\mathrm{He}_s(ix)\mathrm{He}_{a-s}(x),\\
A_2 &:= -3(-1)^a\sum_{s=2}^{a}\binom{a}{s-1}(s-1)\Big(3(s-2)+2\Big)i^{s-2}\mathrm{He}_{s-2}(ix)\mathrm{He}_{a-s}(x),\\
A_3 &:= 5(-1)^a\sum_{s=4}^{a}\binom{a}{s-1}(s-1)(s-2)(s-3)i^s\mathrm{He}_{s-4}(ix)\mathrm{He}_{a-s}(x).
\end{aligned}$$

To simplify A_1, we first establish two formulas. Using (4.65) and $\binom{a+1}{s}=\binom{a}{s}+\binom{a}{s-1}$ in (4.55), we infer that

$$\sum_{s=1}^{a}\binom{a}{s-1}i^s\mathrm{He}_s(ix)\mathrm{He}_{a-s}(x)=\begin{cases} i^a\,\mathrm{He}_a^{(1)}(ix), & \text{if } a\geq 1,\\ 0, & \text{if } a=0.\end{cases} \tag{4.69}$$

Also, using the recurrence (1.5) together with (4.65),

$$
\begin{aligned}
& a\sum_{s=1}^{a}\binom{a-1}{s-1} i^s \mathrm{He}_s(ix)\mathrm{He}_{a-s}(x) \\
& = a\sum_{s=1}^{a}\binom{a-1}{s-1} i^s \Big(ix\mathrm{He}_{s-1}(ix)-(s-1)\mathrm{He}_{s-2}(ix)\Big)\mathrm{He}_{a-s}(x) \\
& = \begin{cases} -x, & \text{if } a=1, \\ 2, & \text{if } a=2, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.70}
$$

For A_1, we use $\binom{a-1}{s-2} = \binom{a}{s-1} - \binom{a-1}{s-1}$ together with (4.69) and (4.70), and find

$$
A_1 = 3(-1)^a a \sum_{s=2}^{a}\binom{a-1}{s-2} i^s \mathrm{He}_s(ix)\mathrm{He}_{a-s}(x) = 3a\, q_{0,a+1}(x) + \begin{cases} -3x & \text{if } a=1, \\ -6 & \text{if } a=2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.71}
$$

Similarly, by (4.55) and (4.69),

$$
\begin{aligned}
A_2 &= 3(-1)^a \sum_{s=2}^{a}\left[3a(a-1)\binom{a-2}{s-3} + 2a\binom{a-1}{s-2}\right] i^{s-2}\mathrm{He}_{s-2}(ix)\mathrm{He}_{a-s}(x) \\
&= 3(1-3a)\left[aq_{0,a-1}(x)\right] + \begin{cases} -3 & \text{if } a=0, \\ 18 & \text{if } a=2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.72}
$$

Simplifying A_3 is a simpler task as it only relies on (4.55), namely

$$
\begin{aligned}
A_3 &= 5(-1)^a a(a-1)(a-2)\sum_{s=0}^{a-4}\binom{a-3}{s} i^s \mathrm{He}_s(ix)\mathrm{He}_{a-4-s}(x) \\
&= \begin{cases} 0, & \text{if } a \in \{0,1,2\} \\ 5a(a-1)(a-2)q_{0,a-3}(x), & \text{if } a \geq 3 \end{cases} \\
&= 5[a(a-1)(a-2)q_{0,a-3}(x)] + \begin{cases} 5(1-x^2), & \text{if } a=0, \\ 5x, & \text{if } a=1, \\ -10, & \text{if } a=2, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.73}
$$

Therefore, by (4.68), (4.71), (4.72) and (4.73), we have

$$\sum_{s=1}^{a}(-1)^{a-s}\binom{a}{s-1}\widehat{p}_{0,s}(x)\mathrm{He}_{a-s}(x)$$
$$= 3a\, q_{0,a+1}(x) + 3(1-3a)[a\, q_{0,a-1}(x)] + 5[a(a-1)(a-2)\, q_{0,a-3}(x)] - \mathfrak{s}_a(x). \tag{4.74}$$

Now (4.48) directly follows from (4.62) and (4.74), which completes the proof. □

In the large n asymptotics of $S_2^{(2)}$, which are obtained in Lemma 4.11 below, the function $\mathcal{G}_0(y; u, a)$, defined in (1.13), will appear inside a logarithm and in a denominator. The next lemma ensures that this function is positive for relevant values of the parameters.

Lemma 4.9 *The function $\mathcal{G}_0(y; u, a)$ given in (1.13) is positive for all $y \in \mathbb{R}$, $u \in \mathbb{R}$, and $a \in \mathbb{N}$.*

Proof Let us write

$$\mathcal{G}_{0,0}(y,a) := (-1)^a\left[p_{0,a}(-\sqrt{2}y)\frac{2-\mathrm{erfc}(y)}{2} - q_{0,a}(-\sqrt{2}y)\frac{e^{-y^2}}{\sqrt{2\pi}}\right],$$
$$\mathcal{G}_{0,1}(y,a) := p_{0,a}(-\sqrt{2}y)\mathrm{erfc}(y) + 2\, q_{0,a}(-\sqrt{2}y)\frac{e^{-y^2}}{\sqrt{2\pi}}.$$

Then by (1.13), we have

$$\mathcal{G}_0(y; u, a) = \mathcal{G}_{0,0}(y,a) + \frac{e^u}{2}\mathcal{G}_{0,1}(y,a). \tag{4.75}$$

Using the definitions (1.8), (1.9), we obtain

$$p'_{0,a+1}(x) = (a+1)p_{0,a}(x), \qquad q'_{0,a+1}(x) = (a+1)q_{0,a}(x) + xq_{0,a+1}(x) - p_{0,a+1}(x). \tag{4.76}$$

The first identity in (4.76) is well-known and easy to prove. The second one follows from two known identities, namely the differentiation rule $(\mathrm{He}_k^{(\nu)})'(x) = (k+\nu)\mathrm{He}_{k-1}^{(\nu)}(x) - \nu\mathrm{He}_{k-1}^{(\nu+1)}(x)$ and the recurrence relation $\mathrm{He}_{k+1}^{(\nu-1)}(x) = x\mathrm{He}_k^{(\nu)}(x) - \nu\mathrm{He}_{k-1}^{(\nu+1)}(x)$, see e.g. [66, eqs (6.3) and (6.6)]. It follows from (4.76) that

$$\frac{d}{dy}\mathcal{G}_{0,0}(y,a+1) = \sqrt{2}(a+1)\mathcal{G}_{0,0}(y,a), \qquad \frac{d}{dy}\mathcal{G}_{0,1}(y,a+1) = -\sqrt{2}(a+1)\mathcal{G}_{0,1}(y,a). \tag{4.77}$$

Let us now show that

$$\mathcal{G}_{0,1}(y,a) > 0, \qquad y \in \mathbb{R}, \tag{4.78}$$

$$\mathcal{G}_{0,1}(y,a) = \sqrt{\frac{2}{\pi}} e^{-y^2} \frac{a!}{(\sqrt{2}y)^{a+1}}\Big(1 + O(y^{-2})\Big), \qquad \text{as } y \to +\infty. \tag{4.79}$$

We shall prove (4.78) and (4.79) by induction on a. For $a = 0$, we have $\mathcal{G}_{0,1}(y,0) = \mathrm{erfc}(y)$ and (4.78), (4.79) follow. Assume now that (4.78) and (4.79) hold for a given a. By combining (4.77) with (4.79), we infer that (4.79) also holds with a replaced by $a+1$; in particular $\mathcal{G}_{0,1}(y,a+1) > 0$ for all sufficiently large $y > 0$. Also, by (4.77) and (4.78), the function $\mathcal{G}_{0,1}(y,a+1)$ is decreasing for $y \in \mathbb{R}$. Since $\mathcal{G}_{0,1}(y,a+1) > 0$ for all sufficiently large $y > 0$, we conclude that $\mathcal{G}_{0,1}(y,a+1) > 0$ for all $y \in \mathbb{R}$.

The proof that $\mathcal{G}_{0,0}(y,a) > 0$ for all $y \in \mathbb{R}$ is similar, so we omit it.

The statement of the lemma now readily follows from (4.75). □

Recall from (3.1) and (4.24) that $M_j = \sqrt{n}(\lambda_j - 1)$ with $\lambda_j := bnr^{2b}/(j+\alpha)$. To obtain the large n asymptotics of $S_2^{(2)}$, we need the following lemma from [21] to approximate sums of the form $\sum_j h(M_j)$ (by contrast, Lemma 2.10 approximates sums of the form $\sum_j f(j/n)$).

Lemma 4.10 ([21, Lemma 3.11]) *Let $h \in C^3(\mathbb{R})$. As $n \to +\infty$, we have*

$$\begin{aligned}
\sum_{j=g_-}^{g_+} h(M_j) &= br^{2b} \int_{-M}^{M} h(t)dt\,\sqrt{n} - 2br^{2b}\int_{-M}^{M} th(t)dt \\
&+ \left(\frac{1}{2} - \theta_-^{(n,M)}\right)h(M) + \left(\frac{1}{2} - \theta_+^{(n,M)}\right)h(-M) \\
&+ \frac{1}{\sqrt{n}}\Bigg[3br^{2b}\int_{-M}^{M} t^2 h(t)dt + \left(\frac{1}{12} + \frac{\theta_-^{(n,M)}(\theta_-^{(n,M)}-1)}{2}\right)\frac{h'(M)}{br^{2b}} \\
&- \left(\frac{1}{12} + \frac{\theta_+^{(n,M)}(\theta_+^{(n,M)}-1)}{2}\right)\frac{h'(-M)}{br^{2b}}\Bigg] \\
&+ \mathcal{O}\Bigg(\frac{1}{n^{3/2}}\sum_{j=g_-+1}^{g_+}\Big((1+|M_j|^3)\tilde{\mathfrak{m}}_{j,n}(h) + (1+M_j^2)\tilde{\mathfrak{m}}_{j,n}(h') \\
&+ (1+|M_j|)\tilde{\mathfrak{m}}_{j,n}(h'') + \tilde{\mathfrak{m}}_{j,n}(h''')\Big)\Bigg),
\end{aligned}$$

where, for $\tilde{h} \in C(\mathbb{R})$ and $j \in \{g_- + 1, \ldots, g_+\}$, $\tilde{\mathfrak{m}}_{j,n}(\tilde{h}) := \max_{x \in [M_j, M_{j-1}]} |\tilde{h}(x)|$.

Lemma 4.11 *As $n \to +\infty$, we have*

$$\begin{aligned}
S_2^{(2)} &= -abr^{2b}M\sqrt{n}\ln n + br^{2b}\sqrt{n}\int_{-M}^{M} h_0(t)dt + a\frac{\theta_-^{(n,M)} + \theta_+^{(n,M)} - 1}{2}\ln n \\
&+ br^{2b}\int_{-M}^{M}\big(h_1(t) - 2th_0(t)\big)dt + \left(\frac{1}{2} - \theta_-^{(n,M)}\right)h_0(M) + \left(\frac{1}{2} - \theta_+^{(n,M)}\right)
\end{aligned}$$

$$h_0(-M) + \mathcal{O}\left(\frac{M^3}{\sqrt{n}} \ln n\right),$$

where

$$\mathcal{H}_0(x) := \frac{r^{a-ab}}{(2b)^a}\,\mathcal{G}_0\Big(-\frac{r^b x}{\sqrt{2}}; u, a\Big), \tag{4.80}$$

$$h_0(x) := \ln\big(\mathcal{H}_0(x)\big), \tag{4.81}$$

$$h_1(x) := \frac{r^{a-(1+a)b}}{(2b)^{a+1}}\,\frac{1}{\mathcal{H}_0(x)}\,\mathcal{G}_1\Big(-\frac{r^b x}{\sqrt{2}}; u, a\Big), \tag{4.82}$$

and the functions $\mathcal{G}_0$ and $\mathcal{G}_1$ are given by (1.13) and (1.14).

Proof Recall the formula (4.26) for $S_2^{(2)}$, namely

$$S_2^{(2)} = \sum_{j:\lambda_j \in I_2} \ln\Bigg(\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})} \times \Bigg[1 + ((-1)^a e^u - 1)\Bigg(\operatorname{erfc}\Big(-\eta_{j,k}\sqrt{\tfrac{a_{j,k}}{2}}\Big) - R_{a_{j,k}}(\eta_{j,k})\Bigg)\Bigg]\Bigg). \tag{4.83}$$

Recall also that for all $j \in \{j : \lambda_j \in I_2\}$, we have $1 - \frac{M}{\sqrt{n}} \le \lambda_j = \frac{bnr^{2b}}{j+\alpha} \le 1 + \frac{M}{\sqrt{n}}$, and $-M \le M_j \le M$.

The expansion (4.31) implies that

$$\sum_{k=0}^{a}\binom{a}{k}\frac{(-r)^{a-k}}{n^{\frac{k}{2b}}}\frac{\Gamma(\frac{2j+2\alpha+k}{2b})}{\Gamma(\frac{2j+2\alpha}{2b})}\Bigg[1 + ((-1)^a e^u - 1)\Bigg(\frac{1}{2}\operatorname{erfc}\Big(-\eta_{j,k}\sqrt{\tfrac{a_{j,k}}{2}}\Big) - R_{a_{j,k}}(\eta_{j,k})\Bigg)\Bigg] \sim \sum_{\ell=0}^{+\infty}\frac{\mathcal{B}_\ell(M_j; a)}{\sqrt{n^\ell}}, \tag{4.84}$$

where

$$\mathcal{B}_\ell(x; a) := \sum_{k=0}^{a}\binom{a}{k}(-1)^{a-k} r^a \mathcal{A}_\ell(x; k). \tag{4.85}$$

It turns out that the first a terms in the expansion (4.84) are 0, i.e.

$$\mathcal{B}_0(\cdot; a) \equiv \mathcal{B}_1(\cdot; a) \equiv \cdots \equiv \mathcal{B}_{a-1}(\cdot; a) \equiv 0. \tag{4.86}$$

To prove this, we use [56, eq 26.8.6], i.e.

$$\sum_{k=0}^{a}\binom{a}{k}(-1)^{a-k}k^\ell = a!\,S(\ell, a), \qquad \ell, a \in \mathbb{N},$$

where we recall that $S(\ell, a)$ is the Stirling number of the second kind. In particular,

$$\sum_{k=0}^{a} \binom{a}{k} (-1)^{a-k} k^{\ell} = \begin{cases} 0 & \text{if } \ell < a, \\ a! & \text{if } \ell = a, \\ (a+1)!\, a/2 & \text{if } \ell = a+1. \end{cases} \tag{4.87}$$

Since $k \mapsto \mathcal{A}_\ell(x; k)$ is a polynomial of degree ℓ by (4.33), the identities (4.86) directly follow from (4.85).

Let us now compute $\mathcal{B}_a(x; a)$ and $\mathcal{B}_{a+1}(x; a)$. By (4.33), (4.85) and (4.87), we have

$$\begin{aligned} \mathcal{B}_a(x; a) = {} & r^a \left((-1)^a + \frac{e^u - (-1)^a}{2} \operatorname{erfc}\left(-\frac{r^b x}{\sqrt{2}} \right) \right) (-1)^a a! \sum_{p=0}^{a} \mathfrak{q}^{(1)}_{a,a,p} x^p \\ & + r^a (e^u - (-1)^a) \frac{e^{-\frac{r^{2b} x^2}{2}}}{\sqrt{2\pi}} (-1)^a a! \sum_{p=0}^{a-1} \mathfrak{q}^{(4)}_{a,a,p} x^p \end{aligned} \tag{4.88}$$

and

$$\begin{aligned} \mathcal{B}_{a+1}(x; a) = {} & r^a \left((-1)^a + \frac{e^u - (-1)^a}{2} \operatorname{erfc}\left(-\frac{r^b x}{\sqrt{2}} \right) \right) (-1)^a a! \\ & \sum_{p=0}^{a+1} \left(\mathfrak{q}^{(1)}_{a+1,a,p} + \frac{a(a+1)}{2} \mathfrak{q}^{(1)}_{a+1,a+1,p} \right) x^p \\ & + r^a (e^u - (-1)^a) \frac{e^{-\frac{r^{2b} x^2}{2}}}{\sqrt{2\pi}} (-1)^a a! \left(\sum_{p=0}^{a+2} \mathfrak{q}^{(4)}_{a+1,a,p} x^p + \frac{a(a+1)}{2} \right. \\ & \left. \sum_{p=0}^{a} \mathfrak{q}^{(4)}_{a+1,a+1,p} x^p \right). \end{aligned} \tag{4.89}$$

More generally, for $\ell \geq 1$, we have

$$\begin{aligned} \mathcal{B}_{a+\ell}(x; a) = {} & r^a \left((-1)^a + \frac{e^u - (-1)^a}{2} \operatorname{erfc}\left(-\frac{r^b x}{\sqrt{2}} \right) \right) (-1)^a a! \\ & \sum_{m=\max(a,1)}^{a+\ell} \sum_{p=0}^{a+\ell} \mathfrak{q}^{(1)}_{a+\ell,m,p} S(m, a) x^p \\ & + r^a (e^u - (-1)^a) \frac{e^{-\frac{r^{2b} x^2}{2}}}{\sqrt{2\pi}} (-1)^a a! \sum_{m=a}^{a+\ell} \sum_{p=0}^{3(a+\ell)-1-2m} \mathfrak{q}^{(4)}_{a+\ell,m,p} S(m, a) x^p. \end{aligned} \tag{4.90}$$

Let us define

$$\mathcal{H}_0(x) := \mathcal{B}_a(x;a), \qquad h_0(x) := \ln(\mathcal{H}_0(x)), \qquad h_1(x) := \frac{\mathcal{B}_{a+1}(x;a)}{\mathcal{B}_a(x;a)}.$$

By combining (4.88) and (4.89) with Lemmas 4.6 and 4.7, we obtain after simplifications that the functions $\mathcal{H}_0(x)$ and $h_1(x)$ can be written as in (4.80) and (4.82).

Since the case $a = 0$ was already done in [19], from here on we focus on the more complicated case $a \geq 1$. We note the following important facts:

(1) By Lemma 4.9, $\mathcal{H}_0(x) > 0$ for all $x \in \mathbb{R}$.
(2) By (4.80) and (4.81), $h_0(x)$ grows logarithmically at $\pm\infty$.
(3) By (4.80) and (4.82), $h_1(x)$ grows linearly at $\pm\infty$.
(4) By (4.90) and (4.88), $\frac{\mathcal{B}_{a+\ell}(x;a)}{\mathcal{B}_a(x;a)} = \mathcal{O}(x^\ell)$ as $x \to \pm\infty$.

(Another reason as to why the case $a = 0$ is significantly simpler than the case $a \geq 1$ stems from the fact that for $a = 0$, the function $h_0(x)$ remains bounded, and furthermore $h_1(x)$ becomes exponentially small as $x \to \pm\infty$.)

After substituting (4.84) in (4.83), we obtain

$$S_2^{(2)} = -\frac{a}{2}\ln n \sum_{j=g_-}^{g_+} 1 + \Sigma_0^{(2)} + \frac{1}{\sqrt{n}}\Sigma_1^{(2)} + \mathcal{O}\Bigg(\frac{1}{n}\sum_{j=g_{k,-}}^{g_{k,+}} M_j^2\Bigg), \tag{4.91}$$

where

$$\Sigma_0^{(2)} := \sum_{j=g_-}^{g_+} h_0(M_j), \qquad \Sigma_1^{(2)} := \sum_{j=g_-}^{g_+} h_1(M_j).$$

The $\mathcal{O}$-term in (4.91) is $\mathcal{O}(\frac{M^3}{\sqrt{n}})$. Also, by Lemma 4.10,

$$-\frac{a}{2}\ln n \sum_{j=g_-}^{g_+} 1 = -abr^{2b}M\sqrt{n}\ln n + a\frac{\theta_-^{(n,M)} + \theta_+^{(n,M)} - 1}{2}\ln n + \mathcal{O}\Bigg(\frac{M^3}{\sqrt{n}}\ln n\Bigg),$$

$$\Sigma_0^{(2)} = br^{2b}\sqrt{n}\int_{-M}^{M} h_0(t)\,dt - 2br^{2b}\int_{-M}^{M} t\,h_0(t)\,dt$$
$$+ \Bigg(\frac{1}{2} - \theta_-^{(n,M)}\Bigg)h_0(M) + \Bigg(\frac{1}{2} - \theta_+^{(n,M)}\Bigg)h_0(-M) + \mathcal{O}\Bigg(\frac{M^3\ln n}{\sqrt{n}}\Bigg),$$

$$\frac{1}{\sqrt{n}}\Sigma_1^{(2)} = br^{2b}\int_{-M}^{M} h_1(t)dt + \mathcal{O}\Bigg(\frac{M^3}{\sqrt{n}}\Bigg), \qquad \frac{1}{n}\sum_{j=g_{k,-}}^{g_{k,+}} M_j^2 = \mathcal{O}\Bigg(\frac{M^3}{\sqrt{n}}\Bigg),$$

as $n \to +\infty$. We now obtain the claim after a computation. □

4.3 Asymptotics of S_2

We are now in a position to compute the large n asymptotics of S_2.

Lemma 4.12 *As $n \to +\infty$,*

$$S_2 = \widehat{C}_1^{(\epsilon)} n + \widehat{C}_2 \sqrt{n} + \widehat{C}_3^{(n,\epsilon)} + \mathcal{O}\Big(\frac{M^3}{\sqrt{n}} \ln n + \frac{\sqrt{n}}{M^5} \Big),$$

where

$$\widehat{C}_1^{(\epsilon)} = \int_{\frac{br^{2b}}{1+\epsilon}}^{br^{2b}} \left(u + a \ln \left(r - \left(\frac{x}{b} \right)^{\frac{1}{2b}} \right) \right) dx + \int_{br^{2b}}^{\frac{br^{2b}}{1-\epsilon}} a \ln \left(\left(\frac{x}{b} \right)^{\frac{1}{2b}} - r \right) dx, \tag{4.92}$$

$$\widehat{C}_2 = br^{2b} \int_{-\infty}^{+\infty} \hat{h}_0(t)\, dt,$$

$$\begin{aligned} \widehat{C}_3^{(n,\epsilon)} &= -\alpha u + \frac{1 + 2\alpha - 2\theta_-^{(n,\epsilon)}}{2} \left(u + a \ln \left(r - r(1+\epsilon)^{-\frac{1}{2b}} \right) \right) \\ &+ \frac{1 - 2\alpha - 2\theta_+^{(n,\epsilon)}}{2} a \ln \left(r(1-\epsilon)^{-\frac{1}{2b}} - r \right) \\ &+ \frac{a}{4} \left\{ \frac{1-a}{1-(1+\epsilon)^{-\frac{1}{2b}}} + \frac{1-a}{(1-\epsilon)^{-\frac{1}{2b}} - 1} + (a - 2b + 4\alpha) \ln \left(\frac{(1-\epsilon)^{-\frac{1}{2b}} - 1}{1 - (1+\epsilon)^{-\frac{1}{2b}}} \right) \right\} \\ &+ br^{2b} \int_{-\infty}^{+\infty} \left\{ h_1(t) - 2t h_0(t) + \frac{a}{4b} \left(1 + 2b + 8b \ln \left(\frac{r|t|}{2b} \right) \right) t + 2ut \chi_{(0,\infty)}(t) \right. \\ &\left. - \frac{(2ab - a^2)t}{4br^{2b}(1+t^2)} \right\} dt. \end{aligned} \tag{4.93}$$

Here

$$\hat{h}_0(x) = h_0(x) - a \ln \left(\frac{r|x|}{2b} \right) - u\, \chi_{(0,\infty)}(x)$$

and h_0, h_1 are given in the statement of Lemma 4.11.

Proof Combining Lemmas 4.1, 4.4 and 4.11 yields

$$\begin{aligned} S_2 &= \widehat{C}_1^{(\epsilon)} + \widehat{C}_2'^{(M)} \sqrt{n} \ln n + \widehat{C}_2^{(M)} \sqrt{n} + \widehat{C}_3'^{(n,M)} \ln n + \widehat{C}_3^{(n,\epsilon,M)} \\ &+ \mathcal{O}\Big(\frac{M^3}{\sqrt{n}} \ln n + \frac{\sqrt{n}}{M^5} \Big), \end{aligned} \tag{4.94}$$

as $n \to +\infty$, where $\widehat{C}_1^{(\epsilon)}$ is given by (4.92). After short (but remarkable) simplifications, we obtain $\widehat{C}_2'^{(M)} = \widehat{C}_3'^{(\epsilon,M)} = 0$. The quantity $\widehat{C}_2^{(M)}$ is given by

$$\widehat{C}_2^{(M)} = br^{2b}M\left(2a\ln\left(\frac{2b}{rM}\right)+2a-u\right)+\frac{a(a-1)b}{M}-\frac{ab(a-1)(2a-3)}{6r^{2b}M^3}$$
$$+br^{2b}\int_{-M}^{M}h_0(t)dt.$$

Using the definition (4.81) of h_0, we verify that

$$h_0(t) = a\ln\left(\frac{r|t|}{2b}\right)+u\chi_{(0,\infty)}(t)+\frac{a(a-1)}{2r^{2b}t^2}-\frac{a(a-1)(2a-3)}{4r^{4b}t^4}+\mathcal{O}(t^{-6}),$$
$$\text{as } t\to\pm\infty, \tag{4.95}$$

from which we conclude

$$\widehat{C}_2^{(M)}\sqrt{n} = \widehat{C}_2\sqrt{n}+\mathcal{O}\left(\frac{\sqrt{n}}{M^5}\right), \qquad \text{as } n\to+\infty,$$

where $\widehat{C}_2$ is given by (4.93). The quantity $\widehat{C}_3^{(n,\epsilon,M)}$ in (4.94) is given by

$$\widehat{C}_3^{(n,\epsilon,M)} = \left(\frac{1}{2}-\theta_-^{(n,M)}\right)\widehat{C}_{3,1}^{(M)}+\left(\frac{1}{2}-\theta_+^{(n,M)}\right)\widehat{C}_{3,2}^{(M)}+\widehat{C}_{3,3}^{(n,\epsilon)}+\widehat{C}_{3,4}^{(M)},$$

where

$$\widehat{C}_{3,1}^{(M)} := h_0(M)-u-a\ln\left(\frac{rM}{2b}\right), \qquad \widehat{C}_{3,2}^{(M)} := h_0(-M)-a\ln\left(\frac{rM}{2b}\right),$$
$$\widehat{C}_{3,3}^{(n,\epsilon)} := \frac{1+2\alpha-2\theta_-^{(n,\epsilon)}}{2}\left(u+a\ln\left(r-r(1+\epsilon)^{-\frac{1}{2b}}\right)\right)$$
$$+\frac{1-2\alpha-2\theta_+^{(n,\epsilon)}}{2}a\ln\left(r(1-\epsilon)^{-\frac{1}{2b}}-r\right)$$
$$-\alpha u+\frac{a}{4}\left\{\frac{1-a}{1-(1+\epsilon)^{-\frac{1}{2b}}}+\frac{1-a}{(1-\epsilon)^{-\frac{1}{2b}}-1}\right.$$
$$\left.+(a-2b+4\alpha)\ln\left(\frac{(1-\epsilon)^{-\frac{1}{2b}}-1}{1-(1+\epsilon)^{-\frac{1}{2b}}}\right)\right\},$$
$$\widehat{C}_{3,4}^{(M)} = br^{2b}\left\{uM^2+\int_{-M}^{M}(h_1(t)-2th_0(t))dt\right\}.$$

It readily follows from (4.95) that

$$\widehat{C}_{3,1}^{(M)} = \mathcal{O}\left(\frac{1}{M^2}\right), \qquad \widehat{C}_{3,2}^{(M)} = \mathcal{O}\left(\frac{1}{M^2}\right), \qquad \text{as } n\to+\infty.$$

We also verify from (4.81), (4.82), (4.95), and

$$\frac{p_{1,a}(x)}{p_{0,a}(x)} = -\frac{a}{2}(1+2b)x - \frac{a}{2}\frac{a+2b(1-2a)}{x} + O(x^{-3}), \qquad \text{as } x \to \pm\infty$$

that

$$h_1(t) - 2th_0(t) = -\frac{a}{4b}\left(1+2b+8b\ln\left(\frac{r|t|}{2b}\right)\right)t - 2ut\chi_{(0,\infty)}(t) + \frac{2ab-a^2}{4br^{2b}t} + \mathcal{O}(t^{-3})$$

as $t \to \pm\infty$. We conclude that

$$\widehat{C}_{3,4}^{(M)} = br^{2b}\int_{-\infty}^{+\infty}\left\{h_1(t) - 2th_0(t) + \frac{a}{4b}\left(1+2b+8b\ln\left(\frac{r|t|}{2b}\right)\right)t + 2ut\chi_{(0,\infty)}(t) - \frac{(2ab-a^2)t}{4br^{2b}(1+t^2)}\right\}dt + \mathcal{O}(M^{-2}), \qquad \text{as } n \to +\infty,$$

and the claim follows. □

5 Proof of Theorem 1.1

By combining (2.4) with Lemmas 3.1, 3.2, 3.3 and 4.12, we obtain

$$\ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3 = C_1 n + C_2\sqrt{n} + C_3 + \mathcal{O}\left(\frac{M^3}{\sqrt{n}}\ln n + \frac{\sqrt{n}}{M^5} + n^{-\frac{1}{2b}}\right),$$

as $n \to +\infty$. Here we obtain the constants C_1, C_2 and C_3 of (1.16) after a long computation using (1.13)–(1.14), a change of variables, and simplifying. Since $M = n^{\frac{1}{8}}(\ln n)^{-\frac{1}{8}}$, the error term is $\mathcal{O}\left(n^{-\frac{1}{2b}} + (\ln n)^{\frac{5}{8}}n^{-\frac{1}{8}}\right)$, which finishes the proof of Theorem 1.1.

Appendix: Uniform Asymptotics of the Incomplete Gamma Function

In this section, we collect some known asymptotic formulas for $\gamma(\tilde{a}, z)$ that are useful for us.

Lemma A.1 (Taken from [56, formula 8.11.2]). *Let $\tilde{a} > 0$ be fixed. As $z \to +\infty$,*

$$\gamma(\tilde{a}, z) = \Gamma(\tilde{a}) + \mathcal{O}(e^{-\frac{z}{2}}).$$

Lemma A.2 (Taken from [61, Section 11.2.4]). *For* $\tilde{a} > 0$ *and* $z > 0$, *we have*

$$\frac{\gamma(\tilde{a}, z)}{\Gamma(\tilde{a})} = \frac{1}{2}\operatorname{erfc}(-\eta\sqrt{\tilde{a}/2}) - R_{\tilde{a}}(\eta), \qquad R_{\tilde{a}}(\eta) = \frac{e^{-\frac{1}{2}\tilde{a}\eta^2}}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\tilde{a}u^2} g(u)du,$$

where erfc *is defined in (1.15),* $g(u) = \frac{dt}{du}\frac{1}{\lambda - t} + \frac{1}{u + i\eta}$,

$$\lambda = \frac{z}{\tilde{a}}, \quad \eta = (\lambda - 1)\sqrt{\frac{2(\lambda - 1 - \ln\lambda)}{(\lambda - 1)^2}}, \quad u = -i(t-1)\sqrt{\frac{2(t - 1 - \ln t)}{(t-1)^2}}, \quad \text{(A.1)}$$

where $\operatorname{sign}(\eta) = \operatorname{sign}(\lambda - 1)$, *and* $\operatorname{sign}(u) = \operatorname{sign}(\operatorname{Im} t)$ *with* $t \in \mathcal{L} := \{\frac{\theta}{\sin\theta}e^{i\theta} : -\pi < \theta < \pi\}$ *and* $u \in \mathbb{R}$ *(in particular* $u = -i(t-1) + \mathcal{O}((t-1)^2)$ *as* $t \to 1$*). Furthermore,*

$$R_{\tilde{a}}(\eta) \sim \frac{e^{-\frac{1}{2}\tilde{a}\eta^2}}{\sqrt{2\pi\tilde{a}}}\sum_{j=0}^{+\infty}\frac{c_j(\eta)}{\tilde{a}^j} \qquad \text{as } \tilde{a} \to +\infty \qquad \text{(A.2)}$$

uniformly for $z \in [0, \infty)$, *where all coefficients* $c_j(\eta)$ *are bounded functions of* $\eta \in \mathbb{R}$ *(i.e. bounded for* $\lambda \in [0, \infty)$*) and given by*

$$c_0 = \frac{1}{\lambda - 1} - \frac{1}{\eta}, \qquad c_j = \frac{1}{\eta}\frac{d}{d\eta}c_{j-1}(\eta) + \frac{\gamma_j}{\lambda - 1}, \ j \geq 1, \qquad \text{(A.3)}$$

where the γ_j *are the Stirling coefficients*

$$\gamma_j = \frac{(-1)^j}{2^j\, j!}\left[\frac{d^{2j}}{dx^{2j}}\left(\frac{1}{2}\frac{x^2}{x - \ln(1+x)}\right)^{j+\frac{1}{2}}\right]_{x=0}.$$

The first few c_j *are* $c_0(\eta) = \frac{1}{\lambda - 1} - \frac{1}{\eta}$ *and*

$$c_1(\eta) = \frac{1}{\eta^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda-1)^2} - \frac{1}{12(\lambda-1)},$$
$$c_2(\eta) = -\frac{3}{\eta^5} + \frac{3}{(\lambda-1)^5} + \frac{5}{(\lambda-1)^4} + \frac{25}{12(\lambda-1)^3} + \frac{1}{12(\lambda-1)^2} + \frac{1}{288(\lambda-1)},$$
$$c_3(\eta) = \frac{15}{\eta^7} - \frac{15}{(\lambda-1)^7} - \frac{35}{(\lambda-1)^6} - \frac{105}{4(\lambda-1)^5} - \frac{77}{12(\lambda-1)^4} - \frac{49}{288(\lambda-1)^3} - \frac{1}{288(\lambda-1)^2} + \frac{139}{51840(\lambda-1)}.$$

In particular, the following hold:

(i) Let $\delta > 0$ *be fixed, and let* $z = \lambda\tilde{a}$. *As* $\tilde{a} \to +\infty$, *uniformly for* $\lambda \geq 1 + \delta$,

$$\gamma(\tilde{a}, z) = \Gamma(\tilde{a})\big(1 + \mathcal{O}(e^{-\frac{\tilde{a}\eta^2}{2}})\big).$$

(ii) Let $z = \lambda\tilde{a}$. *As* $\tilde{a} \to +\infty$, *uniformly for* λ *in compact subsets of* $(0, 1)$,

$$\gamma(\tilde{a}, z) = \Gamma(\tilde{a})\mathcal{O}(e^{-\frac{\tilde{a}\eta^2}{2}}).$$

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