Constructive Approximation
https://doi.org/10.1007/s00365-024-09682-4

CONSTRUCTIVE APPROXIMATION

Non-polynomial q-Askey Scheme: Integral Representations, Eigenfunction Properties, and Polynomial Limits

Jonatan Lenells[1] · **Julien Roussillon**[1]

Received: 20 February 2022 / Revised: 5 January 2024 / Accepted: 26 February 2024

Abstract
We construct a non-polynomial generalization of the q-Askey scheme. Whereas the elements of the q-Askey scheme are given by q-hypergeometric series, the elements of the non-polynomial scheme are given by contour integrals, whose integrands are built from Ruijsenaars' hyperbolic gamma function. Alternatively, the integrands can be expressed in terms of Faddeev's quantum dilogarithm, Woronowicz's quantum exponential, or Kurokawa's double sine function. We present the basic properties of all the elements of the scheme, including their integral representations, joint eigenfunction properties, and polynomial limits.

Keywords q-Askey scheme · Orthogonal polynomial · Confluent limit · Conformal field theory · Virasoro fusion kernel · Ruijsenaars' hypergeometric function · Quantum dilogarithm

Mathematics Subject Classification 33D45 · 33D70 · 33E20 · 81T40

1 Introduction

An important tool in the classification of the different families of classical (basic) hypergeometric orthogonal polynomials is the (q-)Askey scheme [25]. It provides a tree (drawn upside down) of the different families of (q-)orthogonal polynomials, where each downward oriented edge corresponds to an explicit limit transition from one family to another. In particular, each family in the q-Askey scheme satisfies orthogonality relations, a 3-term recurrence relation, and a second-order q-difference

Communicated by Erik Koelink.

✉ Julien Roussillon
julienro@kth.se; jlenells@kth.se

[1] Department of Mathematics, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden

Published online: 11 March 2024

equation. At the top of the q-Askey scheme lies the celebrated family of Askey–Wilson (AW) polynomials [3] which additionally admits a duality relation exchanging its geometric and spectral parameter.

The theory of hypergeometric series and integrals has four levels: rational, trigonometric, hyperbolic and elliptic, with increasing complexity. Moreover, there exist formal limits of the form elliptic → hyperbolic/trigonometric → rational. The associated Gamma functions are the ordinary one (rational), the q-Gamma function (trigonometric, requiring $|q| < 1$), the hyperbolic Gamma function, and the elliptic Gamma function. The Askey scheme and the q-Askey scheme describe the rational and trigonometric levels, respectively. The q-Askey scheme has a trivial extension to the hyperbolic level, with the families now consisting of products of polynomials $P_n \tilde{P}_m$, where P_n (resp. $\tilde{P}_m$) corresponds to the associated family in the q-Askey scheme relative to the base $q = e^{2i\pi b^2}$ (resp. $\tilde{q} = e^{2i\pi b^{-2}}$). However, in this case the orthogonality relations break down when $b \in \mathbb{R}$, which corresponds to the natural hyperbolic regime. Finally, there is no elliptic Askey scheme; however, there is a fairly well understood theory of elliptic hypergeometric biorthogonal functions which generalizes the classical basic hypergeometric orthogonal polynomials. For instance, it was shown in [6] that all families in the q-Askey scheme are obtained as limits of elliptic hypergeometric biorthogonal functions.

Non-polynomial extensions of (q-)Askey schemes have been less extensively studied in the literature. In particular, the first two levels of a trigonometric non-polynomial extension of the q-Askey scheme were introduced in [28, 29]. The top-level family generalizing the AW polynomials consists of a very-well poised ${}_8\phi_7$ basic hypergeometric series called the AW-function (see also [9, 18, 23, 28, 59]). The $q = 1$ analog of the AW-function is the Wilson function, which is a non-polynomial generalization of the Wilson polynomials [17]. The second level consists of the big and little q-Jacobi functions, and both these functions arise as limits of the AW-function.

Our goal in this paper is to provide a complete hyperbolic non-polynomial q-Askey scheme. In fact, we will only be interested in continuous (as opposed to discrete) orthogonal polynomials and we will therefore only consider the part of the q-Askey scheme originating from the Askey–Wilson polynomials shown in Fig. 1. Just like the Askey and q-Askey schemes, the non-polynomial scheme we present has five levels, see Fig. 2. Whereas each element in the q-Askey scheme is a family of polynomials, each element of the non-polynomial scheme is a meromorphic function. These meromorphic functions are given by contour integrals whose integrands are built from Ruijsenaars' hyperbolic gamma function [50] (alternatively, the integrands can be expressed in terms of Faddeev's quantum dilogarithm [14], Woronowicz's quantum exponential [62], or Kurokawa's double sine function [37]). The non-polynomial scheme is a generalization of the q-Askey scheme in the sense that each element in Fig. 1 is obtained from an element at the same level in Fig. 2 when one of the variables is suitably discretized.

Just like in the q-Askey scheme, the elements in the non-polynomial scheme are joint eigenfunctions. Indeed, we will show that each meromorphic function in the non-polynomial scheme satisfies two pairs of difference equations. In the polynomial

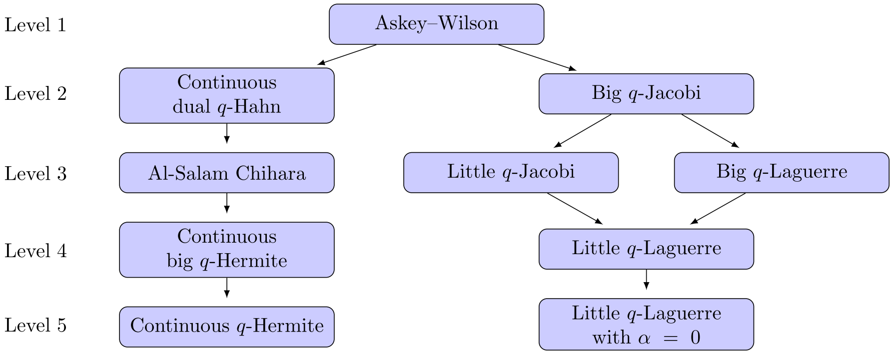

Fig. 1 The part of the q-Askey scheme relevant for the present paper

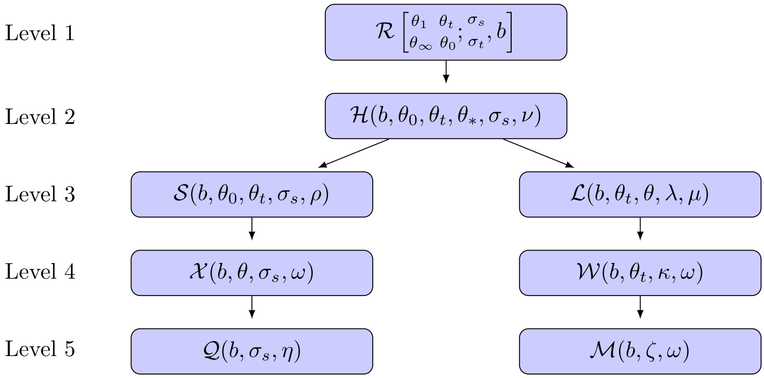

Fig. 2 The elements of the scheme constructed in the present paper. It is a non-polynomial generalization of the scheme in Fig. 1

limit, one of these pairs reduces to the recurrence relation, while the other pair reduces to the q-difference equation.

The first (top) level of the non-polynomial scheme consists of a single function, namely, Ruijsenaars' hypergeometric function $\mathcal{R}$[1] [51]. Below this top level, there are four further levels involving functions which can be obtained from $\mathcal{R}$ by taking various limits; we denote these functions by $\mathcal{H}$, $\mathcal{S}$, $\mathcal{X}$, $\mathcal{Q}$, $\mathcal{L}$, $\mathcal{W}$, and $\mathcal{M}$, where the letters are chosen so that $\mathcal{H}$ is a non-polynomial generalization of the continuous dual q-Hahn polynomials which are denoted by H_n in "Appendix B", $\mathcal{S}$ is a non-polynomial generalization of the Al-Salam Chihara polynomials which are denoted by S_n in "Appendix B", etc.

Each of the functions in the non-polynomial scheme depends on a number of parameters as well as two variables. For example, $\mathcal{H}$ depends on the parameters b, θ_0, θ_t, θ_* as well as the two variables σ_s, ν, while $\mathcal{S}$ depends on the parameters b, θ_0, θ_t as well as the two variables σ_s, ρ, see Fig. 2. As a matter of notation, we will refer to the

[1] Ruijenaars' hypergeometric function is usually referred to as R in the literature. However, we denote the Askey–Wilson polynomials in (B.1) by R_n. Thus, to avoid confusion, we denote Ruijenaars' hypergeometric function by $\mathcal{R}$.

top level of the scheme (which involves $\mathcal{R}$) as the first level, to the next level (which involves $\mathcal{H}$) as the second level, etc. Each function at levels 2–5 will be defined as a limit of the function above it in the scheme. We will show that this limit exists (at least for a certain range of the two variables). We will also derive an integral representation for the function, show that it extends to a meromorphic function of each of the two variables everywhere in the complex plane, and establish two pairs of difference equations (one in each of the two variables). Finally, we will establish the polynomial limit to the corresponding element in the q-Askey scheme. Actually, two of the functions in the non-polynomial scheme, namely $\mathcal{H}$ and $\mathcal{S}$, possess two different polynomial limits: $\mathcal{H}$ reduces to the continuous dual q-Hahn polynomials when ν is discretized and to the big q-Jacobi polynomials when σ_s is discretized; similarly, $\mathcal{S}$ reduces to the Al-Salam Chihara polynomials when ρ is discretized and to the little q-Jacobi polynomials when σ_s is discretized.

The orthogonal polynomials in the Askey and q-Askey schemes are of fundamental importance in a wide variety of fields. Elements in the q-Askey scheme have found applications, for instance, in models of statistical mechanics [11, 56, 60, 61], in representation theory of quantum algebras [1, 4, 15, 30, 31, 33, 34, 42, 43, 63], and in the geometry of Painlevé equations [41]. We expect the functions in the proposed non-polynomial scheme to also be relevant in many different contexts (see Sect. 1.1 for a related discussion). This is certainly true of the top element, Ruijsenaars' $\mathcal{R}$-function. As an example of the broad relevance of $\mathcal{R}$, we note that one of the authors recently showed [48] that $\mathcal{R}$ is equivalent (up to a change of variables) to the Virasoro fusion kernel which is a central object in conformal field theory [10, 45–47]. In fact, it was in the context of conformal field theory that we first conceived of the non-polynomial scheme presented in this paper. First, in [38], we introduced a family of confluent Virasoro fusion kernels $\mathcal{C}_k$ while studying confluent conformal blocks of the second kind of the Virasoro algebra. Later, we realized that the $\mathcal{C}_k$ can be viewed as non-polynomial generalizations of the continuous dual q-Hahn and the big q-Jacobi polynomials, which led us to conjecture that there exists a non-polynomial generalization of the q-Askey scheme with the Virasoro fusion kernel as its top member [39]. In this paper, we prove this conjecture. However, instead of adopting the Virasoro fusion kernel as the top element of the scheme, we use Ruijsenaars' $\mathcal{R}$-function as our starting point. The result of [48] implies that these two choices are equivalent, but we have found that the scheme originating from $\mathcal{R}$ is simpler and mathematically more convenient.

Let us also mention that a more general degeneration scheme of hyperbolic hypergeometric integrals was studied in [8, 9] (see in particular [8, Fig. 5.8]) whose top-family consists of a hyperbolic hypergeometric integral depending on 8 parameters and possessing an E_7-symmetry. It was shown in [9, Sect. 4.6] that Ruijsenaars' $\mathcal{R}$-function is a special case of this function. Although [8] and [9] do not focus on constructing non-polynomial extensions of Askey schemes, the results of these works are relevant for the constructions in this paper. Another relevant work is [5] where a systematic description of limits from elliptic hypergeometric functions to basic hypergeometric functions is described.

1.1 Relations to Other Areas

We discuss some relations that deserve to be studied in more detail in the future.

1.1.1 Relation to Quantum Relativistic Integrable Systems

The function $\mathcal{R}$ was introduced in [49] in the context of relativistic systems of Calogero-Moser type, and studied in greater detail in [51–53]. In particular, after a proper change of parameters, the difference operator $D_{\mathcal{R}}$ defined in (3.6) corresponds to the rank one hyperbolic van Diejen Hamiltonian. Strong coupling (or Toda) limits of $D_{\mathcal{R}}$ and of its higher rank generalizations were considered in [12]. In this paper, we have shown that each element in the non-polynomial scheme is a joint eigenfunction of four difference operators, which, by construction, are confluent limits of $D_{\mathcal{R}}$. Thus it would be desirable to understand if each confluent limit considered in the present article corresponds to a Toda-type limit. In fact, a renormalized version of the function $\mathcal{Q}$ (see (7.6)), which is one of the two elements at the fifth level of the non-polynomial scheme, was studied in [24, 54] and was interpreted as the eigenfunction of a q-Toda type Hamiltonian.

1.1.2 Relation to Two-Dimensional CFTs

As mentioned in the introduction, Ruijsenaars' hypergeometric function is essentially equal to the Virasoro fusion kernel [48]. The latter plays a key role in the conformal bootstrap approach to Liouville conformal field theory on punctured Riemann spheres [45]. The Virasoro fusion kernel also appears in four-dimensional supersymmetric gauge theory as a result of the AGT correspondence [2]. We believe that the various confluent limits considered in the present article correspond to collisions of punctures in Liouville theory. Therefore, we expect the other elements in the non-polynomial scheme to play a role in Liouville theory on different Riemann surfaces with punctures and cusps [16].

1.1.3 Relation to Quantum Groups and Double Affine Hecke Algebras (DAHA)

Families in the Askey scheme have a well-understood group theoretic interpretation [32]. As for the q-Askey scheme, the AW polynomials are well-rooted in the representation theory of quantum groups [43] and of DAHA [44]. Similar interpretations have been found for the AW-function [57, 58]. On the other hand, motivated by the application of quantum groups in two-dimensional conformal field theories, Faddeev introduced in [13] the modular double of $U_q(sl_2(\mathbb{R}))$, denoted $U_{q\tilde{q}}(sl_2(\mathbb{R}))$. Both the functions $\mathcal{R}$ and $\mathcal{Q}$ can be obtained from the viewpoint of harmonic analysis on $U_{q\tilde{q}}(sl_2(\mathbb{R}))$ [7, 24, 45, 46]. More generally, we expect that all the elements of the non-polynomial scheme can be obtained from the viewpoint of harmonic analysis on $U_{q\tilde{q}}(g)$, where g is the Lie algebra of a non-compact Lie group. Examples of such modular doubles were studied in [20–22]. Finally, an understanding of the $\mathcal{R}$-function from the viewpoint of DAHA is currently lacking.

1.2 Outlook

It was shown in [52, Theorem 1.1] that a renormalized version of the function $\mathcal{R}$ has a hidden D_4 symmetry in its four external parameters. It would be interesting to determine what this D_4 invariance becomes after taking the various confluent limits presented in this article.

The $\mathcal{R}$-function can be expressed as a sum of two terms, where each term is proportional to a product of two AW-functions [9, Theorem 6.5] (this is a non-polynomial analog of the hyperbolic extension of the q-Askey scheme described in the introduction). It is therefore natural to study the relation between our non-polynomial hyperbolic scheme and the non-polynomial trigonometric scheme of [27].

A Fourier transform admitting the AW-function as kernel was constructed in [28] (see also [26, 29] for the little and big q-Jacobi functions, respectively). Moreover, in a certain region of the external parameter space, Ruijsenaars constructed in [53] a unitary Hilbert space transform admitting the function $\mathcal{R}$ as a kernel. An important project is to handle the Hilbert space theory of the other difference equations studied in this article. This project would be of particular interest, since the theory of linear (analytic) difference equations is much less understood than the theory of linear discrete difference, or linear differential equations. In particular, it is not known if the existence of joint eigenfunctions implies that they can be promoted to kernels of unitary Hilbert space transforms.

Furthermore, the Askey–Wilson polynomials $R_n(z; \alpha, \beta, \gamma, \delta, q)$ are symmetric under the exchange $z \to z^{-1}$. The non-symmetric Askey–Wilson polynomials, introduced in [44, 55], are Laurent polynomials satisfying a difference/reflection equation. A symmetrization procedure described in [44] can be used to recover the symmetric Askey–Wilson polynomials from their non-symmetric counterparts. More generally, a non-symmetric extension of the q-Askey scheme was initiated in [36, 40]. Finally, a non-symmetric AW-function, as well as its associated Fourier transform, was constructed in [57]. It would be interesting to construct a non-symmetric $\mathcal{R}$-function generalizing the non-symmetric AW-function, and study its transition limits.

1.3 Organization of the Paper

In Sect. 2, we introduce the function $s_b(z)$ which is the basic building block used to define the elements of the non-polynomial scheme. The eight functions $\mathcal{R}, \mathcal{H}, \mathcal{S}, \mathcal{X}, \mathcal{Q}, \mathcal{L}, \mathcal{W}, \mathcal{M}$ that make up the non-polynomial scheme are considered one by one in the eight Sects. 3–10. In Sect. 11, we derive—as an easy application of the non-polynomial scheme—a few duality formulas that relate members of the q-Askey scheme. In the two appendices, we have collected relevant definitions and properties of q-hypergeometric series and of the q-Askey scheme.

1.4 Standing Assumption

Throughout the paper, we make the following assumption.

Assumption 1.1 *(Restrictions on the parameters)* We assume that

$$(b, \theta_\infty, \theta_1, \theta_t, \theta_0, \theta_*, \theta) \in (0, \infty) \times \mathbb{R}^6. \tag{1.1}$$

Assumption 1.1 is made primarily for simplicity; we expect that all our results admit meromorphic continuations to more general values of the parameters, such as $b \notin i\mathbb{R}$ and $(\theta_\infty, \theta_1, \theta_t, \theta_0, \theta_*, \theta) \in \mathbb{C}^6$.

1.5 Notational Conventions

Our choice of parameters for the $\mathcal{R}$-function has its origin in conformal field theory. The parameters $\theta_0, \theta_t, \theta_1, \theta_\infty$ represent the conformal dimensions of the four Virasoro primary fields. This notation is also frequently used in the context of the Painlevé VI equation [19]. Moreover, the parameters σ_s and σ_t are the internal momenta characterizing the Virasoro conformal blocks in the s- and t-channels, respectively [48]. Finally, the parameter b characterizing the quantum deformation parameters $q = e^{2i\pi b^2}$ and $\tilde{q} = e^{2i\pi b^{-2}}$ parametrizes the central charge in conformal field theory as $c = 1 + 6(b + 1/b)^2$.

2 The Function $s_b(z)$

The elements of the non-polynomial scheme presented in this article are given by contour integrals, whose integrands involve the function $s_b(z)$ defined by

$$s_b(z) = \exp\left[i \int_0^\infty \frac{dy}{y} \left(\frac{\sin 2yz}{2\sinh(b^{-1}y)\sinh(by)} - \frac{z}{y} \right) \right], \qquad |\mathrm{Im}\, z| < \frac{Q}{2}, \tag{2.1}$$

where $Q := b + b^{-1}$. The function $s_b(z)$ is closely related to Ruijsenaars' hyperbolic gamma function [50], Faddeev's quantum dilogarithm function [14], Woronowicz's quantum exponential function [62], and Kurokawa's double sine function [37]. More precisely, it is related to Ruijsenaars' hyperbolic gamma function G in [51, Eq. (A.3)] by

$$s_b(z) = G(b, b^{-1}; z). \tag{2.2}$$

It follows from (2.2) and the results in [51] that $s_b(z)$ is a meromorphic function of $z \in \mathbb{C}$ with zeros $\{z_{m,l}\}_{m,l=0}^\infty$ and poles $\{p_{m,l}\}_{m,l=0}^\infty$ located at

$$\begin{aligned} z_{m,l} &= \frac{iQ}{2} + imb + ilb^{-1}, \qquad m, l = 0, 1, 2, \ldots, \qquad \text{(zeros)}, \\ p_{m,l} &= -\frac{iQ}{2} - imb - ilb^{-1}, \qquad m, l = 0, 1, 2, \ldots, \qquad \text{(poles)}. \end{aligned} \tag{2.3}$$

The multiplicity of the zero $z_{m,l}$ in (2.3) is given by the number of distinct pairs $(m_i, l_i) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that $z_{m_i,l_i} = z_{m,l}$. The pole $p_{m,l}$ has the same multiplicity as the zero $z_{m,l}$. In particular, if b^2 is an irrational real number, then all the zeros and poles in (2.3) are distinct and simple. The residue s_b at the simple pole $z = -iQ/2$ is given by

$$\operatorname*{Res}_{z=-\frac{iQ}{2}} s_b(z) = \frac{i}{2\pi}. \tag{2.4}$$

Furthermore, s_b is a meromorphic solution of the following pair of difference equations:

$$\frac{s_b(z+\frac{ib}{2})}{s_b(z-\frac{ib}{2})} = 2\cosh \pi b z, \qquad \frac{s_b(z+\frac{i}{2b})}{s_b(z-\frac{i}{2b})} = 2\cosh \frac{\pi z}{b}. \tag{2.5}$$

Applying the difference equations (2.5) recursively, it can be verified that, for any integer $m \geq 0$,

$$\frac{s_b(x+imb^{\pm 1})}{s_b(x)} = e^{-\frac{\pi b^{\pm 1} m}{2}(2x+ib^{\pm 1}m)} \left(-e^{i\pi b^{\pm 2}} e^{2\pi b^{\pm 1} x}; e^{2i\pi b^{\pm 2}}\right)_m, \tag{2.6}$$

where the q-Pochhammer symbol $(a;q)_m$ is defined in "Appendix A". Finally, the function $s_b(z)$ has the obvious symmetry

$$s_b(z) = s_{b^{-1}}(z) \tag{2.7}$$

and possesses an asymptotic formula which is a consequence of [51, Theorem A.1] and (2.2): For each $\epsilon > 0$,

$$\pm \ln s_b(z) = -\frac{i\pi z^2}{2} - \frac{i\pi}{24}(b^2 + b^{-2}) + O(e^{-\frac{2\pi(1-\epsilon)}{\max(b,b^{-1})}|\operatorname{Re} z|}), \qquad \operatorname{Re} z \to \pm\infty, \tag{2.8}$$

uniformly for $(b, \operatorname{Im} z)$ in compact subsets of $(0,\infty) \times \mathbb{R}$.

3 The Function $\mathcal{R}$

The top element of the non-polynomial q-Askey scheme presented in this paper is Ruijsenaars' hypergeometric function $\mathcal{R}$ [51]. Using the notation of [48], this function can be expressed as

$$\mathcal{R}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; \begin{matrix}\sigma_s \\ \sigma_t\end{matrix}, b\right] = P_{\mathcal{R}}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; \begin{matrix}\sigma_s \\ \sigma_t\end{matrix}, b\right] \int_{\mathcal{C}_{\mathcal{R}}} dx\ I_{\mathcal{R}}\left[x; \begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; \begin{matrix}\sigma_s \\ \sigma_t\end{matrix}, b\right], \tag{3.1}$$

where the prefactor $P_{\mathcal{R}}$ isgiven by

$$P_{\mathcal{R}}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; \begin{matrix}\sigma_s \\ \sigma_t\end{matrix}, b\right] = s_b\left(\frac{iQ}{2} + 2\theta_t\right)$$
$$\times \prod_{\epsilon=\pm 1} s_b\left(\frac{iQ}{2} + \theta_0 + \theta_1 + \epsilon\theta_\infty + \theta_t\right) s_b(\epsilon\sigma_s - \theta_0 - \theta_t) s_b(\epsilon\sigma_t - \theta_1 - \theta_t), \tag{3.2}$$

and the integrand $I_{\mathcal{R}}$ is defined by

$$I_{\mathcal{R}}\left[x; \begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; \begin{matrix}\sigma_s \\ \sigma_t\end{matrix}, b\right] = \frac{1}{s_b(x + \frac{iQ}{2}) s_b(x + \frac{iQ}{2} + 2\theta_t)}$$
$$\times \prod_{\epsilon=\pm 1} \frac{s_b(x + \theta_0 + \theta_t + \epsilon\sigma_s) s_b(x + \theta_1 + \theta_t + \epsilon\sigma_t)}{s_b\left(x + \frac{iQ}{2} + \theta_0 + \theta_1 + \epsilon\theta_\infty + \theta_t\right)}. \tag{3.3}$$

In view of (2.3), the integrand $I_{\mathcal{R}}$ possesses eight semi-infinite sequences of poles in the complex x-plane. With the restriction that $b > 0$ imposed by Assumption 1.1, there are four vertical downward sequences starting at $x = -\theta_0 - \theta_t \pm \sigma_s - \frac{iQ}{2}$ and $x = -\theta_1 - \theta_t \pm \sigma_t - \frac{iQ}{2}$, and four vertical upward sequences starting at $x = 0$, $x = -2\theta_t$, and $x = -\theta_0 - \theta_1 \pm \theta_\infty - \theta_t$. The contour $\mathcal{C}_{\mathcal{R}}$ in (3.1) is any curve from $-\infty$ to $+\infty$ which separates the four upward from the four downward sequences of poles. If in addition to Assumption 1.1, we also assume that $\sigma_s, \sigma_t \in \mathbb{R}$, then the contour of integration $\mathcal{C}_{\mathcal{R}}$ can be chosen to be any curve from $-\infty$ to $+\infty$ lying in the open strip $\operatorname{Im} x \in (-Q/2, 0)$.

Thanks to the symmetry (2.7) of the function s_b, we have

$$\mathcal{R}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; \begin{matrix}\sigma_s \\ \sigma_t\end{matrix}, b^{-1}\right] = \mathcal{R}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; \begin{matrix}\sigma_s \\ \sigma_t\end{matrix}, b\right]. \tag{3.4}$$

Moreover, it follows directly from (3.1) that $\mathcal{R}$ is even in each of the variables σ_s and σ_t, and that it satisfies the following self-duality symmetry:

$$\mathcal{R}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; \begin{matrix}\sigma_s \\ \sigma_t\end{matrix}, b\right] = \mathcal{R}\left[\begin{matrix}\theta_0 & \theta_t \\ \theta_\infty & \theta_1\end{matrix}; \begin{matrix}\sigma_t \\ \sigma_s\end{matrix}, b\right]. \tag{3.5}$$

3.1 Difference Equations

Let $e^{\pm ib\partial_x}$ be the translation operator which formally acts on meromorphic functions $f(x)$ by $e^{\pm ib\partial_x} f(x) = f(x \pm ib)$. Let $D_{\mathcal{R}}$ be the difference operator defined by

$$D_{\mathcal{R}}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; b, \sigma_s\right] = d_{\mathcal{R}}^{+}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; b, \sigma_s\right] e^{ib\partial_{\sigma_s}} + d_{\mathcal{R}}^{+}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; b, -\sigma_s\right] e^{-ib\partial_{\sigma_s}}$$
$$+ d_{\mathcal{R}}^{0}\left[\begin{matrix}\theta_1 & \theta_t \\ \theta_\infty & \theta_0\end{matrix}; b, \sigma_s\right], \tag{3.6}$$

where

$$d_{\mathcal{R}}^{+}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; b, \sigma_s\right] = -\frac{4\prod_{\epsilon=\pm1}\cosh\left(\pi b\left(\frac{ib}{2}+\theta_t+\sigma_s+\epsilon\theta_0\right)\right)\cosh\left(\pi b\left(\frac{ib}{2}+\theta_1+\sigma_s+\epsilon\theta_\infty\right)\right)}{\sinh(2\pi b\sigma_s)\sinh(\pi b(2\sigma_s+ib))} \tag{3.7}$$

and

$$d_{\mathcal{R}}^{0}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; b, \sigma_s\right] = -2\cosh\left(2\pi b(\theta_1+\theta_t+\tfrac{ib}{2})\right) - d_{\mathcal{R}}^{+}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; b, \sigma_s\right] - d_{\mathcal{R}}^{+}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; b, -\sigma_s\right]. \tag{3.8}$$

For $(\sigma_s, \sigma_t) \in \mathbb{C}^2$, the function $\mathcal{R}$ satisfies the following four difference equations [51] (using the notation of [48]):

$$D_{\mathcal{R}}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; b, \sigma_s\right]\mathcal{R}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; \begin{smallmatrix}\sigma_s\\ \sigma_t\end{smallmatrix}, b\right] = 2\cosh(2\pi b\sigma_t)\mathcal{R}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; \begin{smallmatrix}\sigma_s\\ \sigma_t\end{smallmatrix}, b\right], \tag{3.9a}$$

$$D_{\mathcal{R}}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; b^{-1}, \sigma_s\right]\mathcal{R}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; \begin{smallmatrix}\sigma_s\\ \sigma_t\end{smallmatrix}, b\right] = 2\cosh(2\pi b^{-1}\sigma_t)\mathcal{R}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; \begin{smallmatrix}\sigma_s\\ \sigma_t\end{smallmatrix}, b\right], \tag{3.9b}$$

$$D_{\mathcal{R}}\left[\begin{smallmatrix}\theta_0 & \theta_t\\ \theta_\infty & \theta_1\end{smallmatrix}; b, \sigma_t\right]\mathcal{R}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; \begin{smallmatrix}\sigma_s\\ \sigma_t\end{smallmatrix}, b\right] = 2\cosh(2\pi b\sigma_s)\mathcal{R}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; \begin{smallmatrix}\sigma_s\\ \sigma_t\end{smallmatrix}, b\right], \tag{3.9c}$$

$$D_{\mathcal{R}}\left[\begin{smallmatrix}\theta_0 & \theta_t\\ \theta_\infty & \theta_1\end{smallmatrix}; b^{-1}, \sigma_t\right]\mathcal{R}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; \begin{smallmatrix}\sigma_s\\ \sigma_t\end{smallmatrix}, b\right] = 2\cosh(2\pi b^{-1}\sigma_s)\mathcal{R}\left[\begin{smallmatrix}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{smallmatrix}; \begin{smallmatrix}\sigma_s\\ \sigma_t\end{smallmatrix}, b\right]. \tag{3.9d}$$

Note that the difference equations (3.9b), (3.9c), and (3.9d) follow from (3.9a) and the symmetries (3.4)–(3.5) of the function $\mathcal{R}$.

3.2 Polynomial Limit

It was shown in [51] that the function $\mathcal{R}$ reduces to the Askey–Wilson polynomials when one of the variables σ_s and σ_t is suitably discretized. This result was reobtained in the CFT setting in [39]. We now recall the result of [39]. In addition to Assumption 1.1, we need the following assumption.

Assumption 3.1 *(Restriction on the parameters)* Assume that $b > 0$ is such that b^2 is irrational, and that, for $\epsilon, \epsilon' = \pm 1$,

$$\theta_\infty, \theta_t, \operatorname{Re}\sigma_s, \operatorname{Re}\sigma_t \neq 0, \qquad \operatorname{Re}\left(\theta_0 - \theta_1 + \epsilon\sigma_s + \epsilon'\sigma_t\right) \neq 0, \\ \theta_0 + \theta_1 + \epsilon\theta_\infty + \epsilon'\theta_t \neq 0. \tag{3.10}$$

Assumption 3.1 implies that the four increasing and the four decreasing sequences of poles of the integrand in (3.3) do not overlap. The assumption that b^2 is irrational implies that all the poles of the integrand are simple.

Theorem 3.2 *[39, Theorem 4.2] Suppose Assumptions 1.1 and 3.1 are satisfied. Define* $\{\sigma_s^{(n)}\}_{n=0}^\infty \subset \mathbb{C}$ *by*

$$\sigma_s^{(n)} = \theta_0 + \theta_t + \frac{iQ}{2} + ibn. \tag{3.11}$$

Under the parameter correspondence

$$\alpha_R = e^{2\pi b\left(\frac{iQ}{2}+\theta_1+\theta_t\right)}, \quad \beta_R = e^{2\pi b\left(\frac{iQ}{2}+\theta_0-\theta_\infty\right)}, \quad \gamma_R = e^{2\pi b\left(\frac{iQ}{2}-\theta_1+\theta_t\right)},$$
$$\delta_R = e^{2\pi b\left(\frac{iQ}{2}+\theta_0+\theta_\infty\right)}, q = e^{2i\pi b^2}, \tag{3.12}$$

the Ruijsenaars hypergeometric function $\mathcal{R}$ defined in (3.1) satisfies, for each integer $n \geq 0$,

$$\lim_{\sigma_s \to \sigma_s^{(n)}} \mathcal{R}\begin{bmatrix} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{bmatrix}; \begin{matrix} \sigma_s \\ \sigma_t \end{matrix}, b \Bigg] = R_n(e^{2\pi b \sigma_t}; \alpha_R, \beta_R, \gamma_R, \delta_R, q), \tag{3.13}$$

where R_n are the Askey–Wilson polynomials defined in (B.1).

Remark 3.3 [39, Remark 4.3] The result of Theorem 3.2 can be generalized as follows. Instead of considering the limit of $\mathcal{R}$ as σ_s approaches one of the points $\sigma_s^{(n)}$ defined in (3.11), we can consider the limit

$$\sigma_s \to \sigma_s^{(n,m)} := \sigma_s^{(n)} + \frac{im}{b}, \tag{3.14}$$

for any integers $n, m \geq 0$. In this limit, $\mathcal{R}$ reduces to a product of two Askey–Wilson polynomials of the form $R_n \times R_m$. The first polynomial R_n is expressed in terms of the quantum deformation parameter $q = e^{2i\pi b^2}$, while the second polynomial R_m is expressed in terms of $\tilde{q} = e^{2i\pi b^{-2}}$. In the case $m = 0$ treated in Theorem 3.2, the second polynomial reduces to $R_0 = 1$.

A generalization similar to the one described in Remark 3.3 can be formulated for each of the families described in this article, thereby providing a limit from our scheme to the hyperbolic extension of the q-Askey scheme. As a matter of brevity, the details of this construction will be omitted.

4 The Function $\mathcal{H}$

The Askey–Wilson polynomials R_n form the top element of the q-Askey scheme. Just like the elements of the q-Askey scheme are obtained from the polynomials R_n via various limiting procedures, the elements of the non-polynomial scheme we present in this paper are obtained by taking various limits of Ruijsenaars' hypergeometric function $\mathcal{R}$. The second level of the non-polynomial scheme involves the function $\mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu)$, which is defined as the confluent limit $\Lambda \to -\infty$ of the top element $\mathcal{R}$ evaluated at

$$\theta_\infty = \frac{\Lambda - \theta_*}{2}, \qquad \theta_1 = \frac{\Lambda + \theta_*}{2}, \qquad \sigma_t = \frac{\Lambda}{2} + \nu. \tag{4.1}$$

In this section, we derive an integral representation for the function $\mathcal{H}$ and we show that it is a joint eigenfunction of four difference operators, two acting on σ_s and the other two on ν. We also show that $\mathcal{H}$ reduces to the continuous dual q-Hahn and the big q-Jacobi polynomials when ν and σ_s are suitably discretized, respectively. Since these polynomials lie at the second level of the q-Askey scheme, this shows that $\mathcal{H}$ indeed provides a natural non-polynomial generalization of the elements at the second level.

4.1 Definition and Integral Representation

Definition 4.1 The function $\mathcal{H}$ is defined by

$$\mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = \lim_{\Lambda \to -\infty} \mathcal{R}\begin{bmatrix} \frac{\Lambda+\theta_*}{2} & \theta_t & \sigma_s \\ \frac{\Lambda-\theta_*}{2} & \theta_0 & \frac{\Lambda}{2}+\nu \end{bmatrix}; b \Bigg]. \tag{4.2}$$

The next theorem shows that, for each choice of $(b, \theta_0, \theta_t, \theta_*) \in (0, \infty) \times \mathbb{R}^3$, $\mathcal{H}$ is a well-defined and analytic function of

$$(\sigma_s, \nu) \in (\mathbb{C} \setminus \Delta_{\mathcal{H},\sigma_s}) \times \big(\{\operatorname{Im} \nu > -Q/2\} \setminus \Delta_\nu\big), \tag{4.3}$$

where $\Delta_{\mathcal{H},\sigma_s}, \Delta_\nu \subset \mathbb{C}$ are discrete sets of points at which $\mathcal{H}$ may have poles. In particular, $\mathcal{H}$ is a meromorphic function of $\sigma_s \in \mathbb{C}$ and of ν for $\operatorname{Im} \nu > -Q/2$. More precisely, $\Delta_{\mathcal{H},\sigma_s}$ and Δ_ν are given by

$$\Delta_{\mathcal{H},\sigma_s} := \{\pm\sigma_s \mid \sigma_s \in \Delta'_{\mathcal{H},\sigma_s}\}, \tag{4.4a}$$

$$\begin{aligned} \Delta_\nu := &\{\tfrac{\theta_*}{2} \pm \theta_t + \tfrac{iQ}{2} + imb + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{-\tfrac{\theta_*}{2} - \theta_0 + \tfrac{iQ}{2} + imb + ilb^{-1}\}_{m,l=0}^{\infty} \\ &\cup \{\tfrac{\theta_*}{2} + \theta_t - \tfrac{iQ}{2} - imb - ilb^{-1}\}_{m,l=0}^{\infty} \end{aligned} \tag{4.4b}$$

where

$$\begin{aligned} \Delta'_{\mathcal{H},\sigma_s} := &\{\theta_0 \pm \theta_t + \tfrac{iQ}{2} + imb + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{-\theta_* + \tfrac{iQ}{2} + imb + ilb^{-1}\}_{m,l=0}^{\infty} \\ &\cup \{\theta_0 + \theta_t - \tfrac{iQ}{2} - imb - ilb^{-1}\}_{m,l=0}^{\infty}. \end{aligned}$$

The theorem also provides an integral representation for $\mathcal{H}$ for (σ_s, ν) satisfying (4.3). In fact, even if the requirement $\operatorname{Im} \nu > -Q/2$ is needed to ensure convergence of the integral in the integral representation for $\mathcal{H}$, we will show later in this section, with the help of the difference equations satisfied by $\mathcal{H}$, that $\mathcal{H}$ extends to a meromorphic function of $(\sigma_s, \nu) \in \mathbb{C}^2$.

Theorem 4.2 *Suppose that Assumption 1.1 holds. Let $\Delta_{\mathcal{H},\sigma_s}, \Delta_\nu \subset \mathbb{C}$ be the discrete subsets defined in (4.4). Then the limit in (4.2) exists uniformly for (σ_s, ν) in compact subsets of*

$$\Omega_{\mathcal{H}} := (\mathbb{C} \setminus \Delta_{\mathcal{H},\sigma_s}) \times \big(\{Im\, \nu > -Q/2\} \setminus \Delta_\nu\big).$$

Moreover, $\mathcal{H}$ is an analytic function of $(\sigma_s, \nu) \in \Omega_{\mathcal{H}}$ and admits the following integral representation:

$$\mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = P_{\mathcal{H}}(\sigma_s, \nu) \int_{\mathcal{C}_{\mathcal{H}}} dx \; I_{\mathcal{H}}(x, \sigma_s, \nu) \qquad \textit{for } (\sigma_s, \nu) \in \Omega_{\mathcal{H}}, \tag{4.5}$$

where the dependence of $P_{\mathcal{H}}$ and $I_{\mathcal{H}}$ on $b, \theta_0, \theta_t, \theta_*$ is omitted for simplicity,

$$\begin{aligned} P_{\mathcal{H}}(\sigma_s, \nu) &= s_b\left(2\theta_t + \tfrac{iQ}{2}\right) s_b\left(\theta_0 + \theta_* + \theta_t + \tfrac{iQ}{2}\right) s_b\left(\nu - \tfrac{\theta_*}{2} - \theta_t\right) \\ &\quad \times \prod_{\epsilon=\pm 1} s_b\left(\epsilon\sigma_s - \theta_0 - \theta_t\right), \end{aligned} \tag{4.6}$$

$$\begin{aligned} I_{\mathcal{H}}(x, \sigma_s, \nu) &= e^{i\pi x\left(\frac{\theta_*}{2} - \theta_0 + \nu - \frac{iQ}{2}\right)} \\ &\quad \times \frac{s_b\left(x + \frac{\theta_*}{2} + \theta_t - \nu\right) \prod_{\epsilon=\pm 1} s_b(x + \theta_0 + \theta_t + \epsilon\sigma_s)}{s_b\left(x + \frac{iQ}{2}\right) s_b\left(x + 2\theta_t + \frac{iQ}{2}\right) s_b\left(x + \theta_0 + \theta_* + \theta_t + \frac{iQ}{2}\right)}, \end{aligned} \tag{4.7}$$

and the contour $\mathcal{C}_{\mathcal{H}}$ is any curve from $-\infty$ to $+\infty$ which separates the three upward from the three downward sequences of poles. In particular, $\mathcal{H}$ is a meromorphic function of $(\sigma_s, \nu) \in \mathbb{C} \times \{Im\, \nu > -Q/2\}$. If $(\sigma_s, \nu) \in \mathbb{R}^2$, then the contour $\mathcal{C}_{\mathcal{H}}$ can be any curve from $-\infty$ to $+\infty$ lying within the strip $Im\, x \in (-Q/2, 0)$.

Proof Let $(b, \theta_0, \theta_t, \theta_*) \in (0, \infty) \times \mathbb{R}^3$. Using the identity $s_b(z) = 1/s_b(-z)$, it is straightforward to verify that

$$P_{\mathcal{R}}\begin{bmatrix} \frac{\Lambda+\theta_*}{2} & \theta_t \\ \frac{\Lambda-\theta_*}{2} & \theta_0 \end{bmatrix}; \begin{matrix} \sigma_s \\ \frac{\Lambda}{2}+\nu \end{matrix}, b \Big] I_{\mathcal{R}}\begin{bmatrix} x; \begin{matrix} \frac{\Lambda+\theta_*}{2} & \theta_t \\ \frac{\Lambda-\theta_*}{2} & \theta_0 \end{matrix}; \begin{matrix} \sigma_s \\ \frac{\Lambda}{2}+\nu \end{matrix}, b \end{bmatrix} = P_{\mathcal{H}}(\sigma_s, \nu)\, X(x, \Lambda) I_{\mathcal{H}}(x, \sigma_s, \nu), \tag{4.8}$$

where the dependence of $X(x, \Lambda)$ on $b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu$ is omitted for simplicity, and

$$\begin{aligned} X(x, \Lambda) \\ &= e^{i\pi x(\frac{iQ}{2} + \theta_0 - \frac{\theta_*}{2} - \nu)} \frac{s_b\left(\Lambda + \theta_0 + \theta_t + \frac{iQ}{2}\right)}{s_b\left(\Lambda + \frac{\theta_*}{2} + \theta_t + \nu\right)} \frac{s_b\left(x + \Lambda + \frac{\theta_*}{2} + \theta_t + \nu\right)}{s_b\left(x + \Lambda + \theta_0 + \theta_t + \frac{iQ}{2}\right)}. \end{aligned} \tag{4.9}$$

Due to the properties (2.3) of the function s_b, the function $I_{\mathcal{H}}(\cdot, \sigma_s, \nu)$ possesses three increasing sequences of poles starting at $x = 0$, $x = -2\theta_t$ and $x = -\theta_0 - \theta_* - \theta_t$, as well as three decreasing sequences of poles starting at $x = -\frac{iQ}{2} - \frac{\theta_*}{2} - \theta_t + \nu$ and $x = -\frac{iQ}{2} \pm \sigma_s - \theta_0 - \theta_t$. The discrete sets $\Delta_{\mathcal{H},\sigma_s}$ and Δ_ν contain all the values of σ_s and ν, respectively, for which poles in any of the three increasing collide with poles in any of the decreasing sequences. Indeed, consider for example the decreasing sequence starting at $x = -\frac{iQ}{2} - \frac{\theta_*}{2} - \theta_t + \nu$ and the increasing sequence starting at $x = 0$. Poles from these two sequences collide if and only if

$$\nu \in \{\tfrac{\theta_*}{2} + \theta_t + \tfrac{iQ}{2} + imb + ilb^{-1}\}_{m,l=0}^{\infty},$$

giving rise to the first set on the right-hand side of (4.4b).

Similarly, (2.3) implies that $X(\cdot, \Lambda)$ possesses one increasing sequence of poles starting at $x = -\Lambda - \theta_0 - \theta_t$ and one decreasing sequence of poles starting at $x = -\frac{iQ}{2} - \Lambda - \frac{\theta_*}{2} - \theta_t - \nu$. The real parts of the poles in these two sequences tend to $+\infty$ as $\Lambda \to -\infty$. The increasing sequence lies in the half-plane $\operatorname{Im} x \geq 0$ and the decreasing sequence lies in the half-plane $\operatorname{Im} x \leq -\operatorname{Im} \nu - Q/2$.

The sets $\Delta_{\mathcal{H},\sigma_s}$ and Δ_ν also contain all the values of σ_s and ν at which the prefactor $P_{\mathcal{H}}(\sigma_s, \nu)$ has poles. For example, $P_{\mathcal{H}}$ has poles originating from the factor $s_b\left(\nu - \frac{\theta_*}{2} - \theta_t\right)$ at

$$\nu = \tfrac{\theta_*}{2} + \theta_t - \frac{iQ}{2} - imb - ilb^{-1}, \qquad m, l = 0, 1, 2, \ldots,$$

giving rise to the last set on the right-hand side of (4.4b).

Let K_{σ_s} be a compact subset of $\mathbb{C} \setminus \Delta_{\mathcal{H},\sigma_s}$ and let K_ν be a compact subset of $\{\operatorname{Im} \nu > -Q/2\} \setminus \Delta_\nu$. Suppose $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$. Then, the above discussion shows that it is possible to choose a contour $\mathcal{C}_{\mathcal{H}} = \mathcal{C}_{\mathcal{H}}(\sigma_s, \nu)$ from $-\infty$ to $+\infty$ which separates the four upward from the four downward sequences of poles of $X(\cdot, \Lambda) I_{\mathcal{H}}(\cdot, \sigma_s, \nu)$. It also follows that if we let the right tail of $\mathcal{C}_{\mathcal{H}}$ approach the horizontal line $\operatorname{Im} x = -\epsilon$ as $\operatorname{Re} x \to +\infty$, where $\epsilon > 0$ is sufficiently small, then there exists a $N < 0$ such that $\mathcal{C}_{\mathcal{H}}$ can be chosen to be independent of Λ for $\Lambda < -N$. Thus, for such a choice of $\mathcal{C}_{\mathcal{H}}$, (3.1) and (4.8) imply that, for all $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$ and all $\Lambda < -N$,

$$\mathcal{R}\begin{bmatrix} \frac{\Lambda+\theta_*}{2} & \theta_t & \sigma_s \\ \frac{\Lambda-\theta_*}{2} & \theta_0 & \frac{\Lambda}{2}+\nu \end{bmatrix}; b \Bigg] = P_{\mathcal{H}}(\sigma_s, \nu) \int_{\mathcal{C}_{\mathcal{H}}} dx X(x, \Lambda) I_{\mathcal{H}}(x, \sigma_s, \nu). \tag{4.10}$$

If $(\sigma_s, \nu) \in \mathbb{R}^2$, then $\mathcal{C}_{\mathcal{H}}$ can be any curve from $-\infty$ to $+\infty$ lying within the strip $\operatorname{Im} x \in (-Q/2, 0)$.

Utilizing the asymptotic formula (2.8) for the function s_b with $\epsilon = 1/2$, we find that

$$\ln(X(x, \Lambda)) = O\left(e^{-\frac{\pi|\Lambda|}{\max(b, b^{-1})}}\right), \qquad \Lambda \to -\infty, \tag{4.11}$$

uniformly for $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$ and for x in bounded subsets of $\mathcal{C}_{\mathcal{H}}$. We deduce that

$$\lim_{\Lambda \to -\infty} X(x, \Lambda) = 1, \tag{4.12}$$

uniformly for $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$ and x in bounded subsets of $\mathcal{C}_{\mathcal{H}}$.

Using the asymptotic formula (2.8) for s_b with $\epsilon = 1/2$, we find that $I_{\mathcal{H}}$ obeys the estimate

$$I_{\mathcal{H}}(x, \sigma_s, \nu) = \begin{cases} O\left(e^{-2\pi(\frac{Q}{2}+\operatorname{Im}\nu)|\operatorname{Re}x|}\right), & \operatorname{Re}x \to +\infty, \\ O\left(e^{-2\pi Q|\operatorname{Re}x|}\right), & \operatorname{Re}x \to -\infty, \end{cases} \tag{4.13}$$

uniformly for $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$ and $\operatorname{Im}x$ in compact subsets of $\mathbb{R}$. Since the contour $\mathcal{C}_{\mathcal{H}}$ stays a bounded distance away from the increasing and the decreasing pole sequences, we infer that there exists a constant $C_1 > 0$ such that

$$|I_{\mathcal{H}}(x, \sigma_s, \nu)| \leq \begin{cases} C_1 e^{-2\pi(\frac{Q}{2}+\operatorname{Im}\nu)|\operatorname{Re}x|}, & x \in \mathcal{C}_{\mathcal{H}},\ \operatorname{Re}x \geq 0, \\ C_1 e^{-2\pi Q|\operatorname{Re}x|}, & x \in \mathcal{C}_{\mathcal{H}},\ \operatorname{Re}x \leq 0, \end{cases} \tag{4.14}$$

uniformly for $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$. In particular, since $K_\nu \subset \{\operatorname{Im}\nu > -Q/2\}$, $I_{\mathcal{H}}$ has exponential decay on the left and right tails of the contour $\mathcal{C}_{\mathcal{H}}$.

Suppose we can show that there exist constants $c > 0$ and $C > 0$ such that

$$|X(x, \Lambda) I_{\mathcal{H}}(x, \sigma_s, \nu)| \leq C e^{-c|\operatorname{Re}x|} \tag{4.15}$$

uniformly for all $\Lambda < -N$, $x \in \mathcal{C}_{\mathcal{H}}$, and $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$. Then it follows from (4.10), (4.12), and Lebesgue's dominated convergence theorem that the limit in (4.2) exists uniformly for $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$ and is given by (4.5). Since $K_{\sigma_s} \subset \mathbb{C}\setminus\Delta_{\mathcal{H},\sigma_s}$ and $K_\nu \subset \mathbb{C}\setminus\Delta_\nu$ are arbitrary compact subsets, this proves that the limit in (4.2) exists uniformly for (σ_s, ν) in compact subsets of $\Omega_{\mathcal{H}}$ and proves (4.5). Moreover, the analyticity of $\mathcal{H}$ as a function of $(\sigma_s, \nu) \in \Omega_{\mathcal{H}}$ follows from the analyticity of $\mathcal{R}$ together with the uniform convergence on compact subsets. Alternatively, the analyticity of $\mathcal{H}$ in $\Omega_{\mathcal{H}}$ can be inferred directly from the representation (4.5). Indeed, the possible poles of $\mathcal{H}$ lie at such values of (σ_s, ν) at which either the prefactor $P_{\mathcal{H}}$ has a pole or at which the contour of integration gets pinched between two poles of the integrand $I_{\mathcal{H}}$, and the definitions of $\Delta_{\mathcal{H},\sigma_s}$ and Δ_ν exclude both of these situations. Thus to complete the proof of the theorem, it only remains to prove (4.15).

To prove (4.15), we need to estimate the function X defined in (4.9). The asymptotic formula (2.8) for s_b with $\epsilon = 1/2$ implies that there exist constants $C_2, C_3, C_4 > 0$ such that the inequalities

$$\left|\frac{s_b\left(\Lambda + \theta_0 + \theta_t + \frac{iQ}{2}\right)}{s_b\left(\Lambda + \frac{\theta_*}{2} + \theta_t + \nu\right)}\right| \leq C_2 e^{\pi(\frac{Q}{2}-\operatorname{Im}\nu)|\Lambda|}, \qquad \Lambda < -N, \tag{4.16}$$

$$\left|\frac{s_b\left(x + \Lambda + \frac{\theta_*}{2} + \theta_t + \nu\right)}{s_b\left(x + \Lambda + \theta_0 + \theta_t + \frac{iQ}{2}\right)}\right| \leq C_3 e^{-\pi(\frac{Q}{2}-\operatorname{Im}\nu)|\Lambda+\operatorname{Re}x|}, \qquad x \in \mathcal{C}_{\mathcal{H}},\ \Lambda \in \mathbb{R}, \tag{4.17}$$

$$|e^{i\pi x(\frac{iQ}{2}+\theta_0-\frac{\theta_*}{2}-\nu)}| \leq C_4 e^{-\pi(\frac{Q}{2}-\operatorname{Im}\nu)\operatorname{Re}x}, \qquad x \in \mathcal{C}_{\mathcal{H}}, \tag{4.18}$$

hold uniformly for $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$. Combining the above estimates, we infer that there exists a constant C_5 such that

$$|X(x, \Lambda)| \leq C_5 e^{\pi(\frac{Q}{2} - \operatorname{Im}\nu)(|\Lambda| - |\Lambda + \operatorname{Re} x| - \operatorname{Re} x)}, \qquad x \in \mathcal{C}_{\mathcal{H}},\ \Lambda < -N, \tag{4.19}$$

uniformly for $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$. The inequality (4.19) can be rewritten as follows:

$$|X(x, \Lambda)| \leq \begin{cases} C_5 e^{-2\pi(\frac{Q}{2} - \operatorname{Im}\nu)(\Lambda + \operatorname{Re} x)}, & \Lambda + \operatorname{Re} x \geq 0, \\ C_5, & \Lambda + \operatorname{Re} x \leq 0, \end{cases} \quad x \in \mathcal{C}_{\mathcal{H}},\ \Lambda < -N, \tag{4.20}$$

uniformly for $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$.

If $\nu \in K_\nu$ is such that $-Q/2 < \operatorname{Im}\nu \leq Q/2$, then (4.20) implies that $|X|$ is uniformly bounded for all $x \in \mathcal{C}_{\mathcal{H}}$, $\Lambda < -N$, and $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$; hence (4.15) follows from (4.14) in this case. On the other hand, if $\nu \in K_\nu$ is such that $\operatorname{Im}\nu \geq Q/2$, then (4.14) and (4.20) yield the existence of a constant $C_6 > 0$ independent of $x \in \mathcal{C}_{\mathcal{H}}$, $\Lambda < -N$, and $(\sigma_s, \nu) \in K_{\sigma_s} \times K_\nu$ such that

$$\begin{aligned} &|X(x, \Lambda) I_{\mathcal{H}}(x, \sigma_s, \nu)| \\ &\leq \begin{cases} C_6 e^{2\pi(\frac{Q}{2} - \operatorname{Im}\nu)|\Lambda|} e^{-2\pi Q|\operatorname{Re} x|} \leq C_6 e^{-2\pi Q|\operatorname{Re} x|}, & \operatorname{Re} x \geq -\Lambda, \\ C_6 e^{-2\pi(\frac{Q}{2} + \operatorname{Im}\nu)|\operatorname{Re} x|} \leq C_6 e^{-2\pi Q|\operatorname{Re} x|}, & 0 \leq \operatorname{Re} x \leq -\Lambda, \\ C_6 e^{-2\pi Q|\operatorname{Re} x|}, & \operatorname{Re} x \leq 0, \end{cases} \end{aligned} \tag{4.21}$$

which shows (4.15) also in this case. This completes the proof. □

Furthermore, thanks to the symmetry (2.7) of s_b, we have

$$\mathcal{H}(b^{-1}, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu). \tag{4.22}$$

4.2 Difference Equations

We now show that the four difference equations (3.9) satisfied by the function $\mathcal{R}$ survive in the confluent limit (4.2). This implies that the function $\mathcal{H}$ is a joint eigenfunction of four difference operators, two acting on σ_s and the remaining two on ν.

We know from Theorem 4.2 that $\mathcal{H}$ is a well-defined meromorphic function of $(\sigma_s, \nu) \in \mathbb{C} \times \{\operatorname{Im}\nu > -Q/2\}$. The difference equations will first be derived as equalities between meromorphic functions defined on this limited domain. However, the difference equations in ν can then be used to show that: (i) the limit in (4.2) exists for all ν in the whole complex plane away from a discrete subset, (ii) $\mathcal{H}$ is in fact a meromorphic function of (σ_s, ν) in all of $\mathbb{C}^2$, and (iii) the four difference equations hold as equalities between meromorphic functions on $\mathbb{C}^2$, see Proposition 4.5.

4.2.1 First Pair of Difference Equations

Define the difference operator $D_{\mathcal{H}}(b, \sigma_s)$ by

$$D_{\mathcal{H}}(b, \sigma_s) = d_{\mathcal{H}}^{+}(b, \sigma_s)e^{ib\partial_{\sigma_s}} + d_{\mathcal{H}}^{+}(b, -\sigma_s)e^{-ib\partial_{\sigma_s}} + d_{\mathcal{H}}^{0}(b, \sigma_s), \tag{4.23}$$

where $d_{\mathcal{H}}^{0}$ is defined by

$$d_{\mathcal{H}}^{0}(b, \sigma_s) = e^{-\pi b(iQ+\theta_*+2\theta_t)} - d_{\mathcal{H}}^{+}(b, \sigma_s) - d_{\mathcal{H}}^{+}(b, -\sigma_s), \tag{4.24}$$

with

$$\begin{aligned} &d_{\mathcal{H}}^{+}(b, \sigma_s) \\ &\quad = -2e^{-\pi b\left(\sigma_s+\frac{ib}{2}\right)} \cosh(\pi b(\tfrac{ib}{2} + \theta_* + \sigma_s)) \frac{\prod_{\epsilon=\pm1} \cosh\left(\pi b\left(\frac{ib}{2} + \theta_t + \sigma_s + \epsilon\theta_0\right)\right)}{\sinh(\pi b(2\sigma_s + ib)) \sinh(2\pi b\sigma_s)} \end{aligned} \tag{4.25}$$

Proposition 4.3 *For $\sigma_s \in \mathbb{C}$ and $Im\, \nu > -Q/2$, the function $\mathcal{H}$ defined by (4.2) satisfies the following pair of difference equations:*

$$D_{\mathcal{H}}(b, \sigma_s)\, \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = e^{-2\pi b\nu}\mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu), \tag{4.26a}$$

$$D_{\mathcal{H}}(b^{-1}, \sigma_s)\, \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = e^{-2\pi b^{-1}\nu}\mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu). \tag{4.26b}$$

Proof The proof consists of taking the confluent limit (4.2) of the difference equation (3.9a). On the one hand, we have

$$\lim_{\Lambda\to-\infty} \left(e^{\pi b\Lambda} 2\cosh 2\pi b\sigma_t|_{\sigma_t=\frac{\Lambda}{2}+\nu}\right) = e^{-2\pi b\nu}. \tag{4.27}$$

On the other hand, straightforward computations using asymptotics of hyperbolic functions show that the following limits hold:

$$\begin{aligned} \lim_{\Lambda\to-\infty} e^{\pi b\Lambda} d_{\mathcal{R}}^{+} \begin{bmatrix} \frac{\Lambda+\theta_*}{2} & \theta_t \\ & ; b, \pm\sigma_s \\ \frac{\Lambda-\theta_*}{2} & \theta_0 \end{bmatrix} &= d_{\mathcal{H}}^{+}(b, \pm\sigma_s), \\ \lim_{\Lambda\to-\infty} e^{\pi b\Lambda} d_{\mathcal{R}}^{0} \begin{bmatrix} \frac{\Lambda+\theta_*}{2} & \theta_t \\ & ; b, \sigma_s \\ \frac{\Lambda-\theta_*}{2} & \theta_0 \end{bmatrix} &= d_{\mathcal{H}}^{0}(b, \sigma_s), \end{aligned} \tag{4.28}$$

where $d_{\mathcal{R}}^{+}$ and $d_{\mathcal{R}}^{0}$ are defined in (3.7) and (3.8), respectively. Therefore we obtain

$$\lim_{\Lambda\to-\infty} e^{\pi b\Lambda} D_{\mathcal{R}} \begin{bmatrix} \frac{\Lambda+\theta_*}{2} & \theta_t \\ & ; b, \sigma_s \\ \frac{\Lambda-\theta_*}{2} & \theta_0 \end{bmatrix} = D_{\mathcal{H}}(b, \sigma_s), \tag{4.29}$$

where $D_{\mathcal{R}}$ is given in (3.6). By Theorem 4.2, the limit in (4.2) exists whenever $(\sigma_s, \nu) \in \Omega_{\mathcal{H}}$. Thus, the difference equation (4.26a) follows after multiplying (3.9a) by $e^{\pi b\Lambda}$

and utilizing (4.27), (4.29), and the definition (4.2) of $\mathcal{H}$. Finally, (4.26b) follows from (4.26a) and the symmetry (4.22) of $\mathcal{H}$. □

4.2.2 Second Pair of Difference Equations

Define the dual difference operator $\tilde{D}_{\mathcal{H}}(b, \nu)$ by

$$\tilde{D}_{\mathcal{H}}(b,\nu) = \tilde{d}^{+}_{\mathcal{H}}(b,\nu)e^{ib\partial_\nu} + \tilde{d}^{-}_{\mathcal{H}}(b,\nu)e^{-ib\partial_\nu} + \tilde{d}^{0}_{\mathcal{H}}(b,\nu), \tag{4.30}$$

where $\tilde{H}^0_{\mathcal{H}}$ is defined by

$$\tilde{d}^{0}_{\mathcal{H}}(b,\nu) = -2\cosh\left(2\pi b\left(\tfrac{ib}{2} + \theta_0 + \theta_t\right)\right) - \tilde{d}^{+}_{\mathcal{H}}(b,\nu) - \tilde{d}^{-}_{\mathcal{H}}(b,\nu), \tag{4.31}$$

with

$$\begin{aligned}\tilde{d}^{\pm}_{\mathcal{H}}(b,\nu) = &-4e^{2\pi b\nu}e^{\mp\pi b(\theta_0+\theta_t)}\cosh\left(\pi b\left(\tfrac{ib}{2} + \theta_0 \pm (\nu + \tfrac{\theta_*}{2})\right)\right)\\ &\times\cosh\left(\pi b\left(\tfrac{ib}{2} + \theta_t \pm (\nu - \tfrac{\theta_*}{2})\right)\right).\end{aligned} \tag{4.32}$$

Proposition 4.4 *For $\sigma_s \in \mathbb{C}$ and $Im\,(\nu - ib^{\pm 1}) > -Q/2$, the function $\mathcal{H}$ satisfies the following pair of difference equations:*

$$\tilde{D}_{\mathcal{H}}(b,\nu)\,\mathcal{H}(b,\theta_0,\theta_t,\theta_*,\sigma_s,\nu) = 2\cosh(2\pi b\sigma_s)\,\mathcal{H}(b,\theta_0,\theta_t,\theta_*,\sigma_s,\nu), \tag{4.33a}$$

$$\tilde{D}_{\mathcal{H}}(b^{-1},\nu)\,\mathcal{H}(b,\theta_0,\theta_t,\theta_*,\sigma_s,\nu) = 2\cosh(2\pi b^{-1}\sigma_s)\,\mathcal{H}(b,\theta_0,\theta_t,\theta_*,\sigma_s,\nu). \tag{4.33b}$$

Proof It is straightforward to verify that the following limits hold:

$$\begin{aligned}\lim_{\Lambda\to-\infty} d^{+}_{\mathcal{R}}\left[\begin{matrix}\theta_0 & \theta_t\\ \frac{\Lambda-\theta_*}{2} & \frac{\Lambda+\theta_*}{2}\end{matrix}; b,\ \pm(\tfrac{\Lambda}{2}+\nu)\right] &= \tilde{d}^{\pm}_{\mathcal{H}}(b,\nu),\\ \lim_{\Lambda\to-\infty} d^{0}_{\mathcal{R}}\left[\begin{matrix}\theta_0 & \theta_t\\ \frac{\Lambda-\theta_*}{2} & \frac{\Lambda+\theta_*}{2}\end{matrix}; b,\ \tfrac{\Lambda}{2}+\nu\right] &= \tilde{d}^{0}_{\mathcal{H}}(b,\nu),\end{aligned} \tag{4.34}$$

where $d^{+}_{\mathcal{R}}$ and $d^{0}_{\mathcal{R}}$ are defined in (3.7) and (3.8), respectively. We obtain

$$\lim_{\Lambda\to-\infty} D_{\mathcal{R}}\left[\begin{matrix}\theta_0 & \theta_t\\ \frac{\Lambda-\theta_*}{2} & \frac{\Lambda+\theta_*}{2}\end{matrix}; b,\ \tfrac{\Lambda}{2}+\nu\right] = \tilde{D}_{\mathcal{H}}(b,\nu), \tag{4.35}$$

where $D_{\mathcal{R}}$ is defined in (3.6). The difference equation (4.33a) follows from (3.9c), (4.2), (4.35), and Theorem 4.2. Finally, (4.33b) follows from (4.33a) and the symmetry (4.22) of $\mathcal{H}$. □

Using the difference equations (4.33), we can show that $\mathcal{H}$ extends to a meromorphic function of ν everywhere in the complex plane. More precisely, we have the following proposition.

Proposition 4.5 *Let* $(b, \theta_0, \theta_t, \theta_*) \in (0, \infty) \times \mathbb{R}^3$ *and* $\sigma_s \in \mathbb{C} \setminus \Delta_{\mathcal{H},\sigma_s}$. *Then there is a discrete subset* $\Delta \subset \mathbb{C}$ *such that the limit in (4.2) exists for all* $\nu \in \mathbb{C} \setminus \Delta$. *Moreover, the function* $\mathcal{H}$ *defined by (4.2) has a meromorphic continuation to* $(\sigma_s, \nu) \in \mathbb{C}^2$ *and the four difference equations (4.26) and (4.33) hold as equalities between meromorphic functions of* $(\sigma_s, \nu) \in \mathbb{C}^2$.

Proof Consider ν such that $\operatorname{Im} \nu > -Q/2$ but $\operatorname{Im}(\nu - ib) \leq -Q/2$. Solving (3.9c) for

$$e^{-ib\partial_\nu}\left(\mathcal{R}\begin{bmatrix} \frac{\Lambda+\theta_*}{2} & \theta_t & & \sigma_s & \\ \frac{\Lambda-\theta_*}{2} & \theta_0 & ; & \frac{\Lambda}{2}+\nu & , b \end{bmatrix}\right),$$

taking the confluent limit $\Lambda \to -\infty$, and using (4.2), (4.35), and Theorem 4.2, we conclude that the limit in (4.2) exists also for ν in the strip $\{\nu \in \mathbb{C} | -b - Q/2 < \operatorname{Im} \nu \leq -Q/2\} \setminus \Delta_1$, where Δ_1 is a discrete set. By iteration, we conclude that the limit in (4.2) exists for all $\nu \in \mathbb{C} \setminus \Delta$, where Δ is a discrete set. This proves the first assertion. The remaining assertions now follow by repeating the proofs of Propositions 4.3 and 4.4 with $\nu \in \mathbb{C} \setminus \Delta$. □

4.3 First Polynomial Limit

In this subsection, we show that the function $\mathcal{H}$ reduces to the continuous dual q-Hahn polynomials when ν is suitably discretized. In addition to Assumption 1.1, we make the following assumption.

Assumption 4.6 *(Restriction on the parameters)* Assume that $b > 0$ is such that b^2 is irrational, and that

$$\theta_t, \operatorname{Re} \sigma_s \neq 0, \qquad \operatorname{Re}\left(\tfrac{\theta_*}{2} - \nu - \theta_0 \pm \sigma_s\right) \neq 0, \qquad \theta_0 + \theta_* \pm \theta_t \neq 0. \tag{4.36}$$

Assumption 4.6 implies that the three increasing and the three decreasing sequences of poles of the integrand in (4.5) do not overlap. The assumption that b^2 is irrational ensures that all the poles of the integrand are simple and that $q = e^{2i\pi b^2}$ is not a root of unity.

Define $\{\nu_n\}_{n=0}^{\infty} \subset \mathbb{C}$ by

$$\nu_n = \tfrac{\theta_*}{2} + \theta_t + \tfrac{iQ}{2} + inb. \tag{4.37}$$

The sequence $\{\nu_n\}_{n=0}^{\infty}$ is a subset of the set Δ_ν of possible poles of $\mathcal{H}$ defined in (4.4b). The following theorem shows that $\mathcal{H}$ still has a finite limit as $\nu \to \nu_n$ for each $n \geq 0$ and that the limit is given by the continuous dual q-Hahn polynomials. The reason the limit is finite is that the prefactor $P_{\mathcal{H}}$ has a simple zero at each ν_n; this zero cancels the simple pole that the integral in (4.5) has due to the contour of integration being pinched between two poles of the integrand.

Theorem 4.7 (From $\mathcal{H}$ to the continuous dual q-Hahn polynomials) *Let $\sigma_s \in \mathbb{C} \setminus \Delta_{\mathcal{H},\sigma_s}$ and suppose that Assumptions 1.1 and 4.6 are satisfied. Under the parameter correspondence*

$$\alpha_H = e^{2\pi b(\theta_t+\theta_0+\frac{iQ}{2})}, \quad \beta_H = e^{2\pi b(\theta_t-\theta_0+\frac{iQ}{2})}, \quad \gamma_H = e^{2\pi b(\theta_*+\frac{iQ}{2})}, \qquad q = e^{2i\pi b^2}, \tag{4.38}$$

the function $\mathcal{H}$ defined in (4.5) satisfies, for each integer $n \geq 0$,

$$\lim_{\nu \to \nu_n} \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = H_n(e^{2\pi b\sigma_s}; \alpha_H, \beta_H, \gamma_H, q), \tag{4.39}$$

where H_n are the continuous dual q-Hahn polynomials defined in (B.8).

Proof There are two different ways to prove (4.39). The first approach consists of taking the limit $\nu \to \nu_n$ in the integral representation (4.5) for $\mathcal{H}$ for each n; the second approach computes the limit for $n = 0$ and then uses the limit of the difference equation (4.33a) to extend the result to other values of n. The first approach is described in detail in Sect. 10 for the function $\mathcal{M}$. Here we use the second approach.

We first show that the limit in (4.39) exists for $n = 0$ and equals 1. The function $s_b(\nu - \frac{\theta_*}{2} - \theta_t)$ in (4.6) has a simple zero at $\nu_0 = \frac{\theta_*}{2} + \theta_t + \frac{iQ}{2}$. Moreover, in the limit $\nu \to \nu_0$, the pole of the function $s_b(x + \frac{\theta_*}{2} + \theta_t - \nu)$ in (4.7) at $x_0 := -\frac{iQ}{2} - \frac{\theta_*}{2} - \theta_t + \nu$ collides with the pole of $s_b(x + \frac{iQ}{2})^{-1}$ located at $x = 0$, pinching the contour $\mathcal{C}_{\mathcal{H}}$. Therefore, before taking the limit $\nu \to \nu_0$, we deform the contour $\mathcal{C}_{\mathcal{H}}$ into a contour $\mathcal{C}'_{\mathcal{H}}$ which passes below x_0. We obtain

$$\begin{aligned}\mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) &= -2i\pi \; P_{\mathcal{H}}(\sigma_s, \nu) \operatorname*{Res}_{x=x_0} (I_{\mathcal{H}}(x, \sigma_s, \nu)) \\ &+ P_{\mathcal{H}}(\sigma_s, \nu) \int_{\mathcal{C}'_{\mathcal{H}}} dx \; I_{\mathcal{H}}(x, \sigma_s, \nu).\end{aligned} \tag{4.40}$$

Using the residue (2.4), a straightforward computation yields

$$\begin{aligned}-2i\pi \operatorname*{Res}_{x=x_0} (I_{\mathcal{H}}(x, \sigma_s, \nu)) &= e^{i\pi\left(\nu-\theta_0+\frac{\theta_*}{2}-\frac{iQ}{2}\right)\left(\nu-\frac{\theta_*}{2}-\theta_t-\frac{iQ}{2}\right)} \\ &\times \frac{s_b\left(\theta_0 - \frac{\theta_*}{2} + \nu - \frac{iQ}{2} - \sigma_s\right) s_b\left(\theta_0 - \frac{\theta_*}{2} + \nu - \frac{iQ}{2} + \sigma_s\right)}{s_b\left(\theta_0 + \frac{\theta_*}{2} + \nu\right) s_b\left(\nu - \frac{\theta_*}{2} - \theta_t\right) s_b\left(\nu - \frac{\theta_*}{2} + \theta_t\right)}.\end{aligned} \tag{4.41}$$

The right-hand side of (4.41) has a simple pole at $\nu = \nu_0$ due to the factor $s_b(\nu - \frac{\theta_*}{2} - \theta_t)^{-1}$. Moreover, in the limit $\nu \to \nu_0$ the second term in (4.40) vanishes thanks to the zero of $P_{\mathcal{H}}$. Thus we obtain

$$\lim_{\nu \to \nu_0} \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = -2i\pi \lim_{\nu \to \nu_0} \Big(P_{\mathcal{H}}(\sigma_s, \nu) \operatorname*{Res}_{x=x_0} (I_{\mathcal{H}}(x, \sigma_s, \nu)) \Big). \tag{4.42}$$

Using $s_b(x) = s_b(-x)^{-1}$, it is straightforward to verify that the right-hand side equals 1; this proves (4.39).

For each integer $n \geq 0$, let P_n denote the left-hand side of (4.39):

$$P_n := \lim_{\nu \to \nu_n} \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu). \tag{4.43}$$

The same kind of contour deformation used to establish the case $n = 0$ shows that the limit in (4.43) exists for all $n \geq 0$. To show that P_n equals the continuous dual q-Hahn polynomials H_n, we consider the limit $\nu \to \nu_n$ of the difference equation (4.33a). Using the parameter correspondence (4.38), it is straightforward to verify that

$$\lim_{\nu \to \nu_n} \tilde{D}_{\mathcal{H}}(b, \nu) = L_H, \tag{4.44}$$

where the operators $\tilde{D}_{\mathcal{H}}$ and L_H are defined in (4.30) and (B.10), respectively. Hence taking the limit $\nu \to \nu_n$ of the difference equation (4.33a), we see that P_n satisfies

$$L_H P_n = \left(z + z^{-1}\right) P_n, \tag{4.45}$$

where $z = e^{2\pi b \sigma_s}$. Thus the P_n satisfy the same recurrence relation (B.9) as the continuous dual q-Hahn polynomials evaluated at $z = e^{2\pi b \sigma_s}$. Since we have already shown that $P_0 = H_0 = 1$, we infer that $P_n = H_n$ for all $n \geq 0$ by induction, where H_n is evaluated at $z = e^{2\pi b \sigma_s}$. This completes the proof of (4.39). $\square$

4.4 Second Polynomial Limit

In this subsection, we show that $\mathcal{H}$ reduces to the big q-Jacobi polynomials when σ_s is suitably discretized.

Theorem 4.8 (From $\mathcal{H}$ to the big q-Jacobi polynomials) *Let $\nu \in \{Im\, \nu > -Q/2\} \setminus \Delta_\nu$ and suppose that Assumptions 1.1 and 4.6 are satisfied. Under the parameter correspondence*

$$\alpha_J = e^{4\pi b \theta_t}, \quad \beta_J = e^{4\pi b \theta_0}, \quad \gamma_J = e^{2\pi b(\theta_0 + \theta_* + \theta_t)}, \quad x_J = e^{2\pi b(\theta_t + \frac{\theta_*}{2} + \frac{iQ}{2})} e^{-2\pi b \nu},$$
$$q = e^{2i\pi b^2}, \tag{4.46}$$

the function $\mathcal{H}$ defined in (4.5) satisfies, for each integer $n \geq 0$,

$$\lim_{\sigma_s \to \sigma_s^{(n)}} \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = J_n(x_J; \alpha_J, \beta_J, \gamma_J; q), \tag{4.47}$$

where $\sigma_s^{(n)} \in \mathbb{C}$ is given in (3.11) and where J_n are the big q-Jacobi polynomials defined in (B.12).

Proof We first prove that (4.47) holds for $n = 0$. The function $s_b(\sigma_s - \theta_0 - \theta_t)$ in (4.6) has a simple zero located at $\sigma_s = \sigma_s^{(0)} = \theta_0 + \theta_t + \frac{iQ}{2}$. On the other hand, in the limit $\sigma_s = \sigma_s^{(0)}$, the contour $\mathcal{C}_{\mathcal{H}}$ is squeezed between the pole of $s_b(x + \theta_0 + \theta_t - \sigma_s)$ in (4.7) located at $x_0 = -\frac{iQ}{2} + \sigma_s - \theta_0 - \theta_t$ and the pole of $s_b(x + \frac{iQ}{2})^{-1}$ at $x = 0$. Therefore, before taking the limit $\sigma_s \to \sigma_s^{(0)}$, we deform the contour $\mathcal{C}_{\mathcal{H}}$ into a contour $\mathcal{C}'_{\mathcal{H}}$ which passes below x_0. We obtain

$$\mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = -2i\pi \; P_{\mathcal{H}}(\sigma_s, \nu) \operatorname*{Res}_{x=x_0} (I_{\mathcal{H}}(x, \sigma_s, \nu)) \\ + P_{\mathcal{H}}(\sigma_s, \nu) \int_{\mathcal{C}'_{\mathcal{H}}} dx \; I_{\mathcal{H}}(x, \sigma_s, \nu). \tag{4.48}$$

A straightforward computation using (2.4) shows that

$$- 2i\pi \operatorname*{Res}_{x=x_0} I_{\mathcal{H}}(x, \sigma_s, \nu) \\ = e^{i\pi\left(-\theta_0 + \frac{\theta_*}{2} + \nu - \frac{iQ}{2}\right)\left(-\theta_0 - \theta_t - \frac{iQ}{2} + \sigma_s\right)} \frac{s_b\left(2\sigma_s - \frac{iQ}{2}\right) s_b\left(\sigma_s - \theta_0 + \frac{\theta_*}{2} - \nu - \frac{iQ}{2}\right)}{s_b(\theta_* + \sigma_s) s_b(\sigma_s - \theta_0 - \theta_t) s_b(\sigma_s - \theta_0 + \theta_t)}.$$

The right-hand side has a simple pole at $\sigma_s = \sigma_s^{(0)}$ due to the factor $s_b(\sigma_s - \theta_0 - \theta_t)^{-1}$. Moreover, the second term in (4.48) vanishes at $\sigma_s = \sigma_s^{(0)}$ thanks to the zero of $P_{\mathcal{H}}$. Therefore,

$$\lim_{\sigma_s \to \sigma_s^{(0)}} \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = -2i\pi \lim_{\sigma_s \to \sigma_s^{(0)}} \left(P_{\mathcal{H}}(\sigma_s, \nu) \operatorname*{Res}_{x=x_0} (I_{\mathcal{H}}(x, \sigma_s, \nu)) \right). \tag{4.49}$$

A straightforward computation using $s_b(x) = s_b(-x)^{-1}$ shows that the right-hand side equals 1; this proves (4.47) for $n = 0$.

We now use the difference equation (4.26a) to show that (4.47) holds also for $n \geq 1$. Let

$$P_n := \lim_{\sigma_s \to \sigma_s^{(n)}} \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu), \qquad n = 0, 1, \ldots. \tag{4.50}$$

The same kind of contour deformation used for $n = 0$ shows that the limit in (4.47) exists for all $n \geq 0$. Moreover, under the parameter correspondence (4.46), we have

$$\lim_{\sigma_s \to \sigma_s^{(n)}} D_{\mathcal{H}}(b, \sigma_s) = e^{-2\pi b(\theta_t + \frac{\theta_*}{2} + \frac{iQ}{2})} L_J, \tag{4.51}$$

where the operators $D_{\mathcal{H}}$ and L_J are defined in (4.23) and (B.14), respectively. Hence taking the limit $\sigma_s \to \sigma_s^{(n)}$ of (4.26a) and recalling that $x_J = e^{2\pi b(\theta_t + \frac{\theta_*}{2} + \frac{iQ}{2})} e^{-2\pi b\nu}$, we see that P_n satisfies $L_J P_n = x_J P_n$. Thus the P_n satisfy the same recurrence relation (B.13) as the big q-Jacobi polynomials J_n, and the limit (4.47) for $n \geq 1$ follows by induction. ◻

Remark 4.9 It is also possible to define a function $\mathcal{H}'$ by sending $\Lambda \to +\infty$ in (4.2) instead of $\Lambda \to -\infty$:

$$\mathcal{H}'(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = \lim_{\Lambda \to +\infty} \mathcal{R}\begin{bmatrix} \frac{\Lambda+\theta_*}{2} & \theta_t & \sigma_s \\ \frac{\Lambda-\theta_*}{2} & \theta_0 & \frac{\Lambda}{2}+\nu \end{bmatrix}; \tag{4.52}$$

It can be shown that the limit in (4.52) exists for $(\sigma_s, \nu) \in \Omega_{\mathcal{H}}$. Moreover, due to the asymptotic formula (2.8) for s_b, the only difference between $\mathcal{H}'$ and $\mathcal{H}$ resides in the sign of the phases in the representation (4.5). In fact, the following two limits hold:

$$\lim_{\nu \to \nu_n} \mathcal{H}'(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = H_n(e^{2\pi b \sigma_s}; \alpha_H^{-1}, \beta_H^{-1}, \gamma_H^{-1}, q^{-1}), \tag{4.53}$$

$$\lim_{\sigma_s \to \sigma_s^{(n)}} \mathcal{H}'(b, \theta_0, \theta_t, \theta_*, \sigma_s, \nu) = J_n(x_J^{-1}; \alpha_J^{-1}, \beta_J^{-1}, \gamma_J^{-1}; q^{-1}). \tag{4.54}$$

We expect that a similar phenomenon is present for all families of the non-polynomial scheme. For simplicity, we will only study one of the two representatives for each family.

5 The Function $\mathcal{S}$

In this section, we introduce the function $\mathcal{S}(b, \theta_0, \theta_t, \sigma_s, \rho)$ which is one of the two elements at the third level of the non-polynomial scheme, see Fig. 2. The function $\mathcal{S}$ is defined as a confluent limit of the function $\mathcal{H}$. We show that $\mathcal{S}$ is a joint eigenfunction of four difference operators and that it reduces to the Al-Salam Chihara and the little q-Jacobi polynomials, which lie at the third level of the q-Askey scheme, when ρ and σ_s are suitably discretized, respectively.

5.1 Definition and Integral Representation

Let ρ be a new parameter defined in terms of θ_* and ν by

$$\nu = \frac{\theta_*}{2} + \rho. \tag{5.1}$$

Define the open set $\Omega_{\mathcal{S}} \subset \mathbb{C}^2$ by

$$\Omega_{\mathcal{S}} := (\mathbb{C} \backslash \Delta_{\mathcal{S},\sigma_s}) \times (\{\operatorname{Im} \rho > -Q/2\} \backslash \Delta_\rho), \tag{5.2}$$

where the discrete subsets $\Delta_{\mathcal{S},\sigma_s}$ and Δ_ρ are given by

$$\Delta_{\mathcal{S},\sigma_s} := \{\pm\sigma_s \,|\, \sigma_s \in \Delta'_{\mathcal{S},\sigma_s}\},$$
$$\Delta_\rho := \{\pm\theta_t + \tfrac{iQ}{2} + ibm + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{\theta_t - \tfrac{iQ}{2} - ibm - ilb^{-1}\}_{m,l=0}^{\infty},$$

with

$$\Delta'_{\mathcal{S},\sigma_s} := \{\theta_0 \pm \theta_t + \tfrac{iQ}{2} + imb + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{\theta_0 + \theta_t - \tfrac{iQ}{2} - imb - ilb^{-1}\}_{m,l=0}^{\infty}.$$

Definition 5.1 Let $\mathcal{H}$ be defined by (4.2). The function $\mathcal{S}$ is defined for $(\sigma_s, \rho) \in \Omega_{\mathcal{S}}$ by

$$\mathcal{S}(b, \theta_0, \theta_t, \sigma_s, \rho) = \lim_{\theta_* \to -\infty} \mathcal{H}\left(b, \theta_0, \theta_t, \theta_*, \sigma_s, \tfrac{\theta_*}{2} + \rho\right). \tag{5.3}$$

The next theorem shows that $\mathcal{S}$ is a well-defined meromorphic function of $(\sigma_s, \rho) \in \Omega_{\mathcal{S}}$, which has a meromorphic continuation to all of $\mathbb{C}^2$.

Theorem 5.2 *Suppose that Assumption 1.1 is satisfied. The limit in (5.3) exists uniformly for (σ_s, ρ) in compact subsets of $\Omega_{\mathcal{S}}$. Moreover, the function $\mathcal{S}$ can be extended to an analytic function of $(\sigma_s, \rho) \in (\mathbb{C}\setminus\Delta_{\mathcal{S},\sigma_s}) \times (\mathbb{C}\setminus\Delta_\rho)$ and admits the following integral representation:*

$$\begin{aligned}\mathcal{S}(b, \theta_0, \theta_t, \sigma_s, \rho) \\ = P_{\mathcal{S}}(\sigma_s, \rho) \int_{\mathcal{C}_{\mathcal{S}}} dx\, I_{\mathcal{S}}(x, \sigma_s, \rho) \quad \text{for } (\sigma_s, \rho) \in (\mathbb{C}\setminus\Delta_{\mathcal{S},\sigma_s}) \times (\mathbb{C}\setminus\Delta_\rho),\end{aligned} \tag{5.4}$$

where the dependence of $P_{\mathcal{S}}$ and $I_{\mathcal{S}}$ on b, θ_0, θ_t is omitted for simplicity,

$$P_{\mathcal{S}}(\sigma_s, \rho) = s_b\left(2\theta_t + \tfrac{iQ}{2}\right) s_b(\rho - \theta_t) \prod_{\epsilon=\pm 1} s_b(\epsilon\sigma_s - \theta_0 - \theta_t), \tag{5.5}$$

$$\begin{aligned} I_{\mathcal{S}}(x, \sigma_s, \rho) = e^{-\frac{i\pi x^2}{2} - i\pi x(iQ + 2\theta_0 + \theta_t - \rho)} \frac{s_b(x + \theta_t - \rho)}{s_b\left(x + \frac{iQ}{2}\right) s_b\left(x + \frac{iQ}{2} + 2\theta_t\right)} \\ \times \prod_{\epsilon=\pm 1} s_b(x + \theta_0 + \theta_t + \epsilon\sigma_s), \end{aligned} \tag{5.6}$$

and the contour $\mathcal{C}_{\mathcal{S}}$ is any curve from $-\infty$ to $+\infty$ which separates the three decreasing from the two increasing sequences of poles, with the requirement that its right tail satisfies

$$\operatorname{Im} x - \operatorname{Im} \rho < -\delta \quad \text{for all } x \in \mathcal{C}_{\mathcal{S}} \text{ with } \operatorname{Re} x \text{ sufficiently large}, \tag{5.7}$$

for some $\delta > 0$. In particular, $\mathcal{S}$ is a meromorphic function of $(\sigma_s, \rho) \in \mathbb{C}^2$. If $(\sigma_s, \rho) \in \mathbb{R}^2$, then the contour $\mathcal{C}_{\mathcal{S}}$ can be any curve from $-\infty$ to $+\infty$ lying within the strip $\operatorname{Im} x \in (-Q/2, -\delta)$.

Proof The proof is similar to the proof of Theorem 4.2, but there are some differences because the exponent in (5.6) is a quadratic (rather than a linear) polynomial in x. Let $(b, \theta_0, \theta_t) \in (0, \infty) \times \mathbb{R}^2$. It can be verified that

$$P_{\mathcal{H}}\left(\sigma_s, \tfrac{\theta_*}{2}+\rho\right) I_{\mathcal{H}}\left(x, \sigma_s, \tfrac{\theta_*}{2}+\rho\right) = P_{\mathcal{S}}(\sigma_s, \rho) Z(x, \theta_*) I_{\mathcal{S}}(x, \sigma_s, \rho), \quad (5.8)$$

where

$$Z(x, \theta_*) = e^{\frac{i\pi x^2}{2}} e^{i\pi x\left(\theta_0+\theta_*+\theta_t+\frac{iQ}{2}\right)} \frac{s_b\left(\theta_0+\theta_*+\theta_t+\frac{iQ}{2}\right)}{s_b\left(x+\theta_0+\theta_*+\theta_t+\frac{iQ}{2}\right)}. \quad (5.9)$$

Due to the properties (2.3) of the function s_b, the function $I_{\mathcal{S}}(\cdot, \sigma_s, \rho)$ possesses two increasing sequences of poles starting at $x = 0$ and $x = -2\theta_t$, as well as three decreasing sequences of poles starting at $x = -\frac{iQ}{2}+\rho-\theta_t$ and $x = -\frac{iQ}{2} \pm \sigma_s - \theta_0 - \theta_t$. The discrete sets $\Delta_{\mathcal{S},\sigma_s}$ and Δ_ρ contain all the values of σ_s and ρ, respectively, for which poles in any of the two increasing sequences collide with poles in any of the three decreasing sequences. The sets $\Delta_{\mathcal{S},\sigma_s}$ and Δ_ρ also contain all the values of σ_s and ρ at which the prefactor $P_{\mathcal{S}}(\sigma_s, \rho)$ has poles. Furthermore, $Z(x, \theta_*)$ possesses one increasing sequence of poles starting at $x = -\theta_0 - \theta_* - \theta_t$ which lies in the half-plane $\operatorname{Im} x \geq 0$. The real parts of the poles in this sequence tend to $+\infty$ as $\theta_* \to -\infty$.

Let K_{σ_s} and K_ρ be compact subsets of $\mathbb{C}\setminus\Delta_{\mathcal{S},\sigma_s}$ and $\{\operatorname{Im}\rho > -Q/2\}\setminus\Delta_\rho$, respectively. Suppose $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$. Then, the above discussion shows that it is possible to choose a contour $\mathcal{C}_{\mathcal{S}} = \mathcal{C}_{\mathcal{S}}(\sigma_s, \rho)$ from $-\infty$ to $+\infty$ which separates the two upward from the three downward sequences of poles of $Z(\cdot, \theta_*) I_{\mathcal{S}}(\cdot, \sigma_s, \rho)$. Let us choose $\mathcal{C}_{\mathcal{S}}$ so that its right tail approaches the horizontal line $\operatorname{Im} x = -Q/2 - \delta$ as $\operatorname{Re} x \to +\infty$ for some $\delta > 0$. Then there is an $N > 0$ such that $\mathcal{C}_{\mathcal{S}}$ is independent of θ_* for $\theta_* < -N$, and (4.5) and (5.8) imply that, for all $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$ and all $\theta_* < -N$,

$$\mathcal{H}\left(b, \theta_0, \theta_t, \theta_*, \sigma_s, \tfrac{\theta_*}{2}+\rho\right) = P_{\mathcal{S}}(\sigma_s, \rho) \int_{\mathcal{C}_{\mathcal{S}}} dx\, Z(x, \theta_*) I_{\mathcal{S}}(x, \sigma_s, \rho). \quad (5.10)$$

Utilizing the asymptotic formula (2.8), it can be verified that the following limit

$$\lim_{\theta_* \to -\infty} Z(x, \theta_*) = 1, \quad (5.11)$$

holds uniformly for $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$ and for x in bounded subsets of $\mathcal{C}_{\mathcal{S}}$. Moreover, using (2.8) with $\epsilon = 1/2$ we find that $I_{\mathcal{S}}$ obeys the estimates

$$I_{\mathcal{S}}(x, \sigma_s, \rho) = \begin{cases} O\left(e^{2\pi|\operatorname{Re} x|(\operatorname{Im} x - \operatorname{Im}\rho)}\right), & \operatorname{Re} x \to +\infty, \\ O\left(e^{-2\pi Q|\operatorname{Re} x|}\right), & \operatorname{Re} x \to -\infty, \end{cases} \quad (5.12)$$

uniformly for (σ_s, ρ) in compact subsets of $K_{\sigma_s} \times K_\rho$ and $\operatorname{Im} x$ in compact subsets of $\mathbb{R}$. Since the contour $\mathcal{C}_{\mathcal{S}}$ stays a bounded distance away from the poles of the integrand $I_{\mathcal{S}}$, we infer that there exists a constant $C_1 > 0$ such that

$$|I_{\mathcal{S}}(x,\sigma_s,\rho)| \leq C_1 \times \begin{cases} e^{2\pi|\operatorname{Re} x|(\operatorname{Im} x - \operatorname{Im}\rho)}, & \operatorname{Re} x \geq 0, \\ e^{-2\pi Q|\operatorname{Re} x|}, & \operatorname{Re} x \leq 0, \end{cases} \tag{5.13}$$

uniformly for (σ_s, ρ) in compact subsets of $K_{\sigma_s} \times K_\rho$. In particular, the integrand $I_{\mathcal{S}}$ has exponential decay along the left and right tails of the contour $\mathcal{C}_{\mathcal{S}}$.

Suppose we can show that there exist constants $c > 0$ and $C > 0$ such that

$$|Z(x,\theta_*) I_{\mathcal{S}}(x,\sigma_s,\rho)| \leq C e^{-c|\operatorname{Re} x|}, \tag{5.14}$$

uniformly for all $\theta_* < -N$, $x \in \mathcal{C}_{\mathcal{S}}$ and $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$. Then it follows from (5.10), (5.11) and Lebesgue's dominated convergence theorem that the limit in (5.3) exists uniformly for $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$ and is given by (5.4). This proves that the limit in (5.3) exists uniformly for (σ_s, ρ) in compact subsets of $\Omega_{\mathcal{S}}$ and proves (5.4) for $(\sigma_s, \rho) \in \Omega_{\mathcal{S}}$. The analyticity of $\mathcal{S}$ as a function of $(\sigma_s, \rho) \in \Omega_{\mathcal{S}}$ follows from the analyticity of $\mathcal{H}$ together with the uniform convergence on compact subsets.

Let $c_0 < -Q/2$ be a constant. Then we can choose the contour $\mathcal{C}_{\mathcal{S}}$ so that it satisfies the condition in (5.7) for any $\rho \in \mathbb{C}\setminus\Delta_\rho$ with $\operatorname{Im}\rho > c_0$. For such a choice of $\mathcal{C}_{\mathcal{S}}$, the estimate (5.13) ensures that the integral representation (5.4) provides a meromorphic continuation of $\mathcal{S}$ to $(\sigma_s, \rho) \in \mathbb{C} \times \{\operatorname{Im}\rho > c_0\}$ which is analytic for $(\sigma_s, \rho) \in (\mathbb{C}\setminus\Delta_{\mathcal{S},\sigma_s}) \times (\{\operatorname{Im}\rho > c_0\}\setminus\Delta_\rho)$. Since c_0 was arbitrary, it follows that $\mathcal{S}$ admits a meromorphic continuation to $(\sigma_s, \rho) \in \mathbb{C}^2$ which is analytic for $(\sigma_s, \rho) \in (\mathbb{C}\setminus\Delta_{\mathcal{S},\sigma_s}) \times (\mathbb{C}\setminus\Delta_\rho)$.

To complete the proof of the theorem, it only remains to prove (5.14). The asymptotic formula (2.8) for s_b with $\epsilon = 1/2$ implies that there exist constants $C_2, C_3, C_4 > 0$ such that the inequalitites

$$\left|s_b\left(\theta_0+\theta_*+\theta_t+\tfrac{iQ}{2}\right)\right| \leq C_2\, e^{\frac{\pi Q|\theta_*|}{2}}, \qquad \theta_* < -N, \tag{5.15}$$

$$\left|s_b\left(x+\theta_0+\theta_*+\theta_t+\tfrac{iQ}{2}\right)^{-1}\right| \leq C_3\, e^{-\frac{\pi Q|\theta_*+\operatorname{Re} x|}{2}} e^{-\pi(\operatorname{Im} x)|\theta_*+\operatorname{Re} x|}, \qquad x \in \mathcal{C}_{\mathcal{S}},\ \theta_* \in \mathbb{R}, \tag{5.16}$$

$$\left|e^{\frac{i\pi x^2}{2}} e^{i\pi x\left(\theta_0+\theta_*+\theta_t+\frac{iQ}{2}\right)}\right| \leq C_4\, e^{-\pi(\operatorname{Im} x)(\theta_*+\operatorname{Re} x)} e^{-\frac{\pi Q \operatorname{Re} x}{2}}, \qquad x \in \mathcal{C}_{\mathcal{S}}, \tag{5.17}$$

hold uniformly for $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$. Therefore, in view of (5.9), there exists a constant $C_5 > 0$ such that

$$|Z(x,\theta_*)| \leq C_5\, e^{-\pi(\operatorname{Im} x+\frac{Q}{2})(\theta_*+\operatorname{Re} x+|\theta_*+\operatorname{Re} x|)}, \quad x \in \mathcal{C}_{\mathcal{S}},\ \theta_* < -N, \tag{5.18}$$

uniformly for $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$. The inequality (5.18) can be rewritten as follows:

$$|Z(x,\theta_*)| \leq \begin{cases} C_5\, e^{-2\pi(\operatorname{Im} x+\frac{Q}{2})(\theta_*+\operatorname{Re} x)}, & \theta_*+\operatorname{Re} x \geq 0, \\ C_5, & \theta_*+\operatorname{Re} x \leq 0, \end{cases} \quad x \in \mathcal{C}_{\mathcal{S}},\ \theta_* < -N, \tag{5.19}$$

uniformly for $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$. The inequalities (5.13) and (5.19) yield the existence of a constant $C_6 > 0$ independent of $x \in \mathcal{C}_\mathcal{S}$, $\theta_* < -N$ and $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$ such that

$$|Z(x,\theta_*)I_\mathcal{S}(x,\sigma_s,\rho)| \leq \begin{cases} C_6\, e^{-2\pi(\mathrm{Re}\,x)(\frac{Q}{2}+\mathrm{Im}\,\rho)} e^{-2\pi\theta_*(\frac{Q}{2}+\mathrm{Im}\,x)}, & \mathrm{Re}\,x \geq -\theta_*, \\ C_6\, e^{2\pi(\mathrm{Re}\,x)(\mathrm{Im}\,x - \mathrm{Im}\,\rho)}, & 0 \leq \mathrm{Re}\,x \leq -\theta_*, \\ C_6\, e^{-2\pi Q|\mathrm{Re}\,x|}, & \mathrm{Re}\,x \leq 0, \end{cases} \tag{5.20}$$

uniformly for $(\sigma_s, \rho) \in K_{\sigma_s} \times K_\rho$. Since $\mathrm{Im}\,\rho > -Q/2 + \delta$ for $\rho \in K_\rho$ and $\mathrm{Im}\,x < -Q/2 - \delta$ on the right tail of the contour for some $\delta > 0$, this proves (5.14) and completes the proof. □

Furthermore, thanks to the property (2.7) of s_b the function $\mathcal{S}$ has the symmetry

$$\mathcal{S}(b, \theta_0, \theta_t, \sigma_s, \rho) = \mathcal{S}(b^{-1}, \theta_0, \theta_t, \sigma_s, \rho). \tag{5.21}$$

5.2 Difference Equations

By taking the confluent limit (5.3) of the difference equations (4.26) and (4.33) satisfied by $\mathcal{H}$, it follows that the function $\mathcal{S}$ is a joint eigenfunction of four different difference operators, two acting on σ_s and the remaining two acting on ρ. The four difference equations will hold as equalities between meromorphic functions on $\mathbb{C}^2$. Since the derivations are similar to those presented in Sect. 4.2, we state these results without proofs.

5.2.1 First Pair of Difference Equations

Define the difference operator $D_\mathcal{S}(b, \sigma_s)$ by

$$D_\mathcal{S}(b,\sigma_s) = d_\mathcal{S}^+(b,\sigma_s)e^{ib\partial_{\sigma_s}} + d_\mathcal{S}^+(b,-\sigma_s)e^{-ib\partial_{\sigma_s}} + d_\mathcal{S}^0(b,\sigma_s), \tag{5.22}$$

where

$$d_\mathcal{S}^+(b,\sigma_s) = e^{-\pi b(2\sigma_s+iQ)} \frac{\prod_{\epsilon=\pm1}\cosh\left(\pi b\left(\frac{ib}{2}+\epsilon\theta_0+\theta_t+\sigma_s\right)\right)}{\sinh(\pi b(2\sigma_s+ib))\sinh(2\pi b\sigma_s)}, \tag{5.23}$$

$$d_\mathcal{S}^0(b,\sigma_s) = e^{-\pi b(2\theta_t+iQ)} - d_\mathcal{S}^+(b,\sigma_s) - d_\mathcal{S}^+(b,-\sigma_s). \tag{5.24}$$

Proposition 5.3 *For $(\sigma_s, \rho) \in \mathbb{C}^2$, the function $\mathcal{S}$ satisfies the pair of difference equations*

$$D_\mathcal{S}(b,\sigma_s)\,\mathcal{S}(b,\theta_0,\theta_t,\sigma_s,\rho) = e^{-2\pi b\rho}\,\mathcal{S}(b,\theta_0,\theta_t,\sigma_s,\rho), \tag{5.25a}$$

$$D_\mathcal{S}(b^{-1},\sigma_s)\,\mathcal{S}(b,\theta_0,\theta_t,\sigma_s,\rho) = e^{-2\pi b^{-1}\rho}\,\mathcal{S}(b,\theta_0,\theta_t,\sigma_s,\rho). \tag{5.25b}$$

5.2.2 Second Pair of Difference Equations

Define the difference operator $\tilde{D}_{\mathcal{S}}(b,\rho)$ such that

$$\tilde{D}_{\mathcal{S}}(b,\rho) = \tilde{d}^{+}_{\mathcal{S}}(b,\rho)e^{ib\partial_\rho} + \tilde{d}^{-}_{\mathcal{S}}(b,\rho)e^{-ib\partial_\rho} + \tilde{d}^{0}_{\mathcal{S}}(b,\rho), \tag{5.26}$$

where

$$\tilde{d}^{\pm}_{\mathcal{S}}(b,\rho) = -2e^{\pi b\rho}e^{\mp\pi b\left(\frac{ib}{2}+2\theta_0+\theta_t\right)}\cosh\left(\pi b\left(\tfrac{ib}{2}+\theta_t\pm\rho\right)\right), \tag{5.27}$$

$$\tilde{d}^{0}_{\mathcal{S}}(b,\rho) = -2\cosh\left(2\pi b\left(\tfrac{ib}{2}+\theta_0+\theta_t\right)\right) - \tilde{d}^{+}_{\mathcal{S}}(b,\rho) - \tilde{d}^{-}_{\mathcal{S}}(b,\rho). \tag{5.28}$$

Proposition 5.4 *For $(\sigma_s,\rho)\in\mathbb{C}^2$, the function $\mathcal{S}$ satisfies the following pair of difference equations:*

$$\tilde{D}_{\mathcal{S}}(b,\rho)\,\mathcal{S}(b,\theta_0,\theta_t,\sigma_s,\rho) = 2\cosh(2\pi b\sigma_s)\,\mathcal{S}(b,\theta_0,\theta_t,\sigma_s,\rho), \tag{5.29a}$$

$$\tilde{D}_{\mathcal{S}}(b^{-1},\rho)\,\mathcal{S}(b,\theta_0,\theta_t,\sigma_s,\rho) = 2\cosh(2\pi b^{-1}\sigma_s)\,\mathcal{S}(b,\theta_0,\theta_t,\sigma_s,\rho). \tag{5.29b}$$

5.3 Polynomial Limits

Our next two theorems state that $\mathcal{S}$ reduces to the Al-Salam Chihara polynomials when the variable ρ is suitably discretized and to the little q-Jacobi polynomials when σ_s is suitably discretized. The proofs proceed along the same lines as the proofs of Theorem 4.7 and Theorem 4.8 and are therefore omitted.

We make the following assumption which ensures that the poles of the integrand $I_{\mathcal{S}}$ in (5.6) are simple.

Assumption 5.5 Assume that $b>0$ is such that b^2 is irrational. Moreover, assume that

$$\theta_t, \operatorname{Re}\sigma_s \neq 0, \qquad \operatorname{Re}\left(\pm\sigma_s+\rho+\theta_0\right)\neq 0. \tag{5.30}$$

Theorem 5.6 (From $\mathcal{S}$ to the Al-Salam Chihara polynomials) *Let $\sigma_s\in\mathbb{C}\setminus\Delta_{\mathcal{S},\sigma_s}$ and suppose that Assumptions 1.1 and 5.5 are satisfied. Define $\{\rho_n\}_{n=0}^{\infty}\subset\mathbb{C}$ by*

$$\rho_n = \theta_t + \tfrac{iQ}{2} + inb. \tag{5.31}$$

Under the parameter correspondence

$$\alpha_S = e^{2\pi b(\theta_t+\theta_0+\frac{iQ}{2})}, \qquad \beta_S = e^{2\pi b(\theta_t-\theta_0+\frac{iQ}{2})}, \qquad q = e^{2i\pi b^2}, \tag{5.32}$$

the function $\mathcal{S}$ satisfies, for each $n\geq 0$,

$$\lim_{\rho\to\rho_n}\mathcal{S}(b,\theta_0,\theta_t,\sigma_s,\rho) = S_n\left(e^{2\pi b\sigma_s};\alpha_S,\beta_S;q\right), \tag{5.33}$$

where S_n are the Al-Salam Chihara polynomials defined in (B.17).

Theorem 5.7 (From $\mathcal{S}$ to the little q-Jacobi polynomials) *Let $\rho \in \mathbb{C}\setminus\Delta_\rho$ and suppose that Assumptions 1.1 and 5.5 are satisfied. Under the parameter correspondence*

$$\alpha_Y = e^{4\pi b\theta_t}, \qquad \beta_Y = e^{4\pi b\theta_0}, \qquad x_Y = e^{\pi b(iQ+2\theta_t)}e^{-2\pi b\rho}, \qquad q = e^{2i\pi b^2}, \tag{5.34}$$

the function $\mathcal{S}$ satisfies, for each $n \geq 0$,

$$\lim_{\sigma_s \to \sigma_s^{(n)}} \mathcal{S}(b, \theta_0, \theta_t, \sigma_s, \rho) = Y_n\left(x_Y; \alpha_Y, \beta_Y, q\right), \tag{5.35}$$

where $\sigma_s^{(n)}$ is defined in (3.11) and where Y_n are the little q-Jacobi polynomials defined in (B.24).

6 The Function $\mathcal{X}$

In this section, we define the function $\mathcal{X}(b, \theta, \sigma_s, \omega)$ which generalizes continuous big q-Hermite polynomials. It lies at the fourth level of the non-polynomial scheme and is defined as a confluent limit of $\mathcal{S}$. We show that $\mathcal{X}$ is a joint eigenfunction of four difference operators and that it reduces to the continuous big q-Hermite polynomials, which lie at the fourth level of the q-Askey scheme, when ω is suitably discretized.

6.1 Definition and Integral Representation

Let θ and ω be two new parameters defined by

$$\theta_0 = \frac{\theta + \Lambda}{2}, \qquad \theta_t = \frac{\theta - \Lambda}{2}, \qquad \rho = -\frac{\Lambda}{2} + \omega. \tag{6.1}$$

Definition 6.1 The function $\mathcal{X}$ is defined by

$$\mathcal{X}(b, \theta, \sigma_s, \omega) = \lim_{\Lambda \to +\infty} \mathcal{S}\left(b, \tfrac{\theta+\Lambda}{2}, \tfrac{\theta-\Lambda}{2}, \sigma_s, -\tfrac{\Lambda}{2} + \omega\right), \tag{6.2}$$

where $\mathcal{S}$ is given in (5.4).

The next theorem shows that for each choice of $(b, \theta) \in (0, \infty) \times \mathbb{R}$, $\mathcal{X}$ is a well-defined analytic function of $(\sigma_s, \omega) \in (\mathbb{C}\setminus\Delta_{\mathcal{X},\sigma_s}) \times (\mathbb{C}\setminus\Delta_\omega)$, where $\Delta_{\mathcal{S},\sigma_s}, \Delta_\omega \subset \mathbb{C}$ are discrete sets of points where $\mathcal{S}$ may have poles. The proof is omitted since it involves computations which are similar to those presented in the proofs of Theorems 4.2 and 5.2.

Theorem 6.2 *Suppose that Assumption 1.1 is satisfied. The limit (6.2) exists uniformly for (σ_s, ω) in compact subsets of*

$$\Omega_{\mathcal{X}} := (\mathbb{C}\backslash\Delta_{\mathcal{X},\sigma_s}) \times (\mathbb{C}\backslash\Delta_\omega), \tag{6.3}$$

where

$$\Delta_{\mathcal{X},\sigma_s} := \{\pm\sigma_s \,|\, \sigma_s \in \Delta'_{\mathcal{X},\sigma_s}\},$$
$$\Delta_\omega := \{\tfrac{\theta}{2} + \tfrac{iQ}{2} + ibm + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{\tfrac{\theta}{2} - \tfrac{iQ}{2} - ibm - ilb^{-1}\}_{m,l=0}^{\infty},$$

with

$$\Delta'_{\mathcal{X},\sigma_s} := \{\theta + \tfrac{iQ}{2} + imb + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{\theta - \tfrac{iQ}{2} - imb - ilb^{-1}\}_{m,l=0}^{\infty}.$$

Moreover, the function $\mathcal{X}$ admits the following integral representation:

$$\mathcal{X}(b, \theta, \sigma_s, \omega) = P_{\mathcal{X}}(\sigma_s, \omega) \int_{\mathcal{C}_{\mathcal{X}}} dx\; I_{\mathcal{X}}(x, \sigma_s, \omega) \qquad \text{for } (\sigma_s, \omega) \in \Omega_{\mathcal{X}}, \tag{6.4}$$

where

$$P_{\mathcal{X}}(\sigma_s, \omega) = s_b\left(\omega - \tfrac{\theta}{2}\right) \prod_{\epsilon=\pm1} s_b(\epsilon\sigma_s - \theta), \tag{6.5}$$

$$I_{\mathcal{X}}(x, \sigma_s, \omega) = e^{-i\pi x^2} e^{-i\pi x\left(\frac{5\theta}{2} + \frac{3iQ}{2} - \omega\right)} \frac{s_b\left(x + \frac{\theta}{2} - \omega\right)}{s_b(x + \frac{iQ}{2})} \prod_{\epsilon=\pm1} s_b\left(x + \theta + \epsilon\sigma_s\right), \tag{6.6}$$

and the contour $\mathcal{C}_{\mathcal{X}}$ is any curve from $-\infty$ to $+\infty$ which separates the three decreasing from the increasing sequences of poles, with the requirement that its right tail satisfies

$$\mathrm{Im}\, x + \frac{Q}{4} - \frac{\mathrm{Im}\,\omega}{2} < -\delta \qquad \text{for } x \in \mathcal{C}_{\mathcal{X}} \text{ with } \mathrm{Re}\, x \text{ sufficiently large}, \tag{6.7}$$

for some $\delta > 0$. In particular, $\mathcal{X}$ is a meromorphic function of $(\sigma_s, \omega) \in \mathbb{C}^2$. If $(\sigma_s, \omega) \in \mathbb{R}^2$, then the contour $\mathcal{C}_{\mathcal{X}}$ can be any curve from $-\infty$ to $+\infty$ lying within the strip $\mathrm{Im}\, x \in (-Q/2, -Q/4 - \delta)$.

Furthermore, as a consequence of (2.7), $\mathcal{X}$ satisfies

$$\mathcal{X}(b, \theta, \sigma_s, \omega) = \mathcal{X}(b^{-1}, \theta, \sigma_s, \omega). \tag{6.8}$$

6.2 Difference Equations

The function $\mathcal{X}(b, \theta, \sigma_s, \omega)$ is a joint eigenfunction of four difference operators, two acting on σ_s and the remaining two on ω. This follows by taking the confluent limit

(6.2) of the difference equations (5.25) and (5.29) satisfied by $\mathcal{S}$. The proofs of the following two propositions are omitted since they are similar to those presented in Sect. 4.2.

6.2.1 First Pair of Equations

Introduce a difference operator $D_{\mathcal{X}}(b, \sigma_s)$ such that

$$D_{\mathcal{X}}(b, \sigma_s) = d^+_{\mathcal{X}}(b, \sigma_s)e^{ib\partial_{\sigma_s}} + d^+_{\mathcal{X}}(b, -\sigma_s)e^{-ib\partial_{\sigma_s}} + d^0_{\mathcal{X}}(b, \sigma_s), \tag{6.9}$$

where

$$d^+_{\mathcal{X}}(b, \sigma_s) = -e^{-\frac{3}{2}\pi b(2\sigma_s+ib)} \frac{\cosh\left(\pi b\left(\frac{ib}{2} + \theta + \sigma_s\right)\right)}{2\sinh(2\pi b\sigma_s)\sinh(\pi b(2\sigma_s + ib))}, \tag{6.10}$$

$$d^0_{\mathcal{X}}(b, \sigma_s) = e^{-\pi b(\theta+iQ)} - d^+_{\mathcal{X}}(b, \sigma_s) - d^+_{\mathcal{X}}(b, -\sigma_s). \tag{6.11}$$

Proposition 6.3 *For $(\sigma_s, \omega) \in \mathbb{C}^2$, the function $\mathcal{X}$ satisfies the following pair of difference equations:*

$$D_{\mathcal{X}}(b, \sigma_s)\ \mathcal{X}(b, \theta, \sigma_s, \omega) = e^{-2\pi b\omega}\ \mathcal{X}(b, \theta, \sigma_s, \omega) \tag{6.12a}$$

$$D_{\mathcal{X}}(b^{-1}, \sigma_s)\ \mathcal{X}(b, \theta, \sigma_s, \omega) = e^{-2\pi b^{-1}\omega}\ \mathcal{X}(b, \theta, \sigma_s, \omega). \tag{6.12b}$$

6.2.2 Second Pair of Equations

Introduce the difference operator $\tilde{D}_{\mathcal{X}}(b, \omega)$ such that

$$\tilde{D}_{\mathcal{X}}(b, \omega) = \tilde{d}^+_{\mathcal{X}}(b, \omega)e^{ib\partial_\omega} + \tilde{d}^-_{\mathcal{X}}(b, \omega)e^{-ib\partial_\omega} + \tilde{d}^0_{\mathcal{X}}(b, \omega), \tag{6.13}$$

where

$$\tilde{d}^+_{\mathcal{X}}(b, \omega) = e^{-2\pi b(\theta+\frac{iQ}{2})}, \qquad \tilde{d}^-_{\mathcal{X}}(b, \omega) = -2e^{\pi b\left(\frac{ib}{2}+\frac{3\theta}{2}+\omega\right)}\cosh\left(\pi b\left(\frac{ib}{2} + \frac{\theta}{2} - \omega\right)\right),$$
$$\tilde{d}^0_{\mathcal{X}}(b, \omega) = -2\cosh(\pi b(2\theta + ib)) - \tilde{d}^+_{\mathcal{X}}(b, \omega) - \tilde{d}^-_{\mathcal{X}}(b, \omega) = e^{\pi b(\theta+2\omega)}.$$

Proposition 6.4 *For $(\sigma_s, \omega) \in \mathbb{C}^2$, the function $\mathcal{X}$ satisfies the following pair of difference equations:*

$$\tilde{D}_{\mathcal{X}}(b, \omega)\ \mathcal{X}(b, \theta, \sigma_s, \omega) = 2\cosh(2\pi b\sigma_s)\ \mathcal{X}(b, \theta, \sigma_s, \omega), \tag{6.14a}$$

$$\tilde{D}_{\mathcal{X}}(b^{-1}, \omega)\ \mathcal{X}(b, \theta, \sigma_s, \omega) = 2\cosh(2\pi b^{-1}\sigma_s)\ \mathcal{X}(b, \theta, \sigma_s, \omega). \tag{6.14b}$$

6.3 Polynomial Limit

Our next theorem shows that $\mathcal{X}$ reduces to the continuous big q-Hermite polynomials when ω is suitably discretized. We make the following assumption, which implies that all the poles of the integrand $I_{\mathcal{X}}$ are simple and that $q = e^{2i\pi b^2}$ is not a root of unity.

Assumption 6.5 Assume that $b > 0$ is such that b^2 is irrational and that

$$\operatorname{Re}\sigma_s \neq 0, \qquad \operatorname{Re}\left(\omega + \tfrac{\theta}{2} \pm \sigma_s\right) \neq 0. \tag{6.15}$$

The proof of the next theorem is analogous to the proof of Theorem 4.7 and is omitted.

Theorem 6.6 (From $\mathcal{X}$ to the continuous big q-Hermite polynomials) *Let $\sigma_s \in \mathbb{C}\setminus\Delta_{\mathcal{X},\sigma_s}$ and suppose that Assumptions 1.1 and 6.5 are satisfied. Define $\{\omega_n\}_{n=0}^{\infty} \subset \mathbb{C}$ by*

$$\omega_n = \frac{\theta}{2} + \frac{iQ}{2} + ibn. \tag{6.16}$$

Under the parameter correspondence

$$\alpha_X = e^{2\pi b(\theta + \frac{iQ}{2})}, \qquad q = e^{2i\pi b^2}, \tag{6.17}$$

the function $\mathcal{X}$ satisfies, for each $n \geq 0$,

$$\lim_{\omega \to \omega_n} \mathcal{X}(b, \theta, \omega, \sigma_s) = X_n(e^{2\pi b\sigma_s}; \alpha_X, q), \tag{6.18}$$

where X_n are the continuous big q-Hermite polynomials defined in (B.37).

7 The Function $\mathcal{Q}$

The function $\mathcal{Q}(b, \sigma_s, \eta)$ is defined as a confluent limit of $\mathcal{X}$ and is one of the two elements at the fifth and lowest level of the non-polynomial scheme. We show that $\mathcal{Q}$ is a joint eigenfunction of four difference operators, two acting on σ_s and two acting on η. Finally, we show that $\mathcal{Q}$ reduces to the continuous q-Hermite polynomials, which lie at the lowest level of the q-Askey scheme, when η is suitably discretized.

Interestingly, the mechanism behind the polynomial limit for $\mathcal{Q}$ is different from that of all the other polynomial limits in this paper. In all other cases, the simple pole of the contour integral which compensates for the simple zero in the prefactor arises because the contour of integration is squeezed between two poles of the integrand. However, in the case of $\mathcal{Q}$, the simple pole of the contour integral arises because the integrand loses its decay at infinity in the relevant polynomial limit.

The derivation of the difference equations for $\mathcal{Q}$ also presents some novelties compared to the other derivations of difference equations appearing in this paper. More precisely, the first pair of difference equations for $\mathcal{Q}$ obtained by taking the confluent limit of the difference equations for $\mathcal{X}$ are "squares" of the simplest possible difference equations for $\mathcal{Q}$. By finding square roots of the relevant difference operators, we obtain the equations in their simplified form (this is the form that reduces to the standard difference equations for the continuous q-Hermite polynomials in the polynomial limit).

7.1 Definition and Integral Representation

Let η be a new parameter defined as follows:

$$\omega = \frac{\theta}{2} + \eta. \tag{7.1}$$

Moreover, define the normalization factor K by

$$K(\eta, \theta) = e^{2i\pi\left(\eta - \frac{iQ}{2}\right)\left(\theta + \frac{iQ}{2}\right)}. \tag{7.2}$$

The factor K is a non-polynomial analog of the factor α^{-n} in (B.40).

Definition 7.1 The function $\mathcal{Q}$ is defined by

$$\mathcal{Q}(b, \sigma_s, \eta) = \lim_{\theta \to -\infty} \left(K(\eta, \theta)\mathcal{X}(b, \theta, \sigma_s, \tfrac{\theta}{2} + \eta)\right). \tag{7.3}$$

The next theorem shows that $\mathcal{Q}$ is a well-defined and analytic function of $(\sigma_s, \eta) \in \mathbb{C} \times (\{\operatorname{Im} \eta < Q/2\} \setminus \Delta_\eta)$, where Δ_η is a discrete set of points at which $\mathcal{Q}$ may have poles. The theorem also provides an integral representation for $\mathcal{Q}$. Even if the requirement $\operatorname{Im} \eta < Q/2$ is needed to ensure that the integral in the representation for $\mathcal{Q}$ converges, it will follow from the difference equations established later that $\mathcal{Q}$ extends to a meromorphic function of $\eta \in \mathbb{C}$.

Theorem 7.2 *Suppose that Assumption 1.1 is satisfied. The limit (7.3) exists uniformly for (σ_s, η) in compact subsets of*

$$\Omega_{\mathcal{Q}} := \mathbb{C} \times (\{Im\, \eta < Q/2\} \setminus \Delta_\eta), \tag{7.4}$$

where

$$\Delta_\eta := \{-\tfrac{iQ}{2} - imb - ilb^{-1}\}_{m,l=0}^{\infty}. \tag{7.5}$$

Moreover, the function $\mathcal{Q}$ admits the following integral representation:

$$\mathcal{Q}(b, \sigma_s, \eta) = P_{\mathcal{Q}}(\sigma_s, \eta) \int_{\mathcal{C}_{\mathcal{Q}}} dx\ I_{\mathcal{Q}}(x, \sigma_s, \eta) \quad \textit{for } (\sigma_s, \eta) \in \Omega_{\mathcal{Q}}, \tag{7.6}$$

where

$$P_{\mathcal{Q}}(\sigma_s,\eta) = e^{i\pi\left(\frac{1}{6}+\frac{7Q^2}{24}-\frac{\eta^2}{2}+i\eta Q-\sigma_s^2\right)}s_b(\eta), \tag{7.7}$$

$$I_{\mathcal{Q}}(x,\sigma_s,\eta) = e^{-i\pi x^2}e^{2i\pi x(\eta-\frac{iQ}{2})}s_b\left(x+\sigma_s\right)s_b\left(x-\sigma_s\right), \tag{7.8}$$

and the contour of integration $\mathcal{C}_{\mathcal{Q}}$ is any curve from $-\infty$ to $+\infty$ passing above the points $x=\pm\sigma_s - iQ/2$ and with the requirement that its right tail satisfies

$$Im\, x + \frac{Q}{4} - \frac{Im\,\eta}{2} < -\delta \quad \textit{for all } x\in\mathcal{C}_{\mathcal{Q}} \textit{ with } Re\, x \textit{ sufficiently large}, \tag{7.9}$$

for some $\delta>0$. If $(\sigma_s,\eta)\in\mathbb{R}^2$, then the contour $\mathcal{C}_{\mathcal{Q}}$ can be any curve from $-\infty$ to $+\infty$ lying within the strip $Im\, x\in(-Q/2,-Q/4-\delta)$.

Proof The proof will be omitted since it involves computations which are similar to those presented in the proofs of Theorems 4.2 and 5.2. However, we point out that before taking the limit $\theta\to-\infty$ of the integral representation (6.4) for $\mathcal{X}$, one should make the change of variables $x\to x-\theta$ and thus write

$$K(\eta,\theta)P_{\mathcal{X}}(\sigma_s,\tfrac{\theta}{2}+\eta)I_{\mathcal{X}}(x-\theta,\sigma_s,\tfrac{\theta}{2}+\eta) = P_{\mathcal{Q}}(\sigma_s,\eta)Z_{\mathcal{Q}}(x,\theta)I_{\mathcal{Q}}(x,\sigma_s,\eta), \tag{7.10}$$

where

$$\begin{aligned} Z_{\mathcal{Q}}(x,\theta) &:= e^{\frac{i\pi}{6}(3\eta^2+\frac{5Q^2}{4}+6\sigma_s^2-1)}e^{\frac{\pi Q}{2}(x-\theta)}e^{-i\pi\eta x} \\ &\quad\times e^{i\pi\theta(\eta+\theta)}\frac{s_b(x-\eta-\theta)s_b(\sigma_s-\theta)s_b(-\sigma_s-\theta)}{s_b(x-\theta+\frac{iQ}{2})} \end{aligned} \tag{7.11}$$

satisfies $Z_{\mathcal{Q}}(x,\theta)\to 1$ as $\theta\to-\infty$. □

Thanks to (2.7), we have

$$\mathcal{Q}(b,\sigma_s,\eta) = \mathcal{Q}(b^{-1},\sigma_s,\eta). \tag{7.12}$$

Remark 7.3 A close relative of the function $\mathcal{Q}$ has appeared in [24, 54] in the context of quantum relativistic Toda systems. More precisely, the function $\mathcal{H}(a_-,a_+,x,y)$ in [54, Eq. (5.56)] is related to $\mathcal{Q}$ by

$$\mathcal{Q}(b,\sigma_s,\eta) = e^{\frac{1}{4}i\pi\left(\frac{Q^2}{4}+\frac{1}{2}-\eta^2-8\sigma_s^2\right)}g_b(\eta)\,\mathcal{H}(b,b^{-1},-\eta,2\sigma_s), \tag{7.13}$$

where $g_b(z)$ satisfies $s_b(z)=g_b(z)/g_b(-z)$ and is defined by

$$\begin{aligned} g_b(z) &= \exp\left\{\int_0^\infty \frac{dt}{t}\left[\frac{e^{2izt}-1}{4\sinh bt\sinh b^{-1}t}+\frac{1}{4}z^2\left(e^{-2bt}+e^{-\frac{2t}{b}}\right)-\frac{iz}{2t}\right]\right\}, \\ &\quad \text{for Im}\, z > -\frac{Q}{2}. \end{aligned} \tag{7.14}$$

7.2 Difference Equations

We show that $\mathcal{Q}(b,\sigma_s,\eta)$ is a joint eigenfunction of four difference operators, two acting on σ_s and the remaining two on η.

We know from Theorem 7.2 that $\mathcal{Q}$ is a well-defined analytic function of $\sigma_s \in \mathbb{C}$ and a meromorphic function of η for $\mathrm{Im}\,\eta < Q/2$. The difference equations will first be derived as equalities between meromorphic functions defined on this limited domain. However, the difference equations in η can then be used to extend the results to $\eta \in \mathbb{C}$, see Proposition 7.8.

7.2.1 First Pair of Equations

Define the difference operator $D_{\mathcal{Q}}(b,\sigma_s)$ by

$$D_{\mathcal{Q}}(b,\sigma_s) = d^{+}_{\mathcal{Q}}(b,\sigma_s)e^{ib\partial_{\sigma_s}} + d^{+}_{\mathcal{Q}}(b,-\sigma_s)e^{-ib\partial_{\sigma_s}} + d^{0}_{\mathcal{Q}}(b,\sigma_s), \tag{7.15}$$

where

$$d^{+}_{\mathcal{Q}}(b,\sigma_s) = -\frac{e^{-2\pi b(2\sigma_s+ib)}}{4\sinh(2\pi b\sigma_s)\sinh(\pi b(2\sigma_s+ib))}, \tag{7.16}$$

$$d^{0}_{\mathcal{Q}}(b,\sigma_s) = e^{-i\pi bQ} - d^{+}_{\mathcal{Q}}(b,\sigma_s) - d^{+}_{\mathcal{Q}}(b,-\sigma_s). \tag{7.17}$$

Proposition 7.4 *For $\sigma_s \in \mathbb{C}$ and $\mathrm{Im}\,\eta < Q/2$, the function $\mathcal{Q}$ satisfies the following pair of difference equations:*

$$D_{\mathcal{Q}}(b,\sigma_s)\ \mathcal{Q}(b,\sigma_s,\eta) = e^{-2\pi b\eta}\ \mathcal{Q}(b,\sigma_s,\eta), \tag{7.18a}$$

$$D_{\mathcal{Q}}(b^{-1},\sigma_s)\ \mathcal{Q}(b,\sigma_s,\eta) = e^{-2\pi b^{-1}\eta}\ \mathcal{Q}(b,\sigma_s,\eta). \tag{7.18b}$$

Proof It is straightforward to verify that the following limits hold:

$$\lim_{\theta\to-\infty} e^{\pi b\theta} d^{+}_{\mathcal{X}}(b,\pm\sigma_s) = d^{+}_{\mathcal{Q}}(b,\pm\sigma_s), \qquad \lim_{\theta\to-\infty} e^{\pi b\theta} d^{0}_{\mathcal{X}}(b,\sigma_s) = d^{0}_{\mathcal{Q}}(b,\sigma_s), \tag{7.19}$$

where $d^{+}_{\mathcal{X}}$ and $d^{0}_{\mathcal{X}}$ are defined in (6.10) and (6.11), respectively. It follows that

$$\lim_{\theta\to-\infty} e^{\pi b\theta} D_{\mathcal{X}}(b,\sigma_s) = D_{\mathcal{Q}}(b,\sigma_s). \tag{7.20}$$

where $D_{\mathcal{X}}$ is defined in (6.9). The difference equation (7.18a) follows after taking the confluent limit (7.3) of the difference equation (6.12a) and utilizing (7.20). □

In what follows, we show that the function $\mathcal{Q}$ satisfies a pair of difference equations which is more fundamental than (7.18).

Proposition 7.5 *Define the difference operator* $\hat{D}_Q(b,\sigma_s)$ *by*

$$\hat{D}_Q(b,\sigma_s) = \hat{d}_Q^+(b,\sigma_s)e^{\frac{ib}{2}\partial_{\sigma_s}} + \hat{d}_Q^-(b,\sigma_s)e^{-\frac{ib}{2}\partial_{\sigma_s}}, \tag{7.21}$$

with

$$\hat{d}_Q^+(b,\sigma_s) = -\frac{e^{-\pi b\left(2\sigma_s+\frac{iQ}{2}\right)}}{2\sinh(2\pi b\sigma_s)}, \qquad \hat{d}_Q^-(b,\sigma_s) = \hat{d}_Q^+(b,-\sigma_s). \tag{7.22}$$

The following operator identity holds:

$$\left(\hat{D}_Q(b,\sigma_s)\right)^2 = D_Q(b,\sigma_s), \tag{7.23}$$

where D_Q *is defined in (7.15).*

Proof The left-hand side of (7.23) can be written as

$$\begin{aligned}\left(\hat{D}_Q(b,\sigma_s)\right)^2 &= \hat{d}_Q^+(b,\sigma_s)\hat{d}_Q^+(b,\sigma_s+\tfrac{ib}{2})e^{ib\partial_{\sigma_s}} + \hat{d}_Q^-(b,\sigma_s)\hat{d}_Q^-(b,\sigma_s-\tfrac{ib}{2})e^{-ib\partial_{\sigma_s}}\\ &\quad + \hat{d}_Q^+(b,\sigma_s)\hat{d}_Q^-(b,\sigma_s+\tfrac{ib}{2}) + \hat{d}_Q^-(b,\sigma_s)\hat{d}_Q^+(b,\sigma_s-\tfrac{ib}{2})\end{aligned} \tag{7.24}$$

and straightforward computations show that the right-hand side coincides with the operator $D_Q(b,\sigma_s)$. □

We next show that the difference equations (7.18) satisfied by the function $\mathcal{Q}$ can be simplified using the identity (7.23). The simplified equations can be viewed as "square roots" of the equations in (7.18).

Proposition 7.6 *For* $\sigma_s \in \mathbb{C}$ *and* $\operatorname{Im}\eta < Q/2$, *the function* $\mathcal{Q}$ *satisfies*

$$\hat{D}_Q(b,\sigma_s)\ \mathcal{Q}(b,\sigma_s,\eta) = e^{-\pi b\eta}\ \mathcal{Q}(b,\sigma_s,\eta), \tag{7.25a}$$

$$\hat{D}_Q(b^{-1},\sigma_s)\ \mathcal{Q}(b,\sigma_s,\eta) = e^{-\pi b^{-1}\eta}\ \mathcal{Q}(b,\sigma_s,\eta). \tag{7.25b}$$

Proof Let us rewrite (7.6) as

$$\mathcal{Q}(b,\sigma_s,\eta) = \int_{\mathcal{C}_Q} dx\ \psi(x,\sigma_s,\eta)e^{2i\pi x\eta}, \tag{7.26}$$

where

$$\psi(x,\sigma_s,\eta) = e^{i\pi\left(\frac{1}{6}+\frac{7Q^2}{24}-\frac{\eta^2}{2}+i\eta Q-\sigma_s^2\right)} s_b(\eta)e^{-i\pi x^2}e^{\pi Qx}s_b(x+\sigma_s)\,s_b(x-\sigma_s), \tag{7.27}$$

and where the contour $\mathcal{C}_Q$ is such that it passes above the points $x=\pm\sigma_s$, and such that $\operatorname{Im}x < \frac{\operatorname{Im}\eta}{2} - Q/4 - \delta$ for some small but fixed $\delta > 0$ as $\operatorname{Re}x \to +\infty$. The following estimates, which are easily established with the help of (2.8), imply that the integrand has exponential decay on $\mathcal{C}_{\mathcal{Q}}$ as $\operatorname{Re} x \to +\infty$:

$$\psi(x, \sigma_s, \eta) e^{2i\pi x\eta} = \begin{cases} O\left(e^{4\pi(\operatorname{Re} x)(\operatorname{Im} x + \frac{Q}{4} - \frac{\operatorname{Im}\eta}{2})}\right), & \operatorname{Re} x \to +\infty, \\ O\left(e^{2\pi \operatorname{Re} x(\frac{Q}{2} - \operatorname{Im}\eta)}\right), & \operatorname{Re} x \to -\infty, \end{cases} \tag{7.28}$$

uniformly for $(b, \operatorname{Im} x, \sigma_s, \eta)$ in compact subsets of $(0, \infty) \times \mathbb{R} \times \mathbb{C}^2$.

Utilizing the difference equation (2.5), we verify that the following identity holds:

$$\hat{D}_{\mathcal{Q}}(b, \sigma_s)\psi(x, \sigma_s, \eta) = \psi(x - \tfrac{ib}{2}, \sigma_s, \eta). \tag{7.29}$$

Letting the difference operator $\hat{D}_{\mathcal{Q}}$ act inside the contour integral in (7.26) and using (7.29), we obtain

$$\hat{D}_{\mathcal{Q}}(b, \sigma_s)\mathcal{Q}(b, \sigma_s, \eta) = \int_{\mathcal{C}_{\mathcal{Q}}} dx\ \psi(x - \tfrac{ib}{2}, \sigma_s, \eta) e^{2i\pi x\eta}.$$

Performing the change of variables $x = y + ib/2$, we find

$$\hat{D}_{\mathcal{Q}}(b, \sigma_s)\mathcal{Q}(b, \sigma_s, \eta) = e^{-\pi b\eta} \int_{\mathcal{C}_{\mathcal{Q}} - \frac{ib}{2}} dy\ \psi(y, \sigma_s, \eta) e^{2i\pi y\eta}.$$

Deforming the contour back to $\mathcal{C}_{\mathcal{Q}}$, noting that no poles are crossed and that the integrand retains its exponential decay at infinity throughout the deformation, we arrive at (7.25a). The difference equation (7.25b) follows from (7.25a) and the symmetry (7.12) of $\mathcal{Q}$. □

7.2.2 Second Pair of Equations

Define the difference operator $\tilde{D}_{\mathcal{Q}}(b, \eta)$ by

$$\tilde{D}_{\mathcal{Q}}(b, \eta) = e^{ib\partial_\eta} + \left(1 + e^{2\pi b\left(\eta - \frac{ib}{2}\right)}\right) e^{-ib\partial_\eta}. \tag{7.30}$$

Proposition 7.7 *For $\sigma_s \in \mathbb{C}$ and $\operatorname{Im}(\eta + ib^{\pm 1}) < Q/2$, the function $\mathcal{Q}$ satisfies the pair of difference equations*

$$\tilde{D}_{\mathcal{Q}}(b, \eta)\ \mathcal{Q}(b, \sigma_s, \eta) = 2\cosh(2\pi b\sigma_s)\ \mathcal{Q}(b, \sigma_s, \eta), \tag{7.31a}$$

$$\tilde{D}_{\mathcal{Q}}(b^{-1}, \eta)\ \mathcal{Q}(b, \sigma_s, \eta) = 2\cosh(2\pi b^{-1}\sigma_s)\ \mathcal{Q}(b, \sigma_s, \eta). \tag{7.31b}$$

Proof The difference equation (6.14a) can be rewritten as follows:

$$\left(K(\eta, \theta)\tilde{D}_{\mathcal{X}}(b, \tfrac{\theta}{2} + \eta)K(\eta, \theta)^{-1}\right) K(\eta, \theta)\mathcal{X}(b, \theta, \sigma_s, \tfrac{\theta}{2} + \eta)$$

$$= 2\cosh(2\pi b\sigma_s)\; K(\eta,\theta)\mathcal{X}(b,\theta,\sigma_s,\tfrac{\theta}{2}+\eta), \tag{7.32}$$

where the operator $\tilde{D}_{\mathcal{X}}$ is given in (6.13). Moreover, we have

$$\lim_{\theta\to-\infty} K(\eta,\theta)\tilde{D}_{\mathcal{X}}(b,\tfrac{\theta}{2}+\eta)K(\eta,\theta)^{-1} = \tilde{D}_{\mathcal{Q}}(b,\eta). \tag{7.33}$$

Taking the limit $\theta \to -\infty$ of (7.32) and recalling the definition (7.3), we obtain (7.31a). Finally, (7.31b) follows from (7.31a) and the symmetry (7.12) of $\mathcal{Q}$. □

The next proposition shows that $\mathcal{Q}$ extends to a meromorphic function of η everywhere in the complex plane. The proof will be omitted, since it is similar to that of Proposition 4.5.

Proposition 7.8 *Let $b \in (0,\infty)$ and $\sigma_s \in \mathbb{C}$. Then there is a discrete subset $\Delta \subset \mathbb{C}$ such that the limit in (7.3) exists for all $\eta \in \mathbb{C}\backslash\Delta$. Moreover, the function $\mathcal{Q}$ defined by (7.3) is a meromorphic function of $(\sigma_s,\eta) \in \mathbb{C}^2$ and the four difference equations (7.25) and (7.31) hold as equalities between meromorphic functions of $(\sigma_s,\eta) \in \mathbb{C}^2$.*

7.3 Polynomial Limit

In this subsection, we show that the function $\mathcal{Q}$ reduces to the continuous q-Hermite polynomials when η is suitably discretized.

Theorem 7.9 (From $\mathcal{Q}$ to the continuous q-Hermite polynomials) *Let $\sigma_s \in \mathbb{C}$. Suppose that $b > 0$ is such that b^2 is irrational. Define $\{\eta_n\}_{n=0}^{\infty} \subset \mathbb{C}$ by*

$$\eta_n = \frac{iQ}{2} + ibn. \tag{7.34}$$

For each integer $n \geq 0$, the function $\mathcal{Q}$ satisfies

$$\lim_{\eta\to\eta_n} \mathcal{Q}(b,\eta,\sigma_s) = Q_n\left(e^{2\pi b\sigma_s}; e^{2i\pi b^2}\right), \tag{7.35}$$

where Q_n are the continuous q-Hermite polynomials defined in (B.40).

In order to prove Theorem 7.9, we will need the following two lemmas.

Lemma 7.10 *For each $\epsilon > 0$, the integrand $I_{\mathcal{Q}}$ defined in (7.8) obeys the estimates*

$$\ln\left(I_{\mathcal{Q}}(x,\sigma_s,\eta)\right) = \mathcal{A}_{\mathcal{Q}}^{+}(x,\eta) + O\left(e^{-\frac{2\pi(1-\epsilon)}{\max(b,b^{-1})}|Re\, x|}\right), \qquad Re\, x \to +\infty, \tag{7.36a}$$

$$\ln\left(I_{\mathcal{Q}}(x,\sigma_s,\eta)\right) = \mathcal{A}_{\mathcal{Q}}^{-}(x,\eta) + O\left(e^{-\frac{2\pi(1-\epsilon)}{\max(b,b^{-1})}|Re\, x|}\right), \qquad Re\, x \to -\infty, \tag{7.36b}$$

uniformly for $(b, Im\, x, \sigma_s, \eta)$ in compact subsets of $(0,\infty)\times\mathbb{R}\times\mathbb{C}^2$, where

$$\mathcal{A}_{\mathcal{Q}}^{+}(x,\eta) = -2i\pi x^2 + 2i\pi x(\eta - \tfrac{iQ}{2}) - i\pi\left(\sigma_s^2 + \frac{Q^2-2}{12}\right), \tag{7.37a}$$

$$\mathcal{A}_{\mathcal{Q}}^{-}(x,\eta) = 2i\pi x(\eta - \tfrac{iQ}{2}) + i\pi\left(\sigma_s^2 + \tfrac{Q^2-2}{12}\right). \tag{7.37b}$$

Proof The lemma follows from (2.8) and (7.8). □

Lemma 7.11 *Let*

$$\epsilon_{\mathcal{Q}}(x,\eta) = I_{\mathcal{Q}}(x,\sigma_s,\eta) - e^{\mathcal{A}_{\mathcal{Q}}^{-}(x,\eta)}, \tag{7.38}$$

where $\mathcal{A}_{\mathcal{Q}}^{-}$ *is defined in (7.37b). There is a neighborhood* U_0 *of* $\eta_0 = iQ/2$ *and constants* $c > 0$ *and* $M > 0$ *such that*

$$\left|\epsilon_{\mathcal{Q}}(x,\eta)\right| \le Me^{-c|Re\,x|} \qquad \text{for } Re\,x \le 0 \text{ and for } \eta \in U_0 \text{ with } Im\,\eta \le Im\,\eta_0, \tag{7.39}$$

uniformly for $(Im\,x, \sigma_s)$ *in compact subsets of* $\mathbb{R} \times \mathbb{C}$.

Proof Utilizing the estimate (7.36b) with $\epsilon = 1/2$, we find that

$$\begin{aligned} \epsilon_{\mathcal{Q}}(x,\eta) &= e^{\mathcal{A}_{\mathcal{Q}}^{-}(x,\eta)}\left(e^{O\left(\exp(\frac{\pi \operatorname{Re} x}{\max(b,b^{-1})})\right)} - 1\right) \\ &= O\left(e^{\mathcal{A}_{\mathcal{Q}}^{-}(x,\eta)} e^{-\frac{\pi|\operatorname{Re} x|}{\max(b,b^{-1})}}\right), \qquad \operatorname{Re} x \to -\infty, \end{aligned} \tag{7.40}$$

uniformly for $(b, \operatorname{Im} x, \sigma_s, \eta)$ in compact subsets of $(0,\infty) \times \mathbb{R} \times \mathbb{C}^2$. Since

$$e^{\operatorname{Re}\mathcal{A}_{\mathcal{Q}}^{-}(x,\eta)} = e^{2\pi(\operatorname{Re} x)(\frac{Q}{2} - \operatorname{Im}\eta)} e^{-2\pi(\operatorname{Im} x)(\operatorname{Re}\eta)} e^{-2\pi(\operatorname{Im}\sigma_s)(\operatorname{Re}\sigma_s)}, \tag{7.41}$$

the desired conclusion follows. □

Proof of Theorem 7.9 Suppose $b > 0$ and $\sigma_s \in \mathbb{C}$. The function $I_{\mathcal{Q}}(x,\sigma_s,\eta)$ has two downward sequences of poles starting at $x = \pm\sigma_s - iQ/2$. Consider the representation (7.6) for $\mathcal{Q}$ with the contour $\mathcal{C}_{\mathcal{Q}}$ passing above the points $x = \pm\sigma_s - iQ/2$ and satisfying (7.9) on its right tail.

Taking $\epsilon = 1/2$ in the estimate (7.36a), we find that there exists a constant M_1 such that

$$\left|I_{\mathcal{Q}}(x,\sigma_s,\eta)\right| \le M_1\left|e^{\mathcal{A}_{\mathcal{Q}}^{+}(x,\eta)}\right| = M_1 e^{\operatorname{Re}\mathcal{A}_{\mathcal{Q}}^{+}(x,\eta)}, \qquad x \in \mathcal{C}_{\mathcal{Q}},\ \operatorname{Re} x \ge 0, \tag{7.42}$$

uniformly for (b, σ_s, η) in compact subsets of $(0,\infty) \times \mathbb{C}^2$. Using (7.9) and noting that

$$e^{\operatorname{Re}\mathcal{A}_{\mathcal{Q}}^{+}(x,\eta)} = e^{4\pi(\operatorname{Re} x)\left(\operatorname{Im} x + \frac{Q}{4} - \frac{\operatorname{Im}\eta}{2}\right)} e^{-2\pi(\operatorname{Im} x)(\operatorname{Re}\eta)} e^{2\pi(\operatorname{Im}\sigma_s)(\operatorname{Re}\sigma_s)}, \tag{7.43}$$

we infer the existence of a neighborhood U_0 of $\eta_0 = iQ/2$ and constants $c > 0$ and $M > 0$ such that

$$\left|I_{\mathcal{Q}}(x, \sigma_s, \eta)\right| \leq M e^{-c \operatorname{Re} x} \tag{7.44}$$

for all $x \in \mathcal{C}_{\mathcal{Q}}$ with $\operatorname{Re} x \geq 0$ and for all $\eta \in U_0$.

Writing $\mathcal{C}_{\mathcal{Q}} = \mathcal{C}_{\mathcal{Q}}^{+} \cup \mathcal{C}_{\mathcal{Q}}^{-}$, where $\mathcal{C}_{\mathcal{Q}}^{+} = \mathcal{C}_{\mathcal{Q}} \cap \{\operatorname{Re} x \geq 0\}$ and $\mathcal{C}_{\mathcal{Q}}^{-} = \mathcal{C}_{\mathcal{Q}} \cap \{\operatorname{Re} x \leq 0\}$, we can express (7.6) as

$$\begin{aligned}&\mathcal{Q}(b, \sigma_s, \eta)\\ &\quad = P_{\mathcal{Q}}(\sigma_s, \eta) \int_{\mathcal{C}_{\mathcal{Q}}^{+}} dx\ I_{\mathcal{Q}}(x, \sigma_s, \eta) + P_{\mathcal{Q}}(\sigma_s, \eta) \int_{\mathcal{C}_{\mathcal{Q}}^{-}} dx\ I_{\mathcal{Q}}(x, \sigma_s, \eta).\end{aligned} \tag{7.45}$$

The prefactor $P_{\mathcal{Q}}$ defined in (7.7) has a simple zero at $\eta = \eta_0 = iQ/2$. Hence, by (7.44), the first term on the right-hand side of (7.45) vanishes in the limit $\eta \to \eta_0$. On the other hand, by (7.38),

$$\int_{\mathcal{C}_{\mathcal{Q}}^{-}} dx\ I_{\mathcal{Q}}(x, \sigma_s, \eta) = \int_{\mathcal{C}_{\mathcal{Q}}^{-}} dx\ e^{\mathcal{A}_{\mathcal{Q}}^{-}(x,\eta)} + \int_{\mathcal{C}_{\mathcal{Q}}^{-}} dx\ \epsilon_{\mathcal{Q}}(x, \eta). \tag{7.46}$$

In view of Lemma 7.11, there exist constants $M_3, M_4 > 0$ such that

$$\left|\int_{\mathcal{C}_{\mathcal{Q}}^{-}} dx\ \epsilon_{\mathcal{Q}}(x, \eta)\right| \leq M_3 \int_{\mathcal{C}_{\mathcal{Q}}^{-}} dx\ e^{-c|\operatorname{Re} x|} \leq M_4 \tag{7.47}$$

for all η in a small neighborhood of η_0 with $\operatorname{Im} \eta \leq \operatorname{Im} \eta_0$.

By Proposition 7.8, $\mathcal{Q}$ is a meromorphic function of $\eta \in \mathbb{C}$, and so has at most a pole at η_0. To prove (7.35), it is therefore enough to consider the limit $\eta \to \eta_0$ with η such that $\operatorname{Im} \eta < \operatorname{Im} \eta_0 = Q/2$. In the remainder of the proof, we assume $\operatorname{Im} \eta < Q/2$. Then, after multiplication by the prefactor $P_{\mathcal{Q}}$, the second term on the right-hand side of (7.46) vanishes in the limit $\eta \to \eta_0$ as a consequence of (7.47). Furthermore, employing (7.36b) and using that $\operatorname{Im} \eta < \operatorname{Im} \eta_0$ so that the contribution from $-\infty + ia$ vanishes, we obtain

$$\int_{\mathcal{C}_{\mathcal{Q}}^{-}} dx\ e^{\mathcal{A}_{\mathcal{Q}}^{-}(x,\eta)} = \int_{-\infty+ia}^{ia} dx\ e^{\mathcal{A}_{\mathcal{Q}}^{-}(x,\eta)} = e^{i\pi\left(\sigma_s^2 + \frac{Q^2-2}{12}\right)} \frac{e^{-2\pi a(\eta-\eta_0)}}{2i\pi(\eta-\eta_0)}. \tag{7.48}$$

The right-hand side of (7.48) has a simple pole at $\eta = \eta_0$. Therefore, collecting the above conclusions,

$$\lim_{\eta\to\eta_0} \mathcal{Q}(b, \sigma_s, \eta) = \lim_{\eta\to\eta_0} P_{\mathcal{Q}}(\sigma_s, \eta) \int_{\mathcal{C}_{\mathcal{Q}}^{-}} dx\ e^{\mathcal{A}_{\mathcal{Q}}^{-}(x,\eta)}$$

$$= \lim_{\eta\to\eta_0} P_Q(\sigma_s,\eta) e^{i\pi\left(\sigma_s^2+\frac{Q^2-2}{12}\right)} \frac{e^{-2\pi a(\eta-\eta_0)}}{2i\pi(\eta-\eta_0)}, \tag{7.49}$$

where the limits are taken with $\operatorname{Im}\eta < \operatorname{Im}\eta_0$. Utilizing (7.7) and the identity $s_b(z) = s_b(-z)^{-1}$, we obtain

$$\lim_{\eta\to\eta_0} \mathcal{Q}(b,\sigma_s,\eta) = \frac{1}{2i\pi} \lim_{\eta\to\eta_0} \left(\frac{1}{s_b(-\eta)(\eta-\eta_0)} \right). \tag{7.50}$$

Setting $z = -\eta$, recalling that $\eta_0 = iQ/2$, and using (2.4), we find

$$\lim_{\eta\to\eta_0} \mathcal{Q}(b,\sigma_s,\eta) = -\frac{1}{2i\pi} \lim_{z\to -\frac{iQ}{2}} \left(\frac{1}{s_b(z)(z+\frac{iQ}{2})} \right) = 1, \tag{7.51}$$

which proves (7.35) for $n = 0$.

To show (7.35) also for $n \geq 1$, we rewrite the difference equation (7.31a) as follows:

$$\begin{aligned} \mathcal{Q}(b,\sigma_s,\eta+ib) = &-\left(1+e^{2\pi b(\eta-\frac{ib}{2})}\right)\mathcal{Q}(b,\sigma_s,\eta-ib) \\ &+2\cosh(2\pi b\sigma_s)\mathcal{Q}(b,\sigma_s,\eta). \end{aligned} \tag{7.52}$$

Note that $1+e^{2\pi b(\eta-\frac{ib}{2})}$ vanishes for $\eta = \eta_0$. Moreover, the function $\mathcal{Q}(b,\sigma_s,\eta-ib)$ is analytic at $\eta = \eta_0$. Indeed, as η approaches η_0, the contour $\mathcal{C}_Q$ remains above the two decreasing sequences of poles, and, in view of (7.36) (see also (7.43) and (7.41)), the integrand I_Q retains its exponential decay provided that the right tail of the contour is deformed downwards. Therefore, evaluating (7.52) at $\eta = \eta_0$ and using (7.51), we obtain $\mathcal{Q}(b,\sigma_s,\eta_1) = 2\cosh(2\pi b\sigma_s)$. Evaluating the recurrence relation (B.41) satisfied by the continuous q-Hermite polynomials at $n = 0$ and $z = e^{2\pi b\sigma_s}$, we find

$$Q_1\left(e^{2\pi b\sigma_s};q\right) = 2\cosh(2\pi b\sigma_s) = \mathcal{Q}(b,\sigma_s,\eta_1), \tag{7.53}$$

which proves (7.35) for $n = 1$. More generally, suppose that the function $\mathcal{Q}(b,\sigma_s,\eta_n)$ exists for all $n \leq N$ and coincides with the polynomials $Q_n\left(e^{2\pi b\sigma_s};q\right)$. Evaluating (7.52) at $n = N+1$, we obtain

$$\begin{aligned} \mathcal{Q}(b,\sigma_s,\eta_{N+1}) = &-\left(1-q^{N+1}\right)Q_N\left(e^{2\pi b\sigma_s};q\right) \\ &+2\cosh(2\pi b\sigma_s)Q_{N-1}\left(e^{2\pi b\sigma_s};q\right), \end{aligned} \tag{7.54}$$

and hence the recurrence relation (B.41) implies that

$$\mathcal{Q}(b,\sigma_s,\eta_{N+1}) = Q_{N+1}\left(e^{2\pi b\sigma_s};q\right). \tag{7.55}$$

By induction, we conclude that $\mathcal{Q}(b, \sigma_s, \eta_n)$ exists for all $n \geq 0$ and coincides with $Q_n\left(e^{2\pi b\sigma_s}; q\right)$. This completes the proof of the theorem. $\square$

8 The Function $\mathcal{L}$

In this section, we define the function $\mathcal{L}(b, \theta_t, \theta, \lambda, \mu)$ which generalizes the big q-Laguerre polynomials. It is defined as a confluent limit of $\mathcal{H}$ and lies at the third level of the non-polynomial scheme. We show that $\mathcal{L}$ is a joint eigenfunction of four difference operators. Finally, we show that $\mathcal{L}$ reduces to the big q-Laguerre polynomials, which lie at the third level of the q-Askey scheme, when λ is suitably discretized.

8.1 Definition and Integral Representation

Let $\theta, \Lambda, \lambda, \mu$ be defined as follows:

$$\theta_0 = \frac{\theta + \Lambda}{2}, \qquad \theta_* = \frac{\theta - \Lambda}{2}, \qquad \sigma_s = \lambda + \frac{\Lambda}{2}, \qquad \nu = \mu - \frac{\Lambda}{4}. \tag{8.1}$$

Define the open set $\Omega_{\mathcal{L}} \subset \mathbb{C}^2$ by

$$\Omega_{\mathcal{L}} := (\mathbb{C} \backslash \Delta_\lambda) \times (\{\operatorname{Im} \mu > -Q/2\} \backslash \Delta_\mu), \tag{8.2}$$

where the discrete subsets Δ_λ and Δ_μ are given by

$$\begin{aligned}
\Delta_\lambda := &\{\tfrac{iQ}{2} \pm \theta_t + \tfrac{\theta}{2} + ibm + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{\tfrac{iQ}{2} - \tfrac{\theta}{2} + ibm + ilb^{-1}\}_{m,l=0}^{\infty} \\
&\cup \{-\tfrac{iQ}{2} + \theta_t + \tfrac{\theta}{2} - ibm - ilb^{-1}\}_{m,l=0}^{\infty}, \\
\Delta_\mu := &\{\tfrac{iQ}{2} \pm \theta_t + \tfrac{\theta}{4} + ibm + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{\tfrac{iQ}{2} - \tfrac{3\theta}{4} + ibm + ilb^{-1}\}_{m,l=0}^{\infty} \\
&\cup \{-\tfrac{iQ}{2} + \theta_t + \tfrac{\theta}{4} - ibm - ilb^{-1}\}_{m,l=0}^{\infty}.
\end{aligned}$$

Definition 8.1 Let $\mathcal{H}$ be defined by (4.2). The function $\mathcal{L}$ is defined for $(\lambda, \mu) \in \Omega_{\mathcal{L}}$ by

$$\mathcal{L}(b, \theta_t, \theta, \lambda, \mu) = \lim_{\Lambda \to -\infty} \mathcal{H}\left(b, \tfrac{\theta+\Lambda}{2}, \theta_t, \tfrac{\theta-\Lambda}{2}, \lambda + \tfrac{\Lambda}{2}, \mu - \tfrac{\Lambda}{4}\right), \tag{8.3}$$

and is extended meromorphically to $(\lambda, \mu) \in \mathbb{C}^2$.

The next theorem shows that $\mathcal{L}$ is a well-defined meromorphic function of $(\lambda, \mu) \in \mathbb{C}^2$.

Theorem 8.2 *Suppose that Assumption 1.1 is satisfied. The limit in (8.3) exists uniformly for (λ, μ) in compact subsets of $\Omega_{\mathcal{L}}$. Moreover, $\mathcal{L}$ is an analytic function of $(\lambda, \mu) \in (\mathbb{C} \backslash \Delta_\lambda) \times (\mathbb{C} \backslash \Delta_\mu)$ and admits the following integral representation:*

$$\mathcal{L}(b, \theta_t, \theta, \lambda, \mu) = P_{\mathcal{L}}(\lambda, \mu) \int_{\mathcal{C}_{\mathcal{L}}} dx\, I_{\mathcal{L}}(x, \lambda, \mu) \qquad \text{for } (\lambda, \mu) \in (\mathbb{C} \backslash \Delta_\lambda) \times (\mathbb{C} \backslash \Delta_\mu),$$

(8.4)

where

$$P_{\mathcal{L}}(\lambda,\mu) = s_b\left(2\theta_t + \tfrac{iQ}{2}\right) s_b\left(\theta + \theta_t + \tfrac{iQ}{2}\right) s_b\left(\lambda - \tfrac{\theta}{2} - \theta_t\right) s_b\left(\mu - \tfrac{\theta}{4} - \theta_t\right), \tag{8.5}$$

$$I_{\mathcal{L}}(x,\lambda,\mu) = e^{\frac{i\pi x^2}{2}} e^{i\pi x(\frac{\theta}{4}+\theta_t+\lambda+\mu-\frac{iQ}{2})} \times \frac{s_b\left(x + \frac{\theta}{2} + \theta_t - \lambda\right) s_b\left(x + \frac{\theta}{4} + \theta_t - \mu\right)}{s_b\left(x + \frac{iQ}{2}\right) s_b\left(x + 2\theta_t + \frac{iQ}{2}\right) s_b\left(x + \theta + \theta_t + \frac{iQ}{2}\right)}, \tag{8.6}$$

and the contour $\mathcal{C}_{\mathcal{L}}$ is any curve from $-\infty$ to $+\infty$ which separates the three increasing from the two decreasing sequences of poles, with the requirement that its right tail satisfies

$$Im\, x + \tfrac{Q}{2} + Im\,\lambda + Im\,\mu > \delta \qquad \textit{for all } x \in \mathcal{C}_{\mathcal{L}} \textit{ with } Re\, x \textit{ sufficiently large}, \tag{8.7}$$

for some $\delta > 0$. In particular, $\mathcal{L}$ is a meromorphic function of $(\lambda,\mu) \in \mathbb{C}^2$. If $(\lambda,\mu) \in \mathbb{R}^2$, the contour $\mathcal{C}_{\mathcal{L}}$ can be any curve from $-\infty$ to $+\infty$ lying within the strip $Im\, x \in (-Q/2+\delta, 0)$.

Furthermore, the function $\mathcal{L}$ obeys the symmetry

$$\mathcal{L}(b,\theta_t,\theta,\lambda,\mu) = \mathcal{L}(b^{-1},\theta_t,\theta,\lambda,\mu). \tag{8.8}$$

8.2 Difference Equations

The next two propositions, whose proofs are omitted because they are similar to those presented in Sect. 4.2, show that the two pairs of difference equations (4.26) and (4.33) satisfied by the function $\mathcal{H}$ survive in the confluent limit (8.3), implying that $\mathcal{L}(b,\theta_t,\theta,\lambda,\mu)$ is a joint eigenfunction of four difference operators, two acting on λ and the other two on μ. The four difference equations hold as equalities between meromorphic functions of $(\lambda,\mu) \in \mathbb{C}^2$.

8.2.1 First Pair of Difference Equations

Define a difference operator $D_{\mathcal{L}}(b,\lambda)$ such that

$$D_{\mathcal{L}}(b,\lambda) = d^{+}_{\mathcal{L}}(b,\lambda)e^{ib\partial_\lambda} + d^{-}_{\mathcal{L}}(b,\lambda)e^{-ib\partial_\lambda} + d^{0}_{\mathcal{L}}(b,\lambda), \tag{8.9}$$

where

$$d^{+}_{\mathcal{L}}(b,\lambda) = -4e^{-\pi b\left(\frac{\theta}{2}+\theta_t-2\lambda\right)} \cosh\left(\pi b\left(\tfrac{ib}{2} + \tfrac{\theta}{2} + \lambda\right)\right) \cosh\left(\pi b\left(\tfrac{ib}{2} - \tfrac{\theta}{2} + \theta_t + \lambda\right)\right), \tag{8.10}$$

$$d_{\mathcal{L}}^{-}(b,\lambda)=-2e^{\pi b\left(\theta_t-\frac{ib}{2}+3\lambda\right)}\cosh\left(\pi b\left(\tfrac{ib}{2}+\tfrac{\theta}{2}+\theta_t-\lambda\right)\right), \tag{8.11}$$

and

$$d_{\mathcal{L}}^{0}(\lambda)=e^{-\pi b\left(\frac{\theta}{2}+2\theta_t+iQ\right)}-d_{\mathcal{L}}^{+}(b,\lambda)-d_{\mathcal{L}}^{-}(b,\lambda). \tag{8.12}$$

Proposition 8.3 *For $(\lambda,\mu)\in\mathbb{C}^2$, the function $\mathcal{L}$ satisfies the other pair of difference equations:*

$$D_{\mathcal{L}}(b,\lambda)\ \mathcal{L}(b,\theta_t,\theta,\lambda,\mu)=e^{-2\pi b\mu}\ \mathcal{L}(b,\theta_t,\theta,\lambda,\mu), \tag{8.13a}$$

$$D_{\mathcal{L}}(b^{-1},\lambda)\ \mathcal{L}(b,\theta_t,\theta,\lambda,\mu)=e^{-2\pi b^{-1}\mu}\ \mathcal{L}(b,\theta_t,\theta,\lambda,\mu). \tag{8.13b}$$

8.2.2 Second Pair of Difference Equations

Define a difference operator $\tilde{D}_{\mathcal{L}}(b,\mu)$ such that

$$\tilde{D}_{\mathcal{L}}(b,\mu)=\tilde{d}_{\mathcal{L}}^{+}(b,\mu)e^{ib\partial_\mu}+\tilde{d}_{\mathcal{L}}^{-}(b,\mu)e^{-ib\partial_\mu}+\tilde{d}_{\mathcal{L}}^{0}(b,\mu), \tag{8.14}$$

where

$$\tilde{d}_{\mathcal{L}}^{+}(b,\mu)=-4e^{-\pi b\left(\frac{\theta}{2}+\theta_t-2\mu\right)}\cosh\left(\pi b\left(\tfrac{ib}{2}+\tfrac{3\theta}{4}+\mu\right)\right)\cosh\left(\pi b\left(\tfrac{ib}{2}-\tfrac{\theta}{4}+\theta_t+\mu\right)\right), \tag{8.15}$$

$$\tilde{d}_{\mathcal{L}}^{-}(b,\mu)=-2e^{\pi b\left(-\frac{ib}{2}+\frac{\theta}{4}+\theta_t+3\mu\right)}\cosh\left(\pi b\left(\tfrac{ib}{2}+\tfrac{\theta}{4}+\theta_t-\mu\right)\right), \tag{8.16}$$

and

$$\tilde{d}_{\mathcal{L}}^{0}(b,\mu)=e^{-\pi b(\theta+2\theta_t+iQ)}-\tilde{d}_{\mathcal{L}}^{+}(b,\mu)-\tilde{d}_{\mathcal{L}}^{-}(b,\mu). \tag{8.17}$$

Proposition 8.4 *For $(\lambda,\mu)\in\mathbb{C}^2$, the function $\mathcal{L}$ satisfies the pair of difference equations*

$$\tilde{D}_{\mathcal{L}}(b,\mu)\ \mathcal{L}(b,\theta_t,\theta,\lambda,\mu)=e^{-2\pi b\lambda}\ \mathcal{L}(b,\theta_t,\theta,\lambda,\mu), \tag{8.18a}$$

$$\tilde{D}_{\mathcal{L}}(b^{-1},\mu)\ \mathcal{L}(b,\theta_t,\theta,\lambda,\mu)=e^{-2\pi b^{-1}\lambda}\ \mathcal{L}(b,\theta_t,\theta,\lambda,\mu). \tag{8.18b}$$

8.3 Polynomial Limit

Our next theorem, whose proof is similar to that of Theorem 4.7, shows that the function $\mathcal{L}$ reduces to the big q-Laguerre polynomials L_n when λ is suitably discretized.

The following assumption ensures that all the poles of the integrand in (8.4) are distinct and simple.

Assumption 8.5 Assume that $b > 0$ is such that b^2 is irrational and that

$$\theta_t \neq 0, \qquad \theta \pm \theta_t \neq 0, \qquad \mathrm{Re}\left(\tfrac{\theta}{4} - \lambda + \mu\right) \neq 0. \tag{8.19}$$

Theorem 8.6 (From $\mathcal{L}$ to the big q-Laguerre polynomials) *Let* $\mu \in \{Im\,\mu > -Q/2\}\backslash\Delta_\mu$ *and suppose that Assumptions 1.1 and 8.5 are satisfied. Define* $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{C}$ *by*

$$\lambda_n = \frac{\theta}{2} + \theta_t + \tfrac{iQ}{2} + ibn. \tag{8.20}$$

Under the parameter correspondence

$$\alpha_L = e^{4\pi b\theta_t}, \qquad \beta_L = e^{2\pi b(\theta+\theta_t)}, \qquad x_L = e^{\pi b(iQ+\frac{\theta}{2}+2\theta_t)}e^{-2\pi b\mu}, \qquad q = e^{2i\pi b^2}, \tag{8.21}$$

the function $\mathcal{L}$ *satisfies, for each* $n \geq 0$,

$$\lim_{\lambda\to\lambda_n} \mathcal{L}(b, \theta_t, \theta, \lambda, \mu) = L_n(x_L; \alpha_L, \beta_L, q), \tag{8.22}$$

where L_n *are the big* q*-Laguerre polynomials defined in (B.29).*

9 The Function $\mathcal{W}$

In this section, we define the function $\mathcal{W}(b, \theta_t, \kappa, \omega)$ which generalizes the little q-Laguerre polynomials. It is defined as a confluent limit of $\mathcal{L}$ and lies at the fourth level of the non-polynomial scheme. We show that $\mathcal{W}$ is a joint eigenfunction of four difference operators and that it reduces to the little q-Laguerre polynomials, which lie at the fourth level of the q-Askey scheme, when κ is suitably discretized.

9.1 Definition and Integral Representation

Introduce two new parameters κ and ω by

$$\lambda = \frac{\theta}{2} + \kappa, \qquad \mu = -\frac{3\theta}{4} + \omega. \tag{9.1}$$

Definition 9.1 The function $\mathcal{W}(b, \theta_t, \kappa, \omega)$ is defined by

$$\mathcal{W}(b, \theta_t, \kappa, \omega) = \lim_{\theta\to+\infty} \mathcal{L}(b, \theta_t, \theta, \tfrac{\theta}{2} + \kappa, -\tfrac{3\theta}{4} + \omega), \tag{9.2}$$

where $\mathcal{L}$ is defined by (8.4).

The next theorem, whose proof is similar to that of Theorem 4.2 and will be omitted, shows that, for each choice of $(b, \theta_t) \in (0, \infty) \times \mathbb{R}$, $\mathcal{W}$ is a well-defined and analytic function of $(\kappa, \omega) \in (\mathbb{C} \backslash \Delta_\kappa) \times \{\operatorname{Im} \omega < Q/2\}$, where $\Delta_\kappa \subset \mathbb{C}$ is a discrete set of points at which $\mathcal{W}$ may have poles. In particular, $\mathcal{W}$ is a meromorphic function of $\kappa \in \mathbb{C}$ and of ω for $\operatorname{Im} \omega < Q/2$. The theorem also provides an integral representation for $\mathcal{W}$ for $(\kappa, \omega) \in (\mathbb{C} \backslash \Delta_\kappa) \times \{\operatorname{Im} \omega < Q/2\}$. In fact, even if the requirement $\operatorname{Im} \omega < Q/2$ is needed to ensure that the integral representation converges, it will be shown later, with the help of the difference equations, that $\mathcal{W}$ extends to a meromorphic function of $(\kappa, \omega) \in \mathbb{C}^2$.

Theorem 9.2 *Suppose that Assumption 1.1 holds. The limit (9.2) exists uniformly for (κ, ω) in compact subsets of*

$$\Omega_{\mathcal{W}} := (\mathbb{C} \backslash \Delta_\kappa) \times \{Im\, \omega < Q/2\}, \tag{9.3}$$

where

$$\Delta_\kappa := \{\tfrac{iQ}{2} \pm \theta_t + imb + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{-\tfrac{iQ}{2} + \theta_t - imb - ilb^{-1}\}_{m,l=0}^{\infty}. \tag{9.4}$$

Moreover, $\mathcal{W}$ is an analytic function of $(\kappa, \omega) \in \Omega_{\mathcal{W}}$ and admits the following integral representation:

$$\mathcal{W}(b, \theta_t, \kappa, \omega) = P_{\mathcal{W}}(\kappa, \omega) \int_{\mathcal{C}_{\mathcal{W}}} dx\; I_{\mathcal{W}}(x, \kappa, \omega) \qquad \text{for } (\kappa, \omega) \in \Omega_{\mathcal{W}}, \tag{9.5}$$

where

$$P_{\mathcal{W}}(\kappa, \omega) = s_b(\tfrac{iQ}{2} + 2\theta_t) s_b(\kappa - \theta_t), \tag{9.6}$$

$$I_{\mathcal{W}}(x, \kappa, \omega) = e^{\frac{i\pi x^2}{2}} e^{i\pi x(\theta_t + \kappa + 2\omega)} \frac{s_b(x + \theta_t - \kappa)}{s_b\left(x + \frac{iQ}{2}\right) s_b\left(x + 2\theta_t + \frac{iQ}{2}\right)}, \tag{9.7}$$

and the contour $\mathcal{C}_{\mathcal{W}}$ is any curve from $-\infty$ to $+\infty$ which separates the decreasing sequence of poles from the two increasing ones, with the requirement that its right tail satisfies

$$Im\, x + \tfrac{Q}{2} + Im\, \kappa + Im\, \omega > \delta \qquad \text{for all } x \in \mathcal{C}_{\mathcal{W}} \text{ with } Re\, x \text{ sufficiently large}, \tag{9.8}$$

for some $\delta > 0$. If $(\kappa, \omega) \in \mathbb{R}^2$, then $\mathcal{C}_{\mathcal{W}}$ can be any curve from $-\infty$ to $+\infty$ lying within the strip $Im\, x \in (-Q/2 + \delta, 0)$.

Furthermore, thanks to the symmetry (2.7), $\mathcal{W}$ satisfies

$$\mathcal{W}(b^{-1}, \theta_t, \kappa, \omega) = \mathcal{W}(b, \theta_t, \kappa, \omega). \tag{9.9}$$

9.2 Difference Equations

By taking the confluent limit (9.2) of the difference equations (8.13) and (8.18) satisfied by $\mathcal{L}$, we obtain the next two propositions which show that $\mathcal{W}$ is a joint eigenfunction of four difference operators, two acting on κ and the other two on ω.

The difference equations will first be derived as equalities between meromorphic functions of $\kappa \in \mathbb{C}$ and ω with $\operatorname{Im}\omega < Q/2$. We will then use the difference equations in ω to show that (i) the limit in (9.2) exists for all ω in the complex plane away from a discrete subset, (ii) $\mathcal{W}$ is in fact a meromorphic function of (κ, ω) in all of $\mathbb{C}^2$, and (iii) the four difference equations hold as equalities between meromorphic functions on $\mathbb{C}^2$, see Proposition 9.5.

9.2.1 First Pair of Difference Equations

Define the difference operator $D_{\mathcal{W}}(b, \kappa)$ such that

$$D_{\mathcal{W}}(b,\kappa) = d^{+}_{\mathcal{W}}(b,\kappa)e^{ib\partial_\kappa} + d^{-}_{\mathcal{W}}(b,\kappa)e^{-ib\partial_\kappa} + d^{0}_{\mathcal{W}}(b,\kappa), \tag{9.10}$$

where

$$d^{\pm}_{\mathcal{W}}(b,\kappa) = -2e^{3\pi b\kappa}e^{\pm\pi b(\frac{ib}{2}-\theta_t)}\cosh\left(\pi b\left(\tfrac{ib}{2}+\theta_t \pm\kappa\right)\right), \tag{9.11}$$

$$d^{0}_{\mathcal{W}}(b,\kappa) = -d^{+}_{\mathcal{W}}(b,\kappa) - d^{-}_{\mathcal{W}}(b,\kappa). \tag{9.12}$$

Proposition 9.3 *For $\kappa \in \mathbb{C}$ and $\operatorname{Im}\omega < Q/2$, the function $\mathcal{W}$ satisfies the following pair of difference equations:*

$$D_{\mathcal{W}}(b,\kappa)\ \mathcal{W}(b,\theta_t,\kappa,\omega) = e^{-2\pi b\omega}\ \mathcal{W}(b,\theta_t,\kappa,\omega), \tag{9.13a}$$

$$D_{\mathcal{W}}(b^{-1},\kappa)\ \mathcal{W}(b,\theta_t,\kappa,\omega) = e^{-2\pi b^{-1}\omega}\ \mathcal{W}(b,\theta_t,\kappa,\omega). \tag{9.13b}$$

9.2.2 Second Pair of Difference Equations

Introduce the difference operator $\tilde{D}_{\mathcal{W}}(b, \omega)$ by

$$\tilde{D}_{\mathcal{W}}(b,\omega) = \tilde{d}^{+}_{\mathcal{W}}(b,\omega)e^{ib\partial_\omega} + \tilde{d}^{-}_{\mathcal{W}}(b,\omega)e^{-ib\partial_\omega} + \tilde{d}^{0}_{\mathcal{W}}(b,\omega), \tag{9.14}$$

where

$$\tilde{d}^{+}_{\mathcal{W}}(b,\omega) = -2e^{\pi b\left(\omega-\frac{ib}{2}-2\theta_t\right)}\cosh\left(\pi b\left(\omega+\tfrac{ib}{2}\right)\right), \tag{9.15}$$

$$\tilde{d}^{-}_{\mathcal{W}}(b,\omega) = -e^{2\pi b(\theta_t+\omega)}, \tag{9.16}$$

$$\tilde{d}^{0}_{\mathcal{W}}(b,\omega) = 2e^{2\pi b\omega}\cosh(2\pi b\theta_t). \tag{9.17}$$

Proposition 9.4 *For $\kappa \in \mathbb{C}$ and $\mathrm{Im}\,(\omega + ib^{\pm 1}) < Q/2$, the function $\mathcal{W}$ satisfies the following pair of difference equations:*

$$\tilde{D}_{\mathcal{W}}(b, \omega)\, \mathcal{W}(b, \theta_t, \kappa, \omega) = e^{-2\pi b\kappa}\, \mathcal{W}(b, \theta_t, \kappa, \omega), \tag{9.18a}$$

$$\tilde{D}_{\mathcal{W}}(b^{-1}, \omega)\, \mathcal{W}(b, \theta_t, \kappa, \omega) = e^{-2\pi b^{-1}\kappa}\, \mathcal{W}(b, \theta_t, \kappa, \omega). \tag{9.18b}$$

The next proposition is stated without proof since it is similar to that of Proposition 4.5.

Proposition 9.5 *Let $(b, \theta_t) \in (0, \infty) \times \mathbb{R}$ and $\kappa \in \mathbb{C}\setminus\Delta_\kappa$. There is a discrete subset $\Delta \subset \mathbb{C}$ such that the limit in (9.2) exists for all $\omega \in \mathbb{C}\setminus\Delta$. Moreover, the function $\mathcal{W}$ defined by (9.2) is a meromorphic function of $(\kappa, \omega) \in \mathbb{C}^2$ and the four difference equations (9.13) and (9.18) hold as equalities between meromorphic functions of $(\kappa, \omega) \in \mathbb{C}^2$.*

9.3 Polynomial Limit

Our next theorem shows that $\mathcal{W}$ reduces to the little q-Laguerre polynomials when κ is suitably discretized. We omit the proof which is similar to that of Theorem 4.7.

Assumption 9.6 Assume that $b > 0$ is such that b^2 is irrational and

$$\theta_t \neq 0. \tag{9.19}$$

Assumption 9.6 implies that all the poles of the integrand $I_{\mathcal{W}}$ are distinct and simple.

Theorem 9.7 (From $\mathcal{W}$ to the little q-Laguerre polynomials) *Let $\omega \in \mathbb{C}$ be such that $\mathrm{Im}\,\omega < Q/2$ and suppose that Assumptions 1.1 and 9.6 are satisfied. Define $\{\kappa_n\}_{n=0}^\infty \subset \mathbb{C}$ by*

$$\kappa_n = \theta_t + \tfrac{iQ}{2} + ibn. \tag{9.20}$$

Under the parameter correspondence

$$\alpha_W = e^{4\pi b\theta_t}, \qquad x_W = e^{-2\pi b\left(\frac{iQ}{2}+\omega\right)}, \qquad q = e^{2i\pi b^2}, \tag{9.21}$$

the function $\mathcal{W}$ satisfies, for each $n \geq 0$,

$$\lim_{\kappa\to\kappa_n} \mathcal{W}(b, \theta_t, \kappa, \omega) = W_n(x_W; \alpha_W, q), \tag{9.22}$$

where W_n are the little q-Laguerre polynomials defined in (B.33).

10 The Function $\mathcal{M}$

In this section, we define the function $\mathcal{M}(b, \zeta, \omega)$ which generalizes the little q-Laguerre polynomials (B.33) evaluated at $\alpha = 0$. It lies at the fifth and lowest level of the non-polynomial scheme and is defined as a limit of the function $\mathcal{W}$. We show that $\mathcal{M}$ is a joint eigenfunction of four difference operators, two acting on ζ and the other two on ω. Finally, we show that $\mathcal{M}$ reduces to the little q-Laguerre polynomials evaluated at $\alpha = 0$ when ζ is suitably discretized.

10.1 Definition and Integral Representation

Introduce a new parameter ζ such that $\kappa = \theta_t + \zeta$.

Definition 10.1 The function $\mathcal{M}(b, \zeta, \omega)$ is defined by

$$\mathcal{M}(b, \zeta, \omega) = \lim_{\theta_t \to -\infty} \mathcal{W}(b, \theta_t, \theta_t + \zeta, \omega), \tag{10.1}$$

where the function $\mathcal{W}$ is defined in (9.2).

The next theorem shows that, for each choice of $b \in (0, \infty)$, $\mathcal{M}$ is a well-defined and analytic function of

$$(\zeta, \omega) \in \{\operatorname{Im} \omega < Q/2,\ \operatorname{Im}(\zeta + \omega) > 0\} \setminus (\Delta_\zeta \times \mathbb{C}), \tag{10.2}$$

where $\Delta_\zeta \subset \mathbb{C}$ is a discrete set of points at which $\mathcal{M}$ may have poles. More precisely, Δ_ζ is defined by

$$\Delta_\zeta := \{\tfrac{iQ}{2} + ibm + ilb^{-1}\}_{m,l=0}^{\infty} \cup \{-\tfrac{iQ}{2} - imb - ilb^{-1}\}_{m,l=0}^{\infty}. \tag{10.3}$$

The theorem also provides an integral representation for $\mathcal{M}$ for (ζ, ω) satisfying (10.2). The restrictions in (10.2) are needed to ensure convergence of the integral in the integral representation. Nevertheless, it will be shown later, with the help of the difference equations satisfied by $\mathcal{M}$, that $\mathcal{M}$ extends to a meromorphic function of $(\zeta, \omega) \in \mathbb{C}^2$.

Theorem 10.2 *Suppose that Assumption 1.1 holds. Let $\Delta_\zeta \subset \mathbb{C}$ be the discrete subset defined in (10.3). Then the limit (10.1) exists uniformly for (ζ, ω) in compact subsets of*

$$\Omega_{\mathcal{M}} := \{(\zeta, \omega) \in \mathbb{C}^2 \mid Im\, \omega < Q/2,\ Im\,(\zeta + \omega) > 0\} \setminus (\Delta_\zeta \times \mathbb{C}). \tag{10.4}$$

Moreover, $\mathcal{M}$ is an analytic function of $(\zeta, \omega) \in \Omega_{\mathcal{M}}$ and admits the following integral representation:

$$\mathcal{M}(b, \zeta, \omega) = P_{\mathcal{M}}(\zeta, \omega) \int_{\mathcal{C}_{\mathcal{M}}} dx\, I_{\mathcal{M}}(x, \zeta, \omega) \quad \text{for } (\zeta, \omega) \in \Omega_{\mathcal{M}}, \tag{10.5}$$

where

$$P_{\mathcal{M}}(\zeta, \omega) = s_b(\zeta), \tag{10.6}$$

$$I_{\mathcal{M}}(x, \zeta, \omega) = e^{i\pi x\left(\zeta - \frac{iQ}{2} + 2\omega\right)} \frac{s_b(x-\zeta)}{s_b\left(x + \frac{iQ}{2}\right)}, \tag{10.7}$$

and the contour $\mathcal{C}_{\mathcal{M}}$ is any curve from $-\infty$ to $+\infty$ which separates the increasing from the decreasing sequence of poles. If $\mathrm{Im}\,\omega \in (0, Q/2)$ and $\zeta \in \mathbb{R}$, then the contour $\mathcal{C}_{\mathcal{M}}$ can be any curve from $-\infty$ to $+\infty$ lying within the strip $\mathrm{Im}\,x \in (-Q/2, 0)$.

Thanks to the identity (2.7), $\mathcal{M}$ satisfies

$$\mathcal{M}(b^{-1}, \zeta, \omega) = \mathcal{M}(b, \zeta, \omega). \tag{10.8}$$

10.2 Difference Equations

The two pairs of difference equations (9.13) and (9.18) satisfied by the function $\mathcal{W}$ survive in the confluent limit (10.1). This implies that $\mathcal{M}$ is a joint eigenfunction of four difference operators, two acting on ζ and two acting on ω.

We know from Theorem 10.2 that $\mathcal{M}$ is a well-defined holomorphic function of ω for $\mathrm{Im}\,\omega < Q/2$ and is meromorphic in ζ for $\mathrm{Im}\,(\zeta + \omega) > 0$. The difference equations will first be derived as equalities between meromorphic functions defined on this limited domain and then extended to equalities between meromorphic functions on $\mathbb{C}^2$, see Proposition 10.5.

10.2.1 First Pair of Difference Equations

Consider the difference operator $D_{\mathcal{M}}(b, \omega)$ defined by

$$D_{\mathcal{M}}(b, \zeta) = e^{2\pi b\zeta}\left(1 - e^{ib\partial_\zeta}\right). \tag{10.9}$$

Proposition 10.3 *For $\mathrm{Im}\,\omega < Q/2$ and $\mathrm{Im}\,(\zeta + \omega) > 0$, the function $\mathcal{M}$ satisfies the difference equations*

$$D_{\mathcal{M}}(b, \zeta)\, \mathcal{M}(b, \zeta, \omega) = e^{-2\pi b\omega}\, \mathcal{M}(b, \zeta, \omega), \tag{10.10a}$$

$$D_{\mathcal{M}}(b^{-1}, \zeta)\, \mathcal{M}(b, \zeta, \omega) = e^{-2\pi b^{-1}\omega}\, \mathcal{M}(b, \zeta, \omega). \tag{10.10b}$$

Proof The proof consists of taking the confluent limit (10.1) of the difference equation (9.13a). It is straightforward to verify that

$$\lim_{\theta_t \to -\infty} d^+_{\mathcal{W}}(b, \theta_t + \zeta) = -e^{2\pi b\zeta}, \qquad \lim_{\theta_t \to -\infty} d^-_{\mathcal{W}}(b, \theta_t + \zeta) = 0, \tag{10.11}$$

where $d^{\pm}_{\mathcal{W}}$ is defined in (9.11). From (9.12), this implies

$$\lim_{\theta_t \to -\infty} d^0_{\mathcal{W}}(b, \theta_t + \zeta) = e^{2\pi b\zeta}. \tag{10.12}$$

Therefore we obtain

$$\lim_{\theta_t \to -\infty} D_{\mathcal{W}}(b, \theta_t + \zeta) = D_{\mathcal{M}}(b, \zeta), \tag{10.13}$$

where $D_{\mathcal{M}}$ is given in (10.9). By Theorem 10.2, the limit in (10.1) exists whenever $(\zeta, \omega) \in D_{\mathcal{M}}$. Thus, the difference equation (10.10a) follows after utilizing (10.13) and the definition (10.1) of $\mathcal{M}$. Finally, (10.10b) follows from (10.10a) and the symmetry (10.8) of $\mathcal{M}$. □

10.2.2 Second Pair of Difference Equations

Define the dual difference operator $\tilde{D}_{\mathcal{M}}(b, \omega)$ by

$$\tilde{D}_{\mathcal{M}}(b, \omega) = e^{2\pi b\omega} - 2e^{\pi b(\omega - \frac{ib}{2})} \cosh\left(\pi b(\tfrac{ib}{2} + \omega)\right) e^{ib\partial_\omega}. \tag{10.14}$$

Proposition 10.4 *For Im $(\omega + ib^{\pm 1}) < Q/2$ and Im $(\zeta + \omega) > 0$, the function $\mathcal{M}$ satisfies the following pair of difference equations:*

$$\tilde{D}_{\mathcal{M}}(b, \omega)\ \mathcal{M}(b, \zeta, \omega) = e^{-2\pi b\zeta}\ \mathcal{M}(b, \zeta, \omega), \tag{10.15a}$$

$$\tilde{D}_{\mathcal{M}}(b^{-1}, \omega)\ \mathcal{M}(b, \zeta, \omega) = e^{2\pi b^{-1}\zeta}\ \mathcal{M}(b, \zeta, \omega). \tag{10.15b}$$

Proof It is straightforward to show that the following limit holds:

$$e^{2\pi b\theta_t}\, \tilde{D}_{\mathcal{W}}(b, \omega) = \tilde{D}_{\mathcal{M}}(b, \omega), \tag{10.16}$$

where $\tilde{D}_{\mathcal{W}}$ and $\tilde{D}_{\mathcal{M}}$ are defined by (9.14) and (10.14), respectively. By Theorem 10.2, the limit in (10.1) exists whenever $(\zeta, \omega) \in D_{\mathcal{M}}$. Thus, the difference equation (10.15a) follows after multiplying (9.18a) by $e^{2\pi b\theta_t}$ and utilizing (10.16) and the definition (10.1) of $\mathcal{M}$. Finally, (10.15b) follows from (10.15a) thanks to the symmetry (10.8) of $\mathcal{M}$. □

Proposition 10.5 *Let $b \in (0, \infty)$. Then there exist discrete subsets $\Delta, \Delta' \subset \mathbb{C}$ such that the limit in (10.1) exists for all $(\zeta, \omega) \in (\mathbb{C}\setminus\Delta) \times (\mathbb{C}\setminus\Delta')$. Moreover, the function $\mathcal{M}$ defined by (10.5) is a meromorphic function of $(\zeta, \omega) \in \mathbb{C}^2$ and the four difference equations (10.10) and (10.15) hold as equalities between meromorphic functions of $(\zeta, \omega) \in \mathbb{C}^2$.*

Proof The proof utilizes the difference equations in ζ and ω and is similar to the proof of Proposition 4.5. □

10.3 Polynomial Limit

We now show that the function $\mathcal{M}$ reduces to the little q-Laguerre polynomials with $\alpha = 0$ when ζ is suitably discretized.

Theorem 10.6 (From $\mathcal{M}$ to the little q-Laguerre polynomials with $\alpha = 0$) *Assume that $b > 0$ is such that b^2 is irrational. Suppose $\omega \in \mathbb{C}$ satisfies $-Q/2+\delta < Im\,\omega < Q/2$ for some $\delta > 0$. Define $\{\zeta_n\}_{n=0}^{\infty} \subset \mathbb{C}$ by*

$$\zeta_n = \frac{iQ}{2} + ibn. \tag{10.17}$$

For each $n \geq 0$, the function $\mathcal{M}$ satisfies

$$\lim_{\zeta \to \zeta_n} \mathcal{M}(b, \zeta, \omega) = W_n(x_W; 0, q), \tag{10.18}$$

where W_n are the little q-Laguerre polynomials defined in (B.33) and where x_W, q are given in (9.21).

Proof We prove (10.18) by computing the limit $\zeta \to \zeta_n$ of the representation (10.5) for each $n \geq 0$. Let $m, l \geq 0$ be integers and define $x_{m,l} \in \mathbb{C}$ by

$$x_{m,l} = \zeta - \frac{iQ}{2} - imb - \frac{il}{b}. \tag{10.19}$$

The function $s_b(x - \zeta)$ in (10.7) has a simple pole located at $x = x_{m,l}$ for any integers $m, l \geq 0$. In the limit $\zeta \to \zeta_n$, the pole $x_{n,0}$ collides with the pole of $s_b(x + \frac{iQ}{2})$ located at $x = 0$, and the contour $\mathcal{C}_{\mathcal{M}}$ is squeezed between the colliding poles. Hence, before taking the limit $\zeta \to \zeta_n$, we deform $\mathcal{C}_{\mathcal{M}}$ into a contour $\mathcal{C}'_{\mathcal{M}}$ which passes below $x_{n,0}$, thus picking up residue contributions from all the poles $x = x_{m,l}$ which satisfy $\mathrm{Im}\, x_{m,l} \geq \mathrm{Im}\, x_{n,0}$, i.e., from all the poles $x_{m,l}$ such that $mb + \frac{l}{b} \leq nb$. This leads to

$$\begin{aligned}\mathcal{M}(b, \zeta, \omega) = &-2i\pi P_{\mathcal{M}}(\zeta, \omega) \sum_{\substack{m,l \geq 0 \\ mb+\frac{l}{b} \leq nb}} \operatorname*{Res}_{x = x_{m,l}} (I_{\mathcal{M}}(x, \zeta, \omega)) \\ &+ P_{\mathcal{M}}(\zeta, \omega) \int_{\mathcal{C}'_{\mathcal{M}}} dx\, I_{\mathcal{M}}(x, \zeta, \omega).\end{aligned} \tag{10.20}$$

Utilizing the generalized difference equation (2.6) satisfied by the function s_b and the residue formula (2.4), straightforward calculations show that

$$\begin{aligned}&-2i\pi \operatorname*{Res}_{x = x_{m,l}} (I_{\mathcal{M}}(x, \zeta, \omega)) \\ &= e^{-i\pi\left(\frac{b^2m^2}{2} + bm(Q+i(\zeta+2\omega)) + \left(\zeta - \frac{iQ}{2}\right)\left(\frac{iQ}{2} - \zeta - 2\omega\right)\right)} e^{-i\pi b^{-2}\left(\frac{l^2}{2} + ibl(\zeta - ibm - iQ + 2\omega)\right)}\end{aligned}$$

$$\times \frac{1}{\left(e^{-\frac{2il\pi}{b^2}}; e^{\frac{2i\pi}{b^2}}\right)_l \left(e^{-2ib^2m\pi}; e^{2ib^2\pi}\right)_m} \frac{1}{s_b\left(\zeta - \frac{il}{b} - ibm\right)}. \tag{10.21}$$

Because of the factor $s_b(\zeta - \frac{il}{b} - ibm)^{-1}$, we deduce from (2.3) that the right-hand side of (10.21) has a simple pole at $\zeta = \zeta_n$ if the pair (m, l) satisfies $m \in [0, n]$ and $l = 0$, but is regular at $\zeta = \zeta_n$ for all other choices of $m \geq 0$ and $l \geq 0$. On the other hand, the prefactor $P_{\mathcal{M}}$ in (10.6) has a simple zero at $\zeta = \zeta_n$. Therefore, in the limit $\zeta \to \zeta_n$ the second term on the right-hand side of (10.20) vanishes, and the first term is nonzero only if $m \in [0, n]$ and $l = 0$. We conclude that

$$\begin{aligned}
&\lim_{\zeta \to \zeta_n} \mathcal{M}(b, \zeta, \omega) \\
&\quad = \mathcal{M}(b, \zeta_n, \omega) = -2i\pi \lim_{\zeta \to \zeta_n} P_{\mathcal{M}}(\zeta, \omega) \sum_{m=0}^{n} \operatorname*{Res}_{x = x_{m,0}} \left(I_{\mathcal{M}}(x, \zeta, \omega)\right),
\end{aligned} \tag{10.22}$$

or, more explicitly,

$$\begin{aligned}
&\mathcal{M}(b, \zeta_n, \omega) \\
&\quad = \sum_{m=0}^{n} e^{-i\pi b^2\left(\frac{m^2}{2} - mn + n^2\right)} e^{2\pi b\left(\omega(m-n) - \frac{imQ}{4}\right)} \frac{1}{\left(e^{-2ib^2m\pi}; e^{2ib^2\pi}\right)_m} \frac{s_b(ibn + \frac{iQ}{2})}{s_b(ib(n-m) + \frac{iQ}{2})}.
\end{aligned} \tag{10.23}$$

Using (2.6), we obtain

$$\mathcal{M}(b, \zeta_n, \omega) = \sum_{m=0}^{n} e^{\pi b\left(2(m-n)\omega - ibn^2 - imQ\right)} \frac{\left(q^{1+n-m}; q\right)_m}{\left(q^{-m}; q\right)_m}. \tag{10.24}$$

We now apply the q-Pochhammer identity

$$(\alpha; q)_n = (-\alpha)^n q^{\frac{n(n-1)}{2}} \left(\alpha^{-1} q^{1-n}; q\right)_n \tag{10.25}$$

with $\alpha = q^{1+n-m}$ and with $\alpha = q^{-m}$ to find

$$\mathcal{M}(b, \zeta_n, \omega) = \sum_{m=0}^{n} e^{\pi b(2m-n)(2\omega + ibn)} e^{i\pi m\left(b^2 + 2ib\omega - 1\right)} \frac{\left(q^{-n}; q\right)_m}{(q; q)_m}. \tag{10.26}$$

After replacing $m \to n - m$, in the sum and using the parameter correspondence (9.21), we obtain

$$\mathcal{M}(b, \zeta_n, \omega) = \sum_{m=0}^{n} e^{i\pi(2m-n)} e^{-i\pi b^2(2m-n)(n+1)} \frac{\left(q^{-n}; q\right)_{n-m}}{(q; q)_{n-m}} (q x_W)^m. \tag{10.27}$$

Using the q-Pochhammer identity

$$\frac{(\alpha;q)_{n-m}}{(\beta;q)_{n-m}} = \frac{(\alpha;q)_n}{(\beta;q)_n}\frac{(\beta^{-1}q^{1-n};q)_m}{(\alpha^{-1}q^{1-n};q)_m}\left(\frac{\beta}{\alpha}\right)^m, \qquad \alpha,\beta\neq 0, \quad m=0,1,\ldots,n, \tag{10.28}$$

with $\alpha = q^{-n}$ and $\beta = q$, we arrive at

$$\mathcal{M}(b,\zeta_n,\omega) = e^{i\pi n\left(b^2(n+1)-1\right)}\frac{\left(q^{-n};q\right)_n}{(q;q)_n}\sum_{m=0}^{n}\frac{\left(q^{-n};q\right)_m}{(q;q)_m}(qx_W)^m. \tag{10.29}$$

Finally, utilizing the q-Pochhammer identity (10.25) with $\alpha = q^{-n}$, we find

$$e^{i\pi n\left(b^2(n+1)-1\right)}\frac{\left(q^{-n};q\right)_n}{(q;q)_n} = 1. \tag{10.30}$$

Therefore we conclude that

$$\mathcal{M}(b,\zeta_n,\omega) = {}_2\phi_1\left(\left.\begin{matrix} q^{-n},0 \\ 0 \end{matrix}\right| q;qx_W\right) = W_n(x_W;0,q). \tag{10.31}$$

This concludes the proof of (10.18). □

11 A Simple Application: Duality Formulas

As a simple application of the constructions presented above, we show how the non-polynomial scheme can be used to easily obtain various duality formulas which relate members of the q-Askey scheme.

11.1 Duality Formula for the Askey–Wilson Polynomials

The duality formula (B.7) for the Askey–Wilson polynomials is an easy consequence of Theorem 3.2 and the self-duality property (3.5) of the function $\mathcal{R}$ defined in (3.1). Indeed, suppose Assumptions 1.1 and 3.1 are satisfied and define $\{\sigma_t^{(n)}\}_{n=0}^{\infty} \subset \mathbb{C}$ by

$$\sigma_t^{(n)} = \tfrac{iQ}{2} + \theta_1 + \theta_t + ibn. \tag{11.1}$$

Under the parameter correspondence

$$\tilde{\alpha}_R = e^{2\pi b\left(\frac{iQ}{2}+\theta_0+\theta_t\right)}, \quad \tilde{\beta}_R = e^{2\pi b\left(\frac{iQ}{2}+\theta_1-\theta_\infty\right)}, \quad \tilde{\gamma}_R = e^{2\pi b\left(\frac{iQ}{2}-\theta_0+\theta_t\right)},$$
$$\tilde{\delta}_R = e^{2\pi b\left(\frac{iQ}{2}+\theta_1+\theta_\infty\right)}, \quad q = e^{2i\pi b^2},$$

the function $\mathcal{R}$ satisfies, for each integer $n \geq 0$,

$$\lim_{\sigma_t \to \sigma_t^{(n)}} \mathcal{R}\begin{bmatrix} \theta_1 & \theta_t & \sigma_s \\ \theta_\infty & \theta_0 & \sigma_t \end{bmatrix}; b = R_n(e^{2\pi b \sigma_s}; \tilde{\alpha}_R, \tilde{\beta}_R, \tilde{\gamma}_R, \tilde{\delta}_R, q), \tag{11.2}$$

where R_n are the Askey–Wilson polynomials defined in (B.1). Moreover, evaluating $\mathcal{R}$ at $\sigma_s = \sigma_s^{(n)}$ and $\sigma_t = \sigma_t^{(m)}$ with $n, m \in \mathbb{Z}_{\geq 0}$ and using (3.13) and (11.2), we obtain

$$\begin{aligned} \mathcal{R}\begin{bmatrix} \theta_1 & \theta_t & \sigma_s^{(n)} \\ \theta_\infty & \theta_0 & \sigma_t^{(m)} \end{bmatrix}; b &= R_n(e^{2\pi b \sigma_t^{(m)}}; \alpha_R, \beta_R, \gamma_R, \delta_R, q) \\ &= R_m(e^{2\pi b \sigma_s^{(n)}}; \tilde{\alpha}_R, \tilde{\beta}_R, \tilde{\gamma}_R, \tilde{\delta}_R, q). \end{aligned} \tag{11.3}$$

Employing the symmetry $R_n(z; \alpha, \beta, \gamma, \delta, q) = R_n(z^{-1}; \alpha, \beta, \gamma, \delta, q)$ and observing that the parameters in (3.12) satisfy

$$\alpha_R^2 = \frac{\tilde{\alpha}_R \tilde{\beta}_R \tilde{\gamma}_R \tilde{\delta}_R}{q}, \qquad \beta_R = \frac{\tilde{\alpha}_R \tilde{\beta}_R}{\alpha_R}, \qquad \gamma_R = \frac{\tilde{\alpha}_R \tilde{\gamma}_R}{\alpha_R}, \qquad \delta_R = \frac{\tilde{\alpha}_R \tilde{\delta}_R}{\alpha_R}, \tag{11.4}$$

and that

$$e^{-2\pi b \sigma_s^{(n)}} = \alpha_R^{-1} q^{-n}, \qquad e^{-2\pi b \sigma_t^{(m)}} = \tilde{\alpha}_R^{-1} q^{-m}, \tag{11.5}$$

we recover the duality formula (B.7) for the Askey–Wilson polynomials.

11.2 Duality Formula Relating the Continuous Dual *q*-Hahn and the big *q*-Jacobi Polynomials

Suppose that Assumptions 1.1 and 4.6 are satisfied. Evaluating the function $\mathcal{H}$ at $\nu = \nu_n$ and $\sigma_s = \sigma_s^{(m)}$ with $n, m \in \mathbb{Z}_{\geq 0}$ and utilizing the polynomial limits (4.39) and (4.47), we obtain a duality formula relating the polynomials H_n and J_n:

$$\begin{aligned} \mathcal{H}(b, \theta_0, \theta_t, \theta_*, \sigma_s^{(m)}, \nu_n) &= H_n(e^{2\pi b \sigma_s^{(m)}}; \alpha_H, \beta_H, \gamma_H, q) \\ &= J_m(q^{-n}; \alpha_J, \beta_J, \gamma_J; q). \end{aligned} \tag{11.6}$$

Employing the symmetry $H_n(z; \alpha, \beta, \gamma, q) = H_n(z^{-1}; \alpha, \beta, \gamma, q)$ and observing that the parameters in (4.38) and (4.46) are related by

$$\alpha_J = \frac{\alpha_H \beta_H}{q}, \qquad \beta_J = \frac{\alpha_H}{\beta_H}, \qquad \gamma_J = \frac{\alpha_H \gamma_H}{q}, \qquad e^{-2\pi b \sigma_s^{(m)}} = \alpha_H^{-1} q^{-m}, \tag{11.7}$$

we recognize the duality formula (B.16).

Remark 11.1 It is natural to ask whether there exists a well-defined limit of the duality formula (3.5) for $\mathcal{R}$, such that (11.6) follows directly by polynomial specialisation of the resulting duality formula for $\mathcal{H}$. However, we expect that $\mathcal{H}$ does not satisfy a duality relation analogous to (3.5) exchanging the parameters σ_s and ν. One apparent obstruction for the existence of such a duality formula is that, unlike the $\mathcal{R}$-function, the two dual difference operators $\tilde{D}_{\mathcal{H}}(b, \nu)$ and $D_{\mathcal{H}}(b, \sigma_s)$ defined in (4.30) and (4.23), respectively, possess different analytic properties. More precisely, the coefficients of $\tilde{D}_{\mathcal{H}}(b, \nu)$ are entire functions of ν, whereas the coefficients of $D_{\mathcal{H}}(b, \sigma_s)$ are meromorphic functions of σ_s. If our expectation is true, it means that all other families in the non-polynomial scheme also do not exhibit a duality formula of this kind.

11.3 Duality Formula Relating the Little *q*-Jacobi and the Al-Salam Chihara Polynomials

Suppose that Assumptions 1.1 and 5.5 are satisfied. Evaluating $\mathcal{S}(b, \theta_0, \theta_t, \sigma_s, \rho)$ at $\rho = \rho_n$ and $\sigma_s = \sigma_s^{(m)}$ with $n, m \in \mathbb{Z}_{\geq 0}$ and utilizing the limits (5.33) and (5.35), we obtain a duality formula relating the polynomials S_n and Y_n:

$$S_n(e^{2\pi b\sigma_s^{(m)}}; \alpha_S, \beta_S, q) = Y_m(q^{-n}; \alpha_Y, \beta_Y, q), \qquad n, m = 0, 1, 2, \ldots. \tag{11.8}$$

Employing the symmetry $S_n(z; \alpha_S, \beta_S, q) = S_n(z^{-1}; \alpha_S, \beta_S, q)$ and observing that

$$\alpha_Y = \frac{\alpha_S \beta_S}{q}, \qquad \beta_Y = \frac{\alpha_S}{\beta_S}, \qquad e^{-2\pi b\sigma_s^{(m)}} = \alpha_S^{-1} q^{-m}, \tag{11.9}$$

we recover the duality formula (B.28).

Acknowledgements The authors are grateful to the two referees and to the editor for many valuable remarks on an earlier version of the manuscript. Support is acknowledged from the European Research Council, Grant Agreement No. 682537, the Swedish Research Council, Grant No. 2015-05430, and the Ruth and Nils-Erik Stenbäck Foundation.

Funding Open access funding provided by Royal Institute of Technology.

Appendix A: *q*-Hypergeometric Series

The q-hypergeometric series ${}_{s+1}\phi_s$ is defined by

$$ {}_{s+1}\phi_s \left[\begin{matrix} a_1, \ldots, a_{s+1} \\ b_1, \ldots, b_s \end{matrix}; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{s+1}; q)_k}{(b_1, \ldots, b_s, q; q)_k} z^k, \tag{A.1} $$

where the q-Pochammer symbols $(a; q)_n$ and $(a_1, a_2, \ldots, a_m; q)_n$ are given by

$$ (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \quad \text{and} \quad (a_1, a_2, \ldots, a_m; q)_n = \prod_{j=1}^{m} (a_j; q)_n. \tag{A.2} $$

Note that the sum in (A.1) contains only finitely many terms if one of the a_i in the numerator is equal to q^{-n} for some integer $n \geq 1$. Otherwise, the sum converges for $|z| < 1$.

Appendix B: The *q*-Askey Scheme

B.1 Askey–Wilson Polynomials

The Askey–Wilson polynomials R_n, defined by

$$ R_n(z; \alpha, \beta, \gamma, \delta, q) = {}_4\phi_3 \left(\left. \begin{matrix} q^{-n}, \alpha\beta\gamma\delta q^{n-1}, \alpha z, \alpha z^{-1} \\ \alpha\beta, \alpha\gamma, \alpha\delta \end{matrix} \right| q; q \right), \tag{B.1} $$

are the most general polynomials of the q-Askey scheme. The normalization in (B.1) for R_n is related to the standard normalization of [25, Eq. (14.1.1)] by

$$ p_n \left(\tfrac{z+z^{-1}}{2}; \alpha, \beta, \gamma, \delta, q \right) = \alpha^{-n} (\alpha\beta, \alpha\gamma, \alpha\delta; q)_n \; R_n(z; \alpha, \beta, \gamma, \delta, q). \tag{B.2} $$

Since $p_n(x; \alpha, \beta, \gamma, \delta, q)$ is a polynomial of order n in x, R_n is a polynomial of order n in $z + z^{-1}$. The polynomials R_n satisfy the three-term recurrence relation

$$ (L_R R_n)(z; \alpha, \beta, \gamma, \delta, q) = (z + z^{-1}) R_n(z; \alpha, \beta, \gamma, \delta, q), \tag{B.3} $$

where the operator L_R is given by

$$ L_R = a_n^+ T^+ + (\alpha + \alpha^{-1} - a_n^+ - a_n^-) + a_n^- T^-, \tag{B.4} $$

with $T^{\pm} p_n(x) := p_{n\pm 1}(x)$ and

$$a_n^+ = \frac{(1-\alpha\beta q^n)(1-\alpha\gamma q^n)(1-\alpha\delta q^n)\left(1-\alpha\beta\gamma\delta q^{n-1}\right)}{\alpha\left(1-\alpha\beta\gamma\delta q^{2n-1}\right)\left(1-\alpha\beta\gamma\delta q^{2n}\right)},$$
$$a_n^- = \frac{\alpha(1-q^n)\left(1-\beta\gamma q^{n-1}\right)\left(1-\beta\delta q^{n-1}\right)\left(1-\gamma\delta q^{n-1}\right)}{\left(1-\alpha\beta\gamma\delta q^{2n-2}\right)\left(1-\alpha\beta\gamma\delta q^{2n-1}\right)}. \tag{B.5}$$

The Askey–Wilson polynomials also possess a symmetry exchanging the parameters n and z [36]. More precisely, define dual parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ such that

$$\alpha^2 = q^{-1}\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}, \qquad \beta = \frac{\tilde{\alpha}\tilde{\beta}}{\alpha}, \qquad \gamma = \frac{\tilde{\alpha}\tilde{\gamma}}{\alpha}, \qquad \delta = \frac{\tilde{\alpha}\tilde{\delta}}{\alpha}. \tag{B.6}$$

Then, by [36, Eq. (27)],

$$R_n(\alpha^{-1}q^{-m}; \alpha, \beta, \gamma, \delta, q) = R_m(\tilde{\alpha}^{-1}q^{-n}; \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, q), \qquad n, m = 0, 1, 2, \ldots. \tag{B.7}$$

B.2 Continuous dual *q*-Hahn Polynomials

The continuous dual q-Hahn polynomials $H_n(z; \alpha, \beta, \gamma, q)$ are defined by

$$H_n(z; \alpha, \beta, \gamma, q) = R_n(z; \alpha, \beta, \gamma, 0, q) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, \alpha z, \alpha z^{-1} \\ \alpha\beta, \alpha\gamma \end{matrix}\right| q; q\right). \tag{B.8}$$

They satisfy the three-term recurrence relation

$$(L_H H_n)(z; \alpha, \beta, \gamma, q) = (z + z^{-1})\, H_n(z; \alpha, \beta, \gamma, q), \tag{B.9}$$

where the operator L_H is defined by

$$L_H = b_n^+ T^+ + \left(\alpha + \alpha^{-1} - b_n^+ - b_n^-\right) + b_n^- T^-, \tag{B.10}$$

with

$$b_n^+ = \alpha^{-1}(1-\alpha\beta q^n)(1-\alpha\gamma q^n), \qquad b_n^- = \alpha(1-q^n)(1-\beta\gamma q^{n-1}). \tag{B.11}$$

B.3 Big *q*-Jacobi Polynomials

The big q-Jacobi polynomials $J_n(x; \alpha, \beta, \gamma; q)$ are defined by

$$J_n(x; \alpha, \beta, \gamma; q) = \lim_{\lambda \to 0} R_n\left(\frac{x}{\lambda}; \lambda, \frac{\alpha q}{\lambda}, \frac{\gamma q}{\lambda}, \frac{\lambda\beta}{\gamma}, q\right)$$

$$= {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, x \\ \alpha q, \gamma q \end{matrix}\middle| q; q\right). \tag{B.12}$$

They satisfy the three-term recurrence relation

$$L_J J_n(x; \alpha, \beta, \gamma; q) = x J_n(x; \alpha, \beta, \gamma; q), \tag{B.13}$$

where the operator L_J is defined by

$$L_J = c_n^+ T^+ + \left(1 - c_n^+ - c_n^-\right) + c_n^- T^-, \tag{B.14}$$

with

$$\begin{cases} c_n^+ = \dfrac{\left(1-\alpha q^{n+1}\right)\left(1-\alpha\beta q^{n+1}\right)\left(1-\gamma q^{n+1}\right)}{\left(1-\alpha\beta q^{2n+1}\right)\left(1-\alpha\beta q^{2n+2}\right)}, \\ c_n^- = -\alpha\gamma q^{n+1}\dfrac{\left(1-q^n\right)\left(1-\alpha\beta\gamma^{-1}q^n\right)\left(1-\beta q^n\right)}{\left(1-\alpha\beta q^{2n}\right)\left(1-\alpha\beta q^{2n+1}\right)}. \end{cases} \tag{B.15}$$

There exists a duality between the big q-Jacobi and the continuous dual q-Hahn polynomials which is inherited from the duality (B.7) of the Askey–Wilson polynomials. More precisely, by [36, Eq. (44)],

$$\begin{aligned} H_n\left(\alpha^{-1}q^{-m}; \alpha, \beta, \gamma, q\right) &= J_m\left(q^{-n}; q^{-1}\alpha\beta, \alpha\beta^{-1}, q^{-1}\alpha\gamma; q\right), \\ &\qquad n, m = 0, 1, 2, \ldots. \end{aligned} \tag{B.16}$$

B.4 Al-Salam–Chihara Polynomials

The Al-Salam Chihara polynomials $S_n(z; \alpha, \beta, q)$ are defined by

$$\begin{aligned} S_n(z; \alpha, \beta, q) &= H_n(z; \alpha, \beta, 0, q) \\ &= \frac{\alpha^n}{(\alpha\beta; q)_n}(\alpha z; q)_n\, z^{-n}{}_2\phi_1\left(\begin{matrix} q^{-n}, \beta z^{-1} \\ \alpha^{-1}q^{1-n}z^{-1} \end{matrix}\middle| q; \alpha^{-1}qz\right), \end{aligned} \tag{B.17}$$

where H_n are the continuous dual q-Hahn polynomials defined in (B.8). In (B.17), we use the normalization of [36, Eq. (59)]. The polynomials S_n satisfy the three-term recurrence relation

$$L_S S_n(z; \alpha, \beta, q) = \left(z + z^{-1}\right) S_n(z; \alpha, \beta, q), \tag{B.18}$$

where the operator L_S is defined by

$$L_S = \left(\alpha^{-1} - \beta q^n\right) T^+ + (\alpha + \beta)\, q^n + \alpha\left(1 - q^n\right) T^-. \tag{B.19}$$

B.5 Little *q*-Jacobi Polynomials

The Little q-Jacobi polynomials $p_n(x;\alpha,\beta,q)$ are defined by (see [36, Eq. (66)])

$$p_n(x;\alpha,\beta,q) = {}_2\phi_1\left(\left.\begin{matrix} q^{-n}, \alpha\beta q^{n+1} \\ \alpha q \end{matrix}\right| q; qx\right), \tag{B.20}$$

or, equivalently, by [35, Eq. (3.38)]

$$p_n(x;\alpha,\beta,q) = (-q\beta)^{-n} q^{-\frac{n(n-1)}{2}} \frac{(q\beta;q)_n}{(q\alpha;q)_n} {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, q^{n+1}\alpha\beta, q\beta x \\ q\beta, 0 \end{matrix}\right| q; q\right). \tag{B.21}$$

They arise as limits of the big q-Jacobi polynomials in two ways. First, we have (see [36, Eq. (68)])

$$p_n(x;\alpha,\beta,q) = \lim_{\gamma\to\infty} J_n(\gamma q x;\alpha,\beta,\gamma;q). \tag{B.22}$$

Second, we have

$$p_n(x;\alpha,\beta,q) = (-q\beta)^{-n} q^{-\frac{n(n-1)}{2}} \frac{(q\beta;q)_n}{(q\alpha;q)_n} Y_n(q\beta x;\beta,\alpha,q), \tag{B.23}$$

where

$$Y_n(x;\alpha,\beta,q) = \lim_{\gamma\to 0} J_n(x;\alpha,\beta,\gamma,q) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, q^{n+1}\alpha\beta, x \\ \alpha q, 0 \end{matrix}\right| q; q\right). \tag{B.24}$$

The polynomials Y_n satisfy the three-term recurrence relation (see [36, Eqs. (73)–(74)])

$$L_Y Y_n(x;\alpha,\beta,q) = x Y_n(x;\alpha,\beta,q), \tag{B.25}$$

where the operator L_Y is defined by

$$L_Y = y_n^+ T^+ + 1 - y_n^+ - y_n^- + y_n^- T^-, \tag{B.26}$$

with

$$\begin{aligned} y_n^+ &= \frac{\left(1-\alpha q^{n+1}\right)\left(1-\alpha\beta q^{n+1}\right)}{\left(1-\alpha\beta q^{2n+1}\right)\left(1-\alpha\beta q^{2n+2}\right)}, \\ y_n^- &= q^{2n+1}\alpha^2\beta \frac{(1-q^n)(1-\beta q^n)}{\left(1-\alpha\beta q^{2n}\right)\left(1-\alpha\beta q^{2n+1}\right)}. \end{aligned} \tag{B.27}$$

Finally, there exists a duality between the little q-Jacobi polynomials and the Al-Salam Chihara polynomials which is inherited from (B.16). More precisely, from [36, Eq. (75)] we have

$$S_n\left(\alpha^{-1}q^{-m};\alpha,\beta,q\right)=Y_m\left(q^{-n};q^{-n}\alpha\beta,\alpha\beta^{-1},q\right),\qquad n,m=0,1,2,\ldots. \tag{B.28}$$

B.6 Big *q*-Laguerre Polynomials

The big q-Laguerre polynomials $L_n(x;\alpha,\beta;q)$ are defined by setting $\beta=0$ in the big q-Jacobi polynomials (see [25, Eq. (14.11.1)]):

$$\begin{aligned}L_n(x;\alpha,\beta;q)&=J_n(x;\alpha,0,\beta;q)\\&=\frac{1}{\left(\beta^{-1}q^{-n};q\right)_n}\,{}_2\phi_1\left(\left.\begin{matrix}q^{-n},\alpha q x^{-1}\\ \alpha q\end{matrix}\right|q;x\beta^{-1}\right).\end{aligned} \tag{B.29}$$

The polynomials L_n satisfy the three-term recurrence relation

$$L_L L_n(x;\alpha,\beta,q)=x\;L_n(x;\alpha,\beta,q), \tag{B.30}$$

where the operator L_L is defined by

$$L_L=l_n^+T^++(1-l_n^+-l_n^-)+l_n^-T^-, \tag{B.31}$$

with

$$l_n^+=\left(1-\alpha q^{n+1}\right)\left(1-\beta q^{n+1}\right),\qquad l_n^-=-\alpha\beta q^{n+1}\left(1-q^n\right). \tag{B.32}$$

B.7 Little *q*-Laguerre Polynomials

The little q-Laguerre (or Wall) polynomials $W_n(x;\alpha,q)$ are obtained from the big q-Laguerre polynomials as follows:

$$W_n(x;\alpha,q)=\lim_{\beta\to-\infty}L_n(\beta q x;\alpha,\beta;q)={}_2\phi_1\left(\left.\begin{matrix}q^{-n},0\\ \alpha q\end{matrix}\right|q;qx\right). \tag{B.33}$$

They satisfy the recurrence relation

$$L_W W_n(x;\alpha,q)=-xW_n(x;\alpha,q), \tag{B.34}$$

where the operator L_W is defined by

$$L_W=w_n^+T^+-(w_n^++w_n^-)+w_n^-T^-, \tag{B.35}$$

with

$$w_n^+ = q^n \left(1 - \alpha q^{n+1}\right), \qquad w_n^- = \alpha q^n \left(1 - q^n\right). \tag{B.36}$$

B.8 Continuous Big q-Hermite Polynomials

The continuous big q-Hermite polynomials $X_n(z; \alpha, q)$ are defined by

$$X_n(z; \alpha, q) = \lim_{\beta \to 0} S_n(z; \alpha, \beta, q) = \alpha^n z^n {}_2\phi_0 \left(\left. \begin{matrix} q^{-n}, \alpha z \\ - \end{matrix} \right| q; q^n z^{-2} \right), \tag{B.37}$$

where S_n are the Al-Salam-Chihara polynomials defined in (B.17). The normalization in (B.37) is obtained by multiplying $H_n(\frac{1}{2}(z + z^{-1}), \alpha | q)$ in [25, Eq. (14.18.1)] by α^n. They satisfy the recurrence relation

$$R_{X_n} X_n(z; \alpha, q) = \left(z + z^{-1}\right) X_n(z; \alpha, q), \tag{B.38}$$

where the operator R_{X_n} is defined by

$$R_{X_n} = \alpha^{-1} T^+ + \alpha q^n + \alpha \left(1 - q^n\right) T^-. \tag{B.39}$$

B.9 Continuous q-Hermite Polynomials

The continuous q-Hermite polynomials $Q_n(z; q)$ are defined by

$$Q_n(z; q) = \lim_{\alpha \to 0} \alpha^{-n} X_n(z; \alpha, q) = z^n {}_2\phi_0 \left(\left. \begin{matrix} q^{-n}, 0 \\ - \end{matrix} \right| q; q^n z^{-2} \right). \tag{B.40}$$

They satisfy the recurrence relation

$$L_Q Q_n(z; q) = \left(z + z^{-1}\right) Q_n(z; q), \tag{B.41}$$

where the operator L_Q is defined by

$$L_Q = T^+ + \left(1 - q^n\right) T^-. \tag{B.42}$$

References

1. Atakishiyev, M.N., Atakishiyev, N.M., Klimyk, A.U.: Big q-Laguerre and q-Meixner polynomials and representations of the quantum algebra $U_q(su_{1,1})$. J. Phys. A: Math. Gen. **36**, 10335 (2003)
2. Alday, L.F., Gaiotto, D., Tachikawa, Y.: Liouville correlation functions from four-dimensional gauge theories. Lett. Math. Phys. **91**, 167–197 (2010)
3. Askey, R., Wilson, J.: Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, vol. 54. Mem. Am. Math. Soc. (1985)

4. Baseilhac, P., Martin, X., Zhedanov, A.S.: Little and big q-Jacobi polynomials and the Askey-Wilson algebra. Ramanujan J. **51**, 629–648 (2020)
5. van de Bult, F.J., Rains, E.: Basic hypergeometric functions as limits of elliptic hypergeometric functions. SIGMA Symmet. Integrabil. Geom. Methods Appl. **5**, 31 (2009)
6. van de Bult, F.J., Rains, E.: Limits of elliptic hypergeometric biorthogonal functions. J. Approx. Theory **193**, 128–163 (2015)
7. van de Bult, F.J.: Ruijsenaars' hypergeometric function and the modular double of $U_q(sl(2,C))$. Adv. Math. **204**, 53971 (2006)
8. van de Bult, F.J.: Hyperbolic Hypergeometric Functions. Ph. D. thesis, University of Amsterdam (2007)
9. van de Bult, F.J., Rains, E.M., Stokman, J.V.: Properties of generalized univariate hypergeometric functions. Commun. Math. Phys. **275**, 3795 (2007)
10. Collier, S., Gobeil, Y., Maxfield, H., Perlmutter, E.: Quantum Regge trajectories and the Virasoro analytic bootstrap. J. High Energ. Phys. **2019**, 212 (2019)
11. Corteel, S., Williams, L.K.: Tableaux combinatorics for the asymmetric exclusion process and Askey–Wilson polynomials. Duke Math. J. **159**(3), 385–415 (2011)
12. van Diejen, J.F.: Difference Calogero–Moser systems and finite Toda chains. J. Math. Phys. **36**, 1299 (1995)
13. Faddeev, L.: Modular Double of a Quantum Group, World Scientific Series in 21st Century Mathematics, Fifty Years of Mathematical Physics, pp. 523–530 (2016)
14. Faddeev, L., Kashaev, R.: Quantum dilogarithm. Mod. Phys. Lett. **9**, 265–282 (1994)
15. Floreanini, R., LeTourneux, J., Vinet, L.: An algebraic interpretation of the continuous big q-Hermite polynomials. J. Math. Phys. **36**, 5091 (1995)
16. Gaiotto, D., Teschner, J.: Irregular singularities in Liouville theory and Argyres–Douglas type gauge theories, I. J. High Energy Phys. **2012**, 50 (2012). arXiv:1203.1052 [hep-th]
17. Groenevelt, W.: The Wilson function transform scheme. Int. Math. Res. Not **52**, 2779–2817 (2003)
18. Haine, L., Iliev, P.: Askey–Wilson type functions with bound states. Ramanujan J. **11**, 285 (2006)
19. Iorgov, N., Lisovyy, O., Tykhyy, Y.: Painlevé VI connection problem and monodromy of c=1 conformal blocks. J. High Energy Phys. **2013**, 29 (2013)
20. Ip, I.C.H.: Positive Representations of Non-simply-laced Split Real Quantum Groups. J. Algebra **425**, 245–276 (2015)
21. Ip, I.C.H.: Positive representations of split real quantum groups: The universal R operator. Int. Math. Res. Not. **2015**(1), 240–287 (2015)
22. Ip, I.C.H.: Positive representations of split real simply-laced quantum groups. Publ. Res. Inst. Math. Sci. **56**, 603–646 (2020)
23. Ismail, M., Rahman, M.: The associated Askey-Wilson polynomials. Trans. Am. Math. Soc. **328**, 201–237 (1991)
24. Kharchev, S., Lebedev, D., Semenov-Tian-Shansky, M.: Unitary representations of $U_q(sl(2,\mathbb{R}))$, the modular double and the multiparticle q-deformed Toda chains. Commun. Math. Phys. **225**, 573–609 (2002)
25. Koekoek, R., Lesky, P.A., Swarttouw, R.F.: Hypergeometric Orthogonal Polynomials and their q-Analogues. Springer Monographs in Mathematics, Springer, Berlin (2010)
26. Koelink, E., Rosengren, H.: Transmutation kernels for the little q-Jacobi function transform. Rocky Mount. J. Math. **32**(2), 703–738 (2002)
27. Koelink, E., Stokman, J.V.: The Askey–Wilson Function Transform Scheme, Special Functions 2000: Current Perspective and Future Directions. NATO Science Series (Series II: Mathematics, Physics and Chemistry), vol. 30. Springer, Dordrecht
28. Koelink, E., Stokman, J.V.: The Askey–Wilson function transform. Int. Math. Res. Not. **52**, 2779–2817 (2003)
29. Koelink, E., Stokman, J.V.: The big q-Jacobi function transform. Constr. Approx. **19**, 191–235 (2003)
30. Koelink, H.T.: 8 Lectures on quantum groups and q-special functions. Revista Colombiana de Matemáticas **30**(2), 93–180 (1996)
31. Koelink, H.T.: Askey-Wilson polynomials and the quantum SU(2) group: Survey and applications. Acta Applicandae Mathematica **44**, 295–352 (1996)
32. Koornwinder, T.H.: Group theoretic interpretations of Askey's scheme of hypergeometric orthogonal polynomials. In: Alfaro, M., Dehesa, J.S., Marcellan, F.J., Rubio de Francia, J.L., Vinuesa, J. (eds.) Orthogonal polynomials and their applications. Lecture Notes in Mathematics, vol 1329. Springer, Berlin

33. Koornwinder, T.H.: Representations of the twisted SU(2) quantum group and some q- hypergeometric orthogonal polynomials. Proc. Kon. Ned. Akad. van Wetensch. Indag. Math. **51**, 97–117.(1989)
34. Koornwinder, T.H.: Askey-Wilson polynomials as zonal spherical functions on the SU(2) quantum group. SIAM J. Math. Anal. **24**, 795–813 (1993)
35. Koornwinder, T.H.: Compact quantum groups and q-special functions, Representations of Lie Groups and Quantum Groups. In: Baldoni, V., Picardello, (eds.) Pitman Research Notes in Mathematics Series 311, pp. 46–128, Longman Scientific & Technical, New York (1994). arXiv:math/9403216
36. Koornwinder, T.H., Mazzocco, M.: Dualities in the q-Askey scheme and degenerate DAHA. Stud. Appl. Math. **141**, 424–473 (2018)
37. Kurokawa, N.: Multiple sine functions and Selberg zeta functions. Proc. Jpn. Acad. Ser. A Math. Sci. **67**, 61–64 (1991)
38. Lenells, J., Roussillon, J.: Confluent conformal blocks of the second kind. J. High Energ. Phys. **2020**, 133 (2020)
39. Lenells, J., Roussillon, J.: The family of confluent Virasoro fusion kernels and a non-polynomial q-Askey scheme. Adv. Theo. and Math. Phys. **25**(6), 1597–1650 (2021)
40. Mazzocco, M.: Non-Symmetric Basic Hypergeometric Polynomials and Representation Theory for Confluent Cherednik Algebras. SIGMA **10**(2014), 116 (2014)
41. Mazzocco, M.: Confluences of the Painlevé equations, Cherednik algebras and q-Askey scheme. Nonlinearity **29**(9), 2565–2608
42. Noumi, M., Mimachi, K.: Quantum 2-spheres and big q-Jacobi polynomials. Commun. Math. Phys. **128**, 521–531 (1990)
43. Noumi, M., Mimachi, K.: Askey-Wilson polynomials and the quantum group $SU_q(2)$. Proc. Jpn. Acad. Ser. A Math. Sci. **66**(6), 146–149 (1990)
44. Noumi, M., Stokman, J.V.: Askey–Wilson polynomials: an affine Hecke algebra approach, Laredo Lectures on Orthogonal Polynomials and Special Functions, Adv. Theory Spec. Funct. Orthogonal Polynomials, pp. 111–144. Nova Sci. Publ., Hauppauge, NY (2004)
45. Ponsot, B., Teschner, J.: Liouville bootstrap via harmonic analysis on a noncompact quantum group. arXiv:hep-th/9911110 [hep-th]
46. Ponsot, B., Teschner, J.: Clebsch–Gordan and Racah–Wigner coefficients for a continuous series of representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. Commun. Math. Phys. **224**, 613–655 (2001)
47. Ribault, S.: Conformal Field Theory on the Plane. arXiv:1406.4290v5 [hep-th]
48. Roussillon, J.: The Virasoro fusion kernel and Ruijsenaars' hypergeometric function. Lett. Math. Phys. **111**, 7 (2021)
49. Ruijsenaars, S.: Systems of Calogero-Moser type, Particles and fields, pp. 251–352
50. Ruijsenaars, S.: First order analytic difference equations and integrable quantum systems. J. Math. Phys. **38**, 1069 (1997)
51. Ruijsenaars, S.: A generalized hypergeometric function satisfying four analytic difference equations of Askey–Wilson type. Commun. Math. Phys. **206**, 639–690 (1999)
52. Ruijsenaars, S.: A Generalized Hypergeometric Function II. Asymptotics and D4 Symmetry. Commun. Math. Phys. **243**, 389412 (2003)
53. Ruijsenaars, S.: A Generalized Hypergeometric Function III. Associated Hilbert Space Transform. Commun. Math. Phys. **243**, 413448 (2003)
54. Ruijsenaars, S.: A Relativistic Conical Function and its Whittaker Limits. SIGMA **7**, 101 (2011)
55. Sahi, S.: Nonsymmetric Koornwinder Polynomials and Duality. Ann. Math. **150**(1), 267–282 (1999)
56. Sasamoto, T.: One-dimensional partially asymmetric simple exclusion process with open boundaries: orthogonal polynomials approach. J. Phys. A: Math. Gen. **32**, 7109 (1999)
57. Stokman, J.V.: Difference fourier transforms for nonreduced root systems. Selecta Math. (N.S.) **9**, 409–494 (2003)
58. Stokman, J.V.: Askey–Wilson Functions and Quantum Groups. In: Ismail, M.E., Koelink, E. (eds) Theory and Applications of Special Functions. Developments in Mathematics, vol 13. Springer, Boston
59. Suslov, S.K.: Some orthogonal very-well-poised $8\phi7$-functions that generalize Askey–Wilson polynomials. Ramanujan J. **5**, 183–218 (2001)
60. Uchiyama, M.: Two-species asymmetric simple exclusion process with open boundaries. Chaos Solitons Fract. **35**, 2 (2008)
61. Uchiyama, M., Sasamoto, T., Wadati, M.: Asymmetric simple exclusion process with open boundaries and Askey–Wilson Polynomials. J. Phys. A: Math. Gen. **37**, 4985 (2004)
62. Woronowicz, S.L.: Quantum exponential function. Rev. Math. Phys. **12**, 873–920 (2000)

63. Zhedanov, A.S.: “Hidden symmetry” of Askey–Wilson polynomials. Theor. Math. Phys. **89**, 1146–1157 (1991)