



Metric Approximation of Set-Valued Functions of Bounded Variation by Integral Operators

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Abstract

We introduce an adaptation of integral approximation operators to set-valued functions (SVFs, multifunctions), mapping a compact interval $[a, b]$ into the space of compact non-empty subsets of \mathbb{R}^d . All operators are adapted by replacing the Riemann integral for real-valued functions by the weighted metric integral for SVFs of bounded variation with compact graphs. For such a SVF F , we obtain pointwise error estimates for sequences of integral operators at points of continuity, leading to convergence at such points to F . At points of discontinuity of F , we derive estimates, which yield the convergence to a certain set described in terms of the metric selections of F . To obtain these estimates we refine and extend known results on approximation of real-valued functions by integral operators. Our analysis uses recently defined one-sided local quasi-moduli at points of discontinuity and several notions of local Lipschitz property at points of continuity. We also provide a global approach for error bounds. A multifunction F is represented by the set of all its metric selections, while its approximation (its image under the operator) is represented by the set of images of these metric selections under the operator. A bound on the Hausdorff distance between these two sets of single-valued functions in L^1 provides our global estimates. The theory is applied to concrete operators: the Bernstein–Durrmeyer operator and the Kantorovich operator.

Keywords Set-valued functions · Functions of bounded variation · Metric integral · Metric approximation · Integral operators · Positive linear operators · Rate of convergence

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1 Introduction

We study approximation of set-valued functions (SVFs, multifunctions)—functions mapping a compact interval $[a, b] \subset \mathbb{R}$ into the space of compact non-empty subsets of \mathbb{R}^d . These functions appear in different fields such as dynamical systems, control theory, optimization, game theory, differential inclusions, economy, geometric modeling. See the book [2] for foundations of set-valued analysis.

Approximation of SVFs find application in various problems, here we mention two of them. The problem of reconstruction of a 3D object from a finite number of its 2D parallel cross-sections is central in computer tomography. The object may be regarded as a univariate set-valued function and the given cross-sections as its samples. Another possible application appears in parametric optimization problems, where the set of non-unique minimizers depends on the value of the parameter, thus defining a multifunction of the given parameter. This SVF is usually evaluated for a few values of the parameter and should be approximated elsewhere.

Approximation methods for SVFs have been developing in the last decades. Older works deal mostly with convex-valued multifunctions and their approximation based on Minkowski linear combinations, e.g., [5, 9, 13, 15, 24, 25]. In [29], such an adaptation of the classical Bernstein polynomial operator is proved to converge to SVFs with convex compact images (values). Yet, it is shown that this adaptation fails to approximate general SVFs (with general compact not necessarily convex images). Thus approximation methods developed for multifunctions with convex images usually are not suitable for general SVFs.

A first successful attempt to approximate general SVFs from their samples is accomplished by Artstein in [1], where piecewise linear approximants are constructed. This is done by replacing binary Minkowski average of two sets with the metric average, which is further extended in [16] to the metric linear combination of several sets. Based on the metric linear combination, N. Dyn, E. Farkhi and A. Mokhov developed in a series of works [14, 16–18, 20] adaptations of classical sample-based approximation operators to continuous general SVFs. For these adapted operators, termed metric operators, error estimates are obtained, which for most operators are similar to those obtained in the real-valued case. Special attention is given to Bernstein polynomial operators, Schoenberg spline operators and polynomial interpolation operators. Later in [6], the above metric approach is extended to SVFs of bounded variation.

The metric approach is applied in [19] to introduce and study the metric integral for general SVFs of bounded variation. The metric integral is not necessarily convex in contrast to the Aumann integral, which is always convex, even if the integrand is not convex-valued [3]. In [7] the metric integral is extended to the weighted metric integral, which is used, with the Dirichlet kernels as weight functions, to define metric Fourier partial sums for SVFs of bounded variation. The convergence of these partial sums is analyzed at points of continuity of a multifunction as well as at points of discontinuity. An important tool in the analysis at points of discontinuity is the notion of one-sided local quasi-moduli of a function of bounded variation.

In this paper we adapt integral approximation operators for real-valued functions to general SVFs of bounded variation. Previous adaptations of integral operators to SVFs are limited to convex-valued multifunctions and are based on the Aumann integral (see e.g. [4, 10]). Our adaptation is based on the weighted metric integral, and its analysis applies and extends the techniques developed in [7].

The outline of the paper is as follows.

Section 2 gives a short overview of notions we use in the paper, and also discusses different regularity properties of functions with values in a metric space.

In Sect. 3 we obtain pointwise error bounds for the approximation of single-valued functions by integral operators from a rather general class, at points of continuity and discontinuity. These results extend and improve some known results (e.g. [12, 21]). This section provides the tools for the analysis of the case of set-valued functions.

The core part of the paper is Sect. 4, where we construct and study an adaptation of the integral approximation operators from Sect. 3 to SVFs of bounded variation with compact graphs. We investigate the pointwise convergence, in the Hausdorff metric, of sequences of such operators at points of continuity of the multifunction, as well as at points of discontinuity, and derive error bounds.

In Sect. 5 we illustrate our theory by considering examples of two particular integral approximation operators, the Bernstein–Durrmeyer operators and the Kantorovich operators.

In the final Sect. 6 we provide global error bounds. The multifunction F is represented by the set of all its metric selections (see [18] for more information on representations of SVFs), while its approximation (its image under the operator) is represented by the set of images of these metric selections under the operator. A bound of the Hausdorff distance between these two sets of single-valued functions in L^1 is obtained using results from [8].

2 Preliminaries

In this section we introduce some notation and basic notions used in this paper, which are related to sets and set-valued functions. We discuss notions of regularity of functions in metric spaces. We review the notions of metric selections and the weighted metric integral of set-valued functions.

2.1 On Sets

All sets considered from now on are sets in \mathbb{R}^d . We denote by $\mathbf{K}(\mathbb{R}^d)$ the collection of all compact non-empty subsets of \mathbb{R}^d . The metric in \mathbb{R}^d is of the form $\rho(u, v) = |u - v|$, where $|\cdot|$ is any fixed norm on \mathbb{R}^d . Recall that \mathbb{R}^d endowed with this metric is a complete metric space and that all norms on \mathbb{R}^d are equivalent.

To measure the distance between two non-empty sets $A, B \in \mathbb{K}(\mathbb{R}^d)$, we use the Hausdorff metric based on ρ

$$\text{haus}(A, B)_\rho = \max \left\{ \sup_{a \in A} \text{dist}(a, B)_\rho, \sup_{b \in B} \text{dist}(b, A)_\rho \right\},$$

where the distance from a point c to a set D is $\text{dist}(c, D)_\rho = \inf_{d \in D} \rho(c, d)$. It is well known that $\mathbb{K}(\mathbb{R}^d)$ endowed with the Hausdorff metric is a complete metric space [26, 28].

In the following, we keep the metric in \mathbb{R}^d fixed, and omit the notation ρ as a subscript.

We denote by $|A| = \text{haus}(A, \{0\})$ the “norm” of the set $A \in \mathbb{K}(\mathbb{R}^d)$. The set of projections of $a \in \mathbb{R}^d$ on a set $B \in \mathbb{K}(\mathbb{R}^d)$ is $\Pi_B(a) = \{b \in B : |a - b| = \text{dist}(a, B)\}$. A pair (a, b) is called a metric pair of the sets $A, B \in \mathbb{K}(\mathbb{R}^d)$ if

$$(a, b) \in A \times B, \quad a \in \Pi_A(b) \quad \text{or} \quad b \in \Pi_B(a).$$

The collection of all metric pairs of $A, B \in \mathbb{K}(\mathbb{R}^d)$ is denoted by $\Pi(A, B)$.

The notion of metric pairs or similar notions to it appear in several works e.g. [1, 18, 20, 23].

Using metric pairs, we can rewrite

$$\text{haus}(A, B) = \sup\{|a - b| : (a, b) \in \Pi(A, B)\}.$$

We recall the notions of a metric chain and of a metric linear combination [19].

Definition 2.1 [19] Given a finite sequence of sets $A_0, \dots, A_n \in \mathbb{K}(\mathbb{R}^d)$, $n \geq 1$, a metric chain of A_0, \dots, A_n is an $(n + 1)$ -tuple (a_0, \dots, a_n) such that $(a_i, a_{i+1}) \in \Pi(A_i, A_{i+1})$, $i = 0, 1, \dots, n - 1$. The collection of all metric chains of A_0, \dots, A_n is denoted by

$$\text{CH}(A_0, \dots, A_n) = \{(a_0, \dots, a_n) : (a_i, a_{i+1}) \in \Pi(A_i, A_{i+1}), i = 0, 1, \dots, n - 1\}.$$

The metric linear combination of the sets $A_0, \dots, A_n \in \mathbb{K}(\mathbb{R}^d)$, $n \geq 1$, is

$$\bigoplus_{i=0}^n \lambda_i A_i = \left\{ \sum_{i=0}^n \lambda_i a_i : (a_0, \dots, a_n) \in \text{CH}(A_0, \dots, A_n) \right\}, \quad \lambda_0, \dots, \lambda_n \in \mathbb{R}.$$

Remark 2.2 For any $j \in \mathbb{N}$, $0 \leq j \leq n$ and for any $a \in A_j$ there exists a metric chain $(a_0, \dots, a_n) \in \text{CH}(A_0, \dots, A_n)$ such that $a_j = a$. For a possible construction see [16], Figure 3.2.

Note that the metric linear combination depends on the order of the sets, in contrast to the Minkowski linear combination of sets which is defined by

$$\sum_{i=0}^n \lambda_i A_i = \left\{ \sum_{i=0}^n \lambda_i a_i : a_i \in A_i \right\}, \quad n \geq 1.$$

The upper Kuratowski limit of a sequence of sets $\{A_n\}_{n=1}^\infty$ is the set of all limit points of converging subsequences $\{a_{n_k}\}_{k=1}^\infty$, where $a_{n_k} \in A_{n_k}$, $k \in \mathbb{N}$, namely

$$\limsup_{n \rightarrow \infty} A_n = \left\{ a : \exists \{n_k\}_{k=1}^\infty, n_{k+1} > n_k, k \in \mathbb{N}, \exists a_{n_k} \in A_{n_k} \text{ such that } \lim_{k \rightarrow \infty} a_{n_k} = a \right\}.$$

2.2 Notions of Regularity of Functions with Values in a Metric Space

In this paper we consider functions defined on a fixed compact interval $[a, b] \subset \mathbb{R}$ with values in a complete metric space (X, ρ) , where X is either \mathbb{R}^d or $\mathbb{K}(\mathbb{R}^d)$ and ρ is a fixed metric in X .

We recall the notion of the variation of $f : [a, b] \rightarrow X$. Let $\chi = \{x_0, \dots, x_n\}$, $a = x_0 < x_1 < \dots < x_n = b$, be a partition of the interval $[a, b]$ with the norm

$$|\chi| = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i).$$

The variation of f on the partition χ is defined as $V(f, \chi) = \sum_{i=1}^n \rho(f(x_i), f(x_{i-1}))$. The total variation of f on $[a, b]$ is

$$V_a^b(f) = \sup_{\chi} V(f, \chi),$$

where the supremum is taken over all partitions of $[a, b]$.

A function f is said to be of bounded variation on $[a, b]$ if $V_a^b(f) < \infty$. We call functions of bounded variation BV functions and write $f \in \text{BV}[a, b]$. If f is also continuous, we write $f \in \text{CBV}[a, b]$.

For $f \in \text{BV}[a, b]$ the function $v_f : [a, b] \rightarrow \mathbb{R}$, $v_f(x) = V_a^x(f)$ is called the variation function of f . Note that

$$V_z^x(f) = v_f(x) - v_f(z) \quad \text{for } a \leq z < x \leq b,$$

and that v_f is monotone non-decreasing.

For a BV function $f : \mathbb{R} \rightarrow X$ the following property holds (see e.g. Lemma 2.4 in [6]),

$$\int_a^b V_{x-\delta}^{x+\delta}(f) dx \leq 2\delta V_a^b(f). \tag{1}$$

We recall the notion of the local modulus of continuity [27], which is central to the approximation of functions at continuity points.

For $f : [a, b] \rightarrow X$ the local modulus of continuity at $x^* \in [a, b]$ is

$$\begin{aligned} &\omega(f, x^*, \delta) \\ &= \sup \{ \rho(f(x_1), f(x_2)) : x_1, x_2 \in [x^* - \delta/2, x^* + \delta/2] \cap [a, b] \}, \delta > 0. \end{aligned}$$

The following result justifies the uniform convergence of $\omega(f, \cdot, \delta)$ to zero as $\delta \rightarrow 0+$ in a compact interval where f is continuous.

Result 2.3 [22, page 12] *If the function f is defined on $[a, b]$ and $I \subseteq [a, b]$ is a closed interval such that f is continuous at all the points of I , then for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|f(y) - f(x)| < \epsilon$$

for all $x \in I, y \in [a, b]$ and $|y - x| < \delta$.

It follows from the definition of the variation that

Result 2.4 *For a function $f : [a, b] \rightarrow X, f \in \text{BV}[a, b]$,*

$$\omega(f, x^*, \delta) \leq \omega(v_f, x^*, \delta), \quad x^* \in [a, b], \quad \delta > 0.$$

Moreover, f is continuous at $x^* \in [a, b]$ if and only if v_f is continuous at x^* .

Result 2.5 *A function $f : [a, b] \rightarrow X, f \in \text{BV}[a, b]$ is left continuous at $x^* \in (a, b]$ if and only if v_f is left continuous at x^* . The function f is right continuous at $x^* \in [a, b)$ if and only if v_f is right continuous at x^* .*

A function $f : [a, b] \rightarrow X$ of bounded variation with values in a complete metric space (X, ρ) is not necessarily continuous, but has right and left limits at any point x [11]. We denote the one-sided limits by

$$f(x+) = \lim_{t \rightarrow x+0} f(t), \quad f(x-) = \lim_{t \rightarrow x-0} f(t).$$

In [7], we introduced the notion of the left and right local quasi-moduli. For a function $f : [a, b] \rightarrow X$ of bounded variation, the left local quasi-modulus at point x^* is

$$\varpi^-(f, x^*, \delta) = \sup \{ \rho(f(x^*-), f(x)) : x \in [x^* - \delta, x^*) \cap [a, b] \}, \quad \delta > 0, \quad x^* \in (a, b]. \tag{2}$$

Similarly, the right local quasi-modulus is

$$\varpi^+(f, x^*, \delta) = \sup \{ \rho(f(x^*+), f(x)) : x \in (x^*, x^* + \delta] \cap [a, b] \}, \quad \delta > 0, \quad x^* \in [a, b). \tag{3}$$

Clearly, for $f \in \text{BV}[a, b]$ the local quasi-moduli satisfy

$$\lim_{\delta \rightarrow 0+} \varpi^-(f, x^*, \delta) = 0, \quad x^* \in (a, b], \quad \text{and} \quad \lim_{\delta \rightarrow 0+} \varpi^+(f, x^*, \delta) = 0, \quad x^* \in [a, b).$$

Defining

$$\varpi(f, x^*, \delta) = \max\{\varpi^-(f, x^*, \delta), \varpi^+(f, x^*, \delta)\},$$

we obtain

$$\lim_{\delta \rightarrow 0^+} \varpi(f, x^*, \delta) = 0. \tag{4}$$

Lemma 2.6 *Let $f : [a, b] \rightarrow X$, $f \in \text{BV}[a, b]$, then for any $x^* \in (a, b)$ or $[a, b)$, respectively, and $\delta > 0$ we have*

$$\varpi^-(f, x^*, \delta) \leq \varpi^-(v_f, x^*, \delta), \quad \varpi^+(f, x^*, \delta) \leq \varpi^+(v_f, x^*, \delta).$$

Proof The first inequality follows from the fact that for $x^* \in (a, b)$ and $\max\{x^* - \delta, a\} \leq x < x^*$ we have

$$\rho(f(x^*-), f(x)) \leq \lim_{t \rightarrow x^*-0} V_x^t(f) = \lim_{t \rightarrow x^*-0} v_f(t) - v_f(x) = \rho(v_f(x^*-), v_f(x)).$$

Similarly one can show the second inequality. □

Below we discuss several notions of Lipschitz regularity. A function $f : [a, b] \rightarrow X$ is Lipschitz continuous with a constant $\mathcal{L} > 0$, if

$$\rho(f(x), f(y)) \leq \mathcal{L}|x - y|, \quad \forall x, y \in [a, b].$$

Definition 2.7 Let $f : [a, b] \rightarrow X$.

- (a) We say that f is *locally Lipschitz around a point x* with the Lipschitz constant $\mathcal{L} > 0$ if there exists $\delta > 0$ such that

$$\rho(f(x_1), f(x_2)) \leq \mathcal{L}|x_1 - x_2|, \quad \forall x_1, x_2 \in (x - \delta/2, x + \delta/2) \cap [a, b]. \tag{5}$$

- (b) A function f is *locally Lipschitz at a point x* with the Lipschitz constant $\mathcal{L} > 0$ if there exists $\delta > 0$ such that

$$\rho(f(z), f(x)) \leq \mathcal{L}|z - x|, \quad \forall z \in (x - \delta/2, x + \delta/2) \cap [a, b]. \tag{6}$$

We denote by $\text{Lip}\{x, \mathcal{L}\}$ the collection of all functions f satisfying (6).

- (c) A function f is *globally Lipschitz at a point x* if there exists $\mathcal{L} > 0$ such that

$$\rho(f(z), f(x)) \leq \mathcal{L}|z - x|, \quad \forall z \in [a, b]. \tag{7}$$

Remark 2.8 Note that if f is locally Lipschitz around a point x with the Lipschitz constant \mathcal{L} , then $f \in \text{Lip}\{x, \mathcal{L}\}$, but the converse implication does not hold. For example, the function $f(x) = x \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$ is not locally Lipschitz around $x = 0$, while $f \in \text{Lip}\{0, 1\}$.

We say that $f : [a, b] \rightarrow X$ is *bounded* on $[a, b]$ if there exists $y^* \in X$ such that

$$M(f, y^*) = \sup_{x \in [a, b]} \rho(f(x), y^*) < \infty.$$

The following lemmas deal with relations between the above notions.

Lemma 2.9 *If $f \in \text{Lip}\{x, \mathcal{L}\}$ and f is bounded on $[a, b]$, then f is also globally Lipschitz at x .*

Proof By Definition 2.7 (b), there is $\delta > 0$ such that for $|z - x| < \frac{\delta}{2}$ we have $\rho(f(z), f(x)) \leq \mathcal{L}|z - x|$. Also, since f is bounded there is y^* such that

$$\rho(f(z), f(x)) \leq \rho(f(z), y^*) + \rho(y^*, f(x)) \leq 2M(f, y^*).$$

If $|z - x| \geq \frac{\delta}{2}$ we obtain from the above inequality the estimate $\rho(f(z), f(x)) \leq \frac{4M(f, y^*)}{\delta}|z - x|$.

Altogether (7) holds with $\tilde{\mathcal{L}} = \max\left(\mathcal{L}, \frac{4M(f, y^*)}{\delta}\right)$. □

Lemma 2.10 *If $f \in \text{BV}[a, b]$ is locally Lipschitz around a point x with a constant $\mathcal{L} > 0$, then*

- (i) v_f is locally Lipschitz around the point x with the same constant \mathcal{L} ,
- (ii) $v_f \in \text{Lip}\{x, \mathcal{L}\}$.

Proof Assume that f satisfies (5) with some $\delta > 0$. To prove (i), let $x - \frac{\delta}{2} < y < z < x + \frac{\delta}{2}$, then

$$\begin{aligned} v_f(z) - v_f(y) &= V_y^z(f) \\ &= \sup_{\chi} \sum_{j=0}^{n-1} \rho(f(x_j), f(x_{j+1})) \leq \sup_{\chi} \sum_{j=0}^{n-1} \mathcal{L}|x_{j+1} - x_j| = \mathcal{L}|y - z|, \end{aligned}$$

where the supremum is taken over all partitions $\chi = \{y = x_0 < \dots < x_n = z\}$ of $[y, z]$.

By (i) and Remark 2.8 we obtain (ii). □

Lemma 2.11 *Let $f \in \text{BV}[a, b]$.*

- (i) *If v_f is locally Lipschitz around a point x with some $\mathcal{L} > 0$ and $\delta > 0$, then f is locally Lipschitz around x with the same \mathcal{L} and δ .*
- (ii) *If $v_f \in \text{Lip}\{x, \mathcal{L}\}$, then $f \in \text{Lip}\{x, \mathcal{L}\}$.*

Proof (i) For all z, y such that $x - \frac{\delta}{2} < y < z < x + \frac{\delta}{2}$ we have

$$\rho(f(z), f(y)) \leq V_y^z(f) = v_f(z) - v_f(y) \leq \mathcal{L}|z - y|. \tag{8}$$

To prove (ii) we replace either y or z in (8) by x . □

Remark 2.12 The converse implication of Lemma 2.11 (ii) does not hold. Consider, for example, the function $f(x) = x \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Clearly, $f \in \text{Lip}\{0, \mathcal{L}\}$ with $\mathcal{L} = 1$, but $v_f \notin \text{Lip}\{0, \mathcal{L}\}$ for any $\mathcal{L} > 0$.

In the following we consider $f : [a, b] \rightarrow \mathbb{R}^d, f = \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}$. We recall that $|\cdot|$ is a fixed norm on \mathbb{R}^d .

Lemma 2.13 Let $f : [a, b] \rightarrow \mathbb{R}^d$ be Riemann integrable. Then

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Proof Consider a sequence of partitions $\chi_n = \{a = x_0^n < \dots < x_n^n = b\}$ with $\lim_{n \rightarrow \infty} |\chi_n| = 0$. Take some $c_i^n \in [x_{i-1}^n, x_i^n], i = 1, \dots, n$. Using the triangle inequality in the estimate for the Riemann sums we obtain

$$\begin{aligned} & \left| \int_a^b f(x)dx \right| \\ &= \left| \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i^n)(x_i^n - x_{i-1}^n) \right| \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n |f(c_i^n)|(x_i^n - x_{i-1}^n) = \int_a^b |f(x)|dx. \end{aligned}$$

□

We recall the notion of integral moduli of continuity for functions with values in \mathbb{R}^d . We extend a function $f : [a, b] \rightarrow \mathbb{R}^d$ outside $[a, b]$ in a simple way preserving its variation on $[a, b]$

$$f(x) = \begin{cases} f(a), & x < a, \\ f(x), & a \leq x \leq b, \\ f(b), & x > b. \end{cases}$$

For $f : [a, b] \rightarrow \mathbb{R}^d, f = \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}$, we write $f \in L^1[a, b]$ if all its components $f_i,$

$i = 1, \dots, d,$ are in $L^1[a, b]$.

The distance between two functions $f, g \in L^1[a, b]$ is given by

$$\|f - g\|_{L^1} = \int_a^b |f(x) - g(x)|dx.$$

We denote $\|f\|_\infty = \sup_{x \in [a, b]} |f(x)|.$

The first order integral modulus of continuity of $f : [a, b] \rightarrow \mathbb{R}^d$, $f \in L^1[a, b]$ is

$$\vartheta(f, \delta) = \vartheta_1(f, \delta) = \sup_{0 < h \leq \delta} \int_a^b |f(x+h) - f(x)| dx, \quad \delta > 0.$$

The second order integral modulus of continuity is

$$\vartheta_2(f, \delta) = \sup_{0 < h \leq \delta} \int_a^b |f(x+h) - 2f(x) + f(x-h)| dx, \quad \delta > 0.$$

It is easy to see that

$$\vartheta_2(f, \delta) \leq 2\vartheta(f, \delta). \tag{9}$$

Using (1) one can easily obtain that for $f \in \text{BV}[a, b]$

$$\vartheta(f, \delta) \leq \delta V_a^b(f). \tag{10}$$

2.3 Metric Selections and the Weighted Metric Integral of Multifunctions

We consider set-valued functions (SVFs, multifunctions) mapping a compact interval $[a, b] \subset \mathbb{R}$ to $\mathbf{K}(\mathbb{R}^d)$. The graph of a multifunction F is the set of points in \mathbb{R}^{d+1}

$$\text{Graph}(F) = \{(x, y) : y \in F(x), x \in [a, b]\}.$$

It is easy to see that if $F \in \text{BV}[a, b]$ then $\text{Graph}(F)$ is a bounded set and

$$\|F\|_\infty = \sup_{x \in [a, b]} |F(x)| < \infty.$$

We denote the class of SVFs of bounded variation with compact graphs by $\mathcal{F}[a, b]$.

For a set-valued function $F : [a, b] \rightarrow \mathbf{K}(\mathbb{R}^d)$, a single-valued function $s : [a, b] \rightarrow \mathbb{R}^d$ such that $s(x) \in F(x)$ for all $x \in [a, b]$ is called a selection of F .

The notions of the metric selections and of the weighted metric integral are central in our work. We recall their definitions.

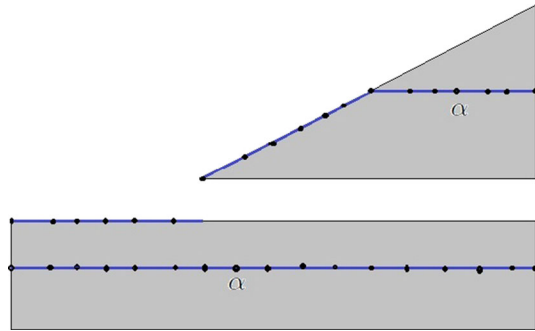
Given a multifunction $F : [a, b] \rightarrow \mathbf{K}(\mathbb{R}^d)$, a partition $\chi = \{x_0, \dots, x_n\} \subset [a, b]$, $a = x_0 < \dots < x_n = b$, and a corresponding metric chain $\phi = (y_0, \dots, y_n) \in \text{CH}(F(x_0), \dots, F(x_n))$ (see Definition 2.1), the *chain function* based on χ and ϕ is

$$c_{\chi, \phi}(x) = \begin{cases} y_i, & x \in [x_i, x_{i+1}), \quad i = 0, \dots, n-1, \\ y_n, & x = x_n. \end{cases}$$

A selection s of F is called a *metric selection*, if there is a sequence of chain functions $\{c_{\chi_k, \phi_k}\}_{k \in \mathbb{N}}$ of F with $\lim_{k \rightarrow \infty} |\chi_k| = 0$ such that

$$s(x) = \lim_{k \rightarrow \infty} c_{\chi_k, \phi_k}(x) \quad \text{pointwisely on } [a, b].$$

Fig. 1 Graph(F) in gray color, two metric selections in blue, two metric chains in black dots



We denote the set of all metric selections of F by $\mathcal{S}(F)$.

Proposition 2.14 *A metric selection s of a multifunction F is constant in any closed interval where the graph of s stays in the interior of Graph(F).*

Proof Denote by $I \subset [a, b]$ a closed interval where the graph of the metric selection s is in the interior of Graph(F), i.e. $\{(x, s(x)) : x \in I\} \subset \text{int}(\text{Graph}(F))$, where int denotes the interior of a set. For $\hat{x} \in I$ let $(\hat{x}, \hat{y}) \in \text{int}(\text{Graph}(F))$. Then for x in a small enough neighborhood of \hat{x} , $\hat{y} \in \text{int}(F(x))$ and $(\hat{y}, \hat{y}) \in \Pi(F(\hat{x}), F(x))$ is the only metric pair containing \hat{y} . Hence, in a small neighborhood of \hat{x} all the metric chains containing \hat{y} are constant with the value \hat{y} . Thus for the given metric selection s with $(\hat{x}, s(\hat{x})) \in \text{int}(\text{Graph}(F))$ there is a small neighborhood of \hat{x} , $O_{s, \hat{x}}$, such that $s(x) = s(\hat{x})$ for $x \in O_{s, \hat{x}}$. Since I is compact and $I \subset \bigcup_{x \in I} O_{s, x}$, it is possible to find for I a finite subcovering of $\bigcup_{x \in I} O_{s, x}$. Since the neighborhoods in this subcovering are open, they are intersecting, and we conclude that s is constant on I . \square

Figure 1 illustrates the above proposition. One can see that the two displayed metric selections of F (one of which is discontinuous) are constant wherever their graphs are in the interior of Graph(F).

Below we quote some results from [19] and [7] which are used in this paper.

Result 2.15 [19, Theorem 3.6] *Let s be a metric selection of $F \in \mathcal{F}[a, b]$. Then $V_a^b(s) \leq V_a^b(F)$ and $\|s\|_\infty \leq \|F\|_\infty$.*

Result 2.16 [19, Corollary 3.7] *Let $F \in \mathcal{F}[a, b]$. Through any point $\alpha \in \text{Graph}(F)$ there exists a metric selection which we denote by s_α . Moreover, F has a representation by metric selections, namely*

$$F(x) = \{s_\alpha(x) : \alpha \in \text{Graph}(F)\}.$$

Result 2.17 [7, Theorem 4.9] *Let $F \in \mathcal{F}[a, b]$, s be a metric selection of F and $x^* \in [a, b]$. Then*

$$\omega(s, x^*, \delta) \leq \omega(v_F, x^*, 2\delta), \quad \delta > 0.$$

In particular, if F is continuous at x^ , then s is continuous at x^* .*

Result 2.18 [7, Theorem 4.13] For $F \in \mathcal{F}[a, b]$, the pointwise limit (if exists) of a sequence of metric selections of F is a metric selection of F .

Result 2.19 [7, Lemmas 6.7, 6.8] Let $F \in \mathcal{F}[a, b]$ and $\delta > 0$. Then any metric selection $s \in \mathcal{S}(F)$ satisfies

$$\begin{aligned} \varpi^-(v_s, x^*, \delta) &\leq \varpi^-(v_F, x^*, 2\delta), \quad x^* \in (a, b], \\ \varpi^+(v_s, x^*, \delta) &\leq \varpi^+(v_F, x^*, \delta), \quad x^* \in [a, b). \end{aligned}$$

A direct consequence of Lemma 2.6 and Result 2.19 is

Corollary 2.20 Let $F \in \mathcal{F}[a, b]$, $s \in \mathcal{S}(F)$ and $\delta > 0$. Then

$$\begin{aligned} \varpi^-(s, x^*, \delta) &\leq \varpi^-(v_F, x^*, 2\delta), \quad x^* \in (a, b], \\ \varpi^+(s, x^*, \delta) &\leq \varpi^+(v_F, x^*, \delta), \quad x^* \in [a, b). \end{aligned}$$

Now we show that the metric selections of F inherit the Lipschitz regularity property at x from v_F .

Lemma 2.21 Let $F \in \mathcal{F}[a, b]$. If $v_F \in \text{Lip}\{x, \mathcal{L}\}$ then $s \in \text{Lip}\{x, 4\mathcal{L}\}$ for all metric selections $s \in \mathcal{S}(F)$.

Proof Since $v_F \in \text{Lip}\{x, \mathcal{L}\}$, there is $\delta > 0$ such that $|v_F(x) - v_F(z)| \leq \mathcal{L}|x - z|$, $\forall z \in (x - \frac{\delta}{2}, x + \frac{\delta}{2}) \cap [a, b]$, implying that $\omega(v_F, x, \eta) \leq \mathcal{L}\eta$ for $\eta \leq \delta$. Thus by Result 2.17 we have for all $z \in (x - \frac{\delta}{4}, x + \frac{\delta}{4}) \cap [a, b]$

$$|s(x) - s(z)| \leq \omega(s, x, 2|x - z|) \leq \omega(v_F, x, 4|x - z|) \leq 4\mathcal{L}|x - z|.$$

□

The metric integral of SVFs is introduced in [19] and extended to the weighted metric integral in [7]. We recall its definition.

For a multifunction $F : [a, b] \rightarrow \mathbf{K}(\mathbb{R}^d)$, a weight function $\kappa : [a, b] \rightarrow \mathbb{R}$ and for a partition $\chi = \{x_0, \dots, x_n\}$, $a = x_0 < x_1 < \dots < x_n = b$, we define the weighted metric Riemann sum of F by

$$\begin{aligned} (\mathcal{M}_\kappa)S_\chi F &= \left\{ \sum_{i=0}^{n-1} (x_{i+1} - x_i)\kappa(x_i)y_i : (y_0, \dots, y_{n-1}) \in \text{CH}(F(x_0), \dots, F(x_{n-1})) \right\} \\ &= \bigoplus_{i=0}^{n-1} (x_{i+1} - x_i)\kappa(x_i)F(x_i). \end{aligned}$$

The *weighted metric integral* of F with the weight function κ is defined as the upper Kuratowski limit of weighted metric Riemann sums

$$(\mathcal{M}_\kappa) \int_a^b \kappa(x)F(x)dx = \limsup_{|\chi| \rightarrow 0} (\mathcal{M}_\kappa)S_\chi F.$$

The set $(\mathcal{M}_\kappa) \int_a^b \kappa(x)F(x)dx$ is non-empty whenever the set-valued function κF has a bounded range.

Result 2.22 [7] *Let $F \in \mathcal{F}[a, b]$ and $\kappa \in \text{BV}[a, b]$. Then the set $(\mathcal{M}_\kappa) \int_a^b \kappa(x)F(x)dx$ is compact and*

$$(\mathcal{M}_\kappa) \int_a^b \kappa(x)F(x)dx = \left\{ \int_a^b \kappa(x)s(x)dx : s \in \mathcal{S}(F) \right\}.$$

Here and in such occasions below we understand that the integral is applied to each

component of $s = \begin{pmatrix} s_1 \\ \vdots \\ s_d \end{pmatrix}$.

3 Pointwise Bounds on the Error for Integral Operators on Real-Valued Functions

In this section we consider a general class of integral operators. For this class of operators convergence results are shown e.g. in [12, Chapter 1]. We obtain pointwise error estimates for any integral operator in this class. Pointwise error bounds at jumps of functions of bounded variation for concrete operators were studied by several authors, see [21] for the Bernstein–Durrmeyer operator and [30] for the Kantorovich operator. We refine the results there and extend them to the general class of integral operators.

Let $\{\mathcal{K}_n(x, t)\}_{n \in \mathbb{N}}$, $\mathcal{K}_n : [a, b] \times [a, b] \rightarrow \mathbb{R}$, be a sequence of functions that are integrable with respect to t for each x . We term the functions \mathcal{K}_n kernels. With the help of the sequence of the kernels we define a sequence of linear integral operators $\{T_n\}_{n \in \mathbb{N}}$ on real-valued functions in $L^\infty[a, b]$ by

$$T_n f(x) = \int_a^b \mathcal{K}_n(x, t) f(t) dt, \quad x \in [a, b], \quad n \in \mathbb{N}. \tag{11}$$

Remark 3.1 If $f : [a, b] \rightarrow \mathbb{R}^d$, then $T_n f$ is obtained by the operation of T_n on each component of f .

Next we introduce the following notation. For $x \in [a, b]$ let

$$\alpha_n(x) = \left| \int_a^b \mathcal{K}_n(x, t) dt - 1 \right| \tag{12}$$

and

$$\beta_n(x, \delta) = \int_{|x-t| \geq \delta} |\mathcal{K}_n(x, t)| dt, \quad \delta > 0. \tag{13}$$

Furthermore, for each $x \in [a, b]$ let $M(x) \in [0, \infty]$ be such that

$$\int_a^b |\mathcal{K}_n(x, t)| dt \leq M(x), \quad n \in \mathbb{N}. \tag{14}$$

3.1 The Case of Continuity Points

The theorem below can be considered as a refinement of Theorem 2.1 in [12, Chapter 1].

Theorem 3.2 *Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded and measurable on $[a, b]$, $x \in [a, b]$ and let $T_n, \alpha_n, \beta_n, M(x)$ be given by (11), (12), (13) and (14) respectively. Then*

(i) *For all $\delta > 0$ and $n \in \mathbb{N}$*

$$|T_n f(x) - f(x)| \leq \omega(f, x, 2\delta)M(x) + \|f\|_\infty 2\beta_n(x, \delta) + |f(x)|\alpha_n(x). \tag{15}$$

(ii) *If $\lim_{n \rightarrow \infty} \alpha_n(x) = 0, \lim_{n \rightarrow \infty} \beta_n(x, \delta) = 0$ for any sufficiently small $\delta > 0, M(x) < \infty,$ and if x is a point of continuity of $f,$ then*

$$\lim_{n \rightarrow \infty} |T_n f(x) - f(x)| = 0. \tag{16}$$

(iii) *If f is continuous at all points of a closed interval $I \subseteq [a, b],$ if $\lim_{n \rightarrow \infty} \alpha_n(x) = 0$ and $\lim_{n \rightarrow \infty} \beta_n(x, \delta) = 0$ uniformly in $x \in I$ for any sufficiently small $\delta > 0,$ and if $M(x)$ is bounded on $I,$ then the convergence is uniform in $I.$*

Proof (i) We have

$$\begin{aligned} & |T_n f(x) - f(x)| \\ &= \left| \int_a^b \mathcal{K}_n(x, t) f(t) dt - \int_a^b \mathcal{K}_n(x, t) f(x) dt + f(x) \int_a^b \mathcal{K}_n(x, t) dt - f(x) \right| \\ &\leq \int_a^b |\mathcal{K}_n(x, t)| |f(t) - f(x)| dt + |f(x)| \left| \int_a^b \mathcal{K}_n(x, t) dt - 1 \right| \\ &= \int_{|x-t| \leq \delta} |\mathcal{K}_n(x, t)| |f(t) - f(x)| dt \\ &\quad + \int_{|x-t| > \delta} |\mathcal{K}_n(x, t)| |f(t) - f(x)| dt + |f(x)|\alpha_n(x). \end{aligned}$$

Using the fact that $|f(t) - f(x)| \leq \omega(f, x, 2\delta)$ in the first integral and that $|f(t) - f(x)| \leq 2\|f\|_\infty,$ we obtain

$$\begin{aligned} & |T_n f(x) - f(x)| \\ &\leq \omega(f, x, 2\delta) \int_{|x-t| \leq \delta} |\mathcal{K}_n(x, t)| dt + 2\|f\|_\infty \int_{|x-t| > \delta} |\mathcal{K}_n(x, t)| dt + |f(x)|\alpha_n(x) \\ &\leq \omega(f, x, 2\delta)M(x) + 2\|f\|_\infty \beta_n(x, \delta) + |f(x)|\alpha_n(x) \end{aligned}$$

which gives (15).

(ii) Fix $\delta > 0$ and let $n \rightarrow \infty$ in (15). With the assumption of (ii), it follows from (15) that

$$\limsup_{n \rightarrow \infty} |T_n f(x) - f(x)| \leq \omega(f, x, 2\delta)M(x).$$

Since this is valid for each $\delta > 0$ and since $\lim_{\delta \rightarrow 0+} \omega(f, x, 2\delta) = 0$ when x is a point of continuity, we obtain (16).

(iii) By Result 2.3, $\omega(f, x, 2\delta)$ tends to zero uniformly in $x \in I$ when $\delta \rightarrow 0+$ and it follows from (15) that the convergence is uniform in I . □

In view of Lemma 2.13, the estimates in the proof of Theorem 3.2 can be repeated verbatim for $f : [a, b] \rightarrow \mathbb{R}^d, f \in \text{BV}[a, b]$, since such f is Riemann integrable. We obtain

Corollary 3.3 *If $f : [a, b] \rightarrow \mathbb{R}^d, f \in \text{BV}[a, b]$ and $\mathcal{K}_n(x, \cdot) \in \text{BV}[a, b]$ for each $x \in [a, b]$, then the estimates in Theorem 3.2 hold.*

Remark 3.4 It is easy to see that if $f : [a, b] \rightarrow \mathbb{R}^d$ is bounded and measurable, then (15) becomes

$$\begin{aligned} |T_n f(x) - f(x)| &\leq C\omega(f, x, 2\delta)M(x) + C\|f\|_\infty 2\beta_n(x, \delta) \\ &\quad + C|f(x)|\alpha_n(x), \quad x \in [a, b], n \in \mathbb{N}, \delta > 0, \end{aligned}$$

where C depends only on the underlying norm $|\cdot|$ in \mathbb{R}^d .

3.2 The Case of Discontinuity Points

In this section we follow and refine the analysis in [21] for real-valued functions.

Let $f : [a, b] \rightarrow \mathbb{R}, f \in \text{BV}[a, b]$. Fix $x \in (a, b)$, following [21] we introduce the function $g_x : [a, b] \rightarrow \mathbb{R}$,

$$g_x(t) = \begin{cases} f(t) - f(x-), & t \in [a, x), \\ f(t) - f(x) = 0, & t = x, \\ f(t) - f(x+), & t \in (x, b]. \end{cases}$$

Note that $f \in \text{BV}[a, b]$ implies that g_x is well defined and is continuous at x . It is easy to check that

$$f(t) = g_x(t) + f(x-)\chi_{[a,x)}(t) + f(x+)\chi_{(x,b]}(t) + f(x)\chi_{\{x\}}(t), \quad t \in [a, b],$$

where χ_A denotes the characteristic function of a set A

$$\chi_A(t) = \begin{cases} 1, & t \in A, \\ 0, & t \notin A. \end{cases}$$

A simple computation shows that for $t \in [a, b]$

$$\begin{aligned}
 f(t) - \frac{1}{2}[f(x+) + f(x-)] &= g_x(t) + \frac{1}{2}[f(x+) - f(x-)](\chi_{(x,b]}(t) - \chi_{[a,x)}(t)) \\
 &\quad + \left(f(x) - \frac{1}{2}[f(x+) + f(x-)] \right) \chi_{\{x\}}(t) \\
 &= g_x(t) + \frac{1}{2}[f(x+) - f(x-)]\text{sign}(t - x) \\
 &\quad + \left(f(x) - \frac{1}{2}[f(x+) + f(x-)] \right) \chi_{\{x\}}(t),
 \end{aligned}$$

where

$$\text{sign}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

Inserting this representation into the operator (11), we obtain

$$\begin{aligned}
 T_n f(x) - \frac{1}{2}[f(x+) + f(x-)] &\int_a^b \mathcal{K}_n(x, t) dt \\
 &= T_n g_x(x) + \frac{1}{2}[f(x+) - f(x-)]T_n(\text{sign}(\cdot - x))(x). \tag{17}
 \end{aligned}$$

Taking into account that $g_x(x) = 0$, the estimate (15) for $T_n g_x(x)$ takes the form

$$|T_n g_x(x)| = |T_n g_x(x) - g_x(x)| \leq \omega(g_x, x, 2\delta)M(x) + 2\|g_x\|_\infty \beta_n(x, \delta). \tag{18}$$

By the definition of the function g_x we easily see that $\|g_x\|_\infty \leq 2\|f\|_\infty$. By the definition of the local modulus of continuity and the local quasi-moduli (see Sect. 2.2), we get

$$\omega(g_x, x, 2\delta) \leq 2 \max \{ \varpi^-(f, x, \delta), \varpi^+(f, x, \delta) \} = 2\varpi(f, x, \delta). \tag{19}$$

Clearly, $\frac{1}{2}|f(x+) - f(x-)| \leq \|f\|_\infty$, since $f \in \text{BV}[a, b]$. Combining this with (17), (18), (19) we arrive at

$$\begin{aligned}
 &\left| T_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \int_a^b \mathcal{K}_n(x, t) dt \right| \\
 &\leq 2\varpi(f, x, \delta)M(x) + \|f\|_\infty (4\beta_n(x, \delta) + |T_n(\text{sign}(\cdot - x))(x)|). \tag{20}
 \end{aligned}$$

Thus we obtain the following result,

Theorem 3.5 *Let $f : [a, b] \rightarrow \mathbb{R}$, $f \in \text{BV}[a, b]$, $x \in (a, b)$ and let $T_n, \alpha_n, \beta_n, M(x)$ be given by (11), (12), (13) and (14) respectively.*

(i) For all $\delta > 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} & \left| T_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| \\ & \leq 2\varpi(f, x, \delta)M(x) + \|f\|_\infty (4\beta_n(x, \delta) + \alpha_n(x) + |T_n(\text{sign}(\cdot - x))(x)|). \end{aligned}$$

(ii) If $\lim_{n \rightarrow \infty} \alpha_n(x) = 0$, $\lim_{n \rightarrow \infty} \beta_n(x, \delta) = 0$ for any sufficiently small $\delta > 0$, $\lim_{n \rightarrow \infty} T_n(\text{sign}(\cdot - x))(x) = 0$, and $M(x) < \infty$, then

$$\lim_{n \rightarrow \infty} \left| T_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| = 0.$$

Proof By the triangle inequality and (20) we have

$$\begin{aligned} & \left| T_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| \\ & \leq \left| T_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \int_a^b \mathcal{K}_n(x, t) dt \right| \\ & \quad + \frac{1}{2} |f(x+) + f(x-)| \left| \int_a^b \mathcal{K}_n(x, t) dt - 1 \right| \\ & \leq 2\varpi(f, x, \delta)M(x) + \|f\|_\infty (4\beta_n(x, \delta) + |T_n(\text{sign}(\cdot - x))(x)|) + \|f\|_\infty \alpha_n(x), \end{aligned}$$

which leads to the first claim. The second claim follows directly from it by applying (4). □

Similarly to Corollary 3.3 for vector-valued functions we obtain

Corollary 3.6 Let $f : [a, b] \rightarrow \mathbb{R}^d$, $f \in \text{BV}[a, b]$, $\mathcal{K}_n(x, \cdot) \in \text{BV}[a, b]$ for each $z \in [a, b]$, then for $x \in (a, b)$

$$\begin{aligned} & \left| T_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| \\ & \leq 2\varpi(f, x, \delta)M(x) + \|f\|_\infty (4\beta_n(x, \delta) + \alpha_n(x) + |T_n(\text{sign}(\cdot - x))(x)|). \end{aligned}$$

4 Pointwise Bounds on the Error for Integral Operators on Set-Valued Functions

Let $F \in \mathcal{F}[a, b]$ and $\mathcal{K}_n(x, \cdot) \in \text{BV}[a, b]$ for any $x \in [a, b]$. Using the concept of the weighted metric integral (see Sect. 2.3), we define

$$T_n F(x) = (\mathcal{M}_{\mathcal{K}_n}) \int_a^b \mathcal{K}_n(x, t) F(t) dt, \quad x \in [a, b], \quad n \in \mathbb{N}, \quad (21)$$

where $\mathcal{K}_n(x, \cdot)$ is a weight function. By Result 2.22 we have for $F \in \mathcal{F}[a, b]$

$$T_n F(x) = \left\{ \int_a^b \mathcal{K}_n(x, t) s(t) dt : s \in \mathcal{S}(F) \right\} = \{T_n s(x) : s \in \mathcal{S}(F)\}, x \in [a, b], \tag{22}$$

where $\mathcal{S}(F)$ is the set of metric selections of F . By Result 2.16 we have as well

$$F(x) = \{s(x) : s \in \mathcal{S}(F)\}, x \in [a, b]. \tag{23}$$

It is easy to obtain from (22) and (23) that

$$\text{haus}(T_n F(x), F(x)) \leq \sup \{|T_n s(x) - s(x)| : s \in \mathcal{S}(F)\}. \tag{24}$$

The arguments leading to (24) are similar to those in the proof of Lemma 6.3 below.

Remark 4.1 Any bounded linear operator defined for single-valued functions can be extended to set-valued functions from the class $\mathcal{F}[a, b]$ by

$$T F(x) = \{T s(x) : s \in \mathcal{S}(F)\}, x \in [a, b], F \in \mathcal{F}[a, b].$$

This approach was used for operators T defined on continuous real-valued functions in [18, Section 8.2].

4.1 The Case of Continuity Points

Recall that by Results 2.15 and 2.17 for each selection $s \in \mathcal{S}(F)$ we have

$$\|s\|_\infty \leq \|F\|_\infty \quad \text{and} \quad \omega(s, x, \delta) \leq \omega(v_F, x, 2\delta), x \in [a, b], \delta > 0.$$

Thus for each $s \in \mathcal{S}(F)$ the estimate in Corollary 3.3 turns into

$$|T_n s(x) - s(x)| \leq \omega(v_F, x, 4\delta)M(x) + \|F\|_\infty(2\beta_n(x, \delta) + \alpha_n(x)). \tag{25}$$

We arrive at

Theorem 4.2 *Let $F \in \mathcal{F}[a, b]$, $x \in [a, b]$, $\{\mathcal{K}_n(x, t)\}_{n \in \mathbb{N}}$ be a sequence of kernels on $[a, b] \times [a, b]$ such that $\mathcal{K}_n(x, \cdot) \in \text{BV}[a, b]$ for any $x \in [a, b]$ and let $T_n, \alpha_n, \beta_n, M(x)$ be defined by (11), (12), (13) and (14) respectively. Then*

(i) *For all $\delta > 0$ and $n \in \mathbb{N}$*

$$\text{haus}(T_n F(x), F(x)) \leq \omega(v_F, x, 4\delta)M(x) + \|F\|_\infty(2\beta_n(x, \delta) + \alpha_n(x)), x \in [a, b].$$

(ii) *If $\lim_{n \rightarrow \infty} \alpha_n(x) = 0, \lim_{n \rightarrow \infty} \beta_n(x, \delta) = 0$ for any sufficiently small $\delta > 0, M(x) < \infty$ and if x is a point of continuity of F , then*

$$\lim_{n \rightarrow \infty} \text{haus}(T_n F(x), F(x)) = 0.$$

(iii) If F is continuous at all points of a closed interval $I \subseteq [a, b]$, if $\lim_{n \rightarrow \infty} \alpha_n(x) = 0$ and $\lim_{n \rightarrow \infty} \beta_n(x, \delta) = 0$ uniformly in $x \in I$ for any sufficiently small $\delta > 0$, and if $M(x)$ is bounded on I , then the convergence is uniform in I .

Proof The statements (i) and (ii) follow from the first two claims of Theorem 3.2 combined with (24) and (25). The proof of the statements (iii) is based on Result 2.4, which implies that v_F is uniformly continuous on compact intervals, and therefore $\omega(v_F, x, 4\delta) \rightarrow 0$ uniformly for $x \in I$. □

4.2 The Case of Discontinuity Points

For $x \in (a, b)$ we define the set (see also [7, Section 6.3])

$$A_F(x) = \left\{ \frac{1}{2} (s(x+) + s(x-)) : s \in \mathcal{S}(F) \right\}.$$

Similarly to (24) we have

$$\text{haus}(T_n F(x), A_F(x)) \leq \sup \left\{ \left| T_n s(x) - \frac{1}{2} (s(x+) + s(x-)) \right| : s \in \mathcal{S}(F) \right\}. \tag{26}$$

Recall that by Result 2.15, $\|s\|_\infty \leq \|F\|_\infty$ and $\frac{1}{2}|s(x+) + s(x-)| \leq \|F\|_\infty$. In view of Corollary 2.20 we have for each $s \in \mathcal{S}(F)$

$$\varpi(s, x, \delta) \leq \varpi(v_F, x, 2\delta).$$

By Corollary 3.6 and the above inequality we have for any $s \in \mathcal{S}(F)$

$$\begin{aligned} & \left| T_n s(x) - \frac{1}{2} [s(x+) + s(x-)] \right| \\ & \leq 2\varpi(v_F, x, 2\delta)M(x) + \|F\|_\infty (4\beta_n(x, \delta) + \alpha_n(x) + |T_n(\text{sign}(\cdot - x))(x)|). \end{aligned}$$

This together with (26), by arguments as in the proof of (ii) of Theorem 3.2, leads to the following result

Theorem 4.3 *Let $F \in \mathcal{F}[a, b]$ and let $\{\mathcal{K}_n(x, t)\}_{n \in \mathbb{N}}$ be a sequence of kernels on $[a, b] \times [a, b]$ such that $\mathcal{K}_n(x, \cdot) \in \text{BV}[a, b]$ for any $x \in [a, b]$ and let $T_n, \alpha_n, \beta_n, M(x)$ be defined by (11), (12), (13) and (14) respectively. Then for any $x \in (a, b)$ the following statements hold.*

(i) For all $\delta > 0$ and $n \in \mathbb{N}$

$$\begin{aligned} & \text{haus}(T_n F(x), A_F(x)) \\ & \leq 2\varpi(v_F, x, 2\delta)M(x) + \|F\|_\infty (4\beta_n(x, \delta) + \alpha_n(x) + |T_n(\text{sign}(\cdot - x))(x)|). \end{aligned}$$

(ii) If $\lim_{n \rightarrow \infty} \alpha_n(x) = 0$, $\lim_{n \rightarrow \infty} \beta_n(x, \delta) = 0$ for any sufficiently small $\delta > 0$, $\lim_{n \rightarrow \infty} T_n(\text{sign}(\cdot - x))(x) = 0$, and $M(x) < \infty$, then

$$\lim_{n \rightarrow \infty} \text{haus}(T_n F(x), A_F(x)) = 0.$$

5 Specific Operators

Here we apply the results of the previous sections to two specific integral operators. In particular, at points of discontinuity we obtain error estimates that combine ideas from [21, 30] with the local quasi-moduli of continuity (2), (3).

5.1 Bernstein–Durrmeyer Operators

For $x \in [0, 1]$, $n \in \mathbb{N}$ the Bernstein basis polynomials are defined as

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \dots, n. \tag{27}$$

Note that $\sum_{k=0}^n p_{n,k}(x) = 1$ and $\int_0^1 p_{n,k}(x) dx = \frac{1}{n+1}$.

The Bernstein–Durrmeyer operator is defined for $f \in L^1[0, 1]$ by

$$\begin{aligned} M_n f(x) &= (n+1) \sum_{k=0}^n p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt \\ &= \int_0^1 \mathcal{K}_n(x, t) f(t) dt, \quad x \in [0, 1], n \in \mathbb{N}, \end{aligned}$$

where

$$\mathcal{K}_n(x, t) = (n+1) \sum_{k=0}^n p_{n,k}(x) p_{n,k}(t). \tag{28}$$

The Bernstein–Durrmeyer operator for a set-valued function $F \in \mathcal{F}[0, 1]$ is

$$M_n F(x) = (\mathcal{M}_{\mathcal{K}_n}) \int_0^1 \mathcal{K}_n(x, t) F(t) dt, \quad x \in [0, 1], \quad n \in \mathbb{N},$$

where $\mathcal{K}_n(x, \cdot)$ is the weight function given in (28).

Now we specify $\alpha_n, \beta_n, M(x)$ in (12), (13) and (14) for the case of the Bernstein–Durrmeyer operator. The properties of the Bernstein basis polynomials $p_{n,k}$ yield

$$\mathcal{K}_n(x, t) \geq 0, \quad \int_0^1 \mathcal{K}_n(x, t) dt = 1,$$

so that $\alpha_n(x) = 0$ and $M(x) = 1, x \in [0, 1]$.

Denote $e_i(t) = t^i, i = 0, 1, 2$. A direct calculation shows that (see e.g. [21])

$$M_n e_0(x) = 1, \quad M_n e_1(x) = \frac{nx + 1}{n + 2}, \quad M_n e_2(x) = \frac{n(n - 1)x^2 + 4nx + 2}{(n + 2)(n + 3)}.$$

This implies

$$M_n((\cdot - x)^2)(x) = \frac{2[(n - 3)x(1 - x) + 1]}{(n + 2)(n + 3)}. \tag{29}$$

Following S. Guo (see Lemma 6 in [21]), we estimate $\beta_n(x, \delta)$ as follows:

$$\begin{aligned} \beta_n(x, \delta) &= \int_{|x-t|\geq\delta} \mathcal{K}_n(x, t) dt \leq \int_{|x-t|\geq\delta} \left(\frac{x-t}{\delta}\right)^2 \mathcal{K}_n(x, t) dt \\ &\leq \frac{1}{\delta^2} \int_0^1 (x-t)^2 \mathcal{K}_n(x, t) dt = \frac{1}{\delta^2} M_n((\cdot - x)^2)(x) \\ &= \frac{1}{\delta^2} \frac{2[(n - 3)x(1 - x) + 1]}{(n + 2)(n + 3)}. \end{aligned}$$

Using the inequality $x(1 - x) \leq \frac{1}{4}, x \in [0, 1]$, we arrive at the estimate $\beta_n(x, \delta) \leq \frac{n + 1}{2\delta^2(n + 2)(n + 3)} \leq \frac{1}{2n\delta^2}$. Thus, for the Bernstein–Durrmeyer operator we have

$$M(x) = 1, \alpha_n(x) = 0, \beta_n(x, \delta) \leq \frac{1}{2n\delta^2}, \quad x \in [0, 1].$$

The case of continuity points

The assumptions of Theorem 3.2 are fulfilled for the Bernstein–Durrmeyer operator. Thus we obtain the following result, where part (i), which provides rate of convergence, is new in this form, while parts (ii) and (iii) are already known.

Corollary 5.1 *Let $f \in L^1[0, 1]$ be bounded on $[0, 1], x \in [0, 1]$. Then*

(i) *For all $\delta > 0$ and $n \in \mathbb{N}$*

$$|M_n f(x) - f(x)| \leq \omega(f, x, 2\delta) + \|f\|_\infty \frac{1}{n\delta^2}.$$

(ii) *If x is a point of continuity of f , then $\lim_{n \rightarrow \infty} |M_n f(x) - f(x)| = 0$.*

(iii) If f is continuous at all points of a closed interval $I \subseteq [0, 1]$, then the convergence is uniform in I .

From Theorem 4.2 for the Bernstein–Durrmeyer operator we get

Corollary 5.2 Let $F \in \mathcal{F}[0, 1]$, $x \in [0, 1]$. Then

(i) For all $\delta > 0$ and $n \in \mathbb{N}$

$$\text{haus}(M_n F(x), F(x)) \leq \omega(v_F, x, 4\delta) + \|F\|_\infty \frac{1}{n\delta^2}.$$

(ii) If x is a point of continuity of F , then $\lim_{n \rightarrow \infty} \text{haus}(M_n F(x), F(x)) = 0$.

(iii) If F is continuous at all points of a closed interval $I \subseteq [0, 1]$, then the convergence is uniform in I .

The case of discontinuity points

Theorem 3.5 for the Bernstein–Durrmeyer operator leads to

$$\begin{aligned} & \left| M_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| \\ & \leq 2\varpi(f, x, \delta) + \|f\|_\infty \left(\frac{2}{n\delta^2} + |M_n(\text{sign}(\cdot - x))(x)| \right) \end{aligned}$$

for $x \in (0, 1)$. S. Guo proved in [21] (see Proof of the Theorem) that

$$|M_n(\text{sign}(\cdot - x))(x)| \leq \frac{13}{2\sqrt{nx(1-x)}}$$

for $x \in (0, 1)$ and n large enough. Thus we get

Corollary 5.3 Let $f : [0, 1] \rightarrow \mathbb{R}$, $f \in \text{BV}[a, b]$ and $x \in (0, 1)$. Then

(i) For all $\delta > 0$ and $n \in \mathbb{N}$ large enough

$$\left| M_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| \leq 2\varpi(f, x, \delta) + \|f\|_\infty \left(\frac{2}{n\delta^2} + \frac{13}{2\sqrt{nx(1-x)}} \right),$$

(ii) $\lim_{n \rightarrow \infty} \left| M_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| = 0$.

Note that (ii) and an estimate for the order of convergence are obtained by S. Guo in [21].

Theorem 4.3 for the Bernstein–Durrmeyer operator results in

Corollary 5.4 Let $F \in \mathcal{F}[0, 1]$, $x \in (0, 1)$. Then

(i) For all $\delta > 0$ and large enough $n \in \mathbb{N}$

$$\text{haus}(M_n F(x), A_F(x)) \leq 2\varpi(v_F, x, 2\delta) + \|F\|_\infty \left(\frac{2}{n\delta^2} + \frac{13}{2\sqrt{nx(1-x)}} \right).$$

(ii) $\lim_{n \rightarrow \infty} \text{haus}(M_n F(x), A_F(x)) = 0$.

5.2 Kantorovich Operators

For $f \in L^1[0, 1]$ the Kantorovich operator is defined by

$$\begin{aligned}
 K_n f(x) &= (n + 1) \sum_{k=0}^n p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \\
 &= \int_0^1 \mathcal{K}_n(x, t) f(t) dt, \quad x \in [0, 1], \quad n \in \mathbb{N},
 \end{aligned}$$

where

$$\mathcal{K}_n(x, t) = (n + 1) \sum_{k=0}^n p_{n,k}(x) \chi_{\left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]}(t)$$

and $p_{n,k}(x)$ are defined in (27).

By formula (21) we can extend this operator to set-valued functions $F \in \mathcal{F}[0, 1]$. Now we specify $\alpha_n, \beta_n, M(x)$ in (12), (13) and (14) for this operator. Using the properties of $p_{n,k}(x)$, we obtain, as in the case of the Bernstein–Durrmeyer operator, that $\mathcal{K}_n(x, t) \geq 0, \int_0^1 \mathcal{K}_n(x, t) dt = 1$, leading to $\alpha_n(x) = 0$ and $M(x) = 1, x \in [0, 1]$.

Calculating $K_n e_i(x), i = 0, 1, 2$ one can get (see e.g. [30])

$$K_n e_0(x) = 1, \quad K_n e_1(x) = \frac{2nx + 1}{2(n + 1)}, \quad K_n e_2(x) = \frac{3n(n - 1)x^2 + 6nx + 1}{3(n + 1)^2},$$

and

$$K_n((\cdot - x)^2)(x) = \frac{3(n - 1)x(1 - x) + 1}{3(n + 1)^2}. \tag{30}$$

Thus, estimating $\beta_n(x, \delta)$ as in the case of the Bernstein–Durrmeyer operator, we obtain

$$\beta_n(x, \delta) \leq \frac{1}{\delta^2} K_n((\cdot - x)^2)(x) = \frac{1}{\delta^2} \frac{3(n - 1)x(1 - x) + 1}{3(n + 1)^2} \leq \frac{1}{4n\delta^2}.$$

The case of continuity points

Theorem 3.2 takes for the Kantorovich operator the following form, where part (i) is new in this form (providing rate of convergence), while parts (ii) and (iii) are well-known.

Corollary 5.5 *Let $f \in L^1[0, 1]$ be bounded on $[0, 1], x \in [0, 1]$. Then*

(i) *For all $\delta > 0$ and $n \in \mathbb{N}$*

$$|K_n f(x) - f(x)| \leq \omega(f, x, 2\delta) + \|f\|_\infty \frac{1}{2n\delta^2}.$$

- (ii) If x is a point of continuity of f , then $\lim_{n \rightarrow \infty} |K_n f(x) - f(x)| = 0$.
- (iii) If f is continuous at all points of a closed interval $I \subseteq [0, 1]$, then the convergence is uniform in I .

Applying Theorem 4.2, we get

Corollary 5.6 *Let $F \in \mathcal{F}[0, 1]$, $x \in [0, 1]$. Then*
 (i) *For all $\delta > 0$ and $n \in \mathbb{N}$*

$$\text{haus}(K_n F(x), F(x)) \leq \omega(v_F, x, 4\delta) + \|F\|_\infty \frac{1}{2n\delta^2}.$$

- (ii) *If x is a point of continuity of F , then $\lim_{n \rightarrow \infty} \text{haus}(K_n F(x), F(x)) = 0$.*
- (iii) *If F is continuous at all points of a closed interval $I \subseteq [0, 1]$, then the convergence is uniform in I .*

The case of discontinuity points

Theorem 3.5 for the Kantorovich operator gives the estimate

$$\begin{aligned} & \left| K_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| \\ & \leq 2\varpi(f, x, \delta) + \|f\|_\infty \left(\frac{1}{n\delta^2} + |K_n(\text{sign}(\cdot - x))(x)| \right) \end{aligned}$$

for $x \in (0, 1)$. An estimate for $|K_n(\text{sign}(\cdot - x))(x)|$ is given in [30]. Closely following the consideration of Zeng and Piriou in [30], we derive here a very similar estimate in a slightly different form. We also replace the estimate $p_{n,k}(x) \leq \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{nx(1-x)}}$ used in [30] and quoted there from an unpublished paper, by the estimate

$$p_{n,k}(x) \leq \frac{5}{2\sqrt{nx(1-x)}} \tag{31}$$

of Guo, which is published in [21] with a full proof.

Let $x \in \left[\frac{\ell}{n+1}, \frac{\ell+1}{n+1} \right)$ with a certain $\ell \in \{0, 1, \dots, n\}$, then

$$\begin{aligned} K_n(\text{sign}(\cdot - x))(x) &= (n+1) \sum_{k=0}^n p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \text{sign}(t-x) dt \\ &= - \sum_{k=0}^{\ell-1} p_{n,k}(x) + \sum_{k=\ell+1}^n p_{n,k}(x) + p_{n,\ell}(x)(n+1) \\ &\quad \times \left(\frac{\ell+1}{n+1} - x - \left(x - \frac{\ell}{n+1} \right) \right) \\ &= 2 \sum_{k=\ell+1}^n p_{n,k}(x) - 1 + 2p_{n,\ell}(x)[(\ell+1) - (n+1)x], \end{aligned}$$

and, since $\ell \leq (n + 1)x < \ell + 1$,

$$|K_n(\text{sign}(\cdot - x))(x)| \leq 2 \left| \sum_{(n+1)x < k \leq n} p_{n,k}(x) - \frac{1}{2} \right| + 2p_{n,\ell}(x).$$

It was proved in [30] (see proof of Lemma 2) that

$$\left| \sum_{nx < k \leq n} p_{n,k}(x) - \frac{1}{2} \right| \leq \frac{0.8(2x^2 - 2x + 1)}{\sqrt{nx(1-x)}} < \frac{1}{\sqrt{nx(1-x)}},$$

where the last inequality is easy to check. Combining this with (31), we obtain

$$\left| \sum_{(n+1)x < k \leq n} p_{n,k}(x) - \frac{1}{2} \right| \leq \frac{7}{2\sqrt{nx(1-x)}}$$

and finally

$$|K_n(\text{sign}(\cdot - x))(x)| \leq \frac{12}{\sqrt{nx(1-x)}}$$

for $x \in (0, 1)$. Thus we obtain

Corollary 5.7 *Let $f : [0, 1] \rightarrow \mathbb{R}$, $f \in \text{BV}[a, b]$ and $x \in (0, 1)$. Then*

(i) *For all $\delta > 0$ and $n \in \mathbb{N}$*

$$\left| K_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| \leq 2\varpi(f, x, \delta) + \|f\|_\infty \left(\frac{1}{n\delta^2} + \frac{12}{\sqrt{nx(1-x)}} \right),$$

(ii) $\lim_{n \rightarrow \infty} \left| K_n f(x) - \frac{1}{2}[f(x+) + f(x-)] \right| = 0.$

Note that (ii) is known (see [30] and the bibliography therein), this is also the case for the order of convergence $1/\sqrt{n}$.

Theorem 4.3 for the Kantorovich operator takes the following form.

Corollary 5.8 *Let $F \in \mathcal{F}[0, 1]$, $x \in (0, 1)$. Then*

(i) *For all $\delta > 0$ and $n \in \mathbb{N}$*

$$\text{haus}(K_n F(x), A_F(x)) \leq 2\varpi(v_F, x, 2\delta) + \|F\|_\infty \left(\frac{1}{n\delta^2} + \frac{12}{\sqrt{nx(1-x)}} \right).$$

(ii) $\lim_{n \rightarrow \infty} \text{haus}(K_n F(x), A_F(x)) = 0.$

5.3 Rate of Convergence of the Bernstein–Durrmeyer Operators and the Kantorovich Operators for Locally Lipschitz Functions

In this section the sequence $\{T_n\}$ is the sequence of Bernstein–Durrmeyer operators $\{M_n\}$ or the sequence of Kantorovich operators $\{K_n\}$.

Using the notion of locally Lipschitz functions we can state the following result.

Theorem 5.9 *Let $F \in \mathcal{F}[0, 1]$ be locally Lipschitz around a point $x \in [0, 1]$. Then*

$$\text{haus}(T_n F(x), F(x)) = O\left(\frac{1}{\sqrt{n}}\right), \quad n \rightarrow \infty.$$

Proof By (24)

$$\text{haus}(T_n F(x), F(x)) \leq \sup\{|T_n s(x) - s(x)| : s \in \mathcal{S}(F)\}.$$

Since F is Lipschitz around the point x with some $\mathcal{L} > 0$, by (ii) of Lemma 2.10, $v_F \in \text{Lip}\{x, \mathcal{L}\}$ and by Lemma 2.21 each metric selection $s \in \mathcal{S}(F)$ satisfies $s \in \text{Lip}\{x, 4\mathcal{L}\}$. By Lemma 2.9 there exists $\tilde{\mathcal{L}} > 0$ such that

$$|s(z) - s(x)| \leq \tilde{\mathcal{L}}|z - x|, \quad \forall z \in [0, 1]. \tag{32}$$

Using Lemma 2.13, (32), the Cauchy-Schwarz inequality and (29) for $\{M_n\}$ or (30) for $\{K_n\}$, we get

$$\begin{aligned} |T_n s(x) - s(x)| &\leq \int_0^1 |s(t) - s(x)| \mathcal{K}_n(x, t) dt \leq \tilde{\mathcal{L}} \int_0^1 |t - x| \mathcal{K}_n(x, t) dt \\ &\leq \tilde{\mathcal{L}} \left(\int_0^1 |t - x|^2 \mathcal{K}_n(x, t) dt \right)^{\frac{1}{2}} \left(\int_0^1 \mathcal{K}_n(x, t) dt \right)^{\frac{1}{2}} \\ &= O\left(\frac{1}{\sqrt{n}}\right), \quad n \rightarrow \infty, \end{aligned}$$

and the statement follows. □

6 Approximation of the Set of Metric Selections in $L^1[a, b]$

In this section we consider the sequence of operators $\{T_n\}$ defined in (11) and study two sets of functions, $\mathcal{S}(F)$ and the set

$$T_n \mathcal{S}(F) = \{T_n s : s \in \mathcal{S}(F)\}.$$

Note that for $F \in \mathcal{F}[a, b]$, the set $\mathcal{S}(F) \subset L^1[a, b]$, since it consists of functions of bounded variation. We assume that $\mathcal{K}_n(\cdot, \cdot) \in C([a, b]^2)$. This condition guaranties that $T_n : \mathcal{S}(F) \rightarrow C[a, b] \subset L^1[a, b]$.

Motivated by (22), we regard the set of functions $T_n\mathcal{S}(F)$ as an approximant to F , represented, in view of Result 2.16, by the set of functions $\mathcal{S}(F)$. We show below that $\mathcal{S}(F)$ and $T_n\mathcal{S}(F)$ are elements of the metric space \tilde{H} of compact non-empty subsets of $L^1[a, b]$, endowed with the Hausdorff metric. We use this metric to measure the approximation error of $\mathcal{S}(F)$ by $T_n\mathcal{S}(F)$.

6.1 Two Compact Sets of Functions

In the the next two lemmas we prove that the sets $\mathcal{S}(F)$, $T_n\mathcal{S}(F)$ are compact in $L^1[a, b]$, thus they are elements of \tilde{H} . It is enough to show that they are sequentially compact, i.e. that every sequence in $\mathcal{S}(F)$ (in $T_n\mathcal{S}(F)$, respectively) has a convergent subsequence with a limit in $\mathcal{S}(F)$ (in $T_n\mathcal{S}(F)$, respectively).

Lemma 6.1 *For $F \in \mathcal{F}[a, b]$ the set $\mathcal{S}(F)$ is compact in $L^1[a, b]$.*

Proof By Result 2.15 any metric selection $s \in \mathcal{S}(F)$ satisfies $\|s\|_\infty \leq \|F\|_\infty$ and $V_a^b(s) \leq V_a^b(F)$. Applying Helly’s selection principle, we conclude that for any sequence of metric selections there is a subsequence converging pointwisely at all points $x \in [a, b]$. By Result 2.18 the pointwise limit function of such a subsequence is a metric selection. Thus by Lebesgue’s Dominated Convergence Theorem this subsequence converges in the L^1 -norm to the same limit metric selection. \square

Lemma 6.2 *Let $F \in \mathcal{F}[a, b]$ and assume that $\mathcal{K}_n(\cdot, \cdot) \in C([a, b]^2)$. Then the set $T_n\mathcal{S}(F)$ is compact in $L^1[a, b]$.*

Proof We have to show that any sequence $\{\sigma_k\}_{k=1}^\infty \subset T_n\mathcal{S}(F)$ has a subsequence converging in the L^1 -norm and that its L^1 -limit is in $T_n\mathcal{S}(F)$.

By definition

$$\sigma_k(x) = T_n s_k(x) = \int_a^b \mathcal{K}_n(x, t) s_k(t) dt,$$

for some $s_k \in \mathcal{S}(F)$.

By Helly’s selection principle there exists a subsequence $\{s_{k_\ell}\}_{\ell=1}^\infty$ such that $\lim_{\ell \rightarrow \infty} s_{k_\ell}(t) = s^\infty(t)$ pointwisely for all $t \in [a, b]$. By Result 2.18, $s^\infty \in \mathcal{S}(F)$. Since \mathcal{K}_n is bounded, we have $\lim_{\ell \rightarrow \infty} \mathcal{K}_n(x, t) s_{k_\ell}(t) = \mathcal{K}_n(x, t) s^\infty(t)$, for all $x, t \in [a, b]$. Clearly, the functions $\mathcal{K}_n(x, t) s_{k_\ell}(t)$, $\ell \in \mathbb{N}$ are dominated by $\|\mathcal{K}_n\|_\infty \|F\|_\infty$. Applying Lebesgue’s Dominated Convergence Theorem we get

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \sigma_{k_\ell}(x) &= \lim_{\ell \rightarrow \infty} \int_a^b \mathcal{K}_n(x, t) s_{k_\ell}(t) dt \\ &= \int_a^b \mathcal{K}_n(x, t) s^\infty(t) dt = \sigma^\infty(x), \quad x \in [a, b]. \end{aligned} \tag{33}$$

Thus, we have that $\{\sigma_{k_\ell}(x)\}_{\ell=1}^\infty$ converges pointwisely to $\sigma^\infty(x)$ for all $x \in [a, b]$. Also the sequence $\{\sigma_{k_\ell}\}_{\ell=1}^\infty$ is dominated by $\|F\|_\infty \|\mathcal{K}_n\|_\infty (b - a)$, indeed

$$\begin{aligned} |\sigma_{k_\ell}(x)| &\leq \int_a^b |\mathcal{K}_n(x, t)| |s_{k_\ell}(t)| dt \\ &\leq \|F\|_\infty \int_a^b |\mathcal{K}_n(x, t)| dt \leq \|F\|_\infty \|\mathcal{K}_n\|_\infty (b - a), \quad x \in [a, b]. \end{aligned}$$

It follows that the sequence of functions $\{\sigma_{k_\ell}(x)\}$, $\ell \in \mathbb{N}$, $x \in [a, b]$ satisfies the assumptions of Lebesgue’s Dominated Convergence Theorem and we get

$$\lim_{\ell \rightarrow \infty} \int_a^b |\sigma_{k_\ell}(x) - \sigma^\infty(x)| dx = 0.$$

To finish the proof, note that by (33) $\sigma^\infty \in T_n \mathcal{S}(F)$, since $s^\infty \in \mathcal{S}(F)$. □

The Hausdorff distance between the sets $\mathcal{S}(F)$ and $T_n \mathcal{S}(F)$ is

$$\text{haus}(\mathcal{S}(F), T_n \mathcal{S}(F)) = \max \left\{ \sup_{\sigma \in T_n \mathcal{S}(F)} \inf_{s \in \mathcal{S}(F)} \|s - \sigma\|_{L^1}, \sup_{s \in \mathcal{S}(F)} \inf_{\sigma \in T_n \mathcal{S}(F)} \|s - \sigma\|_{L^1} \right\}.$$

Lemma 6.3 *Let $F \in \mathcal{F}[a, b]$. For the sets $\mathcal{S}(F)$ and $T_n \mathcal{S}(F)$*

$$\text{haus}(\mathcal{S}(F), T_n \mathcal{S}(F)) \leq \sup_{s \in \mathcal{S}(F)} \|s - T_n s\|_{L^1}.$$

Proof For any fixed $s^* \in \mathcal{S}(F)$ we have

$$\inf_{\sigma \in T_n \mathcal{S}(F)} \|s^* - \sigma\|_{L^1} \leq \|s^* - T_n s^*\|_{L^1} \leq \sup_{s \in \mathcal{S}(F)} \|s - T_n s\|_{L^1}.$$

Similarly, for any fixed $\sigma^* \in T_n \mathcal{S}(F)$ there is $s^* \in \mathcal{S}(F)$ with $T_n s^* = \sigma^*$ and we get

$$\inf_{s \in \mathcal{S}(F)} \|s - \sigma^*\|_{L^1} \leq \|s^* - \sigma^*\|_{L^1} \leq \sup_{s \in \mathcal{S}(F)} \|s - T_n s\|_{L^1}.$$

Now the statement follows easily. □

6.2 Error Estimates

For $e_i(x) = x^i$, denote $\lambda_n = (\max_{i=0,1} \|T_n e_i - e_i\|_{L^1})^{1/2}$. Here we use the integral moduli ϑ and ϑ_2 defined in Sect. 2.2. In light of the Theorem in [8] we get for real-valued functions (as a special case of the statement in [8] for $p = 1$)

Result 6.4 Let T_n be a positive linear operator satisfying $\|T_n g\|_{L^1} \leq \|g\|_{L^1}$ for any $g \in L^1[a, b]$. Then

$$\|T_n f - f\|_{L^1} \leq C_1 \left(\lambda_n^2 \|f\|_{L^1} + \vartheta_2(f, \lambda_n) \right), \quad f \in L^1[a, b]. \tag{34}$$

If, in addition $T_n e_0 \equiv e_0$, then

$$\|T_n f - f\|_{L^1} \leq C_2 \left(\lambda_n^2 \vartheta(f, b - a) + \vartheta_2(f, \lambda_n) \right), \quad f \in L^1[a, b]. \tag{35}$$

Here the constants $C_i, i = 0, 1$ depend only on the interval $[a, b]$.

Applying (34) and (35) to a metric selection $s \in \mathcal{S}(F)$ and using (9) and (10) we get

$$\|T_n s - s\|_{L^1} \leq \tilde{C}_1 \left(\lambda_n^2 \|s\|_{L^1} + 2\vartheta(s, \lambda_n) \right) \leq \tilde{C}_1 \left(\lambda_n^2 \|s\|_{L^1} + 2\lambda_n V_a^b(s) \right),$$

and in the case when $T_n e_0 \equiv e_0$

$$\begin{aligned} \|T_n s - s\|_{L^1} &\leq \tilde{C}_2 \left(\lambda_n^2 \vartheta(s, b - a) + 2\vartheta(s, \lambda_n) \right) \\ &\leq \tilde{C}_2 \left(\lambda_n^2 (b - a) V_a^b(s) + 2\lambda_n V_a^b(s) \right), \end{aligned}$$

where $\tilde{C}_i, i = 0, 1$ depend only on $[a, b]$ and on the underlying norm in \mathbb{R}^d . Since by Result 2.15 $V_a^b(s) \leq V_a^b(F)$, we obtain for any $s \in \mathcal{S}(F)$

$$\sup_{s \in \mathcal{S}(F)} \|T_n s - s\|_{L^1} \leq \tilde{C}_1 \left(\lambda_n^2 \sup_{s \in \mathcal{S}(F)} \|s\|_{L^1} + 2\lambda_n V_a^b(F) \right), \tag{36}$$

and if $T_n e_0 \equiv e_0$

$$\sup_{s \in \mathcal{S}(F)} \|T_n s - s\|_{L^1} \leq \tilde{C}_2 \left(\lambda_n^2 (b - a) + 2\lambda_n \right) V_a^b(F). \tag{37}$$

Thus in view of Lemma 6.3, (36) and (37) we arrive at

Theorem 6.5 Let T_n be a positive linear operator satisfying $\|T_n g\|_{L^1} \leq \|g\|_{L^1}$ for any $g \in L^1[a, b]$ and let $F \in \mathcal{F}[a, b]$. Then

$$\begin{aligned} \text{haus}(\mathcal{S}(F), T_n \mathcal{S}(F)) &\leq \tilde{C}_1 \left(\lambda_n^2 \sup_{s \in \mathcal{S}(F)} \|s\|_{L^1} + 2\lambda_n V_a^b(F) \right) \\ &\leq \tilde{C}_1 \left(\lambda_n^2 \|F\|_\infty (b - a) + 2\lambda_n V_a^b(F) \right). \end{aligned}$$

If, in addition, $T_n e_0 \equiv e_0$, then

$$\text{haus}(\mathcal{S}(F), T_n \mathcal{S}(F)) \leq \tilde{C}_2 \left(\lambda_n^2 (b - a) + 2\lambda_n \right) V_a^b(F).$$

Here $\tilde{C}_i, i = 0, 1$ depend only on $[a, b]$ and on the underlying norm in \mathbb{R}^d .

6.3 Examples

We apply the results of Sect. 6.2 to the Bernstein–Durrmeyer operator M_n and the Kantorovich operator K_n . As was shown in Sect. 5, these operators satisfy $M_n e_0 \equiv e_0, K_n e_0 \equiv e_0$. Moreover,

$$M_n e_1(x) - e_1(x) = \frac{nx + 1}{n + 2} - x = \frac{1 - 2x}{n + 2},$$

$$K_n e_1(x) - e_1(x) = \frac{2nx + 1}{2(n + 1)} - x = \frac{1 - 2x}{2(n + 1)}.$$

Combining this with the second claim of Theorem 6.5 we end up with the following result.

Theorem 6.6 *Let $F \in \mathcal{F}[0, 1]$ and let T_n be the Bernstein–Durrmeyer operator or the Kantorovich operator. Then*

$$\begin{aligned} \text{haus}(\mathcal{S}(F), T_n \mathcal{S}(F)) &\leq CV_a^b(F) \left(\lambda_n^2 (b - a) + 2\lambda_n \right) \\ &= O\left(\frac{1}{\sqrt{n}}\right), \quad n \rightarrow \infty, \end{aligned}$$

with C depending only on the underlying norm in \mathbb{R}^d .

Note that the rate $O\left(\frac{1}{\sqrt{n}}\right)$ here is obtained for any $F \in \mathcal{F}[0, 1]$, while in the pointwise estimates of Sect. 5.3 we obtain this rate under the assumption of local Lipschitz continuity of $F \in \mathcal{F}[0, 1]$.

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