On Discretizing Uniform Norms of Exponential Sums

András Kroó^{1,2}

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Abstract

In this paper, we consider the uniform norm discretization problem for general real multivariate exponential sums $p(\mathbf{w}) = \sum_{0 \leq j \leq n} c_j e^{\langle \mu_j, \mathbf{w} \rangle}$, $\mu_j, \mathbf{w} \in \mathbb{R}^d$. Given arbitrary $0 < \tau \leq 1$ this problem consists in finding discrete point sets $\mathbf{w}_j \in K$, $1 \leq j \leq N$ in the compact domain $K \subset \mathbb{R}^d$, $d \geq 1$ so that for every $p(\mathbf{w})$ as above we have

$$\max_{\mathbf{w} \in K} |p(\mathbf{w})| \leq (1 + \tau) \max_{1 \leq j \leq N} |p(\mathbf{w}_j)|.$$

Using certain new Bernstein–Markov type inequalities for exponential sums it will be verified that for convex polytopes and convex polyhedral cones K in $\mathbb{R}^d$, $d \geq 1$ there exist meshes $\mathbf{w}_1, \dots, \mathbf{w}_N \subset K$ of cardinality

$$N \leq c \left(\frac{n}{\sqrt{\tau}} \right)^d \ln^d \frac{\mu_n^*}{\delta \tau}, \quad \mu_n^* := \max_{1 \leq j \leq n} |\mu_j|$$

for which the above inequality holds for any multivariate exponential sum p with exponents satisfying the separation condition $|\mu_k - \mu_j| \geq \delta$, $j \neq k$, $\delta > 0$. In addition, the optimality of the cardinality estimates will be also discussed.

Keywords Multivariate exponential sums · Discretization of norm · Markov type inequalities · Optimal meshes

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✉ András Kroó
kroo@renyi.hu

¹ Alfréd Rényi Institute of Mathematics, Budapest, Hungary

² Department of Analysis, Budapest University of Technology and Economics, Budapest, Hungary

Introduction

In the past 15–20 years the problem of discretization of uniform and L_q norms in various finite dimensional spaces has been widely investigated. In case of L_q norms for trigonometric polynomials this problem is usually referred to as the *Marcinkiewicz–Zygmund type problem*, on the other hand when uniform norm and algebraic polynomials are considered then the terms *norming sets* or *optimal meshes* are usually used in the literature.

Consider the space $L_q(K)$, $1 \leq q \leq \infty$ endowed with some probability measure on the compact set $K \subset \mathbb{R}^d$. Then, given an $n+1$ -dimensional subspace $U_n \subset L_q(K)$ the Marcinkiewicz–Zygmund type problem for $1 \leq q < \infty$ consists in finding discrete point sets $Y_N = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset K$ such that for any $u \in U_n$ we have

$$c_1 \|u\|_{L_q(K)}^q \leq \frac{1}{N} \sum_{1 \leq j \leq N} |u(\mathbf{x}_j)|^q \leq c_2 \|u\|_{L_q(K)}^q \quad (1)$$

with some constants $c_1, c_2 > 0$ depending only on q, d and K .

Similarly if $q = \infty$ and $U_n \subset C(K)$ the goal is to ensure that for every $u \in U_n$

$$\|u\|_K \leq c(d, K) \max_{1 \leq j \leq N} |u(\mathbf{x}_j)|. \quad (2)$$

Here and throughout this paper $\|p\|_K$ stands for the uniform norm of the function on the compact set $K \subset \mathbb{R}^d$.

The crucial question in the above-mentioned problems is the cardinality N of the discrete point set. Naturally, the goal is to ensure relations (1) and (2) with possibly minimal number of points. Clearly, we must have $N > n$ in order for (1) or (2) to hold, that is cardinality of the mesh cannot be less than the dimension of the space. So ideally one would like to achieve discretization with $N \sim n$ points.

Discretization of uniform and L_q norms for various spaces of algebraic and trigonometric polynomials is widely used in the study of the convergence of Fourier series, Lagrange and Hermite interpolation, positive quadrature formulas, scattered data interpolation, construction of discrete extremal sets of Fekete and Leja type, etc. The first discretization result for the L_q norm of univariate trigonometric polynomials goes back to Marcinkiewicz and Zygmund, see [1]. Various generalizations of Marcinkiewicz–Zygmund type inequalities for multivariate trigonometric and algebraic polynomials can be found in recent papers [2], [3] and [4]. In this paper, we will be concerned primarily with the uniform version (2) of the discretization problem. Some early results on discretization of uniform norm for multivariate algebraic polynomials can be found in [5] and [6] where the discrete sets satisfying (2) were studied. Let us denote by P_n^d the space of real algebraic polynomials in d variables of degree $\leq n$. In [7] it was conjectured that any convex body $K \subset \mathbb{R}^d$ possesses a discrete mesh $Y_N = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset K$ of cardinality $N \sim n^d \sim \dim P_n^d$ for which relation (2) holds. Subsequently, this conjecture was verified in [8] in case $d = 2$.

In this paper, we will consider problem (2) for general real multivariate exponential polynomials of the form

$$p(\mathbf{w}) = \sum_{1 \leq j \leq n} c_j e^{\langle \mu_j, \mathbf{w} \rangle}, \quad \mathbf{w} \in \mathbb{R}^d, \quad c_j \in \mathbb{R} \quad (3)$$

with arbitrary given $\mu_j \in \mathbb{R}^d$, $d \geq 1$. Throughout the paper the “degree” of these exponential polynomials is defined by

$$\mu_n^* := \max_{0 \leq j \leq n} |\mu_j|.$$

We will be seeking the solution of the discretization problem in an *exact form* with constant $c(d, K) = 1 + \tau$, $\forall \tau > 0$ in (2). Naturally in this case we aim for discrete meshes with possibly optimal cardinality with respect to both n and τ . Such exact meshes were introduced in [3], p. 5, see also [9].

One of the main results of this paper (Theorem 3) asserts that when $d = 1$ and $K = [0, 1]$ then for arbitrary $0 < \tau < 1$ there exist meshes $Y_N = \{x_1, \dots, x_N\} \subset [0, 1]$ of cardinality

$$N \leq \frac{cn}{\sqrt{\tau}} \ln \frac{\mu_n^*}{\delta \sqrt{\tau}} \quad (4)$$

so that

$$\|p(x)\|_{[0,1]} \leq (1 + \tau) \max_{1 \leq j \leq N} |p(x_j)|$$

for every exponential polynomial (3) with arbitrary $\mu_j \in \mathbb{R}$, $\mu_{j+1} - \mu_j \geq \delta$, $0 \leq j \leq n - 1$. An important feature of this result is the fact that the number of points in the discrete mesh depends primarily on the number n of exponents $\mu_j \in \mathbb{R}$, while the degree μ_n^* and separation parameter δ of the exponential polynomials appears only in the logarithmic term. So this bound is “almost” degree and exponent independent. In addition, above result provides **universal** discrete meshes which are suitable for all n term univariate exponential sums of given degree and separation. A similar result is shown to hold even on unbounded domain $[0, \infty)$ provided that $\mu_j < 0$ (Theorem 5).

The upper bound (4) for the cardinality of the discrete mesh turns out to be *near optimal* in the sense that it is sharp in general up to the logarithmic term with respect to both n and τ (Theorem 4). Let us note that existence of near optimal discretization meshes for multivariate algebraic polynomials was verified in [10], p. 74 using Fekete points for general compact sets. The crucial property used in [10] was the fact that algebraic polynomials are closed relative to multiplication. This multiplicative property of course does not hold for general exponential polynomials (3) and hence the method of Fekete points is not applicable in our case. Moreover, in contrast with the trigonometric case considered, e.g., in [3], since the exponents $\mu_j \in \mathbb{R}^d$ in the exponential sums (3) are arbitrary the basis functions $e^{\langle \mu_j, \mathbf{w} \rangle}$ are not orthogonal in

our setting and hence this crucial Fourier analytic tool is not available, too. Therefore, we needed to rely on a different method for treating general exponential sums. This method will be based on various new Bernstein and Markov type inequalities for univariate and multivariate exponential sums which are discussed in Sects. 1 and 2, respectively. Section 3 is dealing with the discretization of the uniform norm for univariate exponential sums. Subsequently, in Sect. 4 we will study the discretization problem for multivariate exponential sums (3).

A crucial feature of the results of the present paper is their universality in the sense that the upper bounds for their cardinality are independent of the exponents μ_j of exponential sums (3), as long as they are separated by $|\mu_k - \mu_j| \geq \delta, j \neq k, \delta > 0$ and an upper bound $|\mu_j| \leq M$ is imposed. Therefore, we introduce the following set of n term exponential sums of variable $\mathbf{w} \in \mathbb{R}^d$ with exponents separated by δ and bounded by M

$$\Omega^d(n, \delta, M) := \left\{ \sum_{1 \leq j \leq n} c_j e^{(\mu_j, \mathbf{w})}, \quad c_j \in \mathbb{R}, \quad \mu_j \in \mathbb{R}^d, \quad |\mu_k - \mu_j| \geq \delta, \right. \\ \left. j \neq k, \quad |\mu_j| \leq M, \forall j \right\}.$$

It should be noted that $\Omega^d(n, \delta, M)$ is not a linear subspace.

The main result of Sect. 4 (Theorem 6) asserts that for convex polytopes K in $\mathbb{R}^d$ and any $\mu_j \in \mathbb{R}^d, |\mu_k - \mu_j| \geq \delta, j \neq k, |\mu_j| \leq M, \forall j$ there exist meshes $Y_N = \{\mathbf{w}_1, \dots, \mathbf{w}_N\} \subset K$ of cardinality

$$N \leq c_{K,d} \left(\frac{n}{\sqrt{\tau}} \right)^d \ln^d \frac{M}{\delta \tau}$$

so that for every p given by (3) we have

$$\|p(\mathbf{w})\|_K \leq (1 + \tau) \max_{1 \leq j \leq N} |p(\mathbf{w}_j)|.$$

Throughout this paper, we will always denote by $c(\dots)$ positive constants depending only on the quantities specified in the brackets, while c will stand for absolute constants.

Finally, we would like to emphasize an important advantage of the Bernstein–Markov methods used in this paper: all the discrete meshes are constructed explicitly and the proofs of the main results essentially provide algorithms for their construction.

1 Bernstein and Markov Type Inequalities for Univariate Exponential Sums

In this section, we will use and extend various Bernstein and Markov type inequalities for univariate exponential sums given in the monograph [11]. These inequalities are related to the well-known Newman inequality, see [11], p. 276. First let us mention the following modification of the Newman inequality verified in [11], p. 287.

Given any $\lambda_j \in \mathbb{R}^+, 0 \leq j \leq n, \lambda_0 = 0, \lambda_{j+1} - \lambda_j \geq 1$ and $p(x) = \sum_{0 \leq j \leq n} c_j x^{\lambda_j}, c_j \in \mathbb{R}$ it follows that

$$\|p'\|_{[0,1]} \leq 18 \left(\sum_{0 \leq j \leq n} \lambda_j \right) \|p\|_{[0,1]}. \quad (5)$$

Next, we will need an extension of the Newman type inequality given in [11], p. 301, E.10 by allowing negative exponents and providing explicit constants depending on $[a, b] \subset (0, \infty)$. Knowledge of the explicit constants will be crucial in the multivariate case.

Lemma 1 *Consider generalized polynomials $p(x) = \sum_{0 \leq j \leq n} c_j x^{\lambda_j}, c_j \in \mathbb{R}$ with arbitrary $\lambda_j \in \mathbb{R}, 0 \leq j \leq n, \lambda_{j+1} - \lambda_j \geq 1, \lambda_n^* := \max_{0 \leq j \leq n} |\lambda_j|$. Then, for any $[a, b] \subset (0, \infty)$ we have with some absolute constant $c > 0$*

$$\|p'\|_{[a,b]} \leq \frac{c b n \lambda_n^*}{a(b-a)} \|p\|_{[a,b]}. \quad (6)$$

Proof First let us assume that $\lambda_j \geq j, 0 \leq j \leq n$. Set $m := [2(a+b)/(b-a)]$. Then, it easily follows that

$$a \leq \frac{a+b}{2} \left(\frac{m}{m+1} \right)^2.$$

Set now $q(x) := x^{mn} p(x)$. It follows that for any $y \in [(a+b)/2, b]$ we have $y \left(\frac{m}{m+1} \right)^2 \geq a$ and hence using also [11], p. 301, E.9d we have

$$\|q\|_{[0,y]} \leq \|q\|_{[y(\frac{m}{m+1})^2, y]} \leq \|q\|_{[a,y]}.$$

Hence, rescaling the Newman inequality [11], p. 276 from $[0, 1]$ to $[0, y]$ we obtain

$$\begin{aligned} |q'(y)| &\leq \frac{9}{y} \sum_{0 \leq j \leq n} (\lambda_j + mn) \|q\|_{[0,y]} \leq \frac{18}{a+b} \left(\sum_{0 \leq j \leq n} \lambda_j + \frac{2n(n+1)(a+b)}{b-a} \right) \\ \|q\|_{[a,y]} &\leq \frac{90}{b-a} \left(\sum_{0 \leq j \leq n} \lambda_j \right) \|q\|_{[a,y]}. \end{aligned}$$

Note that $\|q\|_{[a,y]} = |q(\xi)|$ with some $a \leq \xi \leq y$ and therefore

$$y^{-mn} \|q\|_{[a,y]} = y^{-mn} |q(\xi)| \leq \xi^{-mn} |q(\xi)| = |p(\xi)| \leq \|p\|_{[a,y]}.$$

Thus

$$\begin{aligned} |p'(y)| &\leq y^{-mn} |q'(y)| + mn y^{-mn-1} |q(y)| \leq \frac{90}{b-a} \left(\sum_{0 \leq j \leq n} \lambda_j \right) \|p\|_{[a,y]} \\ &\quad + \frac{mn}{y} \|p\|_{[a,y]} \leq \frac{94}{b-a} \left(\sum_{0 \leq j \leq n} \lambda_j \right) \|p\|_{[a,y]}. \end{aligned}$$

On the other hand, if $y \in [a, (a+b)/2]$ then as shown in [11], p. 302

$$|p'(y)| \leq \frac{2n^2}{b-y} \|p\|_{[y,b]} \leq \frac{4n^2}{b-a} \|p\|_{[y,b]}.$$

Thus, combining above estimates yields that whenever $\lambda_j \geq j$, $0 \leq j \leq n$ we have

$$\|p'\|_{[a,b]} \leq \frac{94}{b-a} \left(\sum_{0 \leq j \leq n} \lambda_j \right) \|p\|_{[a,b]}. \quad (7)$$

Now consider arbitrary $\lambda_j \in \mathbb{R}$, $0 \leq j \leq n$, $\lambda_{j+1} - \lambda_j \geq 1$. If $\lambda_0 \geq 0$ then clearly $\lambda_j \geq j$, $0 \leq j \leq n$ holds and the statement of the lemma follows from (7). Thus, we may assume that $\lambda_0 < 0$. Furthermore, if $\lambda_n \leq 0$ then we can use (7) for $g(t) := p(1/t) = \sum_{0 \leq j \leq n} c_j t^{-\lambda_j}$, $t \in [1/b, 1/a]$ yielding that

$$\|g'\|_{[1/b, 1/a]} \leq \frac{cab}{b-a} \left(\sum_{0 \leq j \leq n} |\lambda_j| \right) \|g\|_{[1/b, 1/a]}, \quad c = 94.$$

Hence,

$$\|p'\|_{[a,b]} \leq a^{-2} \|g'\|_{[1/b, 1/a]} \leq \frac{cb}{a(b-a)} \left(\sum_{0 \leq j \leq n} |\lambda_j| \right) \|p\|_{[a,b]}$$

which again implies the needed estimate (6).

Thus, we may assume that $\lambda_0 < 0 < \lambda_n$. Choose any $y \in (a, b)$. Setting $\mu_j := \lambda_j - \lambda_0$ and $g(x) := \sum_{0 \leq j \leq n} c_j x^{\mu_j}$ we have that $\mu_j \geq j$, $0 \leq j \leq n$ and $p(x) = x^{\lambda_0} g(x)$. Thus, applying (7) for g on the interval $[a, y]$ yields

$$|g'(y)| \leq \frac{c}{y-a} \left(\sum_{0 \leq j \leq n} \mu_j \right) \|g\|_{[a,y]} = \frac{c}{y-a} \sum_{0 \leq j \leq n} (\lambda_j - \lambda_0) \|g\|_{[a,y]}.$$

Hence, using that $\lambda_0 < 0$ and thus $\|g\|_{[a,y]} \leq y^{-\lambda_0} \|p\|_{[a,y]}$ we obtain by the last estimate

$$|p'(y)| \leq \frac{|\lambda_0|}{y} |p(y)| + y^{\lambda_0} |g'(y)| \leq \frac{|\lambda_0|}{y} |p(y)| + y^{\lambda_0} \frac{c}{y-a} \sum_{0 \leq j \leq n} (\lambda_j - \lambda_0)$$

$$\|g\|_{[a,y]} \leq \frac{c}{y-a} \left(|\lambda_0| + \sum_{1 \leq j \leq n} (\lambda_j - \lambda_0) \right) \|p\|_{[a,y]}, \quad a < y \leq b. \quad (8)$$

If $y \in [a, b)$ we can set $x = \frac{1}{t}$, $q(t) := p(x) = \sum_{0 \leq j \leq n} c_j t^{-\lambda_j}$ yielding similarly to (8)

$$|q'(1/y)| \leq \frac{c}{\frac{1}{y} - \frac{1}{b}} (\lambda_n + \sum_{0 \leq j \leq n} (\lambda_n - \lambda_j)) \|q\|_{[\frac{1}{b}, \frac{1}{y}]} \leq \frac{cby}{b-y} (\lambda_n + \sum_{0 \leq j \leq n} (\lambda_n - \lambda_j)) \|p\|_{[y,b]}.$$

Hence,

$$|p'(y)| \leq |q'(1/y)| y^{-2} \leq \frac{cb}{a(b-y)} (\lambda_n + \sum_{0 \leq j \leq n-1} (\lambda_n - \lambda_j)) \|p\|_{[y,b]}, \quad a \leq y < b. \quad (9)$$

Finally, using (8) and (9) if $y \in [(a+b)/2, b]$ and $y \in [a, (a+b)/2]$, respectively, we obtain for any $y \in [a, b]$

$$|p'(y)| \leq \frac{cb}{a(b-a)} (|\lambda_0| + \lambda_n + \sum_{1 \leq j \leq n} (\lambda_j - \lambda_0) + \sum_{0 \leq j \leq n-1} (\lambda_n - \lambda_j)) \|p\|_{[a,b]}$$

$$= \frac{cb}{a(b-a)} (n+2)(\lambda_n + |\lambda_0|) \leq \frac{c_0 b}{a(b-a)} n \lambda_n^* \|p\|_{[a,b]}.$$

□

Remark Estimate (6) of Lemma 1 should be compared to the classical Markov inequality stating that

$$\|p'\|_{[a,b]} \leq \frac{2n^2}{b-a} \|p\|_{[a,b]}$$

for any algebraic polynomial p of degree $\leq n$. Here the “square of the degree” provides an upper bound for the derivatives. On the other hand (6) gives a “degree times dimension” type estimate which in general can be of essentially better order.

A standard substitution $x = e^t$ leads to a statement for exponential sums on $[\alpha, \beta] \subset \mathbb{R}$ similar to the above lemma. Moreover, imposing the separation condition $\mu_{j+1} -$

$\mu_j \geq \frac{\delta}{\beta - \alpha}$ for exponents $e^{\mu_j t}$ makes the statement domain independent. We can also iterate the result for higher derivatives as well yielding the next extension of Lemma 1 for exponential sums.

Lemma 2 *Let $[\alpha, \beta] \subset \mathbb{R}$, and let $g(t) = \sum_{0 \leq j \leq n} c_j e^{\mu_j t}$, $c_j \in \mathbb{R}$ be any exponential sum with $\mu_j \in \mathbb{R}$, $0 \leq j \leq n$ satisfying $\mu_{j+1} - \mu_j \geq \frac{\delta}{\beta - \alpha}$, $0 < \delta \leq 1$. Then, with some absolute constant $c > 0$*

$$\|g^{(k)}\|_{[\alpha, \beta]} \leq \left(\frac{cn\mu_n^*}{\delta} \right)^k \|g\|_{[\alpha, \beta]}, \quad k \in \mathbb{N}. \quad (10)$$

Similarly using substitution $x = e^{-\delta t}$ we can derive from (5)

Corollary 1 *Let $g(t) = \sum_{0 \leq j \leq n} c_j e^{-\mu_j t}$, $c_j \in \mathbb{R}$ be any exponential sum with $\mu_j \in \mathbb{R}_+$, $0 \leq j \leq n$ satisfying $\mu_0 = 0$, $\mu_{j+1} - \mu_j \geq \delta$, $0 < \delta \leq 1$. Then, with some absolute constant $c > 0$ we have*

$$|g'(t)| \leq 18e^{-\delta t} \left(\sum_{0 \leq j \leq n} \mu_j \right) \|g\|_{[0, \infty)}, \quad t \in [0, \infty). \quad (11)$$

Now, we proceed by verifying certain Bernstein type inequalities for univariate exponential sums. We will derive the needed inequalities from a basic result given in [11], p. 293, E.4.d according to which for any $g(t) = \sum_{1 \leq j \leq n} c_j e^{-\mu_j t}$, $c_j \in \mathbb{R}$ and $0 < \delta \leq \frac{\beta - \alpha}{2}$ we have

$$\|g'\|_{[\alpha + \delta, \beta - \delta]} \leq \frac{2n - 1}{\delta} \|g\|_{[\alpha, \beta]}, \quad [\alpha, \beta] \subset \mathbb{R}. \quad (12)$$

Obviously, we can iterate the above estimate for higher derivatives using it consequently for $[\alpha + \delta j/k, \beta - \delta j/k]$, $j = 0, 1, \dots, k - 1$ yielding

$$\|g^{(k)}\|_{[\alpha + \delta, \beta - \delta]} \leq \left(\frac{k(2n - 1)}{\delta} \right)^k \|g\|_{[\alpha, \beta]}, \quad [\alpha, \beta] \subset \mathbb{R}, \quad k \in \mathbb{N}. \quad (13)$$

The following Bernstein type inequalities can be derived from estimate (13).

Lemma 3 *For any $g(t) = \sum_{1 \leq j \leq n} c_j e^{-\mu_j t}$, $c_j \in \mathbb{R}$ and $[\alpha, \beta] \subset \mathbb{R}$*

$$|g^{(k)}(t)| \leq \left(\frac{k(\beta - \alpha)(2n - 1)}{(t - \alpha)(\beta - t)} \right)^k \|g\|_{[\alpha, \beta]}, \quad t \in (\alpha, \beta), \quad k \in \mathbb{N}. \quad (14)$$

Furthermore, if $\mu_j \geq 0$, $1 \leq j \leq n$ then

$$|g^{(k)}(t)| \leq \left(\frac{2nk}{t} \right)^k \|g\|_{[0, \infty)}, \quad t > 0, \quad k \in \mathbb{N}. \quad (15)$$

Proof Assume first that $t \in (\alpha, (\alpha + \beta)/2]$. Then, by (13) applied for the interval $[\alpha, 2t - \alpha] \subset [\alpha, \beta]$ and $\delta = t - \alpha$

$$|g^{(k)}(t)| \leq \left(\frac{k(2n-1)}{\delta} \right)^k \|g\|_{[\alpha, \beta]} \leq \left(\frac{k(\beta - \alpha)(2n-1)}{(t - \alpha)(\beta - t)} \right)^k \|g\|_{[\alpha, \beta]}.$$

Similarly, if $t \in [(\alpha + \beta)/2, \beta]$ we can use (13) for the interval $[2t - \beta, \beta] \subset [\alpha, \beta]$ and $\delta = \beta - t$ yielding the same estimate.

Moreover, if $\mu_j \geq 0$, $1 \leq j \leq n$ then given any $t > 0$ we can apply (14) with $\alpha = 0$, $\beta = 2nt$ which immediately yields (15). $\square$

2 Markov Type Inequalities for Multivariate Exponential Sums

In this section, we will extend Markov type estimates given in the previous section to the derivatives of multivariate exponential sums on convex bodies and polyhedral cones. Let us denote by ∇g the gradient of a differentiable function g .

Theorem 1 *Let $K \subset \mathbb{R}^d$, $d \geq 1$ be a convex body with r_K being the radius of its largest inscribed ball. Then, for every exponential sum $g(\mathbf{w}) = \sum_{0 \leq j \leq n} c_j e^{\langle \mu_j, \mathbf{w} \rangle}$, $\mathbf{w} \in \mathbb{R}^d$ with $\mu_j \in \mathbb{R}^d$ satisfying $|\mu_k - \mu_j| \geq \frac{\delta}{r_K}$, $0 \leq j \neq k \leq n$, $0 < \delta \leq 1$ we have with $\mu_n^* := \max_{0 \leq j \leq n} |\mu_n|$ and some absolute constant $c > 0$*

$$\|\nabla g\|_K \leq \frac{cd^3 n^3 \mu_n^*}{\delta} \|g\|_K.$$

In order to verify the above theorem we need first some technical lemmas. Let us denote by $B(0, r)$ the ball of radius r in $\mathbb{R}^d$ centered at the origin. In addition, S^{d-1} stands for the unit sphere in $\mathbb{R}^d$.

Lemma 4 *Assume that $d = 2$ and $K \subset \mathbb{R}^2$ is a convex body satisfying $B(0, r) \subset K \subset B(0, R)$. Then, for any $A \in K$ there exists a triangle $\Delta \subset K$ with vertices A, B, C such that the angle at A is $\geq \frac{r}{\pi R}$ and $\text{dist}(A, [B, C]) \geq \frac{r}{2}$.*

Proof We may assume that $A = (0, y)$, $y \geq 0$. Consider first the case when $A \in B(0, \frac{r}{2})$, i.e., $y \leq \frac{r}{2}$. Set, $B := \frac{r}{2}(\sqrt{3}, -1)$, $C := \frac{r}{2}(-\sqrt{3}, -1)$. Since $B(0, r) \subset K$ and K is convex it follows that triangle Δ with vertices A, B, C is in K . Evidently, $\text{dist}(A, [B, C]) \geq \frac{r}{2}$ and $\alpha \geq \frac{\pi}{3}$, where α is the angle at the vertex A . Now assume $A \notin B(0, \frac{r}{2})$, i.e., $y > \frac{r}{2}$. This time we set $B := \frac{r}{2}(1, 0)$, $C = -B$, $[B, C] \subset K$ and again consider the triangle $\Delta \subset K$ with vertices A, B, C . Then, relation $\text{dist}(A, [B, C]) \geq \frac{r}{2}$ is obvious. Furthermore, for the angle α at the vertex A we clearly have $\sin \frac{\alpha}{2} \geq \frac{r}{2|A-B|} \geq \frac{r}{4R}$ yielding $\alpha \geq \frac{r}{\pi R}$. $\square$

Consider the exponents $e^{\langle \lambda_j, \mathbf{w} \rangle}$, $\lambda_j \in \mathbb{R}^d$, $1 \leq j \leq n$ which satisfy the separation condition $|\lambda_j - \lambda_k| \geq \delta > 0$, $j \neq k$. We will need to know in what way this separation condition is preserved when these exponents are considered on certain lines and hyperplanes. Clearly, the orthogonal projections of λ_j -s into lines $\{t\mathbf{u} : t \in$

$\mathbb{R}\}$, $\mathbf{u} \in S^{d-1}$ is given by $\langle \lambda_j, \mathbf{u} \rangle \mathbf{u}$, where these projections are separated by quantities $|\langle \lambda_j - \lambda_k, \mathbf{u} \rangle|$. Furthermore, when $H_{\mathbf{u}} := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{u} \rangle = 0\}$ is the hyperplane in $\mathbb{R}^d$ with normal $\mathbf{u} \in S^{d-1}$ the exponent $e^{(\lambda, \mathbf{x})}$, $\lambda, \mathbf{x} \in \mathbb{R}^d$ restricted to $\mathbf{x} \in H_{\mathbf{u}}$ transforms into $e^{(\lambda(\mathbf{u}), \mathbf{x})}$, where $\lambda(\mathbf{u}) := \lambda - \langle \lambda, \mathbf{u} \rangle \mathbf{u}$ is the orthogonal projection of λ into $H_{\mathbf{u}}$. We will repeatedly use below the following auxiliary proposition which shows that the separation of the exponents can be preserved in some sense after a proper small perturbation of lines and hyperplanes.

Lemma 5 *Let $\lambda_j \in \mathbb{R}^d$, $1 \leq j \leq n$, $d \geq 2$ satisfy $|\lambda_j - \lambda_k| \geq \delta > 0$, $j \neq k$. Then, for any $\mathbf{w}^* \in S^{d-1}$ any every $\epsilon > 0$ there exist $\mathbf{u}_1, \mathbf{u}_2 \in S^{d-1}$, $|\mathbf{w}^* - \mathbf{u}_j| \leq \epsilon$, $j = 1, 2$ so that with some $c_d > 0$ depending only on d we have*

$$(i) \quad |\langle \lambda_j - \lambda_k, \mathbf{u}_1 \rangle| \geq \frac{c_d \delta \epsilon^{d-1}}{n^2}, \quad \forall j \neq k; \quad (16)$$

$$(ii) \quad |\lambda_j(\mathbf{u}_2) - \lambda_k(\mathbf{u}_2)| \geq \frac{c_d \delta \epsilon}{n^2}, \quad \forall j \neq k. \quad (17)$$

Proof Let $\alpha_d(E)$ stand for the surface area of $E \subset S^{d-1}$. Consider all vectors $\lambda_j - \lambda_k \in \mathbb{R}^d$, $j \neq k$, $1 \leq j, k \leq n$ and denote them by $\mathbf{w}_j$, $1 \leq j \leq N$ where evidently $N = \frac{n(n-1)}{2} < n^2$ and $|\mathbf{w}_j| \geq \delta$, $1 \leq j \leq N$. Denote by $\Omega_\gamma(\mathbf{u}) := \{\mathbf{y} \in S^{d-1} : |\langle \mathbf{u}, \mathbf{y} \rangle| \leq \gamma\}$ the surface ring of width γ around the orthogonal complement of $\mathbf{u} \in S^{d-1}$ on S^{d-1} . Clearly, for every $\mathbf{u} \in S^{d-1}$ we have that $\alpha_d(\Omega_\gamma(\mathbf{u})) \leq c_d \gamma$ where and $c_d > 0$ depends only on d . Furthermore, set

$$\Omega_\gamma := \cup_{1 \leq j \leq N} \Omega_\gamma(\mathbf{w}_j/|\mathbf{w}_j|), \quad \alpha_d(\Omega_\gamma) \leq c_d N \gamma.$$

Now for any $E \subset S^{d-1}$ with $\alpha_d(E) > 0$ set $\gamma := \frac{\alpha_d(E)}{c_d n^2}$. Then, evidently $\alpha_d(\Omega_\gamma) \leq c_d N \gamma < \alpha_d(E)$, i.e., $E \setminus \Omega_\gamma \neq \emptyset$. Hence, choosing any $\mathbf{u}_1 \in E \setminus \Omega_\gamma$ we must have

$$|\langle \mathbf{w}_j/|\mathbf{w}_j|, \mathbf{u}_1 \rangle| > \gamma = \frac{\alpha_d(E)}{c_d n^2}, \quad \forall 1 \leq j \leq N,$$

i.e.,

$$|\langle \mathbf{w}_j, \mathbf{u}_1 \rangle| \geq \frac{\delta \alpha_d(E)}{c_d n^2}, \quad \forall 1 \leq j \leq N.$$

This verifies that for any $E \subset S^{d-1}$ with $\alpha_d(E) > 0$ there exists $\mathbf{u}_1 \in E$ for which we have

$$|\langle \lambda_j - \lambda_k, \mathbf{u}_1 \rangle| \geq \frac{\delta \alpha_d(E)}{c_d n^2}, \quad \forall j \neq k. \quad (18)$$

Now let $E_\epsilon(\mathbf{w}^*) := \{\mathbf{y} \in S^{d-1} : |\mathbf{w}^* - \mathbf{y}| \leq \epsilon\}$ be the ϵ sphere cap centered at $\mathbf{w}^* \in S^{d-1}$. Then, clearly $\alpha_d(E_\epsilon(\mathbf{w}^*)) \geq c_d^* \epsilon^{d-1}$. Hence, applying (18) with $E := E_\epsilon(\mathbf{w}^*)$ verifies the first statement of the lemma.

Furthermore, let

$$B_\epsilon(\mathbf{w}^*) := \left\{ \mathbf{y} \in S^{d-1} : \exists \mathbf{u} \in E_\epsilon(\mathbf{w}^*), \mathbf{y} \perp \mathbf{u} \right\}.$$

Clearly, we have that $\alpha_d(B_\epsilon(\mathbf{w}^*)) \geq \tilde{c}_d \epsilon$. Therefore, by (18) applied with $E = B_\epsilon(\mathbf{w}^*)$ we can choose $\mathbf{u} \in B_\epsilon(\mathbf{w}^*)$ so that

$$|\langle \lambda_j - \lambda_k, \mathbf{u} \rangle| \geq \frac{\tilde{c}_d \delta \epsilon}{c_d n^2}, \quad \forall j \neq k.$$

Since $\mathbf{u} \in B_\epsilon(\mathbf{w}^*)$ it follows that there exists $\mathbf{u}_2 \in E_\epsilon(\mathbf{w}^*)$, $\mathbf{u}_2 \perp \mathbf{u}$. Thus $\mathbf{u} \in H_{\mathbf{u}_2}$ and hence using the above lower bound

$$|\lambda_j(\mathbf{u}_2) - \lambda_k(\mathbf{u}_2)| \geq |\langle \lambda_j(\mathbf{u}_2) - \lambda_k(\mathbf{u}_2), \mathbf{u} \rangle| = |\langle \lambda_j - \lambda_k, \mathbf{u} \rangle| \geq \frac{\tilde{c}_d \delta \epsilon}{c_d n^2}, \quad \forall j \neq k.$$

□

Proof of Theorem 1. Assume first that $K \subset \mathbb{R}^d$ is a convex body satisfying $B(0, r) \subset K \subset B(0, R)$. In order to verify the statement of the theorem we need to give a proper upper bound for the derivative of $g(\mathbf{w}) = \sum_{0 \leq j \leq n} c_j e^{\langle \mu_j, \mathbf{w} \rangle}$ when $|\mu_{j+1} - \mu_j| \geq \delta$ in any direction $\mathbf{u} \in S^{d-1}$ and for any $\mathbf{w} \in K$. Thus clearly we can consider the problem in the 2-dimensional plane spanned by $\mathbf{u}, \mathbf{w}$. This and rotation invariance of the setting yields that it suffices to consider the case $d = 2$ and estimate $\frac{\partial g}{\partial x}(A)$ for any given $A \in K$. By Lemma 4 there exists a triangle $\Delta \subset K$ with vertices A, B, C such that the angle at A is $\geq \frac{r}{\pi R}$ and $\text{dist}(A, [B, C]) \geq \frac{r}{2}$. Let $\mathbf{w}_1^*, \mathbf{w}_2^* \in S^1$ be unit vectors dividing the angle at vertex A into 3 equal parts. Then, using Lemma 5 (i) with $\mathbf{w}^* = \mathbf{w}_1^*$ and $\mathbf{w}^* = \mathbf{w}_2^*$ when $d = 2$, $\epsilon = \frac{r}{12\pi R}$ yields that there exist $D, E \in [B, C]$ such that the angle $DAE \geq \frac{r}{6\pi R}$ and the exponential sum $g(\mathbf{w}) = \sum_{0 \leq j \leq n} c_j e^{\langle \mu_j, \mathbf{w} \rangle}$ along each segment $[A, D]$ and $[A, E]$ reduces to univariate exponential sums $g(t) = \sum_{0 \leq j \leq n} c_j^* e^{t\lambda_j}$, $t \in \mathbb{R}$ with exponents satisfying $\max_{0 \leq j \leq n} |\lambda_j| \leq \mu_n^*$ and separated by

$$|\lambda_j - \lambda_k| \geq \frac{c\delta r}{Rn^2}, \quad j \neq k.$$

In addition, since $\text{dist}(A, [B, C]) \geq \frac{r}{2}$ we have that $|A - D|, |A - E| \geq \frac{r}{2}$. This and the above separation property yield that we can use estimate (10) of Lemma 2 for the exponential sum $g(t) = \sum_{0 \leq j \leq n} c_j^* e^{t\lambda_j}$, $t \in \mathbb{R}$ along segment $[A, D]$ with separation parameter $\delta_1 := \frac{c\delta r^2}{Rn^2}$, and $c > 0$ a proper absolute constant. Thus setting $M_n := \frac{Rn^3 \mu_n^*}{r^2 \delta}$ we obtain by (10)

$$\|D_{\mathbf{u}} g\|_{[A, D]} \leq c_0 M_n \|g\|_{[A, D]}. \quad (19)$$

where $\mathbf{u}$ is the unit vector in the direction of segment $[A, D]$ and $c_0 > 0$ is a proper absolute constant. Analogously,

$$\|D_{\mathbf{w}}g\|_{[A,E]} \leq cM_n \|g\|_{[A,E]}. \quad (20)$$

where $\mathbf{w}$ is the unit vector in the direction of segment $[A, E]$ and $c > 0$ is an absolute constant. Clearly, we may assume that $\mathbf{u} = (0, 1)$. In addition, recalling that the angle $DAE \geq \frac{r}{3\pi R}$, we have $\xi \geq \frac{r}{3\pi R}$ where ξ is the angle between $\mathbf{u}$ and $\mathbf{w}$. Hence, we easily obtain from (19) and (20) that

$$\left| \frac{\partial g}{\partial y}(A) \right| \leq c_0 M_n \|g\|_K, \quad \left| \frac{\partial g}{\partial x}(A) \right| \leq \frac{2cM_n}{\sin \xi} \|g\|_K$$

i.e., we have the next estimate for the gradient $\nabla g(A)$ with some absolute constant $c > 0$

$$|\nabla g(A)| \leq \frac{cR^2 n^3 \mu_n^*}{r^3 \delta} \|g\|_K. \quad (21)$$

Now it remains to apply the well-known John Ellipsoid Theorem [12]. Using this theorem one can show that a certain regular linear transformation of norm $\leq \frac{d}{r_K}$ maps convex body $K \subset \mathbb{R}^d$ into a convex body K_0 satisfying $B(0, 1) \subset K \subset B(0, d)$. Thus we can assume that that $r = 1$ and $R = d$ in (21), this may increase the upper bound at most by a factor of $\frac{d}{r_K}$, see [13], p. 91 for details. This observation together with estimate (21) leads to the final upper bound

$$|\nabla g(A)| \leq \frac{cd^3 n^3 \mu_n^*}{r_K \delta} \|g\|_K.$$

□

Next we will obtain a Markov type inequality for multivariate exponential sums on an unbounded polyhedral cone in $\mathbb{R}_+^d := \{\mathbf{x} = (x_1, \dots, x_d) : x_j \geq 0, 1 \leq j \leq d\}$. A polyhedral cone spanned by $\mathbf{u}_k \in S_+^{d-1} := \mathbb{R}_+^d \cap S^{d-1}$, $1 \leq k \leq m$ is defined as

$$K(\mathbf{u}_1, \dots, \mathbf{u}_m) := \left\{ \sum_{1 \leq k \leq m} a_k \mathbf{u}_k, a_k \geq 0 \right\} \subset \mathbb{R}_+^d.$$

In particular, $K(\mathbf{u}_1, \dots, \mathbf{u}_d) = \mathbb{R}_+^d$ if $\mathbf{u}_k = (\delta_{k,j}, 1 \leq j \leq d)$, $1 \leq k \leq d$ with $\delta_{k,j}$ being the Kronecker delta. Since polyhedral cones are unbounded sets in $\mathbb{R}_+^d$ naturally we will consider exponential sums spanned by $e^{-\langle \mu_j, \mathbf{w} \rangle}$, $\mathbf{w} \in \mathbb{R}_+^d$ with $\mu_j \in \mathbb{R}_+^d$. As above $\mu_n^* := \max_{0 \leq j \leq n} |\mu_j|$.

Theorem 2 Let $K = K(\mathbf{u}_1, \dots, \mathbf{u}_m) \subset \mathbb{R}_+^d$ be a polyhedral cone. Then, for every exponential sum $p(\mathbf{w}) = \sum_{0 \leq j \leq n} c_j e^{-\langle \mu_j, \mathbf{w} \rangle}$, $\mathbf{w} \in \mathbb{R}^d$ with $\mu_j \in \mathbb{R}_+^d$ satisfying $|\mu_k - \mu_j| \geq \delta$, $j \neq k$, $0 < \delta \leq 1$ we have with some constant $c_K > 0$ depending on K

$$|\nabla p(\mathbf{x})| \leq c_K n \mu_n^* \min \left\{ 1, \frac{n^2}{\delta |\mathbf{x}|} \right\} \|p\|_K, \quad \mathbf{x} \in K. \quad (22)$$

Proof Evidently it suffices to verify the theorem for $d = 2$ since in order to estimate the derivative in any direction $\mathbf{u} \in S_+^{d-1}$ we can consider the problem in the 2-dimensional plane spanned by $\mathbf{x}$ and $\mathbf{u}$. In addition, without the loss of generality we may assume that $K = \mathbb{R}_+^2$. (This can be accomplished by a regular linear transformation of the space.) Now consider any $p(\mathbf{w}) = \sum_{0 \leq j \leq n} c_j e^{-\langle \mu_j, \mathbf{w} \rangle}$, $\mathbf{w} \in \mathbb{R}^2$ with $\mu_j \in \mathbb{R}_+^2$ satisfying $|\mu_k - \mu_j| \geq \delta$, $j \neq k$, $0 < \delta \leq 1$. Let $\mathbf{x} \in \mathbb{R}_+^2$. Applying now Lemma 5 (i) with $\mathbf{w}^* = e^{2\pi i/3}$ and $\mathbf{w}^* = e^{5\pi i/6}$ for $d = 2$ and $\epsilon = \frac{\pi}{24}$ we can easily choose two finite segments $I_1, I_2 \subset \mathbb{R}_+^2$ in directions $\mathbf{w}_1, \mathbf{w}_2 \in S^{d-1}$ with endpoints on the coordinate axes so that $\mathbf{x} = I_1 \cap I_2$, $|I_1|, |I_2| \geq |\mathbf{x}|$, $0 < \langle \mathbf{w}_1, \mathbf{w}_2 \rangle < \cos \frac{\pi}{12}$, and

$$|\langle \mu_j - \mu_k, \mathbf{w}_s \rangle| \geq \frac{c\delta}{n^2}, \quad j \neq k, \quad s = 1, 2. \quad (23)$$

Using now univariate Markov type inequality (10) on the intervals I_j , $j = 1, 2$ with δ replaced by $\frac{c\delta|I_j|}{n^2}$ yields the next estimate for the directional derivative

$$|D_{\mathbf{w}_s} p(\mathbf{x})| \leq \frac{cn^3 \mu_n^*}{\delta |\mathbf{x}|} \|p\|_K, \quad s = 1, 2.$$

Since $0 < \langle \mathbf{w}_1, \mathbf{w}_2 \rangle < \cos \frac{\pi}{12}$ this yields similar upper bound for the gradient, as well:

$$|\nabla p(\mathbf{x})| \leq \frac{c_1 n^3 \mu_n^*}{\delta |\mathbf{x}|} \|p\|_K.$$

We can obtain another estimate for the gradient using again Lemma 5 (i) and the Markov type estimate (11). Namely using Lemma 5 (i) with $\mathbf{w}^* = e^{\pi i/6}$ and $\mathbf{w}^* = e^{\pi i/3}$ for $d = 2$ and $\epsilon = \frac{\pi}{24}$ we can choose two rays $r_1, r_2 \subset \mathbb{R}_+^2$ emanating from $\mathbf{x} \in \mathbb{R}_+^2$, $\mathbf{x} = r_1 \cap r_2$ in directions $\mathbf{w}_1, \mathbf{w}_2 \in S_+^1$, $0 < \langle \mathbf{w}_1, \mathbf{w}_2 \rangle < \cos \frac{\pi}{12}$ satisfying (23). Now applying (11) for $t = 0$ on rays r_1, r_2 with common origin at $\mathbf{x} \in \mathbb{R}_+^2$ and using that $0 < \langle \mathbf{w}_1, \mathbf{w}_2 \rangle < \cos \frac{\pi}{12}$ we obtain

$$|\nabla p(\mathbf{x})| \leq c_2 n \mu_n^* \|p\|_K.$$

Combining the two estimates of the gradient verified above easily yields the statement of the theorem. $\square$

3 Discretization of Uniform Norm for Univariate Exponential Sums

Markov type inequalities given in Sects. 1 and 2 provide a straightforward path to estimating cardinality of meshes which discretize uniform norms of exponential sums. This can be accomplished recalling the notion of *fill distance* of sets. The fill distance of a subset D in $K \subset \mathbb{R}^d$ is defined by

$$\rho(D, K) := \sup_{x \in K} \inf_{y \in D} |x - y|.$$

The following proposition provides a simple connection between fill distance and Markov type estimates.

Lemma 6 *Let $K \subset \mathbb{R}^d$ be a convex body and assume that for some function g differentiable on K we have $\|\nabla g\|_K \leq M_g \|g\|_K$. Then, for every compact set $D \subset K$ with $\rho(D, K)M_g < 1$ we have*

$$(1 - M_g \rho(D, K)) \|g\|_K \leq \|g\|_D.$$

In particular, if $\rho(D, K) = \frac{\tau}{(1+\tau)M_g}$, $\tau > 0$ then $\|g\|_K \leq (1 + \tau) \|g\|_D$.

Proof Assume that $\|g\|_K = |g(\mathbf{x})|$, $\mathbf{x} \in K$. Then, there exists $\mathbf{y} \in D$ such that $|\mathbf{x} - \mathbf{y}| \leq \rho(D, K)$. By the convexity of K we have that $[\mathbf{x}, \mathbf{y}] \in K$ and thus

$$\|g\|_K - |g(\mathbf{y})| \leq |g(\mathbf{x}) - g(\mathbf{y})| \leq \|\nabla g\|_K |\mathbf{x} - \mathbf{y}| \leq M_g \rho(D, K) \|g\|_K,$$

i.e.,

$$\|g\|_D \geq |g(\mathbf{y})| \geq (1 - M_g \rho(D, K)) \|g\|_K.$$

□

Above lemma combined with Markov type estimates verified in Sects. 1 and 2 immediately yields certain bounds for the cardinality of discretization meshes for exponential sums. Indeed for any $K \subset \mathbb{R}^d$ one can choose uniformly distributed discrete subsets $Y_N \subset K$ with $\rho(D, K) = \rho$ and cardinality $N \sim \rho^{-d}$. Then, in view of Lemma 6 if a Markov type inequality holds with a factor M_g then relation $\|g\|_K \leq (1 + \tau) \|g\|_D$ can be ensured with discrete sets of cardinality $N \sim (\frac{M_g}{\tau})^d$. Hence, recalling the univariate Markov type estimate of Lemma 2 for $k = 1$ leads to discrete meshes of cardinality $N \sim \frac{n\lambda_n^*}{\delta}$ for univariate exponential sums $g(t) = \sum_{0 \leq j \leq n} c_j e^{\mu_j t}$, $c_j \in \mathbb{R}$ of degree $\mu_n^* = \max_{0 \leq j \leq n} |\mu_j|$ with $\mu_j \in \mathbb{R}$, $0 \leq j \leq n$ satisfying $\mu_{j+1} - \mu_j \geq \delta$, $0 < \delta \leq 1$. Likewise using Theorem 1 leads to discrete meshes on convex bodies $K \subset \mathbb{R}^d$ of cardinality $N \sim (\frac{n^3 \mu_n^*}{\delta})^d$ for exponential sum $p(\mathbf{w}) = \sum_{0 \leq j \leq n} c_j e^{\langle \mu_j, \mathbf{w} \rangle}$, $\mathbf{w} \in \mathbb{R}^d$ with $\mu_j \in \mathbb{R}^d$ satisfying $|\mu_k - \mu_j| \geq \delta$, $j \neq k$, $0 < \delta \leq 1$. However, this rough approach leads to discrete meshes having large cardinality which is far from optimal. The main drawback of

above estimates for cardinality consists in the fact that they depend heavily on the degree μ_n^* of the exponential sums. In Sects. 3 and 4 we will provide a more delicate approach to constructing discretization meshes for exponential sums. This approach will rely not only on Markov but also on Bernstein type inequalities and thus will lead to a finer distribution of meshes around the boundary of the domain considered. This will result in near optimal discretization meshes with only logarithmic dependence of the cardinality on the degree of exponential sums.

The construction of the discrete point sets for univariate exponential sums will be based on the measures

$$\mu_1(E) := \int_E \frac{dx}{x(1-x)}, \quad E \subset (0, 1), \quad (24)$$

and

$$\mu_2(E) := \int_E \frac{dx}{x}, \quad E \subset (0, \infty). \quad (25)$$

As we will see below the discretization meshes for the exponential sums on finite intervals and on $(0, \infty)$ can be chosen to be equidistributed with respect to the measures μ_1 and μ_2 given by (24) and (25), respectively.

Lemma 7 For any $0 < h < 1$ and $m \in \mathbb{N}$ set

$$x_{j,m} := \frac{1}{1 + \frac{1-h}{h} e^{-(j-1)/m}}, \quad 1 \leq j \leq N = N_m := \lceil 2m \ln \frac{1-h}{h} \rceil + 2. \quad (26)$$

Then, $1 - h \leq x_{N,m} \leq 1 - h/e$ and

$$\int_{x_{j,m}}^{x_{j+1,m}} \frac{dx}{x(1-x)} = \frac{1}{m}, \quad 1 \leq j \leq N-1.$$

Proof Clearly,

$$2 \ln \frac{1-h}{h} \leq \frac{N-1}{m} \leq 2 \ln \frac{1-h}{h} + 1$$

and thus we obtain

$$1 - h = \frac{1}{1 + \frac{1-h}{h} e^{-2 \ln \frac{1-h}{h}}} \leq x_{N,m} \leq \frac{1}{1 + \frac{h}{1-h} e^{-1}} \leq 1 - \frac{h}{e}.$$

Finally, since $\frac{1-x_{j,m}}{x_{j,m}} = \frac{1-h}{h} e^{-(j-1)/m}$, $1 \leq j \leq N$ it follows that

$$\int_{x_{j,m}}^{x_{j+1,m}} \frac{dx}{x(1-x)} = \ln \left(\frac{x_{j+1,m}}{1-x_{j+1,m}} \cdot \frac{1-x_{j,m}}{x_{j,m}} \right) = \frac{1}{m}, \quad 1 \leq j \leq N-1.$$

□

Now we will apply Bernstein and Markov type inequalities for univariate exponential sums given in Sect. 1 together with Lemma 6 in order to verify the following basic discretization result for univariate n term exponential sums $g \in \Omega^1(n, \delta, M)$.

Theorem 3 *Let $n \in \mathbb{N}$, $0 < \delta \leq 1$, $M \geq 1$, $[\alpha, \beta] \subset \mathbb{R}$. Then, for each $0 < \tau < 1$ there exists a discrete point set $Y_N = \{x_1, \dots, x_N\} \subset [\alpha, \beta]$ of cardinality*

$$N \leq \frac{cn}{\sqrt{\tau}} \ln \frac{M}{\delta \sqrt{\tau}}$$

with some absolute constant $c > 0$ so that for every exponential sum $g \in \Omega^1(n, \delta, M)$ we have

$$\|g\|_{[\alpha, \beta]} \leq (1 + \tau) \|g\|_{Y_N}.$$

Proof The statement of the theorem is clearly dilation and shift invariant, therefore we can assume without the loss of generality that $[\alpha, \beta] = [0, 1]$. Applying (10) with $k = 2$ yields that

$$\|g''\|_{[0, 1]} \leq M_n^2 \|g\|_{[0, 1]}, \quad M_n := \frac{cnM}{\delta}.$$

The construction of the needed discrete point set will be based on the points $x_{j,m}$, $1 \leq j \leq N$ of Lemma 7 with $h := \frac{\sqrt{\tau}}{M_n}$, and $m := \lceil \frac{8n}{\sqrt{\tau}} \rceil + 1$ in (26). In addition, we set $x_{0,m} := 0$, $x_{N+1,m} := 1$ and let $Y_N := \{x_{j,m}, 0 \leq j \leq N+1\}$. We may assume that $\|g\|_{[0, 1]} = 1 = g(x^*)$. Since $0, 1 \in Y_N$ if x^* coincides with 0 or 1 then $\|g\|_{[0, 1]} = \|g\|_{Y_N}$. Hence, we may assume that $x^* \in (0, 1)$ which implies $g'(x^*) = 0$. Assume first that $x^* \in (0, x_{1,m}) \cup [x_{N,m}, 1)$. Since $x_{1,m} = h$ and $x_{N,m} \geq 1 - h$ it follows that either $|x^* - x_{1,m}| \leq h$ or $|x^* - x_{N,m}| \leq h$. Assume $|x^* - x_{1,m}| \leq h$ (The case $|x^* - x_{N,m}| \leq h$ is totally analogous.) Taking into account that $g'(x^*) = 0$ we have by the Taylor formula

$$1 - |g(x_{1,m})| \leq |g(x^*) - g(x_{1,m})| \leq \frac{1}{2} |x^* - x_{1,m}|^2 \|g''\|_{[0, 1]} \leq \frac{1}{2} h^2 M_n^2 = \frac{\tau}{2}.$$

Thus

$$\|g\|_{Y_N} \geq |g(x_{1,m})| \geq 1 - \frac{\tau}{2} \geq \frac{1}{1 + \tau} = \frac{\|g\|_{[0, 1]}}{1 + \tau}$$

which is the needed estimate.

Now let $x^* \in (x_{j,m}, x_{j+1,m})$ for some $1 \leq j \leq N-1$. Then, applying the Taylor formula with integral remainder and Bernstein type inequality (14) for $k = 2$ and

$[\alpha, \beta] = [0, 1]$ we have

$$\begin{aligned} |g(x^*) - g(x_{j,m})| &\leq (x_{j+1,m} - x_{j,m}) \int_{x_{j,m}}^{x_{j+1,m}} |g''(x)| dx \\ &\leq 16n^2 (x_{j+1,m} - x_{j,m}) \int_{x_{j,m}}^{x_{j+1,m}} \frac{dx}{x^2(1-x)^2} \\ &\leq 16n^2 \frac{x_{j+1,m} - x_{j,m}}{x_{j,m}(1-x_{j+1,m})} \int_{x_{j,m}}^{x_{j+1,m}} \frac{dx}{x(1-x)}. \end{aligned}$$

Setting $x_{j,m} = \frac{1}{1+a_j}$, $a_j := \frac{1-h}{h} e^{-(j-1)/m}$, it follows

$$\frac{x_{j+1,m} - x_{j,m}}{x_{j,m}(1-x_{j+1,m})} = \frac{a_j - a_{j+1}}{a_{j+1}} = e^{\frac{1}{m}} - 1 \leq \frac{2}{m},$$

i.e., using Lemma 7 and relation $m := \lfloor \frac{8n}{\sqrt{\tau}} \rfloor + 1$

$$|g(x^*) - g(x_{j,m})| \leq \frac{32n^2}{m^2} \leq \frac{\tau}{2}.$$

Again this yields the estimate $\|g\|_{Y_N} \geq |g(x_{j,m})| \geq 1 - \frac{\tau}{2} \geq \frac{1}{1+\tau}$ which is the needed lower bound.

It remains now to estimate the cardinality of the discrete point set Y_N . Recall that by Lemma 7

$$N = \left\lceil 2m \ln \frac{1-h}{h} \right\rceil + 2 \leq \frac{cn}{\sqrt{\tau}} \ln \frac{1}{h}$$

where

$$h = \frac{\sqrt{\tau}}{M_n} = \frac{c\delta\sqrt{\tau}}{nM}.$$

Clearly, assumption $\mu_{j+1} - \mu_j \geq \delta$, $\forall j$ means that $n \leq \frac{2M}{\delta}$, i.e.,

$$\ln \frac{1}{h} = \ln \frac{nM}{c\delta\sqrt{\tau}} \leq c_1 \ln \frac{M}{\delta\sqrt{\tau}}.$$

Finally combining this with the above estimate for N we arrive at

$$N \leq \frac{cn}{\sqrt{\tau}} \ln \frac{1}{h} \leq \frac{cn}{\sqrt{\tau}} \ln \frac{M}{\delta\sqrt{\tau}}.$$

□

The proof of Theorem 3 presented yields an algorithm of explicit construction of the discretization meshes for the univariate exponential sums exhibited by the next proposition.

Corollary 2 Let $[\alpha, \beta] = [0, 1]$, $0 < \tau < 1$, $0 < \delta \leq 1$, $M > 0$. Set $x_0 := 0$, $x_1 := h = \frac{c\delta\sqrt{\tau}}{nM}$ with a suitable absolute constant $c > 0$, and $m := \lceil \frac{8n}{\sqrt{\tau}} \rceil + 1$ in (26). This yields a discrete point set $Y_N := \{x_{j,m}, 0 \leq j \leq N+1\} \subset [0, 1]$ of cardinality $\leq \frac{cn}{\sqrt{\tau}} \ln \frac{M}{\delta\sqrt{\tau}}$ such that for every univariate n term exponential sum $g \in \Omega^1(n, \delta, M)$ we have $\|g\|_{[0,1]} \leq (1 + \tau)\|g\|_{Y_N}$.

Theorem 3 gives exact discretization meshes of cardinality $\leq \frac{cn}{\sqrt{\tau}} \ln \frac{M}{\delta\sqrt{\tau}}$ for univariate n term exponential sums $g \in \Omega^1(n, \delta, M)$. A remarkable feature of this upper bound consists in the fact that the degree M and separation parameter δ of exponential sums appear only in the logarithmic term, while $\frac{n}{\sqrt{\tau}}$ provides the main part of the bound. Now we are going to show that this bound for cardinality is in general sharp up to the logarithmic term.

Theorem 4 Let $0 < \tau < 1$ and let $Y_N = \{x_1, \dots, x_N\} \subset [0, 1]$ be such that for every exponential sum $g(x) = \sum_{0 \leq j \leq n} c_j e^{jx}$, $c_j \in \mathbb{R}$ we have $\|g\|_{[0,1]} \leq (1 + \tau)\|g\|_{Y_N}$. Then,

$$N \geq \frac{cn}{\sqrt{\tau}}$$

with some absolute constant $c > 0$.

Proof For any $0 < h < 1$ consider the univariate algebraic polynomial of degree n given by

$$q_n(x) := T_n\left(\frac{2x^2 - h^2 - 1}{1 - h^2}\right),$$

where

$$T_n(x) := \cos n \arccos x = \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right)$$

is the classical Chebyshev polynomial. Then, $|q_n(x)| \leq 1$ for every $h \leq |x| \leq 1$ and

$$\begin{aligned} |q_n(0)| &= T_n\left(\frac{1+h^2}{1-h^2}\right) = \frac{(1+h)^{2n} + (1-h)^{2n}}{2(1-h^2)^n} \\ &\geq \frac{1 + \binom{2n}{2}h^2}{(1-h^2)^n} \\ &\geq \left(1 + n(2n-1)h^2\right)(1+h^2)^n \geq (1+n(2n-1)h^2)(1+nh^2) \geq 1 + 2n^2h^2. \end{aligned}$$

We have constructed a univariate algebraic polynomial q_n of degree n such that $|q_n(x)| \leq 1$ for every $h \leq |x| \leq 1$ and $|q_n(0)| \geq 1 + 2n^2h^2$. Performing a shift and dilation of the variable it is easy to see that for any $[a, b] \subset \mathbb{R}$ and $a < r < r + h < b$ there exists an algebraic polynomial p_n of degree n so that $|p_n(x)| \leq 1$ for every $x \in [a, b] \setminus [r, r + h]$ and $|p_n(r + \frac{h}{2})| \geq 1 + cn^2h^2$ with some $c > 0$ depending only on a, b . Using this observation with $[a, b] = [1, e]$ and substituting $x = e^t$ easily yields that for every subinterval $[\xi, \gamma] \subset [0, 1]$ of length h , $\gamma - \xi = h$ there exists an exponential sum $g^*(t) = \sum_{0 \leq j \leq n} c_j e^{jt}$ such that $|g^*(t)| \leq 1$ for every $t \in [0, 1] \setminus [\xi, \gamma]$ and $|g^*(\frac{\xi + \gamma}{2})| \geq 1 + c_0 n^2 h^2$. Furthermore, since $Y_N \subset [0, 1]$ contains N points for certain $[\xi, \gamma] \subset [0, 1]$ of length $\frac{1}{N+1} = \gamma - \xi$ we must have $(\xi, \gamma) \cap Y_N = \emptyset$. Thus using the exponential sum $g^*(t)$ constructed above for this $[\xi, \gamma]$ with $h := \frac{1}{N+1}$ yields that

$$1 + \frac{c_0 n^2}{(N+1)^2} \leq \|g^*\|_{[0,1]} \leq (1 + \tau) \|g^*\|_{Y_N} \leq 1 + \tau,$$

$$\text{i.e., } N \geq \frac{cn}{\sqrt{\tau}}.$$

□

Near optimal discretization of uniform norm similar to Theorem 3 can be obtained even for exponential sums on **unbounded** domain $[0, \infty)$. This can be achieved applying Markov and Bernstein type estimates (11) and (15) on half axis for exponential sums with negative exponents. In order to present exponent independent results let us introduce the set of n term exponential sums of variable $\mathbf{w} \in \mathbb{R}_+^d$ with “negative” exponents separated by δ and bounded by M

$$\begin{aligned} \Omega_+^d(n, \delta, M) &:= \left\{ \sum_{1 \leq j \leq n} c_j e^{-\langle \mu_j, \mathbf{w} \rangle}, \quad c_j \in \mathbb{R}, \quad \mu_j \in \mathbb{R}_+^d, \quad |\mu_k - \mu_j| \right. \\ &\quad \left. \geq \delta, k \neq j, \quad |\mu_j| \leq M, \forall j \right\}. \end{aligned}$$

Now we give an analogue of Theorem 3s for the domain $[0, \infty)$.

Theorem 5 *Let $n \in \mathbb{N}$, $M \geq 1$, $0 < \delta \leq 1$. Then, for any $0 < \tau < 1$ there exist discrete points sets $Z_N = \{x_1, \dots, x_N\} \subset [0, \infty)$ of cardinality*

$$N \leq \frac{cn}{\sqrt{\tau}} \ln \frac{M}{\delta \tau}$$

so that for every exponential sum $g \in \Omega_+^1(n, \delta, M)$ we have

$$\|g\|_{(0,\infty)} \leq (1 + \tau) \|g\|_{Z_N}.$$

Proof Set

$$x_0 := 0, x_1 := \frac{\tau}{36nM}, x_j := x_1 e^{\frac{(j-1)\sqrt{\tau}}{16n}}, 2 \leq j \leq N, N := \left\lceil \frac{32n}{\sqrt{\tau}} \ln \frac{36nM}{\delta \tau} \right\rceil + 2.$$

Assume $\|g\|_{[0,\infty)} = 1 = g(x^*)$, $x^* \in [0, \infty]$. (Note that the norm could be attained at $x^* = \infty$, as well.) If $x^* = 0 \in Z_N$ then the statement of the theorem holds trivially. So we may assume that $x^* \in (0, \infty]$. Now we distinguish three cases depending on the location of x^* .

Case 1. $x^* \in (0, x_1)$. Then, by (11)

$$|g(x^*) - g(x_1)| \leq 18nMx_1 = \frac{\tau}{2}.$$

Case 2. $x^* \in (x_N, \infty]$. It is easy to see that

$$x_N = x_1 e^{\frac{(N-1)\sqrt{\tau}}{16n}} \geq x_1 \left(\frac{36nM}{\delta\tau} \right)^2 = \frac{36nM}{\delta^2\tau}.$$

Therefore, it follows from (11) that

$$\begin{aligned} |g(x^*) - g(x_N)| &\leq \int_{x_N}^{\infty} |g'(t)| dt \leq 18nM \int_{x_N}^{\infty} e^{-\delta t} dt = \frac{18nM}{\delta} e^{-\delta x_N} \\ &\leq \frac{18nM}{\delta} e^{-\frac{36nM}{\delta\tau}} \leq \tau \sup_{t>0} t e^{-t} = \frac{\tau}{e}. \end{aligned}$$

Case 3. $x^* \in [x_1, x_N]$. Then, $x^* \in [x_j, x_{j+1}]$ for some $1 \leq j \leq N-1$ and $g'(x^*) = 0$. Now we apply Bernstein type inequality (15) for $k=2$ together with the Taylor formula with integral remainder term yielding

$$\begin{aligned} |g(x^*) - g(x_j)| &\leq (x_{j+1} - x_j) \int_{x_j}^{x_{j+1}} |g''(t)| dt \leq 16n^2 (x_{j+1} - x_j) \int_{x_j}^{x_{j+1}} \frac{1}{t^2} dt \\ &\leq \frac{16n^2 (x_{j+1} - x_j)}{x_j} \log \frac{x_{j+1}}{x_j} = n\sqrt{\tau} \left(e^{\frac{\sqrt{\tau}}{16n}} - 1 \right) \leq \frac{e\tau}{16} < \frac{\tau}{2}. \end{aligned}$$

Summarizing above three cases we can see that independently of the location of x^* there always exists an x_j , $1 \leq j \leq N$ for which we have $|g(x^*) - g(x_j)| < \frac{\tau}{2}$. Since $g(x^*) = 1 = \|g\|_{[0,\infty)}$ this means that

$$\|g\|_{Z_N} = \max_{0 \leq j \leq N} |g(x_{j,m})| \geq 1 - \frac{\tau}{2} \geq \frac{1}{1+\tau} = \frac{\|g\|_{[0,1]}}{1+\tau}.$$

It remains to note now that $n\delta \leq M$ and hence we have with some absolute constant $c > 0$

$$N = \left\lceil \frac{32n}{\sqrt{\tau}} \ln \frac{36nM}{\delta\tau} \right\rceil + 2 \leq \frac{cn}{\sqrt{\tau}} \ln \frac{M}{\delta\tau}.$$

□

4 Discretization of Uniform Norm for Multivariate Exponential Sums

Now we proceed to the extension of the discretization results for multivariate exponential sums on convex polytopes in $\mathbb{R}^d$ and polyhedral cones in $\mathbb{R}_+^d$. This extension will be based on multivariate Bernstein-Markov type inequalities of Sect. 2 and discretization methods for univariate sums developed in Sect. 3. The main building block in the multivariate case is the d -dimensional simplex

$$\Delta^d := \left\{ \mathbf{x} \in \mathbb{R}_+^d : \langle \mathbf{u}_0, \mathbf{x} \rangle \leq 1 \right\} \subset \mathbb{R}^d, \quad \mathbf{u}_0 := (1, \dots, 1).$$

Theorem 6 *Let $K \subset \mathbb{R}^d$, $d \geq 2$ be a convex polytope and consider any $0 < \delta, \tau < 1$, $M \geq 1$. Then, given any $\mu_j \in \mathbb{R}^d$, $|\mu_k - \mu_j| \geq \delta$, $j \neq k$, $|\mu_j| \leq M$, $\forall j$ there exist discrete points sets $Y_N \subset K$ of cardinality*

$$N \leq c_{K,d} \left(\frac{n}{\sqrt{\tau}} \ln \frac{M}{\delta \tau} \right)^d$$

such that for every exponential sum $p = \sum_{1 \leq j \leq n} c_j e^{\langle \mu_j, \mathbf{w} \rangle}$, $n \in \mathbb{N}$ we have

$$\|p\|_K \leq (1 + \tau) \|p\|_{Y_N}.$$

Proof Clearly, it suffices to verify the theorem for the d -dimensional simplex Δ^d . We will accomplish this by the induction on the dimension d .

The univariate case $d = 1$ is covered by Theorem 3. So let us assume that the statement of the theorem holds for the $d - 1$ dimensional simplex Δ^{d-1} , $d \geq 2$. It should be noted that a dilation of this simplex alters both parameters δ and M of the exponential sums by the same constant factor. This constant factor clearly cancels out in the upper bound for the cardinality of the discrete mesh, while uniform norms are dilation invariant. This shows that Theorem 6 is dilation invariant.

It follows by Lemma 5 (ii) applied to $\mathbf{w}^* := \mathbf{u}_0/|\mathbf{u}_0|$, $\mathbf{u}_0 = (1, \dots, 1)$ that there exists an ϵ perturbation $\mathbf{u}^* = (b_1, \dots, b_d)$, $|\mathbf{u}_0 - \mathbf{u}^*| < \epsilon$ such that exponents $e^{\langle \mu_j, \mathbf{w} \rangle}$ restricted to the hyperplanes $\{\mathbf{w} \in \mathbb{R}^d : \langle \mathbf{u}^*, \mathbf{w} \rangle = a\}$, $a \in \mathbb{R}_+$ satisfy the separation condition (17) with parameter $\delta_1 := \frac{c_d \delta \epsilon}{n^2}$ and $\mathbf{u}_2 = \mathbf{u}^*$. The choice of $0 < \epsilon < 1/3$ will be specified somewhat later.

Now for any $a > 0$ consider the $d - 1$ dimensional simplex

$$D(a) := \left\{ \mathbf{u} \in \mathbb{R}_+^d : \langle \mathbf{u}^*, \mathbf{u} \rangle = a \right\}.$$

Since $\mathbf{u}^* = (b_1, \dots, b_d)$, $|\mathbf{u}_0 - \mathbf{u}^*| < \epsilon$ implies $1 - \epsilon < b_j < 1 + \epsilon$, $1 \leq j \leq d$ it is easy to see that $D(a) \subset \Delta^d$ if $0 < a \leq 1 - \epsilon$.

Furthermore, let $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the linear map defined by $A\mathbf{u} = (\frac{u_1}{b_1}, \dots, \frac{u_d}{b_d})$, $\mathbf{u} = (u_1, \dots, u_d)$. Then, it is easy to see that $D(1) = A\Delta^{d-1}$, where $D(1) := \{\mathbf{u} \in \mathbb{R}_+^d : \langle \mathbf{u}^*, \mathbf{u} \rangle = 1\}$ and $\Delta^{d-1} = \{\mathbf{w} \in \mathbb{R}_+^d : \langle \mathbf{u}_0, \mathbf{w} \rangle = 1\}$ is the standard $d - 1$ dimensional simplex. Moreover, the substitution $A\mathbf{w} = \mathbf{u}$ transforms

the exponential sum $\sum_{1 \leq j \leq n} c_j e^{(\mu_j, \mathbf{u})} \in \Omega^d(n, \delta, M)$ into $\sum_{1 \leq j \leq n} c_j e^{(A\mu_j, \mathbf{w})}$ where evidently

$$\begin{aligned} |A\mu_j| &\leq \frac{M}{\min_s b_s} \leq \frac{M}{1-\epsilon} \leq \frac{3M}{2}, \quad |A\mu_j - A\mu_k| \geq \frac{|\mu_j - \mu_k|}{\max_s b_s} \\ &\geq \frac{\delta}{1+\epsilon} \geq \frac{3\delta}{4}, \quad j \neq k. \end{aligned} \quad (27)$$

Next let $Z_N^* := \{0 = x_0 < x_1 < x_2 < \cdots < x_N < x_{N+1} = 1\} \subset [0, 1]$ be the discretization nodes for univariate exponential sums specified in Corollary 2. According to Corollary 2 given any δ_2 (to be specified below) these nodes can be chosen so that $x_1 = h := \frac{c\delta_2\sqrt{\tau}}{nM}$, $1-h \leq x_N \leq 1-h/e$, $N \leq \frac{cn}{\sqrt{\tau}} \ln \frac{M}{\delta_2\sqrt{\tau}}$ and for every univariate exponential sum $g \in \Omega^1(n, \delta_2, 2\sqrt{d}M)$, $n \in \mathbb{N}$ we have

$$\|g\|_{[0,1]} \leq (1+\tau)\|g\|_{Z_N^*}. \quad (28)$$

Now consider the $d-1$ dimensional simplices $D_j := D(x_j)$, $1 \leq j \leq N$ which are dilated copies of $D(1)$. As noted above whenever $0 < \epsilon < h/e$, i.e., $x_N \leq 1-h/e \leq 1-\epsilon$ we have $D_j \subset \Delta^d$, $1 \leq j \leq N$. Recall that exponents $e^{(\mu_j, \mathbf{w})}$ restricted to D_j , $1 \leq j \leq N$ satisfy the separation condition (17) with parameter δ_1 (and $\mathbf{u}_2 = \mathbf{u}^*$). In addition, it is easy to see that these restricted exponents are bounded by the same M . Moreover, since simplex $D(1)$ is the image of Δ^{d-1} under linear map A in view of (27) we can use the induction hypothesis for each $d-1$ dimensional simplex D_j with $\frac{M}{\delta_1}$ being replaced by $\frac{\frac{3M}{2}}{\frac{3\delta_1}{4}} = \frac{2M}{\delta_1}$. This yields discrete point sets $Z_j \subset D_j \subset \Delta^d$ of cardinality

$$\text{card} Z_j \leq c_d \left(\frac{n}{\sqrt{\tau}} \ln \frac{2M}{\delta_1 \tau} \right)^{d-1}, \quad 1 \leq j \leq N$$

so that for every exponential sum $p \in \Omega^d(n, \delta, M)$, $n \in \mathbb{N}$ we have

$$\|p\|_{D_j} \leq (1+\tau)\|p\|_{Z_j}, \quad 1 \leq j \leq N. \quad (29)$$

Obviously, the discretization meshes $Z_j \subset D_j \subset \Delta^d$ are dilated copies of each other for distinct values of j .

Now set

$$Y_N := \cup_{1 \leq j \leq N} Z_j \subset \Delta^d.$$

Since $N \leq \frac{c_d n}{\sqrt{\tau}} \ln \frac{M}{\delta_2 \sqrt{\tau}}$ this results in a discrete point set of cardinality

$$\text{card} Y_N = \sum_{1 \leq j \leq N} \text{card} Z_j \leq c_d \left(\frac{n}{\sqrt{\tau}} \right)^d \ln^{d-1} \left(\frac{M}{\delta_1 \tau} \right) \ln \frac{M}{\delta_2 \sqrt{\tau}}. \quad (30)$$

We are going to show now that Y_N provides the needed discretization of the uniform norm of exponential sums on the simplex Δ^d and then will verify its cardinality. By Lemma 5 (i), $\forall \mathbf{u} \in S^{d-1}$, $\exists \mathbf{w} \in S^{d-1}$, $|\mathbf{u} - \mathbf{w}| \leq \epsilon$ so that

$$|\langle \mu_j, \mathbf{w} \rangle - \langle \mu_k, \mathbf{w} \rangle| \geq \frac{c_d \delta \epsilon^{d-1}}{n^2}, j \neq k. \quad (31)$$

Therefore, we can choose a set $\Omega \subset S_+^{d-1}$ with fill distance $\rho(\Omega, S_+^{d-1}) \leq \epsilon$ so that $\forall \mathbf{w} \in \Omega$ relation (31) holds. Now for any fixed $\mathbf{w} \in \Omega$ consider the univariate exponential sum

$$g(x) := p\left(\frac{\mathbf{w}}{\langle \mathbf{u}^*, \mathbf{w} \rangle} x\right), \quad x \in \mathbb{R}_+.$$

Then, g is of the form $g(x) = \sum_{1 \leq j \leq n} c_j e^{\lambda_j x}$, $c_j \in \mathbb{R}$ with $\lambda_j := \frac{\langle \mu_j, \mathbf{w} \rangle}{\langle \mathbf{u}^*, \mathbf{w} \rangle}$. Evidently

$$\frac{1}{2\sqrt{d}} \leq \frac{1-\epsilon}{\sqrt{d}} \leq \langle \mathbf{u}^*, \mathbf{w} \rangle \leq |\mathbf{u}^*| \leq |\mathbf{u}_0| + \epsilon = \sqrt{d} + \epsilon \leq \sqrt{d} + 1.$$

Thus in view of (31) we have

$$|\lambda_j - \lambda_k| = \left| \frac{\langle \mu_j - \mu_k, \mathbf{w} \rangle}{\langle \mathbf{u}^*, \mathbf{w} \rangle} \right| \geq \frac{|\langle \mu_j, \mathbf{w} \rangle - \langle \mu_k, \mathbf{w} \rangle|}{\sqrt{d} + 1} \geq \frac{c_d \delta \epsilon^{d-1}}{n^2}, j \neq k, \\ |\lambda_j| \leq 2\sqrt{d}M.$$

This means that $g \in \Omega^1(n, \delta_2, 2\sqrt{d}M)$, $n \in \mathbb{N}$ if we set $\delta_2 := \frac{c_d \delta \epsilon^{d-1}}{n^2}$ with the constant c_d from the last lower estimate. In addition, $\frac{\mathbf{w}}{\langle \mathbf{u}^*, \mathbf{w} \rangle} x_j \in D_j$, $1 \leq j \leq N$. Hence, (28) is applicable to $g(x)$, i.e., we have by (28) and (29)

$$\|g\|_{[0,1]} \leq (1+\tau)\|g\|_{Z_N^*} = (1+\tau) \max_{1 \leq j \leq N} \left| p\left(\frac{\mathbf{w}}{\langle \mathbf{u}^*, \mathbf{w} \rangle} x_j\right) \right| \\ \leq (1+\tau) \max_{1 \leq j \leq N} \|p\|_{D_j} \leq (1+\tau)^2 \max_{1 \leq j \leq N} \|p\|_{Z_j} = (1+\tau)^2 \|p\|_{Y_N}.$$

This means that

$$\left| p\left(\frac{\mathbf{w}}{\langle \mathbf{u}^*, \mathbf{w} \rangle} x\right) \right| \leq (1+\tau)^2 \|p\|_{Y_N}, \quad \forall \mathbf{w} \in \Omega, \quad \forall x \in [0, 1],$$

that is

$$\|p\|_{D^*} \leq (1+\tau)^2 \|p\|_{Y_N}, \quad \text{where } D^* := \left\{ \frac{\mathbf{w}}{\langle \mathbf{u}^*, \mathbf{w} \rangle} x : \mathbf{w} \in \Omega, \quad x \in [0, 1-\epsilon] \right\}. \quad (32)$$

It should be noted that since $\mathbf{u}^* = (b_1, \dots, b_d)$, $b_j \geq 1 - \epsilon$ it follows that $(1 - \epsilon)\langle \mathbf{u}_0, \mathbf{w} \rangle \leq \langle \mathbf{u}^*, \mathbf{w} \rangle$, i.e., $D^* \subset \Delta^d$. Now in order to extend the above upper bound to the simplex Δ^d we need to estimate the fill distance $\rho(D^*, \Delta^d)$. Since $\rho(\Omega, S_+^{d-1}) \leq \epsilon$ clearly for any $\mathbf{y} \in \Delta^d \setminus 0$ we can choose $\mathbf{w} \in \Omega$ so that $\mathbf{w} = \frac{\mathbf{y}}{|\mathbf{y}|} + \mathbf{w}_\epsilon$ where $|\mathbf{w}_\epsilon| \leq \epsilon$. Now setting $x_0 := (1 - \epsilon)\langle \mathbf{u}_0, \mathbf{y} \rangle \in [0, 1 - \epsilon]$ we have with $\mathbf{w}_0 := \frac{\mathbf{w}}{\langle \mathbf{u}^*, \mathbf{w} \rangle} x_0 \in D^*$ and $\mathbf{y} = |\mathbf{y}|\mathbf{w} + O(\epsilon)$

$$\langle \mathbf{u}^*, \mathbf{w} \rangle (\mathbf{w}_0 - \mathbf{y}) = (1 - \epsilon)\mathbf{w}\langle \mathbf{u}_0, \mathbf{y} \rangle - \mathbf{y}\langle \mathbf{u}^*, \mathbf{w} \rangle = |\mathbf{y}|\mathbf{w}\langle \mathbf{u}_0 - \mathbf{u}^*, \mathbf{w} \rangle + O(\epsilon).$$

Since $|\mathbf{u}_0 - \mathbf{u}^*| < \epsilon$ and

$$\langle \mathbf{u}^*, \mathbf{w} \rangle \geq (1 - \epsilon)\langle \mathbf{u}_0, \mathbf{w} \rangle \geq (1 - \epsilon)|\mathbf{w}| = 1 - \epsilon,$$

we clearly obtain from the last two estimates that $|\mathbf{w}_0 - \mathbf{y}| \leq c_d \epsilon$, i.e.,

$$\rho(D^*, \Delta^d) \leq c_d \epsilon. \quad (33)$$

Now recall that by Theorem 1 is the Markov factor of the simplex Δ^d is bounded by $\frac{c_d n^3 M}{\delta}$. In view of Lemma 6 this means that if we set $\epsilon := \frac{c_d^* \delta \tau}{n^3 M}$ with a proper constant $c_d^* > 0$ then relation (33) will yield that $\|p\|_{\Delta^d} \leq (1 + \tau)\|p\|_{D^*}$. This together with (32) leads to the upper bound $\|p\|_{\Delta^d} \leq (1 + \tau)^3 \|p\|_{Y_N}$. Clearly, since τ is arbitrary we can modify τ by a constant factor in order to arrive at $\|p\|_{\Delta^d} \leq (1 + \tau)\|p\|_{Y_N}$, which is the required upper bound.

Now it remains to verify the cardinality of the discrete set Y_N . Since $\epsilon = \frac{c_d^* \delta \tau}{n^3 M}$ it follows that

$$\delta_1 \sim \frac{\delta^2 \tau}{n^5 M}, \quad \delta_2 \sim \frac{\delta^d \tau^{d-1}}{n^{3d-1} M^{d-1}}.$$

Applying this together with (30) easily implies

$$\text{card} Y_N \leq c_d \left(\frac{n}{\sqrt{\tau}} \right)^d \ln^d \left(\frac{Mn}{\delta \tau} \right).$$

Furthermore, recall that $|\mu_k - \mu_j| \geq \delta$, $j \neq k$ and $|\mu_j| \leq M$, $1 \leq j \leq n$. Hence, all open balls of radius $\delta/2$ centered at μ_j -s are disjoint while all of them are contained in a ball of radius $M + \delta$ yielding that $n \leq c_d \left(\frac{M}{\delta} \right)^d$. Therefore,

$$N \leq c(K, d) \left(\frac{n}{\sqrt{\tau}} \ln \frac{M}{\delta \tau} \right)^d.$$

This completes the proof of the theorem. $\square$

Next we extend the discretization result of Theorem 6 to polyhedral cones given by

$$K(\mathbf{u}_1, \dots, \mathbf{u}_m) := \left\{ \sum_{1 \leq k \leq m} a_k \mathbf{u}_k, a_k \geq 0 \right\}, \quad \mathbf{u}_k \in S_+^{d-1}, \quad 1 \leq k \leq m.$$

Theorem 7 *Let $K := K(\mathbf{u}_1, \dots, \mathbf{u}_m) \subset \mathbb{R}_+^d$, $d \geq 2$ be a polyhedral cone and consider arbitrary $0 < \tau, \delta < 1$, $M \geq 1$. Then, given any $\mu_j \in \mathbb{R}_+^d$, $|\mu_k - \mu_j| \geq \delta$, $j \neq k$, $|\mu_j| \leq M$, $\forall j$ there exist discrete points sets $Y_N \subset K$ of cardinality*

$$\text{card} Y_N \leq c_{K,d} \left(\frac{n}{\sqrt{\tau}} \ln \frac{M}{\delta \tau} \right)^d$$

such that for every exponential sum $p = \sum_{1 \leq j \leq n} c_j e^{-\langle \mu_j, \mathbf{w} \rangle}$ we have

$$\|p\|_K \leq (1 + \tau) \|p\|_{Y_N}.$$

Proof The proof of this theorem will be based on the discretization for univariate exponential sums on $[0, \infty)$ (Theorem 5), and discretization for multivariate exponential sums on convex polytopes (Theorem 6). In addition, the Markov type estimate provided by Theorem 2 will be applied, as well.

First note that for any $\mathbf{u} \in S_+^{d-1}$ the set

$$D_a(\mathbf{u}) := K(\mathbf{u}_1, \dots, \mathbf{u}_m) \cap \left\{ \mathbf{w} \in \mathbb{R}^d, \langle \mathbf{u}, \mathbf{w} \rangle = a \right\}, \quad a > 0$$

is a $d - 1$ dimensional convex polytope. Lemma 5 (ii) permits certain ϵ perturbations of $\mathbf{u}_0 := \frac{1}{\sqrt{d}}(1, \dots, 1) \in S_+^{d-1}$ so that the new separation parameter for the exponents on perturbed polytopes $D_a(\mathbf{u}^*)$, $\mathbf{u}^* \in S_+^{d-1}$, $|\mathbf{u}^* - \mathbf{u}_0| < \epsilon$ is $\delta_1 = \frac{c_d \delta \epsilon}{n^2}$. We will specify the choice of $0 < \epsilon < \frac{1}{2\sqrt{d}}$ somewhat later. In addition, the upper bound M for the size of exponents is also preserved when restricting them to the hyperplanes $\{\mathbf{w} \in \mathbb{R}^d, \langle \mathbf{u}^*, \mathbf{w} \rangle = a\}$. Now we can apply Theorem 6 for $d - 1$ dimensional convex polytopes $D_a(\mathbf{u}^*)$, $a > 0$ with this δ_1 which implies that each of these polytopes possesses discrete point sets $Y_a \subset D_a(\mathbf{u}^*)$ of cardinality $\leq c_{K,d} \left(\frac{n}{\sqrt{\tau}} \ln \frac{M}{\delta_1 \tau} \right)^{d-1}$ providing discretization of the uniform norm for the exponential sums $p = \sum_{1 \leq j \leq n} c_j e^{-\langle \mu_j, \mathbf{w} \rangle}$ on $D_a(\mathbf{u}^*)$, i.e.,

$$\|p\|_{D_a(\mathbf{u}^*)} \leq (1 + \tau) \|p\|_{Y_a}, \quad a > 0. \quad (34)$$

Furthermore by Theorem 5 for any given $0 < \delta_2 \leq 1$ (to be specified below) there exist discrete points sets $Z_N = \{a_1, \dots, a_N\} \subset [0, \infty)$ of cardinality $N \leq \frac{cn}{\sqrt{\tau}} \ln \frac{M}{\delta_2 \tau}$, so that for any exponential sum $g \in \Omega_+^1(n, \delta_2, 2dM)$ we have

$$\|g\|_{(0, \infty)} \leq (1 + \tau) \|g\|_{Z_N}. \quad (35)$$

With these choice of nodes $a_1, \dots, a_N$ consider now the convex polytopes $D_{a_k}(\mathbf{u}^*)$, $1 \leq k \leq N$. Moreover set

$$D := \cup_{1 \leq k \leq N} D_{a_k}(\mathbf{u}^*), \quad Y_N := \cup_{1 \leq k \leq N} Y_{a_k} \cup \{0\}, \quad Y_N \subset K.$$

Clearly,

$$\text{Card} Y_N \leq c(K, d) \left(\frac{n}{\sqrt{\tau}} \right)^d \ln^{d-1} \left(\frac{M}{\delta_1 \tau} \right) \ln \left(\frac{\mu_n^*}{\delta_2 \tau} \right). \quad (36)$$

Furthermore by (34)

$$\|p\|_D = \max_{1 \leq k \leq N} \|p\|_{D_{a_k}(\mathbf{u}^*)} \leq (1 + \tau) \|p\|_{Y_N}. \quad (37)$$

Now with a proper choice of ϵ and δ_2 we need to extend the above upper bound for $\|p\|_K$, and to verify the cardinality of Y_N . By Lemma 5 (i), $\forall \mathbf{u} \in S^1$, $\exists \mathbf{w} \in S^1$, $|\mathbf{u} - \mathbf{w}| \leq \epsilon$ so that

$$|\langle \mu_j, \mathbf{w} \rangle - \langle \mu_k, \mathbf{w} \rangle| \geq \frac{c_d \delta \epsilon^{d-1}}{n^2}, \quad j \neq k. \quad (38)$$

Therefore, setting $K^* := K \cap S_+^{d-1}$ we can choose a set $\Omega \subset K^*$ with fill distance $\rho(\Omega, K^*) \leq \epsilon$ so that $\forall \mathbf{w} \in \Omega$ relation (38) holds. Now set $\delta_2 := \frac{c_d \delta \epsilon^{d-1}}{n^2}$ where c_d is the constant in (38). By (38) with this choice of δ_2 for any fixed $\mathbf{w} \in \Omega$ the univariate exponential sums

$$g(x) := p \left(\frac{\mathbf{w}}{\langle \mathbf{u}^*, \mathbf{w} \rangle} x \right) = \sum_{1 \leq j \leq n} c_j e^{-\lambda_j x}, \quad x \in \mathbb{R}_+, \quad \lambda_j := \frac{\langle \mu_j, \mathbf{w} \rangle}{\langle \mathbf{u}^*, \mathbf{w} \rangle} > 0$$

satisfy the separation condition $|\lambda_j - \lambda_k| \geq \delta_2$, $j \neq k$. In addition since $|\mathbf{u}^* - \mathbf{u}_0| < \epsilon < \frac{1}{2\sqrt{d}}$ where $\mathbf{u}_0 := \frac{1}{\sqrt{d}}(1, \dots, 1) \in S_+^{d-1}$ it follows that $\langle \mathbf{u}^*, \mathbf{w} \rangle \geq \frac{1}{2d}$, i.e., $0 < \lambda_j \leq 2dM$, $1 \leq j \leq n$. This means that $g \in \Omega_+^1(n, \delta_2, 2dM)$.

Hence, noting that $\frac{\mathbf{w}}{\langle \mathbf{u}^*, \mathbf{w} \rangle} a_k \in D_{a_k}(\mathbf{u}^*)$, $1 \leq k \leq N$ it follows by (35) and (37) that

$$\begin{aligned} \|g\|_{(0, \infty)} &\leq (1 + \tau) \|g\|_{Z_N} = (1 + \tau) \max_{1 \leq k \leq N} |g(a_k)| \\ &= (1 + \tau) \max_{1 \leq k \leq N} \left| p \left(\frac{\mathbf{w}}{\langle \mathbf{u}^*, \mathbf{w} \rangle} a_k \right) \right| \leq (1 + \tau) \|p\|_D \leq (1 + \tau)^2 \|p\|_{Y_N}. \end{aligned}$$

Clearly, this means that

$$|p(t\mathbf{w})| \leq (1 + \tau)^2 \|p\|_{Y_N}, \quad \forall t > 0, \quad \forall \mathbf{w} \in \Omega. \quad (39)$$

Now we set $\epsilon := \frac{\delta \tau}{4c_K n^3 M}$, $0 < \epsilon < \frac{1}{2}$, where c_K is the constant in the Markov type inequality (22) for the polyhedral cone K . Furthermore, let $\|p\|_K = |p(\mathbf{x})|$, $\mathbf{x} \in K$.

Since $0 \in Y_N$ we may assume that $\mathbf{x} \neq 0$. Then, using that $\rho(\Omega, K^*) \leq \epsilon$ for the given $\mathbf{x}/|\mathbf{x}| \in K^*$ there exists $\mathbf{w} \in \Omega$ so that $|\mathbf{x} - |\mathbf{x}|\mathbf{w}| \leq \epsilon|\mathbf{x}|$. Note that $\mathbf{x}, |\mathbf{x}|\mathbf{w} \in K$ hence by the convexity of K we can find a certain $\mathbf{y} := s\mathbf{x} + (1-s)|\mathbf{x}|\mathbf{w} \in K$, $0 \leq s \leq 1$ for which

$$|p(\mathbf{x}) - p(|\mathbf{x}|\mathbf{w})| \leq |\mathbf{x} - |\mathbf{x}|\mathbf{w}||\nabla p|(\mathbf{y}) \leq \epsilon|\mathbf{x}||\nabla p|(\mathbf{y}).$$

In addition,

$$|\mathbf{y}| \geq |\mathbf{x}| - |\mathbf{x} - \mathbf{y}| \geq |\mathbf{x}| - \epsilon|\mathbf{x}| \geq (1 - \epsilon)|\mathbf{x}| \geq \frac{|\mathbf{x}|}{2}.$$

Using the last two bounds together with (22) we obtain

$$\begin{aligned} |p(\mathbf{x}) - p(|\mathbf{x}|\mathbf{w})| &\leq \epsilon|\mathbf{x}||\nabla p|(\mathbf{y}) \leq c_K \epsilon |\mathbf{x}| n M \min \left\{ 1, \frac{n^2}{\delta |\mathbf{y}|} \right\} \|p\|_K \\ &\leq c_K \epsilon |\mathbf{x}| n M \cdot \frac{2n^2}{\delta |\mathbf{x}|} \|p\|_K = \frac{2\epsilon c_K n^3 \mu_n^*}{\delta} \|p\|_K = \frac{\tau}{2} \|p\|_K. \end{aligned}$$

Since $\|p\|_K = |p(\mathbf{x})|$ the last estimate together with (39) yields

$$\|p\|_K \leq \frac{1}{1 - \tau/2} |p(|\mathbf{x}|\mathbf{w})| \leq \frac{(1 + \tau)^2}{1 - \tau/2} \|p\|_{Y_N} \leq (1 + \tau)^3 \|p\|_{Y_N}.$$

Since in the last upper bound $0 < \tau < 1$ is arbitrary it is clearly equivalent to having proved $\|p\|_K \leq (1 + \tau)\|p\|_{Y_N}$.

It remains now to verify the cardinality of the discrete point set Y_N . Recalling that $\epsilon = \frac{\delta\tau}{4c_K n^3 M}$ it follows that

$$\delta_1 \sim \frac{\delta^2 \tau}{n^5 M}, \quad \delta_2 \sim \frac{\delta^d \tau^{d-1}}{n^{3d-1} M^{d-1}}.$$

In addition, as seen in the proof of Theorem 6 conditions $|\mu_k - \mu_j| \geq \delta$, $j \neq k$, $|\mu_j| \leq M$ yield that $n \leq c_{K,d} \left(\frac{M}{\delta}\right)^d$. Applying these observations together with (36) we arrive at

$$\text{card} Y_N \leq c_d \left(\frac{n}{\sqrt{\tau}} \ln \frac{Mn}{\delta\tau} \right)^d \leq c_{K,d} \left(\frac{n}{\sqrt{\tau}} \ln \frac{M}{\delta\tau} \right)^d.$$

This completes the proof of the theorem. $\square$

Remark It should be noted that Theorems 3 and 5 provide exponent independent universal meshes for n term exponential sums in the univariate case. On the other hand their multivariate extensions given by Theorems 6 and 7 yield meshes dependent on the exponents in the sum. This dependence is related to the necessity of preserving separation of exponents when restricting them to hyperplanes, see Lemma 5. On the

other hand imposing some additional regularity assumptions on the distribution of exponents can make it possible to choose proper perturbed directions in Lemma 5 suitable for the corresponding set of exponents. Then, application of above methods would lead to construction of universal exponent independent meshes in multivariate case.

Finally, let us address the question of sharpness of the upper bounds for the cardinality of discrete multidimensional meshes presented in Theorems 6 and 7. It is easy to see that Theorem 4 can be extended to the multivariate case. Namely for any compact set $K \subset \mathbb{R}^d$ of positive Lebesgue measure and any discrete mesh $Y_N \subset K$ such that $\|p^*\|_K \leq (1 + \tau)\|p^*\|_{Y_N}$ for every $p^*(\mathbf{w}) = \sum_{j_1+\dots+j_d \leq n} c_{\mathbf{j}} e^{(\mathbf{j}, \mathbf{w})}$, $c_{\mathbf{j}} \in \mathbb{R}$, $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{N}^d$ it follows that $\text{Card} Y_N \geq c(K, d) \left(\frac{n}{\sqrt{\tau}}\right)^d$. This shows that the upper bounds of Theorems 6 and 7 are sharp with respect to the parameter τ up to the logarithmic term. However, this leaves open the question of sharpness with respect to the number of terms in exponential sums which is of order $\sim n^d$ for p^* as above. On the other hand it should be also mentioned that the number of distinct exponents in $\Omega^d(n, \delta, M)$ and $\Omega_+^d(n, \delta, M)$ with, say, uniform spacing is of order $\sim (\frac{M}{\delta})^d$. Clearly, the cardinality of discrete meshes in Theorems 6 and 7 might be relatively small compared to this.

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