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New Orthogonality Relations for Super-Jack Polynomials and an Associated Lassalle–Nekrasov Correspondence

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Abstract

The super-Jack polynomials, introduced by Kerov, Okounkov and Olshanski, are polynomials in $n+m$ variables, which reduce to the Jack polynomials when $n = 0$ or $m = 0$ and provide joint eigenfunctions of the quantum integrals of the deformed trigonometric Calogero–Moser–Sutherland system. We prove that the super-Jack polynomials are orthogonal with respect to a bilinear form of the form $(p, q) \mapsto (L_p q)(0)$, with L_p quantum integrals of the deformed rational Calogero–Moser–Sutherland system. In addition, we provide a new proof of the Lassalle–Nekrasov correspondence between deformed trigonometric and rational harmonic Calogero–Moser–Sutherland systems and infer orthogonality of super-Hermite polynomials, which provide joint eigenfunctions of the latter system.

Keywords Orthogonal polynomials · Super-Jack polynomials · Calogero–Moser–Sutherland systems · Lassalle–Nekrasov correspondence

Mathematics Subject Classification 33C52 · 81Q80 · 81R12

Contents

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1 Introduction

As is well-known, the (symmetric) Jack polynomials $P_\lambda^{(\theta)}(x_1,\dots,x_N)$, labelled by a partition λ and depending rationally on a parameter θ, have numerous remarkable properties. In particular, they form an orthogonal system with respect to the weight function

$$\Delta_N(x;\theta)=\prod_{1\le i<j\le N}[(1-x_i/x_j)(1-x_j/x_i)]^\theta \tag{1}$$

on the N-torus; and they are joint eigenfunctions of the trigonometric Calogero–Moser–Sutherland operator

$$\mathcal{L}_N(x;\theta)=\sum_{i=1}^N\left(x_i\frac{\partial}{\partial x_i}\right)^2+\theta\sum_{1\le i<j\le N}\frac{x_i+x_j}{x_i-x_j}\left(x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}\right) \tag{2}$$

and its quantum integrals. For historical accounts of Henry Jack, his construction of the polynomials that now bear his name as well as many more of their properties; see e.g. the proceedings [24] and Macdonald's book [29]. We note that the parameter θ is the inverse of the parameter α used, in particular, in [29].

In addition to the root system generalisations introduced by Olshanetsky and Perelomov [34], the operator (2) allows for non-symmetric integrable generalisations, such as

$$\mathcal{L}_{n,m}(x,y;\theta)=\mathcal{L}_n(x;\theta)-\theta\mathcal{L}_m(y;1/\theta)-\sum_{i=1}^n\sum_{j=1}^m\frac{x_i+y_j}{x_i-y_j}\left(x_i\frac{\partial}{\partial x_i}+\theta y_j\frac{\partial}{\partial y_j}\right). \tag{3}$$

This operator was first introduced in the $m=1$ case by Chalykh, Feigin and Veselov [10], who proved integrability, expressed the operator in terms of a particular deformation of the root system A_n and introduced the terminology *deformed Calogero–Moser–Sutherland operator*. Shortly thereafter, Sergeev [38, 39] wrote down the operator (3) for general m and showed that it has the so-called super-Jack polynomials $SP_\lambda^{(\theta)}((x_1,\dots,x_n),(y_1,\dots,y_m))$, introduced a few years earlier by Kerov, Okounkov and Olshanski [26], as eigenfunctions. These results on integrability and eigenfunctions were then extended to arbitrary $m>1$ and all quantum integrals, respectively, by Sergeev and Veselov [40, 41].

At an early stage, it became clear that the orthogonality of the Jack polynomials with respect to the weight function (1) could not be directly generalised to the super-Jack polynomials. Indeed, starting from Sergeev's work and proceeding formally, one is naturally led to the weight function

$$\Delta_{n,m}(x,y;\theta)=\frac{\Delta_n(x;\theta)\Delta_m(y;1/\theta)}{\prod_{i=1}^n\prod_{j=1}^m(1-x_i/y_j)(1-y_j/x_i)},$$

which is manifestly singular along the hyperplane $x_i=y_j$ for all $1\le i\le n$ and $1\le j\le m$ and any value of θ.

Together with Atai and Langmann [1], we circumvented this problem by integrating x and y over tori with different radii. Even though this meant dealing with a complex-valued weight function, we could prove that the resulting sesquilinear form is Hermitian on the space spanned by the super-Jack polynomials. Moreover, we identified its kernel as the subspace spanned by the $SP_\lambda^{(\theta)}$ with $(m^n)\not\subset\lambda$, and showed that the form descends to a positive definite inner product on the corresponding factor space. As a consequence, we obtained a Hilbert space interpretation of the deformed Calogero–Moser–Sutherland operator (3). These results were essentially all inferred from orthogonality relations, including an explicit formula for (quadratic) norms, for super-Jack polynomials.

In this paper, we prove orthogonality of the super-Jack polynomials with respect to another rather different bilinear form. Initially, we will define it in terms of deformed rational Calogero–Moser–Sutherland operators, but, for suitable values of θ, it also has the integral representation

$$\begin{aligned}(p,q)_{n,m}\\ &=M_{n,m}^{-1}\int_{\mathbb{R}^n+i\xi}\int_{\mathbb{R}^m+i\eta}\big(e^{-L_{n,m}/2}p\big)(x,y)\big(e^{-L_{n,m}/2}q\big)(x,y)\frac{e^{-x^2/2+\theta^{-1}y^2/2}}{A_{n,m}(x,y)}dxdy,\end{aligned}\tag{4}$$

with

$$A_{n,m}(x,y)=\prod_{1\le i<j\le n}(x_i-x_j)^{-2\theta}\cdot\prod_{1\le i<j\le m}(y_i-y_j)^{-2/\theta}\cdot\prod_{i=1}^n\prod_{j=1}^m(x_i-y_j)^2,$$

and where $M_{n,m}$ is a normalisation constant, $\xi\in\mathbb{R}^n$ and $\eta\in\mathbb{R}^m$ are chosen such that all singularities are avoided, $x^2=x_1^2+\dots+x_n^2$, $y^2=y_1^2+\dots+y_m^2$ and $L_{n,m}$ denotes the rational limit of (3).

When $m=0$ this form amounts to a restriction of the A_{n-1}-instance of Dunkl's [15] bilinear form $[p,q]_\theta=(p(D)q)(0)$ to symmetric polynomials p,q. Here, $p(D)$

denotes the operators obtained from $p(x)$ by substituting for x_i a Dunkl operator $D_{i,n}$ (see Eq. (16) below). An integral representation similar to (4) was established in the same paper by Dunkl. Corresponding orthogonality results for (non-symmetric) Jack polynomials were later deduced by Baker and Forrester [5] as well as Rösler [36].

Restricting attention further to $\theta = 0$, we recover the bilinear form $[p, q]_\partial := (p(\partial)q)(0)$, where $p(\partial) = p(\partial/\partial x_1, \ldots, \partial/\partial x_n)$, for which Macdonald [28] proved $[p, q]_\partial = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (e^{-\Delta/2} p)(x)(e^{-\Delta/2} q)(x) e^{-x^2/2} dx$. In this special case, Jack polynomials are simply symmetric monomials m_λ and their orthogonality on the n-torus T^n simply amounts to $\int_{T^n} m_\mu(x) m_\lambda(x^{-1}) dx = |S_n(\mu)| \cdot \delta_{\mu\lambda}$, with $\delta_{\mu\lambda}$ the Kronecker delta, $S_n(\mu)$ the orbit of μ under the standard action of S_n and dx the normalised Haar measure on T^n.

Moreover, if $m = 1$ and $\theta \in -\mathbb{N}$, then $(\cdot, \cdot)_{n,m}$ amounts to a (restricted) instance of the bilinear form on quasi-invariants introduced and studied by Feigin and Veselov [20, 21], with an integral representation of the form (4) obtained in Ref. [18].

From (4) and our orthogonality results for the super-Jack polynomials, we infer that the polynomials

$$SH_\lambda^{(\theta)}(x, y) := e^{-L_{n,m}/2} SP_\lambda^{(\theta)}(x, y) \tag{5}$$

form an orthogonal system on $(\mathbb{R}^n + i\xi) \times (\mathbb{R}^m + i\eta)$ with respect to the weight function $e^{-x^2/2+\theta^{-1}y^2/2}/A_{n,m}(x, y)$. These polynomials coincide with the so-called super-Hermite polynomials introduced in Ref. [11] (see Prop. 6.7); and for $m = 0$ they amount to the generalised Hermite polynomials first introduced by Lassalle [27] and studied in further detail by Baker and Forrester [4] and van Diejen [13].

In recent joint work with Feigin and Veselov [19], we showed that $e^{-L_{n,m}/2}$ intertwines quantum integrals of deformed Calogero–Moser–Sutherland operators of trigonometric and rational harmonic type, and so (5) can be interpreted as a correspondence between joint eigenfunctions of these two integrable systems. Motivated by Lassalle's construction of generalised Hermite polynomials and Nekrasov's [31] discovery that the ordinary A type trigonometric and rational harmonic Calogero–Moser–Sutherland systems are essentially equivalent, we proposed for such a correspondence the terminology *Lassalle–Nekrasov correspondence*.

Our results entail that this particular example of a Lassalle–Nekrasov correspondence between deformed Calogero–Moser–Sutherland systems is isometric in the sense that the operator $e^{-L_{n,m}/2}$ becomes an isometry when its domain is equipped with the bilinear form $(\cdot, \cdot)_{n,m}$ and its codomain with the form given by the right-hand side of (4) with $e^{-L_{n,m}/2}$ removed. In the undeformed case $m = 0$ this can be seen already in Ref. [15]; see also [36].

We conclude this introduction with an outline of the remainder of the paper. In Sect. 2, we review definitions and results pertaining to (super-)Jack polynomials and (deformed) rational Calogero–Moser–Sutherland operators that we make use of. Readers familiar with these matters may wish to skip ahead to Sect. 3, where particular instances of generalised hypergeometric series associated with Jack polynomials, Jack symmetric functions or super-Jack polynomials are recalled and shown to be joint eigenfunctions of (deformed) rational Calogero–Moser–Sutherland quantum integrals.

In Sect. 4, we introduce the relevant bilinear form and establish a number of its basic properties, identify its reproducing kernel as a generalised hypergeometric series and make the integral representation (4) precise. We infer orthogonality relations for super-Jack polynomials in Sect. 5 and our results on the Lassalle–Nekrasov correspondence between deformed trigonometric and rational harmonic Calogero–Moser–Sutherland systems are contained in Sect. 6. In the final Sect. 7, we provide a brief outlook on possible directions for future research; and in Appendix A, we study convergence properties of the pertinent generalised hypergeometric series associated with super-Jack polynomials.

Notation

We use the convention $\mathbb{N} = \{1, 2, \ldots\}$ and write $\mathbb{Z}_{\geq 0}$ for $\mathbb{N} \cup \{0\}$. To a large extent, we follow Macdonald's book [29] for notation and terminology from the theory of symmetric functions. One notable exception is our use of the parameter θ, which is the inverse of the parameter α.

2 Preliminaries

In this section, we specify our basic notations and review terminology and results relating to (super-)Jack polynomials and (deformed) rational Calogero–Moser–Sutherland operators that we rely on in our construction of a bilinear form in Sect. 4.

2.1 Symmetric Functions

Throughout the paper, we shall work over the complex numbers $\mathbb{C}$. Therefore, we write simply Λ_N for the algebra of symmetric polynomials in N independent variables with complex coefficients and Λ_N^k for the subspace consisting of homogeneous symmetric polynomials of degree k.

Given an infinite sequence $x = (x_1, x_2, \ldots)$ of independent variables, we recall that a homogeneous symmetric function of degree k with complex coefficients can be viewed as a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha,$$

where the sum extends over all sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ of non-negative integers such that $\alpha_1 + \alpha_2 + \cdots = k$, each $c_\alpha \in \mathbb{C}$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ and $f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots)$ for any permutation σ of $\mathbb{N}$; see e.g. Chapter 7 in Stanley's book [45].

We let Λ^k denote the space of all (complex) homogenous symmetric functions of degree k. If $f \in \Lambda^k$ and $g \in \Lambda^l$, it is clear that $fg \in \Lambda^{k+l}$, which implies that

$$\Lambda := \Lambda^0 \oplus \Lambda^1 \oplus \cdots$$

has a natural structure of a graded $\mathbb{C}$-algebra, called the ($\mathbb{C}$-)algebra of symmetric functions.

Partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of k, i.e. sequences of non-negative integers $\lambda_1, \lambda_2, \ldots$ such that $\lambda_1 + \lambda_2 + \cdots = k$, naturally label bases in Λ^k. There are numerous important examples and we use, in particular, the following:

(1) Monomial symmetric functions:

$$m_\lambda = \sum_\alpha x^\alpha,$$

with the sum extending over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of the parts of $\lambda = (\lambda_1, \lambda_2, \ldots)$.

(2) Power sum symmetric functions:

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

with $p_0 \equiv 1$ and

$$p_r = x_1^r + x_2^r + \cdots \quad (r \in \mathbb{N}).$$

2.2 Jack Symmetric Functions

We recall that, as λ runs through all partitions λ of weight $k \in \mathbb{Z}_{\geq 0}$, the (monic) Jack symmetric functions $P_\lambda^{(\theta)}$ form an orthogonal basis in Λ^k with respect to the the scalar product given by

$$\langle p_\lambda, p_\mu \rangle = \theta^{-l(\lambda)} z_\lambda \delta_{\lambda\mu}, \tag{6}$$

where $l(\lambda)$ denotes the length of λ and $z_\lambda = \prod_{i \geq 1} i^{m_i} \cdot m_i!$ with $m_i = m_i(\lambda)$ the multiplicity of i in λ.

More specifically, they are characterized by the following two properties:

(1) $P_\lambda^{(\theta)}(x) = m_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda\mu}^{(\theta)} m_\mu(x)$,
(2) $\left\langle P_\lambda^{(\theta)}, P_\mu^{(\theta)} \right\rangle = 0$ whenever $\lambda \neq \mu$,

where $<$ refers to the dominance order. (Note that, since the dominance order is only a partial order, it is far from obvious that such symmetric functions actually exist.) The coefficients $u_{\lambda\mu}^{(\theta)}$ are known to be rational functions of θ, with poles occurring only at negative rational values of θ. To ensure that such poles are not encountered and also that we avoid the singularity at $\theta = 0$ in (6), we consider throughout the paper only values of θ not of the form

$$\theta = i/j, \quad i = -\mathbb{Z}_{\geq 0}, \quad j \in \mathbb{N}. \tag{7}$$

The Jack symmetric functions were first introduced by Jack [23], while the above characterisation is due to Macdonald [29].

When setting $x_i = 0$ for $i > N \in \mathbb{N}$, the Jack symmetric function $P_\lambda^{(\theta)}(x)$ specialises to the Jack polynomial $P_\lambda^{(\theta)}(x_1, \dots, x_N)$ in the remaining N variables $x_1, \dots, x_N$.

For later reference, we record Stanley's [44] (see also [29]) quadratic norms formula

$$\langle P_\lambda^{(\theta)}, P_\lambda^{(\theta)} \rangle = \frac{1}{b_\lambda^{(\theta)}}, \quad b_\lambda^{(\theta)} = \prod_{s \in \lambda} \frac{a(s) + \theta l(s) + \theta}{a(s) + 1 + \theta l(s)}, \tag{8}$$

and specialisation formula

$$\epsilon_X \left(P_\lambda^{(\theta)} \right) = \prod_{s \in \lambda} \frac{\theta X + a'(s) - \theta l'(s)}{a(s) + \theta l(s) + \theta}, \tag{9}$$

where $\epsilon_X : \Lambda \to \mathbb{C}[X]$ denotes the homomorphism given by

$$\epsilon_X : p_r \mapsto X \ (r \in \mathbb{N}). \tag{10}$$

In particular, the value at $x = 1^N$ is obtained by setting $X = N$.

2.3 Super-Jack Polynomials

For $n, m \in \mathbb{Z}_{\geq 0}$, we let

$$P_{n,m} = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m].$$

From [40, 41], we recall its subalgebra $\Lambda_{n,m}$, consisting of all polynomials $p(x, y)$ that are symmetric in $x = (x_1, \dots, x_n)$, symmetric in $y = (y_1, \dots, y_m)$ and satisfy the quasi-invariance condition

$$\left(\frac{\partial}{\partial x_i} + \theta \frac{\partial}{\partial y_j} \right) p(x, y) = 0 \tag{11}$$

on each hyperplane $x_i = y_j$ with $i = 1, \dots, n$ and $j = 1, \dots, m$; the (for generic values of θ) surjective homomorphism

$$\varphi_{n,m} : \Lambda \to \Lambda_{n,m}, \ p_r \mapsto p_{r,\theta}(x, y) \ (r \in \mathbb{N}), \tag{12}$$

given in terms of the deformed power (or Newton) sums

$$p_{r,\theta}(x, y) = \sum_{i=1}^{n} x_i^r - \frac{1}{\theta} \sum_{j=1}^{m} y_j^r;$$

and the fact that the kernel of $\varphi_{n,m}$ is spanned by the Jack symmetric functions $P_\lambda^{(\theta)}$ labelled by the partitions $\lambda \notin H_{n,m}$, where

$$H_{n,m} = \{\lambda \in \mathscr{P} \mid \lambda_{n+1} \leq m\},$$

(with $\mathscr{P}$ denoting the set of all partitions).

For $\lambda \in H_{n,m}$, the super-Jack polynomial $SP_\lambda^{(\theta)}(x, y) \in \Lambda_{n,m}$ is given by

$$SP_\lambda^{(\theta)}(x, y) = \varphi_{n,m}\big(P_\lambda^{(\theta)}\big). \tag{13}$$

We note that they were originally introduced by Kerov, Okounkov Olshanski [26] in the setting of infinitely many variables and further studied by Sergeev and Veselov [40, 41] in the present context of $n + m$ variables.

2.4 Rational Calogero–Moser–Sutherland Operators

It is readily seen that when substituting $x_i \to e^{i2\pi x_i/\mu}$, rescaling by $-(2\pi/\mu)^2$ and taking the period $\mu \to \infty$ the trigonometric Calogero–Moser–Sutherland operator (2) degenerates to its rational counterpart

$$L_N := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\theta \sum_{1\leq i<j\leq N} \frac{1}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right). \tag{14}$$

We rely on the observation that the algebra of symmetric quantum integrals of this operator is given by the differential operators

$$L_{p,N} := \mathrm{Res}\big(p\big(D_{1,N}, \ldots, D_{N,N}\big)\big) \quad (p \in \Lambda_N), \tag{15}$$

where Res stands for restriction to Λ_N and $D_{i,N}$ $(i = 1, \ldots, N)$ are Dunkl differential-difference operators [14], given by

$$D_{i,N} = \frac{\partial}{\partial x_i} + \theta \sum_{j\neq i} \frac{1}{x_i - x_j}(1 - \sigma_{ij}) \tag{16}$$

with the transposition σ_{ij} acting on a function $f(x)$ in $x = (x_1, \ldots, x_N)$ by exchanging variables x_i and x_j. Letting

$$L_N^{(r)} = L_{p_r(x_1,\ldots,x_N)} \quad (r \in \mathbb{N}),$$

where $L_N^{(2)} = L_{x_1^2+\cdots+x_N^2} = L_N$, we note that the algebra of symmetric quantum integrals is (freely) generated by $L_N^{(1)}, \ldots, L_N^{(N)}$.

The above important observation is due to Heckman [22], who worked in the more general setting of an arbitrary root system, whereas (14)–(16) are associated with A_{N-1}.

For further details on (rational) Calogero–Moser–Sutherland systems and their integrability, see e.g. [16, 34, 37].

2.5 Dunkl Operator at Infinity

We proceed to recall Sergeev and Veselov's [43] notion of a Dunkl operator in infinitely many variables as well as corresponding expressions for rational Calogero–Moser–Sutherland operators at infinity.

Introducing a generator p_0, replacing the dimension N, the algebra $\bar{\Lambda} := \Lambda[p_0]$ and homomorphisms

$$\varphi_N : \bar{\Lambda} \to \Lambda_N,\ p_r \mapsto x_1^r + \cdots + x_N^r\ (r \in \mathbb{Z}_{\geq 0}, N \in \mathbb{N})$$

are considered. Using a further generator x, the infinite-dimensional Dunkl operator $D_\infty : \bar{\Lambda}[x] \to \bar{\Lambda}[x]$ is given by

$$D_\infty = \partial + \theta \Delta,$$

with the derivation ∂ in $\bar{\Lambda}[x]$ characterised by Leibniz rule and

$$\partial(x) = 1,\ \ \partial(p_r) = r x^{r-1}\ \ (r \in \mathbb{Z}_{\geq 0}),$$

and the operator $\Delta : \bar{\Lambda}[x] \to \bar{\Lambda}[x]$ defined by

$$\Delta(x^r f) = \Delta(x^r) f,\ \ \Delta(1) = 0,\ \ \Delta(x^r) = \sum_{s=0}^{r-1} x^{r-1-s} p_s - r x^{r-1}\ \ (f \in \bar{\Lambda}, r \in \mathbb{Z}_{\geq 0}).$$

The above definition is motivated by the fact that the homomorphism $\bar{\Lambda}[x] \to \Lambda_N[x_i]$ that maps $p_r \mapsto x_1^r + \cdots + x_N^r$ $(r \in \mathbb{Z}_{\geq 0})$ and $x \mapsto x_i$ $(i = 1, \ldots, N)$ intertwines D_∞ and $D_{i,N}$.

Introducing also the linear 'symmetrisation' operator $E : \bar{\Lambda}[x] \to \bar{\Lambda}$ by

$$E(x^r f) = E(x^r) f,\ \ E(1) = 1,\ \ E(x^r) = p_r,\ \ (f \in \bar{\Lambda}, r \in \mathbb{Z}_{\geq 0}),$$

rational Calogero–Moser–Sutherland integrals at infinity $L^{(r)} : \bar{\Lambda} \to \bar{\Lambda}$ $(r \in \mathbb{Z}_{\geq 0})$ are given by

$$L^{(r)} = \operatorname{Res} E \circ D_\infty^r, \tag{17}$$

with Res denoting restriction to $\bar{\Lambda}$. Combining the intertwining relation between D_∞ and $D_{i,N}$ with (15), it is readily seen that the diagram

$$\begin{array}{ccc} \bar{\Lambda} & \xrightarrow{L^{(r)}} & \bar{\Lambda} \\ \big\downarrow \varphi_N & & \big\downarrow \varphi_N \\ \Lambda_N & \xrightarrow{L_N^{(r)}} & \Lambda_N \end{array} \tag{18}$$

is commutative for all $r \in \mathbb{Z}_{\geq 0}$; and, as a straightforward consequence, it follows that $[L^{(r)}, L^{(s)}] = 0$ $(r, s \in \mathbb{Z}_{\geq 0})$.

2.6 Deformed Rational Calogero–Moser–Sutherland Operators

By setting

$$x_{n+i} = y_i \quad (i = 1, \ldots, m) \tag{19}$$

and using the 'parity' function

$$p(i) := \begin{cases} 0, & i = 1, \ldots, n \\ 1, & i = n+1, \ldots, n+m \end{cases} \tag{20}$$

the rational limit of (3) takes the simple and convenient form

$$\begin{aligned} L_{n,m} := & \sum_{i=1}^{n+m} (-\theta)^{p(i)} \frac{\partial^2}{\partial x_i^2} \\ & -2 \sum_{1 \leq i < j \leq n+m} \frac{(-\theta)^{1-p(i)-p(j)}}{x_i - x_j} \left((-\theta)^{p(i)} \frac{\partial}{\partial x_i} - (-\theta)^{p(j)} \frac{\partial}{\partial x_j} \right). \end{aligned} \tag{21}$$

We shall make use of a recursive formula for its quantum integrals and a connection with Calogero–Moser–Sutherland operators at infinity, both due to Sergeev and Veselov [40, 43].

Specifically, taking $\partial_i^{(1)} = (-\theta)^{p(i)} \frac{\partial}{\partial x_i}$, differential operators of order $r > 1$ are defined recursively by

$$\partial_i^{(r)} = \partial_i^{(1)} \partial_i^{(r-1)} - \sum_{j \neq i} \frac{(-\theta)^{1-p(j)}}{x_i - x_j} \left(\partial_i^{(r-1)} - \partial_j^{(r-1)} \right) \tag{22}$$

and the quantum integrals are given by

$$L_{n,m}^{(r)} = \sum_{i=1}^{n+m} (-\theta)^{-p(i)} \partial_i^{(r)} \quad (r \in \mathbb{N}), \tag{23}$$

with $L_{n,m}^{(2)} = L_{n,m}$.

Moreover, substituting $\mathbb{Z}_{\geq 0}$ for $\mathbb{N}$ in (12) produces a homomorphism $\varphi_{n,m} : \bar{\Lambda} \to \Lambda_{n,m}$ mapping $p_0 \mapsto \varphi_{n,m}(p_0) := n - m/\theta$. From Thm. 3.3 in Ref. [43], we recall that the diagram

$$\begin{array}{ccc} \bar{\Lambda} & \xrightarrow{L^{(r)}} & \bar{\Lambda} \\ \downarrow{\scriptstyle \varphi_{n,m}} & & \downarrow{\scriptstyle \varphi_{n,m}} \\ \Lambda_{n,m} & \xrightarrow{L^{(r)}_{n,m}} & \Lambda_{n,m} \end{array} \tag{24}$$

is commutative for all $r \in \mathbb{N}$ and commutativity of the deformed Calogero–Moser–Sutherland operators (23), i.e.

$$\left[L^{(r)}_{n,m}, L^{(s)}_{n,m}\right] = 0 \ \ (r, s \in \mathbb{N}),$$

follows from that of the operators (17).

We proceed to describe a Harish-Chandra type isomorphism mapping the algebra of quantum integrals

$$Q_{n,m} := \mathbb{C}\left[L^{(1)}_{n,m}, L^{(2)}_{n,m}, \dots\right] \tag{25}$$

of the deformed rational Calogero–Moser–Sutherland system onto $\Lambda_{n,m}$. That such an identification of $Q_{n,m}$ with $\Lambda_{n,m}$ is possible was first observed by Sergeev and Veselov [40].

We write $V_{n,m}$ for $\mathbb{C}^{n+m}$ equipped with the bilinear form

$$B_{n,m}(u, v) := \sum_{i=1}^{n} u_i v_i - \theta \sum_{i=1}^{m} u_{n+i} v_{n+i}$$

and $\mathscr{D}_{n,m}$ for the algebra of differential operators with constant (complex) coefficients generated by $\frac{\partial}{\partial x_i}$ $(i = 1, \dots, n)$ and $\frac{\partial}{\partial y_i}$ $(i = 1, \dots, m)$. Given $p \in P_{n,m}$, we let

$$\partial(p) = p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, -\theta\frac{\partial}{\partial y_1}, \dots, -\theta\frac{\partial}{\partial y_m}\right).$$

Identifying $P_{n,m}$ with $S(V_{n,m})$ and $\mathscr{D}_{n,m}$ with $S(V^*_{n,m})$, the map $p \mapsto \partial(p)$ amounts to the isomorphism $S(V_{n,m}) \to S(V^*_{n,m})$ induced by $B_{n,m}$. (Here, we have used the notation $S(V)$ for the symmetric algebra of V.)

Letting $\mathscr{R}_{n,m}$ be the algebra of rational functions generated by $(x_i - x_j)^{-1}$ $(1 \leq i < j \leq n)$, $(x_i - y_j)^{-1}$ $(1 \leq i \leq n,\ 1 \leq j \leq m)$ and $(y_i - y_j)^{-1}$ $(1 \leq i < j \leq m)$, we introduce the algebra of differential operators $\mathscr{D}_{n,m}[\mathscr{R}_{n,m}]$, generated by the derivatives $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ with coefficients in $\mathscr{R}_{n,m}$, and note that $Q_{n,m} \subset \mathscr{D}_{n,m}[\mathscr{R}_{n,m}]$.

For $N \in \mathbb{N}$, we write $\mathfrak{N}(N, \mathbb{Z}_{\geq 0})$ for the set of strictly upper triangular $N \times N$ matrices with non-negative integer entries. Using again the convention $x_{n+i} = y_i$

($i = 1, \ldots, m$), we observe that any $L \in \mathscr{D}_{n,m}[\mathscr{R}_{n,m}]$ has a representation of the form

$$L = \sum_{M \in \mathfrak{N}(n+m, \mathbb{Z}_{\geq 0})} \prod_{1 \leq i < j \leq n+m} (x_i - x_j)^{-M_{ij}} \cdot \partial(\ell_M)$$

for some $\ell_M \in P_{n,m}$. Hence, we can define a homomorphism $\psi_{n,m} : \mathscr{D}_{n,m}[\mathscr{R}_{n,m}] \to P_{n,m}$ by setting $\psi_{n,m}(L) = \ell_0$. From (22) to (23), it is readily inferred that

$$\psi_{n,m}\big(L_{n,m}^{(r)}\big) = \sum_{i=1}^{n+m} (-\theta)^{-p(i)} x_i^r = p_{r,\theta}(x, y). \tag{26}$$

Since the deformed power sums $p_{r,\theta}$ generate $\Lambda_{n,m}$, it follows that $\psi_{n,m}$ maps $Q_{n,m}$ onto $\Lambda_{n,m}$. In Sect. 3.3, we shall provide a rather different description of this map and, as a consequence, infer injectivity.

3 Generalised Hypergeometric Series

Our main results rely on generalised hypergeometric series in two sequences of variables associated with either Jack polynomials, Jack symmetric functions or super-Jack polynomials.

3.1 Associated with Jack Polynomials

In the case of Jack polynomials in N variables $x = (x_1, \ldots, x_N)$, such series appeared, in particular, in Macdonald's widely circulated informal working paper [30]. They take a particularly simple form when expressed in terms of Kaneko's normalisation [25]

$$C_\lambda^{(\theta)}(x) = \frac{|\lambda|!}{\prod_{s \in \lambda}(a(s) + 1 + \theta l(s))} P_\lambda^{(\theta)}(x), \tag{27}$$

also characterised by

$$\sum_{|\lambda|=k} C_\lambda^{(\theta)}(x) = p_1(x)^k \quad (k \in \mathbb{Z}_{\geq 0}).$$

Indeed, the simplest such series, with no additional parameters beyond θ and the only one we require, is given by

$$F_N^{(\theta)}(x, y) = \sum_{d=0}^{\infty} F_{N,d}^{(\theta)}(x, y), \quad F_{N,d}^{(\theta)}(x, y) = \sum_{|\lambda|=d} \frac{1}{|\lambda|!} \frac{C_\lambda^{(\theta)}(x) C_\lambda^{(\theta)}(y)}{C_\lambda^{(\theta)}(1^N)}, \tag{28}$$

where $F_{N,d}^{(\theta)}(x, y)$ is the homogeneous part of degree d in both the x_i and the y_j. (In Ref. [30], this series is denoted ${}_0F_0(x, y; 1/\theta)$). From (9) and (27), it is clear that the individual terms in these series are well-defined whenever θ is not a negative rational number or zero. Moreover, for $\theta > 0$, the infinite series is known to converge locally uniformly on $\mathbb{C}^N \times \mathbb{C}^N$ and therefore define an entire function; see e.g. Props. 3.10–11 in Ref. [5] or Thm. 6.5 in Ref. [8].

As indicated in Ref. [33] (see also [8] for a more detailed explanation), the generalised hypergeometric series $F_N^{(\theta)}(x, y)$ is equal to the A_{N-1} instance of Opdam's [35] multivariable Bessel functions associated with root systems. In particular, this equality manifests itself in the following joint eigenfunction property.

Proposition 3.1 *Assume that θ is not a negative rational number or zero. For each $p \in \Lambda_N$, we have*

$$L_{p,N}(x)F_N^{(\theta)}(x, y) = p(y)F_N^{(\theta)}(x, y).$$

When $p = p_r$ with $r = 1, 2$, this result can be found in Ref. [4], whereas an eigenfunction property equivalent to the general case is sketched in [33]. For arbitrary $p \in \Lambda_N$, a proof of the proposition is readily inferred from results by Baker and Forrester [5] on a non-symmetric generalised hypergeometric series $\mathscr{K}_A(x, y)$, defined in Eq. (3.17) in Ref. [5]. More specifically, from its joint eigenfunction property

$$D_{i,N}(x)\mathscr{K}_A(x, y) = y_i \mathscr{K}_A(x, y) \quad (i = 1, \dots, N)$$

and symmetrisation property

$$\sum_{\sigma \in S_N} \mathscr{K}_A(x, \sigma y) = N! F_N^{(\theta)}(x, y),$$

obtained in Thm. 3.8(c) and Prop. 3.11 in Ref. [5], respectively, it follows that

$$\begin{aligned} L_p(x)F_N^{(\theta)}(x, y) &= p\big(D_{1,N}(x), \dots, D_{N,N}(x)\big)F_N^{(\theta)}(x, y) \\ &= \frac{1}{N!}\sum_{\sigma \in S_N} p\big(D_{1,N}(x), \dots, D_{N,N}(x)\big)\mathscr{K}_A(x, \sigma y) \\ &= \frac{1}{N!}\sum_{\sigma \in S_N} p(y_{\sigma^{-1}(1)}, \dots, y_{\sigma^{-1}(N)})\mathscr{K}_A(x, \sigma y) \\ &= p(y_1, \dots, y_N)F_N^{(\theta)}(x, y). \end{aligned}$$

In terms of homogeneous components, the result reads as follows.

Corollary 3.2 *For all $d, k \in \mathbb{Z}_{\geq 0}$ and $p \in \Lambda_N^k$, we have*

$$L_p(x)F_{N,d}^{(\theta)}(x, y) = p(y)F_{N,d-k}^{(\theta)}(x, y),$$

with $F_{N,d-k}^{(\theta)} \equiv 0$ when $d < k$ and the above assumptions on θ in place.

3.2 Associated with Jack Symmetric Functions

In the context of symmetric functions, generalised hypergeometric series were introduced in Ref. [11]. For our purposes, it suffices to consider the simplest instance, involving only the parameters θ and p_0, and recall its homogeneous components:

$$F_d^{(\theta,p_0)}(x,y) = \sum_{|\lambda|=d} \frac{1}{|\lambda|!} \frac{C_\lambda^{(\theta)}(x)C_\lambda^{(\theta)}(y)}{\epsilon_{p_0}\big(C_\lambda^{(\theta)}\big)} \quad (d \in \mathbb{Z}_{\geq 0}), \tag{29}$$

where, as before, ϵ_{p_0} denotes the homomorphism $\Lambda \to \mathbb{C}[p_0]$ given by $p_r \mapsto p_0$ ($r \in \mathbb{N}$). Just as in the previous section, θ not being a negative rational number or zero ensures that this series is well-defined.

The following infinite-dimensional generalisation of Cor. 3.2 is now readily identified and proved.

Proposition 3.3 *Take $\theta \in \mathbb{C}$ not of the form (7). For all $r, d \in \mathbb{Z}_{\geq 0}$, we have*

$$L^{(r)}(x)F_d^{(\theta,p_0)}(x,y) = p_r(y)F_{d-r}^{(\theta,p_0)}(x,y), \tag{30}$$

with $F_{d-r}^{(\theta,p_0)} \equiv 0$ when $d < r$.

Proof Since $L^{(r)}$ preserves $\bar{\Lambda}$ and lowers the degree by r, it is clear that the left-hand side of (30) is identically zero whenever $r > d$. Hence, fixing $r, d \in \mathbb{Z}_{\geq 0}$ such that $r \leq d$, we consider the symmetric function

$$f(x,y) := L^{(r)}(x)F_d^{(\theta,p_0)}(x,y) - p_r(y)F_{d-r}^{(\theta,p_0)}(x,y),$$

which amounts to a polynomial in $p_r(x)$, $p_r(y)$, $1 \leq r \leq d$, with coefficients depending rationally on p_0.

At this point, we choose $N \geq d$. We note that (9) and $l'(s) = i - 1 < \ell(\lambda) \leq |\lambda|$, where $s = (i, j) \in \lambda$, ensures that $f(x, y)$ has no pole at $p_0 = N$. From (18) and Cor. 3.2, we can thus infer that $\varphi_N^{(x)}\varphi_N^{(y)} f(x,y) = 0$. (Of course $F_d^{(\theta,p_0)}(x,y) \notin \bar{\Lambda}$, but this minor technical snag is easily resolved by clearing denominators or, as discussed in Sect. 2.4 in Ref. [11], extending the definition of φ_N to all elements in $\mathbb{C}(p_0) \otimes \Lambda$ that lack a pole at $p_0 = N$.) By algebraic independence of the $\varphi_N(p_r)$, $1 \leq r \leq d$, it follows that each coefficient of $f(x, y)$ must vanish at $p_0 = N$. Since this is the case for all $N \geq d$, the coefficients vanish identically and $f(x, y) \equiv 0$. □

3.3 Associated with Super-Jack Polynomials

In this section, we work with sequences of $n + m$ variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ as well as $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_m)$. Introducing the renormalised super-Jack polynomials

$$SC_\lambda^{(\theta)}(x,y) := \frac{|\lambda|!}{\prod_{s\in\lambda}(a(s) + 1 + \theta l(s))} SP_\lambda^{(\theta)}(x,y), \tag{31}$$

we perform the substitutions $p_0 \mapsto n - m/\theta$, $C_\lambda^{(\theta)}(x) \mapsto SC_\lambda^{(\theta)}(x, y)$ and $C_\lambda^{(\theta)}(y) \mapsto SC_\lambda^{(\theta)}(z, w)$ in (29) to obtain

$$SF_{n,m;d}^{(\theta)}(x, y; z, w) := \sum_{\substack{\lambda \in H_{n,m} \\ |\lambda| = d}} \frac{1}{|\lambda|!} \frac{SC_\lambda^{(\theta)}(x, y) SC_\lambda^{(\theta)}(z, w)}{SC_\lambda^{(\theta)}(1^{n+m})} \quad (d \in \mathbb{Z}_{\geq 0}). \tag{32}$$

Before proceeding further, a few remarks are in order: First, the above substitutions are essentially given by the homomorphism $\varphi_{n,m}$ (cf. the remark in the proof of Prop. 3.3); second, summation over $d \in \mathbb{Z}_{\geq 0}$ yields

$$\begin{aligned} SF_{n,m}^{(\theta)}(x, y; z, w) &:= \sum_{d=0}^{\infty} SF_{n,m;d}^{(\theta)}(x, y; z, w) \\ &= \sum_{\lambda \in H_{n,m}} \frac{1}{|\lambda|!} \frac{SC_\lambda^{(\theta)}(x, y) SC_\lambda^{(\theta)}(z, w)}{SC_\lambda^{(\theta)}(1^{n+m})}, \end{aligned} \tag{33}$$

which is identical with ${}_0\mathscr{S}\mathscr{F}_0(x, y; z, w)$ in Section 6.3 of [11]; and, third, to ensure that each $SF_{n,m;d}^{(\theta)}(x, y; z, w)$ is well-defined, we need to avoid the parameter values

$$\theta = \frac{i}{j}, \quad i \in -\mathbb{Z}_{\geq 0},\ j \in \mathbb{N} \text{ or } 1 \leq i \leq m,\ 1 \leq j \leq n, \tag{34}$$

since we might encounter a pole of a coefficient in $SC_\lambda^{(\theta)}$ when θ is a negative rational number and $SC_\lambda^{(\theta)}(1^{n+m})$ could vanish when θ takes one of the finitely many positive rational values specified above.

The methods used in Ref. [5, 8] to study convergence properties of $F_N^{(\theta)}$ do not directly apply to $SF_{n,m}^{(\theta)}$. However, using an approach from Desrosiers and Liu [12], it is possible to establish convergence for generic $\theta > 0$. A precise statement and proof can be found in Appendix A. For the remaining parameter values, we offer two interpretations: Either view $SF_{n,m}^{(\theta)}$ as a formal power series or consider each homogeneous component separately, so that convergence is not an issue.

By combining Prop. 3.3 with (24), we establish

$$L_{n,m}^{(r)}(x, y) SF_{n,m;d}^{(\theta)}(x, y; z, w) = p_{r,\theta}(z, w) SF_{n,m;d-r}^{(\theta)}(x, y; z, w) \quad (r, d \in \mathbb{Z}_{\geq 0}),$$

where $SF_{d-r}^{(\theta)} \equiv 0$ if $d < r$. As a direct consequence of (25), (26) and (33), we thus obtain the following lemma.

Lemma 3.4 *For $\theta \in \mathbb{C}$ not of the form (34) and $L \in Q_{n,m}$, we have*

$$L(x, y) SF_{n,m}^{(\theta)}(x, y; z, w) = \psi_{n,m}(L)(z, w) SF_{n,m}^{(\theta)}(x, y; z, w).$$

Using this joint eigenfunction property, it is straightforward to verify that the Harish-Chandra homomorphism $\psi_{n,m}$ maps $Q_{n,m}$ *one-to-one* onto $\Lambda_{n,m}$. To this end, let $L \in Q_{n,m}$ be such that $\psi_{n,m}(L) = 0$. Since the $SP_\lambda^{(\theta)}$ ($\lambda \in H_{n,m}$) span $\Lambda_{n,m}$, it follows that L vanishes on $\Lambda_{n,m}$. As long as the Vandermonde polynomial

$$V_{n+m}(x, y) := \prod_{1 \le i < j \le n} (x_i - x_j) \cdot \prod_{1 \le i < j \le m} (y_i - y_j) \cdot \prod_{i=1}^{n} \prod_{i=1}^{m} (x_i - y_j) \neq 0,$$

we can use $f_r := p_{r,\theta} - p_{r,\theta}(x, y)$ with $r = 1, \ldots, n+m$ as coordinate functions (centered) at $(x, y) \in \mathbb{C}^n \times \mathbb{C}^m$. (This is easily seen by computing the Jacobian determinant $\partial(p_{1,\theta}, \ldots, p_{n+m,\theta})/\partial(x_1, \ldots, x_n, y_1, \ldots, y_m)$.) Then, we can write

$$L = \sum c_\alpha \partial_{f_1}^{\alpha_1} \cdots \partial_{f_{n+m}}^{\alpha_{n+m}}, \tag{35}$$

where the coefficient functions $c_\alpha \neq 0$ only for a finite subset of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_{n+m}) \in \mathbb{Z}_{\ge 0}^{n+m}$. Now, assume that L is a non-trivial differential operator. Then, there exists $\alpha \in \mathbb{Z}_{\ge 0}^{n+m}$ of minimal weight $|\alpha| = \alpha_1 + \cdots + \alpha_{n+m}$ such that $c_\alpha \neq 0$. However, since L amounts to the zero operator on $\Lambda_{n,m}$, we have

$$0 = L f_1^{\alpha_1} \cdots f_{n+m}^{\alpha_{n+m}} = c_\alpha \alpha!,$$

which contradicts the above assumption.

We can thus conclude that $\psi_{n,m} : Q_{n,m} \to \Lambda_{n,m}$ is an isomorphism, write

$$L_p = \psi_{n,m}^{-1}(p) \quad (p \in \Lambda_{n,m}) \tag{36}$$

and reformulate Lemma 3.4 as the following 'deformed' analogue of Prop. 3.1.

Proposition 3.5 *For each $p \in \Lambda_{n,m}$, we have*

$$L_p(x, y) SF_{n,m}^{(\theta)}(x, y; z, w) = p(z, w) SF_{n,m}^{(\theta)}(x, y; z, w),$$

as long as the θ-values (34) are avoided.

Just as in the Jack polynomial case (cf. Cor. 3.2), the corresponding result for the homogenous components $SF_{n,m;d}^{(\theta)}$ immediately follows.

Corollary 3.6 *For all $d, k \in \mathbb{Z}_{\ge 0}$ and $p \in \Lambda_{n,m}^k$, we have*

$$L_p(x, y) SF_{n,m;d}^{(\theta)}(x, y; z, w) = p(z, w) F_{n,m;d-k}^{(\theta)}(x, y; z, w),$$

with $SF_{n,m;d-k}^{(\theta)} \equiv 0$ when $d < k$ and θ not of the form (34).

4 The Bilinear Form

We are now ready to introduce the relevant bilinear form on $\Lambda_{n,m}$.

Definition 4.1 Assuming that $\theta \in \mathbb{C}$ is not a negative rational number or zero, we define a (complex) bilinear form on $\Lambda_{n,m}$ by

$$(p, q)_{n,m} = (L_p q)(0) \ \ (p, q \in \Lambda_{n,m}).$$

Writing L^* for the adjoint of $L \in Q_{n,m}$, we proceed to formulate and prove some basic properties of $(\cdot, \cdot)_{n,m}$.

Proposition 4.2 *Excluding the values of* θ *given in (34), we have*

(1) $(p, q)_{n,m} = L_p(x, y)L_q(z, w)SF^{(\theta)}_{n,m}(x, y; z, w)|_{x=y=z=w=0}$,
(2) $(p, q)_{n,m} = 0$ *whenever* $p, q \in \Lambda_{n,m}$ *are homogenous of different degrees,*
(3) $(\cdot, \cdot)_{n,m}$ *is symmetric,*
(4) $L^*_p = p$.

Proof (1) By degree considerations, it is readily seen that the polynomial $L_p(x, y)$ $L_q(z, w)SF^{(\theta)}_{n,m;d}(x, y; z, w)$ vanishes at $x = y = z = w = 0$ unless $\deg p = \deg q = d$, in which case Cor. 3.6 entails

$$\begin{aligned} L_p(x, y)L_q(z, w)SF^{(\theta)}_{n,m;d}(x, y; z, w) &= L_p(x, y)q(x, y)SF^{(\theta)}_{n,m;0} \\ &= (L_p q)(0)SF^{(\theta)}_{n,m;0} \end{aligned}$$

and, since $SF^{(\theta)}_{n,m;0} = 1$, the claim follows.
(2) Follows immediately from the definition of $(\cdot, \cdot)_{n,m}$ and the fact that L_p is homogenous of degree $-\deg p$.
(3) Clear from (33) and Property (1).
(4) By definition, we have

$$(s, L_p q)_{n,m} = (L_s L_p q)(0) = (L_{sp} q)(0) = (ps, q)_{n,m}$$

for all $p, q, s \in \Lambda_{n,m}$.

□

As is clear from (1) in Prop. 4.2, there is close connection between the generalised hypergeometric series $SF^{(\theta)}_{n,m}$ and the bilinear form $(\cdot, \cdot)_{n,m}$. This connection if further clarified in the following proposition, which identifies $SF^{(\theta)}_{n,m}$ as the reproducing kernel of $(\cdot, \cdot)_{n,m}$.

Proposition 4.3 *For each* $p \in \Lambda_{n,m}$*, we have*

$$\big(p(x, y), SF^{(\theta)}_{n,m}(x, y; z, w)\big)_{n,m} = p(z, w), \tag{37}$$

where we assume that θ *is not of the form (34).*

Proof The result is an immediate consequence of Def. 4.1, Prop. 3.5 and the observation that

$$SF^{(\theta)}_{n,m}(0^n, 0^m; z, w) = SF^{(\theta)}_{n,m}(x, y; 0^n, 0^m) = SF^{(\theta)}_{n,m}(0^n, 0^m; 0^n, 0^m) = 1,$$

which, in turn, is clear from (33). □

Corollary 4.4 *Assuming that θ is not of the form (34), the bilinear form $(\cdot, \cdot)_{n,m}$ is nondegenerate.*

We continue to make precise the integral representation (4) for the bilinear form $(\cdot, \cdot)_{n,m}$. First of all, we note that the operator $e^{-L_{n,m}/2}$ has a well defined action on $\Lambda_{n,m}$. Indeed, since $L_{n,m}$ is homogeneous of degree -2 on $\Lambda_{n,m}$, it is locally nilpotent, and so if $p \in \Lambda_{n,m}$ has degree $d \in \mathbb{Z}_{\geq 0}$, then

$$e^{-L_{n,m}/2} p := \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{(-1)^k}{2^k k!} L^k_{n,m} p.$$

Requiring that $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ satisfy

$$\xi_n > \cdots > \xi_1, \quad \eta_m > \cdots > \eta_1, \quad \xi_i \neq \eta_j \ \ (1 \leq i \leq n, 1 \leq j \leq m), \tag{38}$$

so that we are working with $\mathrm{Im}(x_i - x_j) < 0$ $(1 \leq i < j \leq n)$ and $\mathrm{Im}(y_i - y_j) < 0$ $(1 \leq i < j \leq m)$, we fix the branch of $A_{n,m}(x, y)$ by taking the principal value of z^ρ for $z \in \mathbb{C}^*$ and $\rho \in \mathbb{C}$. From Def. 4.1, it is clear that $(1, 1)_{n,m} = 1$, which requires

$$M_{n,m} = \int_{\mathbb{R}^n + i\xi} \int_{\mathbb{R}^m + i\eta} \frac{e^{-x^2/2 + \theta^{-1} y^2/2}}{A_{n,m}(x, y)} dx dy.$$

This generalised Macdonald–Mehta integral was computed in [18]. Specifically, taking $t_i \to t_i/\sqrt{-\rho}$ and setting $\rho = \theta$ in Eq. (33), we infer from Prop. 6.1 that

$$M_{n,m} = C_{n,m} \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{j - i\theta} \cdot \prod_{i=1}^{n} \frac{\Gamma(1-\theta)}{\Gamma(1 - i\theta)} \cdot \prod_{j=1}^{m} \frac{\Gamma(1 - 1/\theta)}{\Gamma(1 - j/\theta)}, \tag{39}$$

where

$$C_{n,m} = (2\pi)^{\frac{n+m}{2}} (-\theta)^{n/2 + n(n-1)/(2\theta)} \exp\left(-i\pi \left(\frac{n(n-1)\theta}{2} + \frac{m(m-1)}{2\theta}\right)\right) \tag{40}$$

For suitable values of θ, we can now establish the validity of (4).

Proposition 4.5 *Assuming that $\mathrm{Re}\,\theta < 0$ and θ is not a negative rational number, the integral representation (4), with $M_{n,m}$ given by (39)–(40), for the bilinear form $(\cdot, \cdot)_{n,m}$ holds true as long as $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ satisfy the restrictions in (38).*

Proof Under our assumptions on θ and ξ, η, it is clear that the integral in the right-hand side of (4) is convergent

We note that the bilinear form $(\cdot, \cdot)_{n,m}$ is uniquely determined by the following properties:

(1) $(1, 1)_{n,m} = 1$,
(2) $(p, q)_{n,m} = (q, p)_{n,m}$,
(3) $(ps, q)_{n,m} = (s, L_p q)_{n,m}$,

for arbitrary $p, q, s \in \Lambda_{n,m}$. Indeed, by linearity and Property (2), it suffices to consider homogenous $p, q \in \Lambda_{n,m}$ such that $\deg p \geq \deg q$, and, from Property (3), we get

$$(p, q)_{n,m} = (1, L_p q)_{n,m}.$$

If $\deg p > \deg q$, the right-hand side is clearly zero, and in the remaining case $\deg p = \deg q$, it follows from Property (1) that

$$(1, L_p q)_{n,m} = (L_p q)(0)(1, 1)_{n,m} = (L_p q)(0),$$

(where the first equality is due to $\deg(L_p q) = 0$). Hence, writing $(p, q)'_{n,m}$ for the right-hand side of (4), it suffices to establish Properties (1)-(3) for the resulting bilinear form $(\cdot, \cdot)'_{n,m}$ on $\Lambda_{n,m}$.

Properties (1) and (2) are obvious.

To prove Property (3), we rewrite the weight function in terms of the convenient notation (19)–(20):

$$e^{-\frac{1}{2}\sum_{i=1}^{n+m}(-\theta)^{-p(i)}x_i^2} \cdot \prod_{1\leq i<j\leq n+m} (x_i - x_j)^{-2(-\theta)^{1-p(i)-p(j)}}.$$

Then, we use (22)–(23) to verify, by direct computations, that the corresponding formal adjoint of $L^{(r)}_{n,m}$ is given by

$$\widehat{L}^{(r)}_{n,m} = \sum_{i=1}^{n+m} (-\theta)^{-p(i)} \widehat{\partial}^{(r)}_i,$$

with $\widehat{\partial}^{(1)}_i = x_i - \partial^{(1)}_i$ and

$$\partial^{(r)}_i = \widehat{\partial}^{(1)}_i \widehat{\partial}^{(r-1)}_i + \sum_{j\neq i} \frac{(-\theta)^{1-p(j)}}{x_i - x_j} \left(\widehat{\partial}^{(r-1)}_i - \widehat{\partial}^{(r-1)}_j\right) \tag{41}$$

for $r > 1$.

Since $L_{p_{r,\theta}} = L^{(r)}_{n,m}$ and the deformed power sums $p_{r,\theta}(x, y)$ generate $\Lambda_{n,m}$, the desired Property (3) will follow once we prove that

$$e^{L_{n,m}/2} \widehat{L}^{(r)}_{n,m} e^{-L_{n,m}/2} = p_{r,\theta}(x, y). \tag{42}$$

Following the approach in Sect. 3, we first establish such a conjugation formula for the operators

$$\widehat{L}_{p,N} = \mathrm{Res}\big(p\big(x_1 - D_{1,N}, \ldots, x_N - D_{N,N}\big)\big) \quad (p \in \Lambda_N)$$

in Λ_N, then lift it to Λ and finally restrict the result to $\Lambda_{n,m}$.

Using the commutation relations

$$[D_{i,N}, x_j] = \begin{cases} 1 + \theta \sum_{k \neq i} \sigma_{ik}, & j = i \\ -\theta \sigma_{ij}, & j \neq i \end{cases}$$

a straightforward computation yields

$$\left[\sum_{i=1}^{N} D_{i,N}^2, x_j\right] = 2D_{j,N},$$

which entails

$$e^{\frac{1}{2}\sum_{i=1}^{N} D_{i,N}^2} x_j e^{-\frac{1}{2}\sum_{i=1}^{N} D_{i,N}^2} = e^{\frac{1}{2}\mathrm{ad}\left(\sum_{i=1}^{N} D_{i,N}^2\right)} x_j = x_j + D_{j,N},$$

where we have used the standard formula $\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X}$. It follows that

$$\begin{aligned} & e^{L_N/2}\widehat{L}_{p,N} e^{-L_N/2} \\ & = \mathrm{Res}\left(e^{\frac{1}{2}\sum_{i=1}^{N} D_{i,N}^2} p\big(x_1 - D_{1,N}, \ldots, x_N - D_{N,N}\big) e^{-\frac{1}{2}\sum_{i=1}^{N} D_{i,N}^2}\right) \\ & = p(x_1, \ldots, x_N). \end{aligned} \tag{43}$$

Consulting the proof of Thm. 2.2 in [43], it is readily seen how to lift the operators $\widehat{L}_N^{(r)} := \widehat{L}_{p_r,N}$ ($r \in \mathbb{Z}_{\geq 0}$) to $\bar{\Lambda}$. Specifically, introducing the operators

$$\widehat{L}^{(r)} := \mathrm{Res}\, E \circ (x - D_\infty)^r : \bar{\Lambda} \to \bar{\Lambda} \quad (r \in \mathbb{N}),$$

we have the commutative diagram

$$\begin{array}{ccc} \bar{\Lambda} & \xrightarrow{\widehat{L}^{(r)}} & \bar{\Lambda} \\ \varphi_N \downarrow & & \downarrow \varphi_N \\ \Lambda_N & \xrightarrow{\widehat{L}_N^{(r)}} & \Lambda_N \end{array} \tag{44}$$

for each $r \in \mathbb{N}$. Since $f \in \bar{\Lambda}$ satisfies $\varphi_N(f) = 0$ for all $N \in \mathbb{N}$ if and only if $f \equiv 0$ (cf. Lemma 2.3 in [43]), we infer from (18) and (43)–(44) that

$$e^{L^{(2)}/2}\widehat{L}^{(r)} e^{-L^{(2)}/2} = p_r \quad (r \in \mathbb{Z}_{\geq 0}).$$

Finally, by easily identifiable modifications of the discussion in Sect. 3 of [43], we find that the diagram

$$\begin{array}{ccc} \bar{\Lambda} & \xrightarrow{\widehat{L}^{(r)}} & \bar{\Lambda} \\ {\scriptstyle \varphi_{n,m}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{n,m}} \\ \Lambda_{n,m} & \xrightarrow{\widehat{L}^{(r)}_{n,m}} & \Lambda_{n,m} \end{array} \tag{45}$$

is commutative for all $r \in \mathbb{N}$, and consequently that (42) holds true. □

Remark 4.1 Note that the integral representation (4) is independent of the specific choice of parameter values. When altering ξ, η such that a hyperplane $\xi_i = \eta_j$ is crossed, it would seem that we pick up a residue term and thus alter the representation. However, the quasi-invariance conditions (11) ensure that any such residue vanishes.

5 Orthogonality Relations

Having proved in Prop. 4.3 that $SF^{(\theta)}_{n,m}$ is the reproducing kernel of $(\cdot,\cdot)_{n,m}$, the desired orthogonality relations for the super-Jack polynomials are easily inferred from the definition in (33) of $SF^{(\theta)}_{n,m}$.

More specifically, setting $p(x, y) = SC^{(\theta)}_{\mu}(x, y)$ in (37), substituting the latter series expansion in (33) and comparing coefficients, we obtain

$$\big(SC^{(\theta)}_{\mu}, SC^{(\theta)}_{\lambda}\big)_{n,m} = \delta_{\mu\lambda}|\lambda|!SC^{(\theta)}_{\lambda}(1^{n+m}) \quad (\mu, \lambda \in H_{n,m}), \tag{46}$$

where $\delta_{\lambda\mu}$ denotes the Kronecker delta. Keeping the definitions of the two homomorphisms ϵ_X (10) and $\varphi_{n,m}$ (12) in mind, it becomes clear from (13) that

$$SP^{(\theta)}_{\lambda}(1^{n+m}) = \epsilon_{n-m/\theta}\big(P^{(\theta)}_{\lambda}\big),$$

where by $\epsilon_{n-m/\theta}$ we mean the homomorphism $\Lambda \to \mathbb{C}$ given by ϵ_X followed by evaluation at $X = n - m/\theta$. Substituting (31) in (46), we can now use (8)–(9) to rewrite the right-hand side of the resulting equation in terms of the generalised Pochhammer symbol

$$(a)^{(\theta)}_{\lambda} = \prod_{i=1}^{\ell(\lambda)} (a - \theta(i-1))_{\lambda_i}, \tag{47}$$

with $(a)_m$ the ordinary Pochhammer symbol, and the inverse $b^{(\theta)}_{\lambda}$ of the quadratic norm of $P^{(\theta)}_{\lambda}$, thus arriving at the following result.

Theorem 5.1 *As long as θ is not a negative rational number or zero, we have*

$$\big(SP_\mu^{(\theta)}, SP_\lambda^{(\theta)}\big)_{n,m} = \delta_{\mu\lambda}\frac{(\theta n - m)_\lambda^{(\theta)}}{b_\lambda^{(\theta)}} \ \ (\mu, \lambda \in H_{n,m}).$$

We conclude this section by sketching an alternative approach to this orthogonality result, starting from the bilinear form on Λ given by

$$(p, q)_{p_0} = \epsilon_0(L_p q),$$

with the homomorphisms $p \mapsto L_p$ and $\epsilon_0 : \Lambda \to \mathbb{C}$ characterised by $p_r \mapsto L^{(r)}$ and $\epsilon(p_r) = 0$, respectively, for $r \in \mathbb{N}$, and where we think of p_0 as a complex parameter. Proceeding as above, it is readily seen that $F^{(\theta,p_0)}(x, y) := \sum_{d=0}^\infty F_d^{(\theta,p_0)}(x, y)$ is the reproducing kernel of $(\cdot,\cdot)_{p_0}$, which, in turn, implies

$$(P_\mu, P_\lambda)_{p_0} = \delta_{\mu\lambda}\frac{(\theta p_0)_\lambda^{(\theta)}}{b_\lambda^{(\theta)}}.$$

Setting $p_0 = n - m/\theta$, we note that $(\theta p_0)_\lambda^{(\theta)} = (\theta n - m)_\lambda^{(\theta)} = 0$ if and only if $(n + 1, m + 1) \in \lambda$ or equivalently $\lambda \notin H_{n,m}$. (To be precise, the ‘only if’ part of this claim holds true as long as we avoid the θ-values (34).) In other words, the kernel of $(\cdot,\cdot)_{n-m/\theta}$ equals

$$K_{n,m} := \operatorname{span}\Big\{P_\lambda^{(\theta)} \mid \lambda \notin H_{n,m}\Big\}.$$

From [41] (see Thm. 2), we recall that $K_{n,m}$ is also the kernel of $\varphi_{n,m} : \Lambda \to \Lambda_{n,m}$, so that $(\cdot,\cdot)_{n-m/\theta}$ descends to a non-degenerate bilinear form on the factor space $\Lambda/K_{n,m} \cong \Lambda_{n,m}$ that amounts to $(\cdot,\cdot)_{n,m}$.

6 Lassalle–Nekrasov Correspondence

In this section, we provide a new proof of the Lassalle–Nekrasov correspondence between the deformed trigonometric and rational harmonic Calogero–Moser–Sutherland systems; and, in addition, we show that the correspondence is isometric in a natural sense.

Using the notation (19)–(20), we recall from [40, 41] that if (22) is modified such that $\partial_i^{(1)} = (-\theta)^{p(i)} x_i \partial/\partial x_i$ and

$$\partial_i^{(r)} = \partial_i^{(1)}\partial_i^{(r-1)} - \frac{1}{2}\sum_{j\neq i}(-\theta)^{1-p(j)}\frac{x_i + x_j}{x_i - x_j}\big(\partial_i^{(r-1)} - \partial_j^{(r-1)}\big)$$

for $r > 1$, the differential operators

$$\mathcal{L}_{n,m}^{(r)} = \sum_{i=1}^{n+m} (-\theta)^{-p(i)} \partial_i^{(r)} \quad (r \in \mathbb{N}) \tag{48}$$

where $\mathcal{L}_{n,m}^{(2)} = \mathcal{L}_{n,m}$, pairwise commute and are simultaneously diagonalised by the super-Jack polynomials. As we shall see below, the Lassalle–Nekrasov correspondence implies that corresponding quantum integrals for the rational harmonic system are given by

$$\begin{aligned}\mathscr{L}_{n,m}^{(r)} &= \mathcal{L}_{n,m}^{(r)} + \frac{1}{2}\big[\mathcal{L}_{n,m}^{(r)}, L_{n,m}\big] + \frac{1}{2^2 2!}\big[\big[\mathcal{L}_{n,m}^{(r)}, L_{n,m}\big], L_{n,m}\big] \\ &\quad + \cdots + \frac{1}{2^r r!}\big[\cdots\big[\mathcal{L}_{n,m}^{(r)}, L_{n,m}\big], \ldots, L_{n,m}\big] \quad (r \in \mathbb{N}),\end{aligned} \tag{49}$$

where

$$\mathscr{L}_{n,m}^{(1)} = \sum_{i=1}^{n+m} x_i \frac{\partial}{\partial x_i} - L_{n,m}.$$

First, we deduce an alternative description of the map $e^{-L_{n,m}/2} : \Lambda_{n,m} \to \Lambda_{n,m}$. From Prop. 3.5, it is clear that

$$\begin{aligned}G_{n,m}^{(\theta)}(x, y; z, w) &:= SF_{n,m}^{(\theta)}(x, y; z, w) e^{-p_{2,\theta}(z,w)/2} \\ &= e^{-L_{n,m}(x,y)/2} SF_{n,m}^{(\theta)}(x, y; z, w),\end{aligned}$$

and so (5) and (33) entail the generating function expansion

$$G_{n,m}^{(\theta)}(x, y; z, w) = \sum_{\lambda \in H_{n,m}} \frac{b_\lambda(\theta)}{(\theta n - m)_\lambda^{(\theta)}} SH_\lambda^{(\theta)}(x, y) SP_\lambda^{(\theta)}(z, w),$$

cf. (8)–(9), (31) and (47). Hence, invoking Thm. 5.1, we obtain the following proposition.

Proposition 6.1 *Assuming θ is not of the form (34), we have*

$$e^{-L_{n,m}(x,y)/2} p(x, y) = \big(G_{n,m}^{(\theta)}(x, y; z, w), p(z, w)\big)_{n,m}$$

for each $p \in \Lambda_{n,m}$.

For suitable θ-values, we proceed to introduce an additional bilinear form on $\Lambda_{n,m}$, which can be viewed as a natural generalisation to the deformed case of the L^2 inner product over $\mathbb{R}^N$ with weight function $e^{-x^2/2} \cdot \prod_{1 \le i < j \le N} |x_i - x_j|^{2\theta}$.

Definition 6.2 Assuming that $\theta \in \mathbb{C}$ is not a negative rational number, that it satisfies $\operatorname{Re}\theta < 0$ and $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$ satisfy (38), we define a bilinear form on $\Lambda_{n,m}$ by

$$\{p, q\}_{n,m} = M_{n,m}^{-1} \int_{\mathbb{R}^n+i\xi} \int_{\mathbb{R}^m+i\eta} p(x, y)q(x, y)\frac{e^{-x^2/2+\theta^{-1}y^2/2}}{A_{n,m}(x, y)} dxdy \ \ (p, q \in \Lambda_{n,m}),$$

where the value of $M_{n,m}$ is given by (39)–(40).

We are now ready to state and prove the main result of this section.

Theorem 6.3 *For θ not of the form (34), the map $e^{-L_{n,m}/2} : \Lambda_{n,m} \to \Lambda_{n,m}$ intertwines the deformed trigonometric and rational harmonic Calogero–Moser–Sutherland operators given by (48) and (49), respectively. More precisely, the diagram*

$$\begin{array}{ccc} \Lambda_{n,m} & \xrightarrow{e^{-L_{n,m}/2}} & \Lambda_{n,m} \\ \mathcal{L}_{n,m}^{(r)} \big\downarrow & & \big\downarrow \mathscr{L}_{n,m}^{(r)} \\ \Lambda_{n,m} & \xrightarrow{e^{-L_{n,m}/2}} & \Lambda_{n,m} \end{array}$$

is commutative for all $r \in \mathbb{N}$.

If, in addition, $\operatorname{Re}\theta < 0$, we have

$$\left\{e^{-L_{n,m}/2}p, e^{-L_{n,m}/2}q\right\}_{n,m} = (p, q)_{n,m} \ \ (p, q \in \Lambda_{n,m}).$$

Proof Using the formula $\operatorname{Ad}_{e^X} = e^{\operatorname{ad}_X}$, with X the operator of multiplication by $p_{2,\theta}/2$, as well as the fact that ad_X lowers the order of a differential operator by at least one, we deduce

$$e^{p_{2,\theta}/2}\mathcal{L}_{n,m}^{(r)}e^{-p_{2,\theta}/2} = \mathcal{L}_{n,m}^{(r)} + \frac{1}{2}\left[p_{2,\theta}, \mathcal{L}_{n,m}^{(r)}\right] + \frac{1}{2^2 2!}\left[p_{2,\theta}, \left[p_{2,\theta}, \mathcal{L}_{n,m}^{(r)}\right]\right]$$
$$+ \cdots + \frac{1}{2^r r!}\left[p_{2,\theta}, \ldots, \left[p_{2,\theta}, \mathcal{L}_{n,m}^{(r)}\right] \cdots\right].$$

As a direct consequence of the definition in (33) of $SF_{n,m}^{(\theta)}$ and the fact that the super-Jack polynomials are joint eigenfunctions of the operators $\mathcal{L}_{n,m}^{(r)}$, we get

$$\mathcal{L}_{n,m}^{(r)}(x, y)SF_{n,m}^{(\theta)}(x, y; z, w) = \mathcal{L}_{n,m}^{(r)}(z, w)SF_{n,m}^{(\theta)}(x, y; z, w).$$

Combining the previous two formulae with Prop. 3.5, we infer

$$\begin{aligned} &\mathcal{L}_{n,m}^{(r)}(z, w)G_{n,m}^{(\theta)}(x, y; z, w) \\ &= e^{-p_{2,\theta}(z,w)/2}\left(e^{p_{2,\theta}/2}\mathcal{L}_{n,m}^{(r)}e^{-p_{2,\theta}/2}\right)(z, w)SF_{n,m}^{(\theta)}(x, y; z, w) \\ &= \mathscr{L}_{n,m}^{(r)}(x, y)G_{n,m}^{(\theta)}(x, y; z, w). \end{aligned}$$

Since Thm. 5.1 and the pertinent joint eigenfunction property imply

$$\big(\mathcal{L}^{(r)}_{n,m} p, q\big)_{n,m} = \big(p, \mathcal{L}^{(r)}_{n,m} q\big)_{n,m} \quad (p, q \in \Lambda_{n,m}),$$

it follows from Prop. 6.1 and our reasoning thus far that

$$\begin{aligned}\big(\mathscr{L}^{(r)}_{n,m} e^{-L_{n,m}/2}\big)(p) &= \big(\mathscr{L}^{(r)}_{n,m}(x, y) G^{(\theta)}_{n,m}(x, y; z, w), p(z, w)\big)_{n,m}\\ &= \big(G^{(\theta)}_{n,m}(x, y; z, w), \mathcal{L}^{(r)}_{n,m}(z, w) p(z, w)\big)_{n,m}\\ &= (e^{-L_{n,m}/2} \mathcal{L}^{(r)}_{n,m})(p)\end{aligned}$$

for all $r \in \mathbb{N}$ and $p \in \Lambda_{n,m}$.

Finally, if $\operatorname{Re}\theta < 0$, it is clear from Prop. 4.5 and Def. 6.2 that the map $e^{-L_{n,m}/2}$: $\Lambda_{n,m} \to \Lambda_{n,m}$ becomes an isometry when the domain is equipped with the bilinear form $(\cdot, \cdot)_{n,m}$ and the codomain with $\{\cdot, \cdot\}_{n,m}$. □

Remark 6.1 This is precisely the Lassalle–Nekrasov correspondence we had in mind, and, while the first part of the result should be compared with Thm. 6 in [19], the above proof runs in parallel with that of Thm. 4 in [19], which pertains to the ordinary undeformed case.

Remark 6.2 As detailed in Thm. 8 in [19], it is readily inferred that the deformed rational harmonic Calogero–Moser–System is integrable. This was first proved independently by Desrosiers and the author [11] and Feigin [17]; see also Berest and Chalykh [6].

Corollary 6.4 *For $\theta \in \mathbb{C}$ satisfying $\operatorname{Re}\theta < 0$ while not being equal to a negative rational number, we have*

$$\big\{SH^{(\theta)}_{\mu}, SH^{(\theta)}_{\lambda}\big\}_{n,m} = \delta_{\mu\lambda} \frac{(\theta n - m)^{(\theta)}_{\lambda}}{b^{(\theta)}_{\lambda}} \quad (\mu, \lambda \in H_{n,m}).$$

Proof Taking $p = SH^{(\theta)}_{\mu}$ and $q = SH^{(\theta)}_{\lambda}$ in Thm. 6.3 and invoking Thm. 5.1, we immediately obtain the claim. □

7 Outlook

In this paper, we have worked with deformed Calogero–Moser–Sutherland operators and corresponding eigenfunctions associated with deformations of root systems of type A. It seems plausible that our results can be generalised to the BC case, with the super-Jack polynomials being replaced by the super-Jacobi polynomials introduced in [42]. Indeed, the constructions and results from [5, 11, 43] that we have relied on are available also in the BC case.

We also note that Feigin's [17] approach to integrability of deformed Calogero–Moser–Sutherland operators, using special representations of rational Cherednik algebras, might provide a different way to establish the present results and point the way towards further generalisations; and the recent orthogonality results in [2] on super-Macdonald polynomials hint at generalisations to the difference case.

Due, in particular, to parameter-independent singularities of eigenfunctions, deformed Calogero–Moser–Sutherland systems were for a long time seen as problematic to interpret within (quantum) physics. However, in recent years, the deformed trigonometric and even elliptic systems have naturally appeared in a quantum field theory formulation of the ordinary systems [3, 7] as well as in the context of super-symmetric gauge theories [9, 32]. It would be interesting to explore possible connections to the results presented in this paper, not least the Lassalle–Nekrasov correspondence between deformed trigonometric and rational harmonic systems.

It is also interesting to note that, in contrast to the situation in [1], the bilinear forms introduced in Defs. 4.1 and 6.2 are nondegenerate for generic θ-values and it is not obvious whether a natural positive definite inner product can be extracted, e.g. by restricting attention to particular parameter values and a subspace of $\Lambda_{n,m}$. Insights into this problem might provides clues on potential (physical) interpretations of the results obtained in this paper and vice versa.

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Appendix A. On Convergence of Generalised Hypergeometric Series

In this appendix, we study convergence properties of the generalised hypergeometric series $SF^{(\theta)}_{n,m}(x, y; z, w)$ (33), restricting, for simplicity, attention to $\theta > 0$. Specifically, we prove the proposition below by following the approach of Desrosiers and Liu [12] (see Appendix B), who considered generalised hypergeometric series ${}_pSF^{(\alpha)}_q(a_1, \dots, a_p; b_1, \dots, b_q; x, y)$, depending on $p + q$ parameters in addition to $\alpha = 1/\theta$, where the special case $p = q = 0$ corresponds to the specialisation $(z, w) = (1^n, 1^m)$ of $SF^{(\theta)}_{n,m}(x, y; z, w)$.

Proposition A.1 *Assume that $\theta > 0$ is such that $\theta \neq i/j$ for any $1 \leq i \leq m$, $1 \leq j \leq n$. Then $SF^{(\theta)}_{n,m}(x, y; z, w)$ is analytic for all $(x, y), (z, w) \in \mathbb{C}^n \times \mathbb{C}^m$.*

Proof Letting $\chi^{\lambda}_{\mu}(\theta)$ denote the coefficient of p_μ in the power sum expansion of the Jack symmetric function $P^{(\theta)}_{\lambda}$, we get from (12) and (13) that

$$SP^{(\theta)}_{\lambda}(x, y) = \sum_{\mu} \chi^{\lambda}_{\mu}(\theta) p_{\mu,\theta}(x, y),$$

where the sum extends over partitions μ such that $|\mu| = |\lambda|$. Just as in Lemma B.1 in [12], we use the Cauchy-Schwartz inequality to deduce the bound

$$\left|SP^{(\theta)}_{\lambda}(x, y)\right|^2 \leq \left(\sum_{\mu} \chi^{\lambda}_{\mu}(\theta)^2 \theta^{-\ell(\mu)} z_\mu\right) \cdot \left(\sum_{\mu} |p_{\mu,\theta}(x, y)|^2 \theta^{\ell(\mu)} z^{-1}_{\mu}\right)$$

and, since the former sum amounts to the norm of P_λ with respect to the scalar product $\langle\cdot, \cdot\rangle$ (cf. (6)), Stanley's formula (8) amounts to

$$\sum_{\mu} \chi^{\lambda}_{\mu}(\theta)^2 \theta^{-\ell(\mu)} z_\mu = \frac{1}{b^{(\theta)}_{\lambda}}.$$

Moreover, we have

$$\begin{aligned} |p_{\mu,\theta}(x, y)|^2 \theta^{\ell(\mu)} &\leq p_{\mu,-\theta}((|x_1|, \ldots, |x_n|), (|y_1|, \ldots, |y_m|))^2 \theta^{\ell(\mu)} \\ &\leq ||(x, y)||^{2|\mu|}_{\infty} \cdot \left(\sqrt{\theta} n + m/\sqrt{\theta}\right)^{2\ell(\mu)}. \end{aligned}$$

Using $\ell(\mu) \leq |\mu| = |\lambda|$ and $\sum_{|\mu|=|\lambda|} z^{-1}_{\mu} = 1$, we thus arrive at the bound

$$|SP^{(\theta)}_{\lambda}(x, y)| \leq \frac{1}{\sqrt{b^{(\theta)}_{\lambda}}} \left(||(x, y)||_\infty \cdot \left(\sqrt{\theta} n + m/\sqrt{\theta}\right)\right)^{|\lambda|},$$

which, when combined with (9) with $X = n - m/\theta$ and (31) yields

$$\begin{aligned} S^{(\theta)}(x, y; z, w) &:= \sum_{\lambda \in H_{n,m}} \left| \frac{1}{|\lambda|!} \frac{SC^{(\theta)}_{\lambda}(x, y) SC^{(\theta)}_{\lambda}(z, w)}{SC^{(\theta)}_{\lambda}(1^{n+m})} \right| \\ &\leq \sum_{\lambda \in H_{n,m}} \frac{\left(||(x, y)||_\infty \cdot ||(z, w)||_\infty \cdot \left(\sqrt{\theta} n + m/\sqrt{\theta}\right)^2\right)^{|\lambda|}}{\prod_{s\in\lambda} |\theta(n - l'(s)) - (m - a'(s))|}. \end{aligned}$$

For a partition $\lambda \in H_{n,m}$, we let

$$e(\lambda) = (\langle\lambda_1 - m\rangle, \cdots, \langle\lambda_n - m\rangle), \quad s(\lambda) = (\langle\lambda'_1 - n\rangle, \cdots, \langle\lambda'_m - n\rangle),$$

with $\langle a\rangle = \max(0, a)$, so that $e(\lambda)$ and $s(\lambda)$ correspond to the set of boxes located to the 'east' and 'south', respectively, of the $m \times n$ rectangle (m^n) in the diagram of λ, cf. Fig. 1 in [1]. Then, we have

$$\prod_{s\in\lambda}|\theta(n-l'(s))-(m-a'(s))| \\ =(\theta n)^{(\theta)}_{e(\lambda)}\cdot(\theta)^{|s(\lambda)|}(m/\theta)^{(1/\theta)}_{s(\lambda)}\cdot\prod_{s\in\lambda\cap(m^n)}|\theta(n-l'(s))-(m-a'(s))|.$$

As long as the restrictions (34) are in place, each factor in the product over boxes in $\lambda\cap(m^n)$ in the right-hand side is non-zero and independent of λ, which implies that the product is uniformly bounded below by some positive constant. Furthermore, under our assumption $\theta>0$, it is clear from (47) that

$$(\theta n)^{(\theta)}_{e(\lambda)}\geq\min\{1,\theta\}^{|e(\lambda)|}\cdot e(\lambda)!,\quad (\theta)^{|s(\lambda)|}(m/\theta)^{(1/\theta)}_{s(\lambda)}\geq\min\{1,\theta\}^{|s(\lambda)|}\cdot s(\lambda)!.$$

Hence, we can find constants $C,r>0$ such that

$$S^{(\theta)}(x,y;z,w)\leq C\big(1+\big(||(x,y)||_\infty\cdot||(z,w)||_\infty\big)^{mn}\big) \\ \cdot\sum_{(\mu,\nu)\in\mathbb{Z}^n_{\geq0}\times\mathbb{Z}^m_{\geq0}}\frac{\big(r\cdot||(x,y)||_\infty\cdot||(z,w)||_\infty\big)^{|\mu|+|\nu|}}{\mu!\nu!}.$$

Since the sum converges locally uniformly to $\exp((n+m)r\cdot||(x,y)||_\infty\cdot||(z,w)||_\infty)$, the assertion follows. □

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