Pointwise Remez inequality

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Abstract

The standard well-known Remez inequality gives an upper estimate of the values of polynomials on $[-1, 1]$ if they are bounded by 1 on a subset of $[-1, 1]$ of fixed Lebesgue measure. The extremal solution is given by the rescaled Chebyshev polynomials for one interval. Andrievskii asked about the maximal value of polynomials at a fixed point, if they are again bounded by 1 on a set of fixed size. We show that the extremal polynomials are either Chebyshev (one interval) or Akhiezer polynomials (two intervals) and prove Totik–Widom bounds for the extremal value, thereby providing a complete asymptotic solution to the Andrievskii problem.

Keywords Remez inequality · Chebyshev and Akhiezer polynomials · Totik–Widom bounds · Comb domains

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1 Introduction

Finding exact constants is one of the most desirable goals in constructive approximation. Many classical results of this sort are collected in the addendum to the Akhiezer

Dedicated to A. Aptekarev on the occasion of his 65th birthday.

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book [3]. We are happy to mention in this connection [7]. This paper deals with a certain clarification of the famous exact Remez inequality, that is, we provide the exact constant in the Andrievskii problem defined below.

Based on several results by Erdélyi, Saff and himself [5,6,15,16], Andrievskii posed the following problem.

Problem 1.1 *Let $\mathcal{P}_n$ be the collection of polynomials of degree at most n . Let E be a closed subset of $[-1, 1]$, and $|E|$ denote its Lebesgue measure. For $x_0 \in [-1, 1]$, define*

$$M_n(x_0, E) = \sup\{|P_n(x_0)| : P_n \in \mathcal{P}_n, |P_n(x)| \leq 1 \text{ for } x \in E\}. \quad (1.1)$$

For $\delta \in (0, 1)$, find

$$L_{n,\delta}(x_0) = \sup\{M_n(x_0, E) : E \text{ such that } |E| \geq 2 - 2\delta\}. \quad (1.2)$$

Let us comment the setting with the following three evident remarks:

- (1) $L_{n,\delta}(x_0)$ is even, thus we will consider only $x_0 \in [-1, 0]$.
- (2) $\mathcal{L}_{n,\delta} := \sup_{x_0 \in [-1,0]} L_{n,\delta}(x_0)$ is the famous Remez constant [24]. It is attained at the endpoint -1 by the Chebyshev polynomial $R_{n,\delta}(x)$ for the interval $[-1+2\delta, 1]$, i.e.,

$$\mathcal{L}_{n,\delta} = \mathfrak{T}_n\left(\frac{1+\delta}{1-\delta}\right), \quad R_{n,\delta}(x) = \mathfrak{T}_n\left(\frac{\delta-x}{1-\delta}\right),$$

where $\mathfrak{T}_n$ denotes the standard Chebyshev polynomial of degree n associated with $[-1, 1]$,

$$\mathfrak{T}_n(x) = \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right).$$

We will henceforth call $R_{n,\delta}$ the Remez polynomial.

- (3) Clearly, for Problem 1.1, $R_{n,\delta}(x)$ cannot be extremal for all $x_0 \in [-1, 0]$ as soon as δ is sufficiently small. Let $\delta = 1/2$. Then, $R_{n,1/2}(0) = 1$, while for the so-called Akhiezer polynomial $A_{2m,\delta}(x)$ we get $A_{2m,1/2}(0) = \mathfrak{T}_m(5/3) > 1$. Recall [1] or [4, Chapter 10]

$$A_{2m,\delta}(x) = \mathfrak{T}_m\left(\frac{1+\delta^2-2x^2}{1-\delta^2}\right)$$

is the even Chebyshev polynomial on the set $[-1, 1] \setminus (-\delta, \delta)$.

Andrievskii raised his question on several international conferences, including Jaen Conference on Approximation Theory, 2018. The third remark highlights that the problem is non-trivial. We found it highly interesting and in this paper we provide its complete asymptotic solution.

In [1,2], Akhiezer studied various extremal problems for polynomials bounded on two disjoint intervals $E(\alpha, \delta) = [-1, 1] \setminus (\alpha - \delta, \alpha + \delta)$, $\delta - 1 < \alpha \leq 0$, see also [3, Appendix Section 36 and Section 38 in German translation].

Definition 1.2 We say that a polynomial $A_{n,\alpha,\delta}(x)$ is the Akhiezer polynomial for $E(\alpha, \delta)$ with respect to an *internal gap* if it solves the following extremal problem

$$A_{n,\alpha,\delta}(x_0) = \sup\{|P_n(x_0)| : P_n \in \mathcal{P}_n, |P_n(x)| \leq 1 \text{ for } x \in E(\alpha, \delta)\}, \quad (1.3)$$

where $x_0 \in (\alpha - \delta, \alpha + \delta)$.

We point out that $A_{2m,\delta}$ as introduced above corresponds to the symmetric case $\alpha = 0$ and even $n = 2m$. We describe properties of Akhiezer polynomials in Sect. 2.1. Note, in particular, that the extremal property of $A_{n,\alpha,\delta}(x)$ does not depend on which point $x_0 \in (\alpha - \delta, \alpha + \delta)$ was fixed in (1.3).

In Sect. 3, we prove the following theorem.

Theorem 1.3 *The extremal value $L_{n,\delta}(x_0)$ is assumed either on the Remez polynomial $R_{n,\delta}(x)$ or on an Akhiezer polynomial $A_{n,\alpha,\delta}(x)$ with a suitable α .*

Our main result, the asymptotic behavior of $L_{n,\delta}(x_0)$, is presented in Sect. 4. Nowadays, the language of potential theory is so common and widely accepted, see, e.g., [6,11,13,19,27], that we will formulate our asymptotic result using this terminology.

Let $G_{\alpha,\delta}(z, z_0)$ be the Green function in the domain $\Omega := \mathbb{C} \setminus E(\alpha, \delta)$ with respect to $z_0 \in \Omega$. In particular, we set $G_{\alpha,\delta}(z) = G_{\alpha,\delta}(z, \infty)$. Respectively, $G_\delta(z)$ is the Green function of the domain $\mathbb{C} \setminus E(\delta)$ with respect to infinity, where $E(\delta) = [-1 + 2\delta, 1]$. It is well known, that

$$R_{n,\delta}(x) = \frac{e^{nG_\delta(x)} + e^{-nG_\delta(x)}}{2}, \quad x \in [-1, -1 + 2\delta], \quad (1.4)$$

and

$$\lim_{\alpha \rightarrow -1+\delta} G_{\alpha,\delta}(x) = G_\delta(x), \quad x \in (-1, -1 + 2\delta).$$

Lemma 1.4 *Let $\Phi_\delta(x)$ be the upper envelope*

$$\Phi_\delta(x) = \sup_{\alpha \in (\delta-1, 0]} G_{\alpha,\delta}(x). \quad (1.5)$$

If for x_0 the supremum is attained for some internal point $\alpha \in (\delta - 1, 0]$, then $x_0 = x_0(\alpha)$ is a unique solution of the equation

$$\partial_\alpha G_{\alpha,\delta}(x) = 0. \quad (1.6)$$

Otherwise, it is attained as $\alpha \rightarrow \delta - 1$ and $\Phi_\delta(x_0) = G_\delta(x_0)$.

Note that the non-trivial claim in Lemma 1.4 is the uniqueness of $x_0(\alpha)$. We will provide a proof of this fact in Sect. 4.2. Thus, we show that the upper envelope (1.5) is in fact the upper envelope of just two curves. One of them is the graph of $G_\delta(x)$ given in terms of elementary functions, see (4.8). The other one is obtained in the following way: first we find the unique solution $x_0(\alpha)$ of (1.6) and then we form the curve $(x_0(\alpha), G_{\alpha,\delta}(x_0(\alpha)))$. Although each individual Green function $G_{\alpha,\delta}(x)$ can be expressed in terms of elliptic functions [4, § 55 equation (4)], due to the implicit definition of $x_0(\alpha)$ by means of (1.6), we don't believe that a parametric description of the upper envelope in terms of elliptic functions exists. However, for any fixed $\delta > 0$ it can be solved numerically, as we demonstrate in Example 4.5 including a graph of the curve $(x_0(\alpha), G_{\alpha,\delta}(x_0(\alpha)))$, cf. Fig. 3. Note that the endpoints of this curve are also given explicitly.

Our main theorem below shows that the asymptotics of $L_{n,\delta}(x_0)$ are described by the upper envelope $\Phi_\delta(x_0)$.

Theorem 1.5 *The following limit exists*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log 2L_{n,\delta}(x_0) = \Phi_\delta(x_0).$$

To be more precise, if $\Phi_\delta(x_0) = G_\delta(x_0)$, then

$$\log 2L_{n,\delta}(x_0) = n\Phi_\delta(x_0) + o(1). \quad (1.7)$$

If $\Phi_\delta(x_0) = G_{\delta,\alpha}(x_0)$ for some $\alpha \in (\delta - 1, 0]$, then the Totik–Widom bound

$$\log 2L_{n,\delta}(x_0) = n\Phi_\delta(x_0) + O(1) \quad (1.8)$$

holds.

For the classical Chebyshev problem related to a fixed set E , the asymptotics are described in terms of the Green function associated with this set. In contrast to this, $\Phi_\delta(x_0)$, respectively, $E(\alpha_0, \delta)$ for some extremal α_0 depend on x_0 in a quite sophisticated way. We present an asymptotic diagram which allows to describe completely the asymptotic behavior of $L_{n,\delta}(x_0)$ for all $x_0 \in [-1, 1]$ and a given δ .

The organization of the paper is as follows. In Sect. 2, we prove part 1 of Theorem 1.3 and present technical tools that will be used throughout the paper. The structure of Chebyshev polynomials can be revealed by their representation in terms of conformal mappings on so-called comb domains. However, the relation between the parameters describing a given set E and the related comb domain are highly non-trivial. We provide a comprehensive description of Akhiezer polynomials and their dependence on the position α . In Sect. 2.3, we finish the proof of Theorem 1.3. The proof of Theorem 1.5 and a description of the upper envelope (1.5) are given in Sect. 4.

2 Preliminaries

In [26], the sharp constant in the Remez inequality for trigonometric polynomials on the unit circle was given. The proof was based on the following two steps:

- (i) The structure of possible extremal polynomials was revealed with the help of their comb representations,
- (ii) The principle of harmonic measure (a monotonic dependence on a domain extension) allows to get an extremal configuration for the comb parameters in the given problem.

We also start with recalling comb representations for extremal polynomials, see Sect. 2.1. We refer the reader to [17,25] and the references therein for more information about the use of comb mappings in approximation theory. However, we doubt that the step (ii) is applicable to the Andrievskii problem. That is, that comparably simple arguments from potential theory such as the principle of harmonic measure, would also allow us to bring a certain fixed configuration (comb) to an extremal one. Instead, we develop here an *infinitesimal approach*, closely related to the ideas of Loewner chains [12,22].

Using this method, we will prove in Sect. 3 that the extremal solution for $L_{n,\delta}(x_0)$ is either a Chebyshev or an Akhiezer polynomial. For this reason, we continue in Sect. 2.3 with the complete discussion of Akhiezer polynomials $A_{n,\alpha,\delta}$, see (1.3). Recall that $A_{n,\alpha,\delta}$ is extremal on two given intervals with respect to points $x_0 \in (\alpha - \delta, \alpha + \delta)$.¹ Generically, $A_{n,\alpha,\delta}^{-1}([-1, 1])$ is a union of *three* intervals—the set contains also an additional interval outside of $[-1, 1]$. We demonstrate our infinitesimal approach showing dependence of this additional interval on α for fixed n and δ . In the language of comb domains, we will observe a rather involved dependence of the comb parameters in a simple monotonic motion of α . The domain can undergo all possible variations: to narrow, expand, or a combination of both, see Theorem 2.14. Essentially, this is the base for our belief that simple arguments, in the spirit of (ii), in the Andrievskii problem are hardly possible.

If for fixed x_0 the extremal polynomial is a Remez polynomial, there is no additional interval outside of $[-1, 1]$. Intuitively, the same considerations as were used in [26] should work. However, a technical difference prevents direct applications of the principle of harmonic measure. Namely, the Lebesgue measure on the circle corresponds to the harmonic measure evaluated at 0, while in the sense of potential theory the Lebesgue measure on $\mathbb{R}$ corresponds to the Martin or Phragmén–Lindelöf measure, [20]. Although we are convinced that a limiting process would allow to overcome this technical issue, we provide an alternative proof below; cf. Lemma 2.5.

¹ We would like to point out that up to some trivial degenerations, $A_{n,\alpha,\delta}$ is different to the polynomial of degree n that has maximal leading coefficient and is bounded by one on the set $E(\alpha, \delta)$.

2.1 Comb Representation for Extremal Polynomials

By a regular comb, we mean a half strip with a system of vertical slits

$$\Pi = \{z : \Re z \in (\pi n_-, \pi n_+), \Im z > 0\} \setminus \{z = \pi k + iy : y \in (0, h_k), k \in (n_-, n_+)\}$$

where $n_+ - n_- = n \in \mathbb{N}$ and the k 's are integers. We call $h_k, h_k \geq 0$, the height of the k -th slit and point out that the degeneration $h_k = 0$ is possible. Let $\theta : \mathbb{C}_+ \rightarrow \Pi$ be a conformal mapping of the upper half-plane onto a regular comb such that $\theta(\infty) = \infty$. Then,

$$T_n(z) = \cos \theta(z)$$

defines a polynomial of degree n . Let $E \subset [-1, 1]$ be compact and $x_0 \in [-1, 1] \setminus E$. Moreover, let (a, b) denote the maximal open interval in $\mathbb{R} \setminus E$ that contains x_0 . Then, using the technique of Markov correction terms we obtain the following representation of the extremizer of (1.1)

Theorem 2.1 ([25, 7.5. Basic theorem], [14, Theorem 3.2.]) *Under the normalization $T_n(x_0) > 0$, there exists a unique extremal polynomial for (1.1) and it only depends on the gap (a, b) and not on the particular point $x_0 \in (a, b)$. Moreover, let $\theta : \mathbb{C}_+ \rightarrow \Pi$ be a comb mapping onto a regular comb Π and E be such that:*

- (i) $h_0 > 0$ and $a = \theta^{-1}(-0), b = \theta^{-1}(+0)$,
- (ii) E contains at least one of the points $\theta^{-1}(k\pi \pm 0)$, for all $k \in (n_-, n_+)$,
- (iii) E contains at least one of the points $\theta^{-1}(\pi n_-), \theta^{-1}(\pi n_+)$,

Then,

$$T_n(z) = \cos \theta(z) \tag{2.1}$$

is an extremal polynomial for E and the interval (a, b) . Vice versa, if T_n is an extremal polynomial for a set E and an interval (a, b) , then there exists a regular comb with these properties such that (2.1) holds.

Let us now specify to the case of Akhiezer polynomials, where $E = E(\alpha, \delta) = [-1, 1] \setminus (\alpha - \delta, \alpha + \delta)$. By the above theorem, the so-called n -extension $E_n = E_n(\alpha, \delta) := A_{n,\alpha,\delta}^{-1}([-1, 1])$ can be of the following types:

- (i) there is an additional interval to the right of 1,
- (ii) E is extended at 1,
- (iii) E is extended at -1 ,
- (iv) there is an additional interval to the left of -1 .

The corresponding comb-mapping realization is collected in the following Corollary:

Corollary 2.2 *For a fixed set $E = E(\alpha, \delta)$ and degree n the extremal polynomial $A_{n,\alpha,\delta}(z)$ of (1.1) is of the form (2.1). The unique comb $\Pi_{n,\alpha,\delta}$ has one of four possible shapes shown in Figs. 1 and 2. Moreover, the following normalizations distinguish the cases*

- (i) $\theta^{-1}(\pi n_-) = -1$, $\theta^{-1}(\pi(n_+ - 1) - 0) = 1$, see Fig. 1, left,
- (ii) $\theta^{-1}(\pi n_- - 0) = -1$, $\theta(1) \in [\pi(n_+ - 1), \pi n_+]$ see Fig. 1, right,
- (iii) $\theta(-1) \in [\pi(n_-), \pi(n_- + 1)]$, $\theta^{-1}(\pi n_- - 0) = -1$, see Fig. 2, left,
- (iv) $\theta^{-1}(\pi(n_- + 1) + 0) = -1$, $\theta^{-1}(\pi n_+) = 1$, see Fig. 2, right.

Remark 2.3 Let us denote the additional interval for the cases (i) and (iv) by I_n . These cases include the limit cases $h_{n_+ - 1} = \infty$ and $h_{n_- + 1} = \infty$, respectively. That is, the extremal polynomial is of the degree $n - 1$. Note that $A_{n,\alpha,\delta}$ has a zero on I_n . The aforementioned degeneration corresponds to “the zero being at ∞ .” On the other hand, also the degenerations $h_{n_+ - 1} = 0$ and $h_{n_- + 1} = 0$ are possible, which allow for a smooth transition to the cases (ii) and (iii), respectively.

Fig. 1 Comb domains for the cases (i) and (ii)

Fig. 2 Comb domains for the cases (iii) and (iv)

2.2 Reduction to Remez Polynomials (Proof of Theorem 1.3, Part 1)

As we have mentioned in the beginning of this section, we cannot use the ideas of the principle of harmonic measure directly. We overcome this technical problem by using transition functions from the theory of Loewner chains [12,22]. Let us consider an arbitrary regular comb $\Pi = \Pi(0)$ as in Theorem 2.1 and let us fix a slit with height $h_k > 0$. Let $h_k(\epsilon)$ be strictly monotonically decreasing such that $h_k(0) = h_k$ and $\Pi(\epsilon)$ be the comb which coincides with Π , but h_k is reduced to $h_k(\epsilon)$. Let θ and θ_ϵ be the corresponding comb mappings. Then, clearly $\Pi \subset \Pi_\epsilon$ and thus the transition function

$$w(z, \epsilon) = w_\epsilon(z) := \theta_\epsilon^{-1}(\theta(z))$$

is well defined and is an analytic map from $\mathbb{C}_+$ into $\mathbb{C}_+$. If we define $I(\epsilon) := \theta^{-1}([\pi k + ih_k(\epsilon), \pi k + ih_k])$ and $c_k(\epsilon) = \theta_\epsilon^{-1}(\pi k + ih_k(\epsilon))$, then $w_\epsilon : \mathbb{R} \setminus I(\epsilon) \rightarrow \mathbb{R} \setminus \{c_k(\epsilon)\}$ is one-to-one and onto. The arc $J(\epsilon) := \theta_\epsilon^{-1}([\pi k + ih_k(\epsilon), \pi k + ih_k])$ lies, except its endpoint in $\mathbb{C}_+$ and $w_\epsilon : \mathbb{C}_+ \rightarrow \mathbb{C}_+ \setminus J(\epsilon)$ is conformal.

Lemma 2.4 *The Nevanlinna function w_ϵ admits the representation*

$$w(z, \epsilon) = z + \int_{I(\epsilon)} \frac{1 - z^2}{1 - \xi^2} \frac{d\sigma_\epsilon(\xi)}{\xi - z}. \quad (2.2)$$

Proof It follows from the above, that w_ϵ is a Nevanlinna function, i.e., is an analytic function with positive imaginary part on $\mathbb{C}_+$, and that the measure in its integral representation is supported on $I(\epsilon)$. That is, we can write

$$w_\epsilon(z) = A_\epsilon z + B_\epsilon + \int_{I(\epsilon)} \frac{d\sigma_\epsilon(\xi)}{\xi - z}, \quad A_\epsilon \geq 0, B_\epsilon \in \mathbb{R}.$$

Moreover, it satisfies the normalization conditions

$$w_\epsilon(-1) = -1, \quad w_\epsilon(1) = 1, \quad w_\epsilon(\infty) = \infty,$$

which yields

$$B_\epsilon = 1 - A_\epsilon - \int_{I(\epsilon)} \frac{d\sigma_\epsilon(\xi)}{\xi - 1}, \quad A_\epsilon = 1 + \int_{I(\epsilon)} \frac{d\sigma_\epsilon(\xi)}{1 - \xi^2}. \quad (2.3)$$

From this, we obtain (2.2). $\square$

With the aid of this lemma, one could already recover Remez' result, that for points on the boundary the extremal configuration is an interval.

Lemma 2.5 *Let*

$$E = [b_0, 1] \setminus \bigcup_{j=1}^g (a_j, b_j), \quad x_0 \in [-1, b_0), \quad |E| = 2 - 2\delta$$

and $T_n(x, E)$ be the extremal polynomial for x_0 as described in Theorem 2.1. Then,

$$R_{n,\delta}(x_0) > T_n(x_0, E).$$

Proof For an elementary proof even in the multi-dimensional case, we refer to [8, 9, Lemma 2]. $\square$

Remark 2.6 Continuing the heuristics provided at the beginning of this section, we give an interpretation of this result in terms of the harmonic measure. Assume that θ_ϵ is a conformal map for which one of the slit heights was reduced, $E_\epsilon = \theta_\epsilon^{-1}([\pi n_-, \pi n_+])$ and w_ϵ the corresponding transition function. Lemma 2.5 follows easily, provided that $|E_\epsilon| > |E|$ and $w_\epsilon(x_0) > x_0$. Let us assume we were interested in the harmonic measure of E instead of its Lebesgue measure. Then, the above properties have a probabilistic interpretation: namely, the first one corresponds to the probability that a particle which starts at a point which is close to infinity in the domain Π_ϵ and Π and terminates on the base of the comb. From this perspective, it is clear that this increases if one of the slit heights is decreased. Similarly, writing the second one as $w_\epsilon(x_0) + 1 > x_0 + 1$, it corresponds to the probability that a particle terminates in $[\theta(-1), \theta(x_0)]$ in Π and Π_ϵ , respectively. Again, this explains, why the value should be increased. This can be made rigorous by using Lemma 2.4.

The above interpretation uses the fact that there is no additional portion of E outside of $[-1, 1]$. For Akhiezer polynomials, this is not true. This makes the problem essentially more delicate and we extend our tools by using the infinitesimal variation $\dot{w}$ as defined in (2.4); cf. also Remark 2.15.

2.3 Transformation of the Akhiezer Polynomials as α is moving

We believe that the observations presented in this section are of independent interest. Our goal is to describe the transformation of the comb as the interval starting from the center moves continuously to the left. This should correspond to a continuous transformation of the comb, which seems at the first sight impossible, since the base of the slits are positioned only at integers. We show in the following how the cases (i), (ii), (iii), (iv) can be connected by a continuous transformation: let us start with case (i) in the degenerated configuration $h_{n_+-1} = \infty$. Then, we start decreasing h_{n_+-1} until we reach $h_{n_+-1} = 0$. Now we are in the situation that $\theta(1) = \pi(n_+ - 1)$ and we continue with case (ii). That is, $\theta(1)$ increases until it reaches $\theta(1) = \pi n_+$. All the time $\theta(-1) = \pi n_-$. Now we continue with case (iii) and increase $\theta(-1)$ until the point $\theta(-1) = \pi(n_- + 1)$. We continue with case (iv) and increase h_{n_++1} from $h_{n_++1} = 0$ to $h_{n_++1} = \infty$. We have arrived at our initial configuration but the base of the comb was shifted from position $(\pi n_-, \pi(n_+ - 1))$ to position $(\pi(n_- + 1), \pi n_+)$.

We believe it is helpful to understand these transformations also on the level of E_n . Our initial configuration corresponds to $E = E_n$. We view this as the additional interval is hidden at ∞ . Starting case (i) corresponds to the motion that the position of the additional interval decreases from $+\infty$ until it joins E . This corresponds to the change from case (i) to (ii). When it is fully absorbed, we arrive at case (iii), i.e., E

starts to be extended to the left until the extension separates and starts to decrease to $-\infty$. And the circle starts from the beginning.

Theorem 2.7 *Moving (a, b) to the left corresponds to a succession of continuous transformations of the comb as described above. If n is odd, we start with case (i) and $h_{n+1} = \infty$. If n is even, we start with case (iii) and $\theta(-1) = \pi n_-$.*

Remark 2.8 When $\alpha = 0$ by symmetry, the extremal polynomial is even and all critical points of the extremal polynomial are equally distributed on $[-1, -\delta]$ and $[\delta, 1]$. The above procedure describes how all critical points are moved from the interval $[-1, \alpha - \delta]$ to $[\alpha + \delta, 1]$ as α is decreasing. In the limit when α approaches $-1 + \delta$, all critical points will be on $[-1 + 2\delta, 1]$ and the extremal polynomial will transform into the Remez polynomial $R_{n,\delta}$.

As we have already indicated at the end of Section 2.2 the proof of Theorem 2.7 requires in addition to the transition function w_ϵ its infinitesimal transform $\dot{w}$. Let us introduce the notation

$$w'(z, \epsilon) = \partial_z w(z, \epsilon), \quad \dot{w}(z, \epsilon) = \partial_\epsilon w(z, \epsilon) \quad \text{and} \quad \dot{w}(z) = \dot{w}(z, 0). \quad (2.4)$$

Lemma 2.9 *Under an appropriate choice of ϵ , $w(z, \epsilon)$ is differentiable with respect to ϵ and there exists $\sigma_k > 0$ such that*

$$\dot{w}(z) := \lim_{\epsilon \rightarrow 0} \frac{w(z, \epsilon) - w(z, 0)}{\epsilon} = \frac{1 - z^2}{1 - c_k^2} \frac{\sigma_k}{c_k - z}, \quad (2.5)$$

where $c_k = c_k(0) = \theta^{-1}(\pi k + i h_k)$.

Proof We note that if $k = n_- + 1$ or $k = n_+ - 1$ and we are in the situation of Theorem 2.1 (i) or (iv), then due to (2.3) $A_\epsilon < 1$ and $A_\epsilon > 1$ otherwise. In any case, we obtain from the properties of w_ϵ that A_ϵ is strictly monotonic. Therefore, if we fix ϵ_0 and define $\beta_0 = \left| \int_{I(\epsilon_0)} \frac{d\sigma_{\epsilon_0}(\xi)}{1 - \xi^2} \right|$, then $\epsilon \mapsto \left| \int_{I(\epsilon)} \frac{d\sigma_\epsilon(\xi)}{1 - \xi^2} \right|$ maps $[0, \epsilon_0]$ continuously and bijectively on $[0, \beta_0]$. Through a reparametrization we can achieve that $\left| \int_{I(\epsilon')} \frac{d\sigma_{\epsilon'}(\xi)}{1 - \xi^2} \right| = \frac{\epsilon' \beta_0}{\epsilon_0}$. Thus, the measures $\frac{d\sigma_{\epsilon'}(\xi)}{(1 - \xi^2)\epsilon'}$ are normalized and we get by passing to a subsequence

$$\lim_{k \rightarrow \infty} \frac{w(z, \epsilon'_k) - z}{\epsilon'_k} = \lim_{k \rightarrow \infty} \int_{I(\epsilon'_k)} \frac{1 - z^2}{1 - \xi^2} \frac{1}{\xi - z} \frac{d\sigma_{\epsilon'_k}(\xi)}{\epsilon'_k} = \int \frac{1 - z^2}{1 - \xi^2} \frac{d\sigma_\infty(\xi)}{\xi - z}.$$

Since the support of $\sigma_{\epsilon'_k}$ shrinks to $\{c_k\}$, we get that $\sigma_\infty = \sigma_k \delta_{c_k}$, for $\sigma_k / |1 - c_k^2| = \beta_0 / \epsilon_0$. This concludes the proof. $\square$

We are also interested in the case of growing slit heights. This corresponds to

$$u_\epsilon(x) = w_\epsilon^{-1}(x),$$

for some transition function as defined above. Note that since $c_k(\epsilon) \rightarrow c_k$ as $\epsilon \rightarrow 0$, this inverse is well defined on $\mathbb{R}$ away from a vicinity of c_k and we conclude since $w'(z, 0) = 1$ that

$$\dot{u}(x) = -\dot{w}(x).$$

The following lemma discusses the case (i). We show that decreasing h_{n_+-1} and simultaneously increasing h_0 appropriately allows us to move the gap $(\alpha - \delta, \alpha + \delta)$ to the left without changing its size. Let us set $(a, b) = (\alpha - \delta, \alpha + \delta)$ let c be the critical point in (a, b) and c_+ be the critical point outside $[-1, 1]$, i.e., $c = \theta^{-1}(ih_0)$, $c_+ = \theta^{-1}(\pi(n_+ - 1) + ih_{n_+-1})$.

Lemma 2.10 *Let $E(\alpha, \delta)$ with $\alpha \leq 0$ be such that the corresponding extremal polynomial $A_{n,\alpha,\delta}(z)$ corresponds to the case (i). Then, the infinitesimal variation $\dot{w}(x)$ generated by decreasing the height h_{n_+-1} under the constraint of a constant gap length ($\delta = \text{const}$) leads to an increase in h_0 . Moreover, in this variation the gap is moving to the left, that is, $\alpha(\epsilon)$ is decreasing.*

Proof Let w_{n_+-1} and w_0 be transforms corresponding to a variation of the slit heights h_{n_+-1} and h_0 . In (2.5), we chose $\sigma_{n_+-1} > 0$ and determine the sign of σ_0 by the condition of constant gap length. Thus, the total transform corresponds to $w(z, \epsilon) = w_{n_+-1}(w_0(z, \epsilon), \epsilon)$ and hence, by the previous computations, we find that

$$\dot{w}(x) = \frac{1-x^2}{1-c^2} \frac{\sigma_0}{c-x} + \frac{1-x^2}{1-c_+^2} \frac{\sigma_+}{c_+-x},$$

with $\sigma_+ > 0$. The value σ_0 is determined due to the constraint $w(b, \epsilon) - w(a, \epsilon) = b - a$, i.e., $\dot{w}(b) = \dot{w}(a)$. Thus, we obtain

$$\begin{aligned} 0 &= \left(\frac{1-b^2}{c-b} - \frac{1-a^2}{c-a} \right) \frac{\sigma_0}{1-c^2} + \left(\frac{1-b^2}{c_+-b} - \frac{1-a^2}{c_+-a} \right) \frac{\sigma_+}{1-c_+^2} \\ &= (b-a) \left(\frac{1+ab-c(a+b)}{(c-a)(c-b)} \frac{\sigma_0}{1-c^2} + \frac{1+ab-c_+(a+b)}{(c_+-a)(c_+-b)} \frac{\sigma_+}{1-c_+^2} \right). \end{aligned} \quad (2.6)$$

Let

$$\ell(x) = 1 + ab - (a+b)x. \quad (2.7)$$

Since $a + b = 2\alpha \leq 0$, ℓ is non-decreasing. Moreover, since $a, b > -1$, $\ell(-1) = (1+a)(1+b) > 0$ and hence, $\ell(c) > 0$ and $\ell(c_+) > 0$. Using that the numerator is negative for both summands, we find that $\sigma_+ > 0$ implies $\sigma_0 < 0$. In other words, the compression of the height h_{n_+-1} leads to a growth of h_0 . Since both summands are negative,

$$\dot{w}(a) = \frac{1-a^2}{1-c^2} \frac{\sigma_0}{c-a} + \frac{1-b^2}{1-c_+^2} \frac{\sigma_+}{c_+-a} < 0.$$

Consequently, $\dot{w}(b) = \dot{w}(a) < 0$, and we find that the ends of the interval (a_ϵ, b_ϵ) are moving to the left. This concludes the proof. $\square$

Case (iv) is similar to case (i). However, we will encounter certain specific phenomena. First of all, we will increase the value of h_{n+1} to move the given gap to the left. But in order to fulfill the constraint of fixed gap length both an increasing or a decreasing of h_0 is possible.

Lemma 2.11 *Let $E(\alpha, \delta)$ with $\alpha \leq 0$ be such that the corresponding extremal polynomial $A_{n,\alpha,\delta}(z)$ corresponds to the case (iv). Let ℓ be defined as in (2.7) and*

$$\eta := \frac{1}{2} \left(\alpha + \frac{1 - \delta^2}{\alpha} \right), \quad (\eta < -1), \quad (2.8)$$

be its zero. If $c_- < \eta$, the infinitesimal variation $\dot{w}(x)$ generated by increasing the height h_{n+1} under the constraint of a constant gap length ($\delta = \text{const}$) leads to an increase in h_0 and it leads to a decrease in h_0 if $c_- \in (\eta, -1)$. In both cases, the gap is moving to the left, that is, $\alpha(\epsilon)$ is decreasing.

Proof As before, we find that the infinitesimal variation is of the form

$$\dot{w}(x) = \frac{1 - x^2}{1 - c_-^2} \frac{\sigma_0}{c_- - x} + \frac{1 - x^2}{1 - c_-^2} \frac{\sigma_-}{c_- - x}, \quad \sigma_- < 0.$$

Respectively, the constraint $\dot{w}(b) = \dot{w}(a)$ corresponds to

$$\frac{\ell(c)}{(c - a)(c - b)} \frac{\sigma_0}{1 - c^2} + \frac{\ell(c_-)}{(c_- - a)(c_- - b)} \frac{\sigma_-}{1 - c_-^2} = 0.$$

We have that $\ell(c_-) < 0$ for $c_- < \eta$. Since $c_- < -1 < a < c < b$, this implies that $\sigma_0 < 0$. As before, we conclude that $\dot{w}(a) < 0$, and the interval is moving to the left. If $c_- \in (\eta, -1)$, then $\ell(c_-) > 0$ and therefore $\sigma_0 > 0$. In this case, $\dot{w}(b) < 0$ and again the interval is moving to the left. Finally, if $c_- = \eta$, we have $\ell(c_-) = 0$ and therefore (2.6) implies that $\sigma_0 = 0$. $\square$

2.3.1 The cases (ii) and (iii)

We will discuss case (ii) and case (iii) simultaneously. Recall that case (ii) corresponds to an extension of E to the right and case (iii) to an extension to the left, i.e., $\theta(1) \in (\pi(n_+ - 1), \pi n_+)$ or $\theta(-1) \in (\pi n_-, \pi(n_- + 1))$. Let $\Pi \equiv \Pi(\epsilon)$ but let us increase the normalization point $\theta_\epsilon(\pm 1)$. Let $w_\epsilon^\pm(z) = \theta_\epsilon^{-1}(\theta(z))$ be the corresponding transition functions.

Lemma 2.12 *Let $w_\epsilon^\pm$ be defined as above. Then, there exists $\rho_\epsilon^\pm > 0$, such that*

$$w_\epsilon^+(z) = -1 + \rho_\epsilon^+(z + 1), \quad w_\epsilon^-(z) = 1 + \rho_\epsilon^-(z - 1), \quad \rho_\epsilon^+ < 1, \rho_\epsilon^- > 1 \quad (2.9)$$

and

$$\dot{w}^\pm(z) = \tau^\pm(1 \pm z), \quad \tau^\pm < 0. \quad (2.10)$$

Proof We only prove the claim for w_ϵ^+ . Since $\Pi(\epsilon) \equiv \Pi(0)$, w_ϵ is just an affine transformation and using $w_\epsilon(-1) = -1$ and $w_\epsilon(\infty) = \infty$ we find (2.9). Since $\theta_\epsilon(1) > \theta(1)$, we obtain that $w_\epsilon(1) = \theta_\epsilon^{-1}(\theta(1)) < \theta_\epsilon^{-1}(\theta_\epsilon(1)) = 1$ and thus $\rho_\epsilon < 1$. Therefore,

$$\dot{w}(z) = \lim_{\epsilon \rightarrow \infty} \frac{w_\epsilon(z) - z}{\epsilon} = (z + 1) \lim_{\epsilon \rightarrow \infty} \frac{\rho_\epsilon - 1}{\epsilon} = \tau < 0.$$

□

Lemma 2.13 *Let $E(\alpha, \delta)$ with $\alpha \leq 0$ be such that the corresponding extremal polynomial $A_{n,\alpha,\delta}(z)$ corresponds to the case (ii) (case (iii)). Then, the infinitesimal variation $\dot{w}(x)$ generated by increasing $\theta(1)$ (increasing $\theta(-1)$) under the constraint of a constant gap length ($\delta = \text{const}$) leads to an increase (decrease) of h_0 . Moreover, in this variation the gap is moving to the left, that is, $\alpha(\epsilon)$ is decreasing.*

Proof We only prove the case (ii). We have

$$\dot{w}(x) = \tau(x + 1) + \frac{1 - x^2}{1 - c^2} \frac{\sigma_0}{c - x}, \quad \tau < 0. \quad (2.11)$$

Applying the constraint $\dot{w}(a) = \dot{w}(b)$, we obtain

$$0 = (b - a) \left(\tau + \frac{1 + ab - (a + b)c}{(c - a)(c - b)} \frac{\sigma_0}{1 - c^2} \right).$$

Since $\ell(c) > 0$, this implies $\sigma_0 < 0$ and thus h_0 is increasing. Moreover, $\dot{w}(a) < 0$, which concludes the proof. □

We summarize our results in the following theorem:

Theorem 2.14 *Let η be defined by (2.8). Then, we have:*

- (i) h_0 is increasing,
- (ii) h_0 is increasing,
- (iii) h_0 is decreasing,
- (iv) if $c_- < \eta$, then h_0 is decreasing, if $c_- > \eta$, then h_0 is increasing.

In all cases α is decreasing.

Remark 2.15 We have seen in the proof of Lemma 2.5 that $w_\epsilon(x_0)$ increased monotonically, if some other slit height was decreased. Theorem 2.14, in particular case (iii) and case (iv) show that such a monotonicity is lacking for Akhiezer polynomials, which makes the situation essentially different to the Remez extremal problem.

3 Reduction to Akhiezer Polynomials (Proof of Theorem 1.3, Part 2)

The goal of this section is to finish the proof of Theorem 1.3. That is, if x_0 is in an internal gap, then the extremal polynomial is an Akhiezer polynomial. Recall that in contrast to Section 2.2 now it is possible that there is an extension outside of $[-1, 1]$, moreover, this is a generic position. All possible types of extremizer were described in Corollary 2.2 and the discussion above the corollary. Thus, it remains to show that on $[-1, 1]$ the extremal configuration in fact only has one gap, i.e., the extremal set is of the form $E(\alpha, \delta)$ for some $\alpha \in (-1 + \delta, 1 - \delta)$.

First, we will show that E has at most two gaps on $[-1, 1]$. Let $T_n(z, E)$ denote the extremizer of (1.1) for the set E .

Lemma 3.1 *Let $E \subset [-1, 1]$ and let x_0 be in an internal gap of E . Then, there exists a two-gap set $\tilde{E}$, such that*

$$T_n(x_0, E) = T_n(x_0, \tilde{E}), \quad |E| = |\tilde{E}|. \quad (3.1)$$

Proof We will only prove the case that there are no boundary gaps. Moreover, let us assume that E is already maximal, i.e., $T_n^{-1}([-1, 1], E) \cap [-1, 1] = E$. Let us write

$$E = [-1, 1] \setminus \bigcup_{j=1}^g (a_j, b_j),$$

and let us denote the gap which contains x_0 by (a, b) . Let Π be the comb related to E and let us assume that the slit corresponding to (a, b) is placed at 0 and let c denote the critical point of T_n in (a, b) . Assume that there are two additional gaps (a_1, b_1) and (a_2, b_2) with slit heights h_k and critical points c_k , $k = 1, 2$. We will vary the slit heights h_k and h such that $w_\epsilon(x_0) = x_0$ and $|E_\epsilon| = |E|$. Therefore, we get

$$\dot{w}(z) = \frac{z^2 - 1}{c_1^2 - 1} \frac{\sigma_1}{c_1 - z} + \frac{z^2 - 1}{c_2^2 - 1} \frac{\sigma_2}{c_2 - z} + \frac{z^2 - 1}{c^2 - 1} \frac{\sigma}{c - z}. \quad (3.2)$$

The constraints

$$\sum_{j=1}^g (\dot{w}(b_j) - \dot{w}(a_j)) = 0, \quad \dot{w}(x_0) = 0 \quad (3.3)$$

will guarantee that (3.1) is satisfied. Our goal is to find $\sigma_1 > 0$ and σ, σ_2 , such that (3.3) is satisfied. Let us define

$$H(z) = \frac{(z^2 - 1)(z - x_0)}{(z - c_1)(z - c_2)(z - c)}.$$

Due to the second constraint in (3.3), we have

$$\dot{w}(z) = f(z)H(z), \quad (3.4)$$

where $f(z)$ is linear. Thus $f(z) = K(z - \xi)$ or $f(z) = K$. We need to check that we can always find ξ such that the first constraint in (3.3) is satisfied. If $\sum_{j=1}^g (H(b_j) - H(a_j)) = 0$, we set $f(z) = K$. If $\sum_{j=1}^g (H(b_j) - H(a_j)) \neq 0$, we define

$$\xi = \frac{\sum_{j=1}^g (b_j H(b_j) - a_j H(a_j))}{\sum_{j=1}^g (H(b_j) - H(a_j))}.$$

In any case, we define $\dot{w}$ by (3.4) and $\sigma_1, \sigma_2, \sigma$ as the residues of this function at c_1, c_2, c , respectively. Now we have to distinguish two cases. If $\xi \neq c_1$, we can choose K so that $\sigma_1 > 0$ and decrease h_1 . If at some point $\xi = c_1$, we can choose K so that σ_2 will be decreased. Note that in this case h_1 remains unchanged. In particular, it won't increase again. Hence, this procedure allows us to "erase" all but one additional gap. The case of boundary gaps works essentially in the same way, only using instead of the variations used above, variations as described in Lemma 2.12. $\square$

Ending Proof of Theorem 1.3 First, assume that the extremizer is in a generic position, that is, there is an extra interval $I_n \subset \mathbb{R} \setminus [-1, 1]$. We have $T_n(z, E)$ with

$$E = [-1, 1] \setminus ((a, b) \cup (a_1, b_1)) \quad (3.5)$$

and $|T_n(z, E)| \leq 1$ for $z \in E \cup I_n$. The corresponding comb has three slits, which heights we denote by h_{out}, h, h_1 . Our goal is to show that we can reduce the value h_1 . Varying all three values, we get that the corresponding infinitesimal variation as described by expression (3.2), with the parameters $\sigma_{out}, \sigma, \sigma_1$. Therefore, we still can satisfy the two constraints in (3.3), and choose one of the parameters positive. Since the direction of the variation of the heights h_{out} and h is not essential for us, we choose $\sigma_1 > 0$, and therefore reduce the size of h_1 .

In a degenerated case we can use variations, which were described in Section 2.3. This is possible as long as either $\theta(1)$ or $\theta(-1)$ can be moved in both directions. Thus, it remains to consider the case that E is of the form (3.5), but the corresponding comb has only two non-trivial teeth of the heights h and h_1 . WLOG we assume that $b < a_1$. We have

$$T_n(z, E) = \cos \theta(z), \quad \theta(-1) = \pi n_-, \quad \theta(1) = \pi n_+.$$

We will apply the third variation, see Fig. 2, left. That is, we will reduce the value h_1 and move the preimage of -1 in the positive direction. According to (2.10), see also (2.11), we obtain

$$\dot{w}(z) = -\tau(z-1) + \frac{1-z^2}{1-c_1^2} \frac{\sigma_1}{c_1-z}, \quad \tau < 0, \quad \sigma_1 > 0.$$

Let us point out that with an arbitrary choice of the parameter $\sigma_1 > 0$ and $\tau < 0$ we get an increasing function. Therefore,

$$\dot{w}(1) - \dot{w}(b_1) > 0, \quad \dot{w}(a_1) - \dot{w}(b) > 0, \quad \dot{w}(a) - \dot{w}(-1) > 0.$$

Thus, with a small variation ϵ of this kind we get

$$E_\epsilon = [w_\epsilon(-1), w_\epsilon(1)] \setminus ((w_\epsilon(a), w_\epsilon(b)) \cup (w_\epsilon(a_1), w_\epsilon(b_1)))$$

with

$$\begin{aligned} |E_\epsilon| &= (w_\epsilon(1) - w_\epsilon(b_1)) + (w_\epsilon(a_1) - w_\epsilon(b)) + (w_\epsilon(a) - w_\epsilon(-1)) \\ &> (1 - b_1) + (a_1 - b) + (a - (-1)) = |E|. \end{aligned}$$

On the other hand $\dot{w}(z)$ has a trivial zero $z = 1$ and a second one, which we denote by y_* . Since

$$\tau + \frac{1 + y_*}{1 - c_1^2} \frac{\sigma_1}{c_1 - y_*} = 0,$$

we get

$$y_* = y_*(\rho_1, \rho_2) = \rho_1(-1) + \rho_2 c_1,$$

where

$$\rho_2 = \frac{-\tau}{-\tau + \frac{\sigma_1}{1 - c_1^2}}, \quad \rho_1 = 1 - \rho_2, \quad \rho_{1,2} > 0.$$

Thus, with different values of $\tau < 0$ and $\sigma_1 > 0$, we can get an arbitrary value $y_* \in (-1, c_1)$.

Assume that $x_0 < c$. We choose ρ_1, ρ_2 such that $y_* > c$ (recall our assumption $b < a_1$, therefore this is possible). Then, $\dot{w}(x_0) < 0$. For a small ϵ , we get $w_\epsilon(x_0) < x_0$. By definition, $T_n(w_\epsilon(x_0), E_\epsilon) = T_n(x_0, E)$. Since in this range $T_n(w_\epsilon(x_0), E_\epsilon)$ is increasing, we get $T_n(x_0, E_\epsilon) > T_n(x_0, E)$, that is $T_n(z, E)$ was not an extremal polynomial for the Andrievskii problem.

If $x_0 \in (c, b)$, we choose $y_* < c$. Then, $\dot{w}(x_0) > 0$. We repeat the same arguments, having in mind that in this range $T_n(z, E)$ is decreasing. Thus, we arrive to the same contradiction $T_n(x_0, E_\epsilon) > T_n(x_0, E)$. $\square$

4 Asymptotics (Proof of Theorem 1.5)

The goal of this section is to prove Theorem 1.5 and to give a description of the upper envelope (1.5).

4.1 Totik–Widom bounds

We need an asymptotic result for Akhiezer polynomials. Recall that $E(\alpha, \delta) = [-1, 1] \setminus (\alpha - \delta, \alpha + \delta)$ and that $A_{n,\alpha,\delta}$ denotes the associated Akhiezer polynomial.

Moreover, we denote $\hat{E}_n(\alpha, \delta) = A_{n,\alpha,\delta}^{-1}([-1, 1]) = E(\alpha, \delta) \cup I_n$ and $\hat{\Omega}_n = \overline{\mathbb{C}} \setminus \hat{E}_n$. We have described the shape of the additional interval I_n in Theorem 2.1 and the discussion below. The following Lemma is known in a much more general context [11, Proposition 4.4.].

Lemma 4.1 *Let α, δ be fixed and $A_{n,\alpha,\delta}$ be the associated Akhiezer polynomial. Let $x_n \in I_n$ denote the single zero of $A_{n,\alpha,\delta}$ outside of $[-1, 1]$. For any $y \in (\alpha - \delta, \alpha + \delta)$*

$$\log 2|A_{n,\alpha,\delta}(y)| = nG(y, \infty, \hat{\Omega}_n) + o(1). \quad (4.1)$$

If we pass to a subsequence such that $\lim_{k \rightarrow \infty} x_{n_k} = x_\infty \in (\mathbb{R} \cup \{\infty\}) \setminus (-1, 1)$, then

$$\lim_{k \rightarrow \infty} n_k(G(y, \infty; \Omega(\alpha, \delta)) - G(y, \infty; \hat{\Omega}_{n_k})) = G(y, x_\infty, \Omega(\alpha, \delta)), \quad (4.2)$$

where $\Omega(\alpha, \delta) = \overline{\mathbb{C}} \setminus E(\alpha, \delta)$.

Proof of Theorem 1.5 We start with the case that the sup in (1.5) is attained at some internal point $\alpha_0 \in (-1 + \delta, 0]$. Let T_{n,δ,x_0} be the extremizer of (1.2) and set $\hat{E}_n = T_{n,\delta,x_0}^{-1}([-1, 1])$ and $\hat{\Omega}_n = \overline{\mathbb{C}} \setminus \hat{E}_n$. Due to Theorem 1.3, $\hat{E}_n$ is either $[-1 + 2\delta, 1]$ or $\hat{E}_n = E(\alpha_n, \delta) \cup I_n$ for some α_n and some additional interval outside of $[-1, 1]$. In the following, we will denote $G(x, \infty, \Omega) = G(x, \Omega)$ and we note that by definition $G(x, \Omega(\alpha, \delta)) = G_{\alpha,\delta}(x)$. Put $E_n = \hat{E}_n \cap [-1, 1]$ and $\Omega_n = \overline{\mathbb{C}} \setminus E_n$. Due to extremality of α_0 , we have

$$G(x_0, \Omega(\alpha_0, \delta)) \geq G(x_0, \Omega_n).$$

Since $\hat{\Omega}_n \subset \Omega_n$, we get

$$G(x_0, \Omega_n) \geq G(x_0, \hat{\Omega}_n).$$

Using (4.1) or (1.4) and the extremality property of $T_{n,\delta,x_0}(x_0)$, we get

$$\begin{aligned} nG(x_0, \hat{\Omega}_n) &= \log 2|T_{n,\delta,x_0}(x_0)| + o(1) \\ &\geq \log 2|T_n(x_0, E(\alpha_0, \delta))| + o(1). \end{aligned}$$

On the other hand, equation (4.2) yields

$$\log 2|T_n(x_0, E(\alpha_0, \delta))| \geq nG(x_0, \Omega(\alpha_0, \delta)) - C + o(1),$$

where

$$C = \sup_{x \in (\mathbb{R} \cup \{\infty\})} G(x, x_0, \Omega(\alpha_0, \delta)). \quad (4.3)$$

By definition, $|T_{n,\delta,x_0}(x_0)| = L_n(x_0)$ and therefore combining all inequalities finishes the proof of (1.8). The proof for the boundary case is essentially the same. Only in the

last step, due to representation (1.4), there is no extra constant C (due to the fact that there is no extension of the domain) and we get (1.7). $\square$

4.2 The Asymptotic Diagram

In this section, we introduce an asymptotic diagram, which provides a description of the upper envelope $\Phi_\delta(x)$. First of all, we prove Lemma 1.4.

Proof of Lemma 1.4 Recall the explicit representation of Green functions for two intervals as elliptic integrals. For $a = \alpha - \delta$, $b = \alpha + \delta$, we have

$$G_{\alpha,\delta}(x) = \int_a^x \frac{(c - \xi)d\xi}{\sqrt{(\xi^2 - 1)(\xi - a)(\xi - b)}}, \quad (4.4)$$

where

$$c = c(\alpha) = \frac{\int_a^b \frac{\xi d\xi}{\sqrt{(\xi^2 - 1)(\xi - a)(\xi - b)}}}{\int_a^b \frac{d\xi}{\sqrt{(\xi^2 - 1)(\xi - a)(\xi - b)}}}. \quad (4.5)$$

If for fixed x_0 the sup is attained at an internal point, then clearly

$$\partial_\alpha G_{\alpha,\delta}(x) = 0 \quad (4.6)$$

holds. Thus, it remains to show that (4.6) has a unique solution $x_0(\alpha)$. Due to (4.4), we have

$$\partial_\alpha \log \partial_x G_{\alpha,\delta}(x) = \frac{\dot{c}}{c - x} - \frac{1}{2} \frac{\dot{a}}{a - x} - \frac{1}{2} \frac{\dot{b}}{b - x}.$$

Since $\dot{a} = \dot{b} = 1$, we get

$$\partial_x \partial_\alpha G_{\alpha,\delta}(x) = \left\{ \dot{c} + \frac{(c - x)(x - \alpha)}{(x - a)(x - b)} \right\} \frac{1}{\sqrt{(x^2 - 1)(x - a)(x - b)}}. \quad (4.7)$$

Note that

$$\inf_{x \in (a,b)} \frac{(c - x)(x - \alpha)}{(x - a)(x - b)} > -1.$$

Since the distance $|c(\alpha) - b(\alpha)|$ monotonically increases with $|\alpha|$, we have $\dot{c}(\alpha) > 1$. Therefore, we get $\partial_x \partial_\alpha G_{\alpha,\delta}(x) > 0$ in (4.7). That is, $\partial_\alpha G_{\alpha,\delta}(x)$ is increasing for $x \in (a, b)$. Moreover, $\lim_{x \rightarrow a} \partial_\alpha G_{\alpha,\delta}(x) = -\infty$ and $\lim_{x \rightarrow b} \partial_\alpha G_{\alpha,\delta}(x) = +\infty$ and we obtain that a zero $x_0(\alpha)$ of the function $\partial_\alpha G_{\alpha,\delta}(x)$ in (a, b) exists and is unique. $\square$

Thus, the limiting behavior of $L_{n,\delta}(x_0)$, $n \rightarrow \infty$, can be distinguished by a diagram with two curves, which we describe in the following proposition, see also Example 4.5.

Proposition 4.2 *In the range $x \in [-1, 0]$, $\Phi_\delta(x)$ represents the upper envelope of the following two curves. The first one is given explicitly*

$$y = G_\delta(x) = \log \left(\frac{\delta - x}{1 - \delta} + \sqrt{\left(\frac{\delta - x}{1 - \delta} \right)^2 - 1} \right), \quad x \in [-1, -1 + 2\delta], \quad (4.8)$$

and the second one is given in parametric form

$$x = x_0(\alpha), \quad y = G_{\alpha,\delta}(x_0(\alpha)), \quad \alpha \in (-1 + \delta, 0]. \quad (4.9)$$

Moreover, the end points of the last curve are given explicitly by

$$\left(0, \frac{1}{2} \log \frac{1 + \delta}{1 - \delta} \right) \text{ for } \alpha = 0, \quad (-1 + 2\delta, 0) \text{ for } \alpha \rightarrow -1 + \delta. \quad (4.10)$$

Proof According to Lemma 1.4, if $\Phi_\delta(x)$ is assumed at the end point $\alpha \rightarrow -1 + \delta$, then it is the Green function in the complement to the interval $[-1 + 2\delta, 1]$, which has a well-known representation (4.8). Alternatively, $x = x_0(\alpha)$ and $\Phi_\delta(x) = G_{\alpha,\delta}(x_0(\alpha))$ for a certain α in the range, what is (4.9).

Further, for $\alpha = 0$, $G_{0,\delta}(x)$ is the Green function related to two symmetric intervals. Due to the symmetry $x_0(0) = 0$ and $G_{0,\delta}(x)$ can be reduced to the Green function of a single interval $[\delta^2, 1]$. We get

$$G_{0,\delta}(0) = \frac{1}{2} \log \frac{1 + \delta}{1 - \delta}.$$

Thus, it remains to prove the last statement of the proposition. Our proof is based on expressing $\partial_\alpha G_{\alpha,\delta}$ in terms of elliptic integrals and manipulating those. It will be convenient to make a standard substitution in (4.4). Let $\xi(\psi, \alpha) = \alpha - \delta \cos \psi$, then

$$G_{\alpha,\delta}(x) = \int_{\phi(x,\alpha)}^{\pi} \frac{\xi(\psi, \alpha) - c(\alpha)}{\sqrt{1 - \xi(\psi, \alpha)^2}} d\psi, \quad \phi(x, \alpha) = \arccos \frac{\alpha - x}{\delta}. \quad (4.11)$$

Differentiating (4.11), we get

$$\partial_\alpha G_{\alpha,\delta}(x) = I_1(x, \alpha) + I_2(x, \alpha) - \dot{c}(\alpha) I_3(x, \alpha)$$

where

$$I_1(x, \alpha) = \int_{\phi(x, \alpha)}^{\pi} \frac{1 - c(\alpha)\xi(\psi, \alpha)}{(1 - \xi(\psi, \alpha)^2)\sqrt{1 - \xi(\psi, \alpha)^2}} d\psi$$

$$I_2(x, \alpha) = \frac{x - c(\alpha)}{\sqrt{(1 - x^2)(\delta^2 - (x - \alpha)^2)}}, \quad I_3(x, \alpha) = \int_{\phi(x, \alpha)}^{\pi} \frac{d\psi}{\sqrt{1 - \xi(\psi, \alpha)^2}}.$$

Let $\epsilon = \epsilon(\alpha)$ be such that $\epsilon \rightarrow 0$ as $\alpha \rightarrow -1 + \delta$, to be chosen later on, and let

$$x(\alpha) = \alpha - \delta \cos(\pi - \epsilon). \quad (4.12)$$

Direct estimations show that

$$I_1(x(\alpha), \alpha) = o(1), \quad I_2(x(\alpha), \alpha) = \frac{a_1(\alpha)}{\epsilon} + o(1),$$

where $\lim_{\alpha \rightarrow -1+\delta} a_1(\alpha) > 0$. The integral $I_3(x(\alpha), \alpha)$ also tends to zero, but we will need a more precise decomposition

$$I_3(x(\alpha), \alpha) = b_1(\alpha)\epsilon + \frac{b_2(\alpha)}{3}\epsilon^3 + O(\epsilon^5) \quad (4.13)$$

with $\lim_{\alpha \rightarrow -1+\delta} b_1(\alpha) > 0$. Indeed, we have

$$I_3 = \int_{\phi(x(\alpha), \alpha)}^{\pi} \frac{d\psi}{\sqrt{1 - \xi(\psi, \alpha)^2}} = \int_0^{\epsilon} \frac{d\psi}{\sqrt{1 - (\alpha + \delta \cos \psi)^2}}.$$

Therefore, for ψ sufficiently small we can use the expansion

$$\frac{1}{\sqrt{1 - (\alpha + \delta \cos \psi)^2}} = b_1(\alpha) + b_2(\alpha)\psi^2 + O(\psi^4),$$

where $O(\psi^4)$ is uniform in α . We get (4.13). In the same time,

$$\lim_{\alpha \rightarrow -1+\delta} b_1(\alpha) = \frac{1}{\sqrt{1 - (-1 + 2\delta)^2}} > 0.$$

Collecting all three terms, we obtain

$$\begin{aligned} \partial_{\alpha} G_{\alpha, \delta}(x(\alpha)) &= \frac{a_1(\alpha)}{\epsilon} - \dot{c}(\alpha) \left(b_1(\alpha)\epsilon + \frac{b_2(\alpha)}{3}\epsilon^3 \right) + o(1) \\ &= \varkappa_1(\alpha) \left(\frac{\varkappa(\alpha)}{\epsilon} - \dot{c}(\alpha) \left(\frac{\epsilon}{\varkappa(\alpha)} + \frac{\varkappa_2(\alpha)}{3}\epsilon^3 \right) \right) + o(1), \end{aligned} \quad (4.14)$$

where $\varkappa_1(\alpha) = \sqrt{a_1(\alpha)b_1(\alpha)}$, $\varkappa(\alpha) = \sqrt{a_1(\alpha)/b_1(\alpha)}$ and $\varkappa_2(\alpha)$ is chosen appropriately. Recall that $\varkappa_1(\alpha)$ and $\varkappa(\alpha)$ have positive and finite limits as $\alpha \rightarrow -1 + \delta$.

Similar manipulations with elliptic integrals show that $\dot{c}(\alpha) \rightarrow +\infty$ as $\alpha \rightarrow -1 + \delta$. In fact, the rate of this divergence is (see Appendix)

$$\dot{c}(\alpha) \simeq (\varepsilon \log \varepsilon)^{-2}, \quad \alpha = -1 + \delta(1 + \frac{1}{2}\varepsilon^2), \quad \varepsilon \rightarrow 0,$$

but such accuracy is not needed for our purpose.

Having these estimations, we can find a suitable interval $[x_-(\alpha), x_+(\alpha)]$, with the limit

$$\lim_{\alpha \rightarrow -1+\delta} x_{\pm}(\alpha) = -1 + 2\delta, \quad (4.15)$$

such that $\partial_{\alpha} G_{\alpha,\delta}(x)$ changes its sign in it and therefore this interval contains the unique solution of the equation $\partial_{\alpha} G_{\alpha,\delta}(x) = 0$.

Define

$$\epsilon_{\pm}(\alpha) = \frac{\varkappa(\alpha)}{\sqrt{\dot{c}(\alpha)} \pm 1}.$$

Since $\dot{c}(\alpha)\epsilon_{\pm}(\alpha)^3 = o(1)$, (4.14) gets the form

$$\begin{aligned} \varkappa_1(\alpha)^{-1} \partial_{\alpha} G(x_{\pm}(\alpha), \alpha) &= \sqrt{\dot{c}(\alpha)} \pm 1 - \frac{\dot{c}(\alpha)}{\sqrt{\dot{c}(\alpha)} \pm 1} + o(1) \\ &= \frac{\pm 2\sqrt{\dot{c}(\alpha)} + 1}{\sqrt{\dot{c}(\alpha)} \pm 1} + o(1), \end{aligned} \quad (4.16)$$

where $x_{\pm}(\alpha)$ is defined by (4.12) for $\epsilon = \epsilon_{\pm}(\alpha)$. For $\dot{c}(\alpha)$ sufficiently large, we obtain in (4.16) both positive and negative values, and simultaneously we have (4.15). Consequently, $x_0(\alpha) \rightarrow -1 + 2\delta$. $\square$

Corollary 4.3 *Let δ_* be a unique solution of the equation*

$$\delta_*^2 = \frac{1 - \delta_*}{1 + \delta_*}, \quad \delta_* \in (0, 1),$$

numerically $\delta_ = 0.543689\dots$. Then, for $\delta < \delta_*$ $\Phi_{\delta}(x)$ does not coincide identically with $G_{\delta}(x)$ (in its range $x \in (-1, 0]$).*

On the other hand, for an arbitrary $\delta > 0$ there exists $x_(\delta) > -1$ such that $\Phi_{\delta}(x)$ and $G_{\delta}(x)$ coincide in $[-1, x_*(\delta))$.*

Proof The first claim follows by a direct comparison of $G_{\delta}(0)$ and $G_{0,\delta}(0)$, see (4.8) and (4.10).

For a fixed δ , we define

$$x_*(\delta) = \inf_{\alpha \in (-1+\delta, 0]} x_0(\alpha) \quad (4.17)$$

Fig. 3 The asymptotic diagram for $\delta = 0.4$

By (4.10) and continuity, $x_*(\delta) > -1$. Thus, the curve (4.9) does not intersect the range $[-1, x_*(\delta))$. $\square$

Remark 4.4 We do not claim here that $[-1, x_*(\delta))$ with $x_*(\delta)$ given by (4.17) is the biggest possible interval on which $\Phi_\delta(x) = G_\delta(x)$, see Example 4.5 for details.

Example 4.5 A numerical example of the asymptotic diagram for $\delta = 0.4$ is given in Fig. 3 (diagrams for other values of $\delta < 0.5$ look pretty the same). Let $x_s(\delta)$ be the switching point between two (Remez and Akhiezer) extremal configurations, i.e.,

$$\Phi_\delta(x_s) = G_\delta(x_s(\delta)) = G_{\alpha_s, \delta}(x_0(\alpha_s)), \quad x_0(\alpha_s) = x_s(\delta).$$

Recall that $x_*(\delta)$ was defined in (4.17). On the diagram, we can observe the following four regions: $(-1, x_*(\delta))$, $(x_*(\delta), x_s(\delta))$, $(x_s(\delta), -1 + 2\delta)$ and $(-1 + 2\delta, 0)$. Note that we discuss the case $-1 + 2\delta < 0$, i.e., $\delta < 0.5 < \delta_*$.

- $x \in (-1 + 2\delta, 0)$. In this case $x \in (\alpha - \delta, \alpha + \delta)$ implies that such an interval is a subset of $(-1, 1)$ even in the leftmost position $\alpha = -\delta + x$. Therefore, the function $G_{\alpha, \delta}(x)$ for a fixed x and $\alpha \in (x - \delta, x + \delta)$ attains its maximum at some internal point and we get the case $\partial_\alpha G_{\alpha, \delta}(x) = 0$.
- $x \in (x_s(\delta), -1 + 2\delta)$. As soon as $x < -1 + 2\delta$ the left boundary for a possible value of α is given by $\alpha - \delta = -1$. Respectively, the supremum of $G_{\alpha, \delta}(x)$ for a fixed x can be attained either at an internal point $\alpha \in (-1 + \delta, x + \delta)$ or as the limit at the left end point. In this range, it is still attained at an internal point. Note that besides the local maximum the function gets a local minimum (the second branch of the curve (4.9) with the same coordinate $x_0(\alpha) = x$).

- c) $x \in (x_*(\delta), x_s(\delta))$. For such x , the function $G_{\alpha,\delta}(x)$ has still its local maximum and minimum, but the biggest value is attained at the boundary point $\alpha = -1 + \delta$, i.e., $\Phi_\delta(x) = G_\delta(x)$.
- d) $x \in (-1, x_*(\delta))$. At $x = x_*(\delta)$ the points of local maximum and minimum of the function $G_{\alpha,\delta}(x)$ collide, that is, in fact, they produce an inflection point. The function $G_{\alpha,\delta}(x)$ become monotonic in this region. Its supremum is the limit at the boundary point $\alpha = -1 + \delta$, see the second claim in Corollary 4.3.

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Appendix A. Lemma on the limit of $\dot{c}(\alpha)$

Lemma A.1 Set $\alpha = -1 + \delta(1 + \frac{1}{2}\epsilon^2)$. Then, $\dot{c}(\alpha)$ tends to $+\infty$ as $\epsilon \rightarrow 0$ with the rate

$$\dot{c}(\alpha) \simeq (\epsilon \log \epsilon)^{-2}.$$

Proof By (4.5), we have

$$\alpha - c = \frac{\int_a^b \frac{(\alpha - \xi) d\xi}{\sqrt{(\delta^2 - (\xi - \alpha)^2)(1 - \xi^2)}}}{\int_a^b \frac{d\xi}{\sqrt{(\delta^2 - (\xi - \alpha)^2)(1 - \xi^2)}}}.$$

Making the change of variables

$$\xi = \xi(\alpha, \phi) = \alpha - \delta \cos \phi, \quad \phi \in (0, \pi)$$

we get

$$\alpha - c = \frac{\int_0^\pi \frac{\delta^2 \cos \phi \sin \phi d\phi}{\delta \sin \phi \sqrt{1 - x^2}}}{\int_0^\pi \frac{\delta \sin \phi d\phi}{\delta \sin \phi \sqrt{1 - x^2}}} = \delta \frac{\int_0^\pi \frac{\cos \phi d\phi}{\sqrt{1 - x^2}}}{\int_0^\pi \frac{d\phi}{\sqrt{1 - x^2}}}.$$

Thus, introducing

$$u(\alpha) = \int_0^\pi \frac{\cos \phi d\phi}{\sqrt{1-\xi^2}}, \quad v(\alpha) = \int_0^\pi \frac{d\phi}{\sqrt{1-\xi^2}},$$

we have

$$1 - \dot{c} = \frac{\delta}{v^2} \det \begin{bmatrix} \dot{u} & \dot{v} \\ u & v \end{bmatrix}.$$

Since $\partial_\alpha \xi(\phi, \alpha) = 1$, we have

$$\dot{u}(\alpha) = \int_0^\pi \frac{\xi \cos \phi d\phi}{(1-\xi^2)\sqrt{1-\xi^2}}, \quad \dot{v}(\alpha) = \int_0^\pi \frac{\xi d\phi}{(1-\xi^2)\sqrt{1-\xi^2}}.$$

Using the definition of $\xi(\alpha, \phi)$, we get

$$\begin{aligned} \delta \det \begin{bmatrix} \dot{u} & \dot{v} \\ u & v \end{bmatrix} &= -\det \left(\begin{bmatrix} \dot{u} & \dot{v} \\ u & v \end{bmatrix} \begin{bmatrix} -\delta & 0 \\ \alpha + 1 & 1 \end{bmatrix} \right) \\ &= -\det \begin{bmatrix} \int_0^\pi \frac{\xi(1+\xi)d\phi}{(1-\xi^2)\sqrt{1-\xi^2}} & \int_0^\pi \frac{\xi d\phi}{(1-\xi^2)\sqrt{1-\xi^2}} \\ \int_0^\pi \frac{(1+\xi)d\phi}{\sqrt{1-\xi^2}} & \int_0^\pi \frac{d\phi}{\sqrt{1-\xi^2}} \end{bmatrix}. \end{aligned}$$

Finally,

$$\dot{c} = 1 + \frac{1}{v^2} \det \begin{bmatrix} \int_0^\pi \frac{\xi d\phi}{(1-\xi)\sqrt{1-\xi^2}} & \int_0^\pi \frac{\xi d\phi}{(1-\xi^2)\sqrt{1-\xi^2}} \\ \int_0^\pi \frac{1+\xi}{\sqrt{1-\xi^2}} d\phi & \int_0^\pi \frac{d\phi}{\sqrt{1-\xi^2}} \end{bmatrix} = 1 + \frac{1}{v^2} \det \begin{bmatrix} I_1 & I_2 \\ I_3 & v \end{bmatrix}. \quad (\text{A.1})$$

Now we insert $\alpha = -1 + \delta(1 + \frac{1}{2}\epsilon^2)$, $\epsilon \rightarrow 0$. For a sufficiently small ϕ_0 , we have

$$1 + \xi = 1 + \alpha - \delta \cos \phi = \frac{\delta}{2}(\phi^2 + \epsilon^2) + O(\phi^4), \quad \phi \in (0, \phi_0).$$

Recall that $\xi(\pi, \alpha) = b$. Therefore, the following limit

$$\lim_{\epsilon \rightarrow 0} \int_{\phi_0}^\pi \frac{d\phi}{\sqrt{1-\xi^2}}$$

exists. Thus, we have

$$\begin{aligned} v &= \int_0^\pi \frac{d\phi}{\sqrt{1-\xi^2}} \simeq \int_0^{\phi_0} \frac{d\phi}{\sqrt{1+\xi}} \simeq \int_0^{\phi_0} \frac{d\phi}{\sqrt{\delta(\epsilon^2 + \phi^2)}} \\ &\simeq \int_0^{\phi_0/\epsilon} \frac{dt}{\sqrt{1+t^2}} \simeq -\log \epsilon. \end{aligned}$$

Also

$$I_1 = \int_0^\pi \frac{\xi d\phi}{(1-\xi)\sqrt{1-\xi^2}} \simeq \int_0^{\phi_0} \frac{d\phi}{\sqrt{1+\xi}} \simeq -\log \epsilon.$$

Moreover, for I_3 we get

$$\lim_{\epsilon \rightarrow 0} \int_0^\pi \sqrt{\frac{1+\xi}{1-\xi}} d\phi = \int_0^\pi \sqrt{\frac{\delta(1-\cos\phi)}{2-\delta+\delta\cos\phi}} d\phi > 0.$$

As before, we can split up I_2 and get

$$\begin{aligned} I_2 &\simeq \int_0^{\phi_0} \frac{\xi d\phi}{(1-\xi^2)\sqrt{1-\xi^2}} \simeq - \int_0^{\phi_0} \frac{d\phi}{((\phi^2 + \epsilon^2))^{3/2}} \\ &= -\frac{1}{\epsilon^2} \int_0^{\phi_0/\epsilon} \frac{dt}{(1+t^2)^{3/2}} \\ &\simeq -\frac{1}{\epsilon^2}. \end{aligned}$$

Collecting all terms and inserting it into (A.1) yield the claim. $\square$

References

1. Achyzer, N.: [N.I. Akhiezer], Über einige Funktionen, die in gegebenen Intervallen am wenigsten von Null abweichen, *Izv. Kazan, Fiz.-Mat. Obshch.* (3) **3** (1928), 1–69
2. Achyzer, N.: Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen, I, II, III, *Izv. Akad. Nauk SSSR*, **1932**, 1163–1202; **1933**, 309–344, 499–536
3. Akhiezer, N.I.: *Lectures on Approximation Theory*, 2nd rev. ed., Nauka, Moscow, 1965; German transl., Akademie-Verlag, Berlin, 1967; Engl transl. of 1st ed., Ungar, New York, 1956
4. Akhiezer, N.I.: *Elements of the theory of elliptic functions*, Translations of Mathematical Monographs, vol. 79, American Mathematical Society, Providence, RI, 1990, Translated from the second Russian edition by H. H. McFaden
5. Andrievskii, V.: Pointwise Remez-type inequalities in the unit disk. *Constr. Approx.* **22**(3), 297–308 (2005)
6. Andrievskii, V.: Local Remez-type inequalities for exponentials of a potential on a piecewise analytic arc. *J. Anal. Math.* **100**, 323–336 (2006)
7. Aptekarev, A.I., Draux, A., Tulyakov, D.N.: On asymptotics of the sharp constants of the Markov-Bernstein inequalities for the Sobolev spaces. *Lobachevskii J. Math.* **39**(5), 609–622 (2018)
8. Brudnyi, Ju.A., Ganzburg, M.I.: A certain extremal problem for polynomials in n variables, *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 344–355. (Russian),
9. Brudnyi, Ju.A., Ganzburg, M.I.: A certain extremal problem for polynomials in n variables, Engl. trans. in *Math USSR-Izv.* **7** (1973), 345–356
10. Christiansen, J.S., Simon, B., Zinchenko, M.: Asymptotics of Chebyshev polynomials, I: subsets of $\mathbb{R}$. *Invent. Math.* **208**(1), 217–245 (2017)
11. Christiansen, J.S., Simon, B., Yuditskii, P., Zinchenko, M.: Asymptotics of Chebyshev polynomials, II: DCT subsets of $\mathbb{R}$. *Duke Math. J.* **168**(2), 325–349 (2019)
12. Conway, J.B.: *Functions of one complex variable. II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, New York, 1995
13. Eichinger, B.: Szegő-Widom asymptotics of Chebyshev polynomials on circular arcs. *J. Approx. Theory* **217**, 15–25 (2017)

14. Eichinger, B., Yuditskii, P.: The Ahlfors problem for polynomials. *Mat. Sb.* **209**(3), 34–66 (2018)
15. Erdélyi, T.: Remez-type inequalities on the size of generalized polynomials, *J. London Math. Soc.* (2) **45** (1992), no. 2, 255–264
16. Erdélyi, T., Li, X., Saff, E.B.: Remez- and Nikolskii-type inequalities for logarithmic potentials. *SIAM J. Math. Anal.* **25**(2), 365–383 (1994)
17. Eremenko, A., Yuditskii, P.: Comb functions, Recent advances in orthogonal polynomials, special functions, and their applications, *Contemp. Math.*, vol. 578, Amer. Math. Soc., Providence, RI, 2012, pp. 99–118
18. Garnett, J.B., Marshall, D.E.: Harmonic measure, *New Mathematical Monographs*, vol. 2, Cambridge University Press, Cambridge, 2008, Reprint of the 2005 original
19. Kalmykov, S., Nagy, B., Totik, V.: Bernstein- and Markov-type inequalities for rational functions. *Acta Mathematica* **219**(1), 21–63 (2017)
20. Koosis, P.: The logarithmic integral. I, *Cambridge Studies in Advanced Mathematics*, vol. 12, Cambridge University Press, Cambridge, 1998, Corrected reprint of the 1988 original
21. Landkof, N.S., Foundations of modern potential theory, Springer-Verlag, New York-Heidelberg,: Translated from the Russian by A. P. 180. P. Doohovskoy, *Die Grundlehren der mathematischen Wissenschaften, Band* (1972)
22. Pommerenke, C.: Univalent functions, Vandenhoeck & Ruprecht, Göttingen, 1975, With a chapter on quadratic differentials by Gerd Jensen, *Studia Mathematica/Mathematische Lehrbücher, Band XXV*
23. Ransford, T.: Potential theory in the complex plane, *London Mathematical Society Student Texts*, vol. 28. Cambridge University Press, Cambridge (1995)
24. Remes, E.: Sur une propriété extrémale des polynômes de Tchebychef. *Commun. Inst. Sci. Math. et Méc.* **13**, 93–95 (1936)
25. Sodin, M., Yuditskii, P.: Functions that deviate least from zero on closed subsets of the real axis. *Algebra i Analiz* **4**(2), 1–61 (1992)
26. Tikhonov, S., Yuditskii, P.: Sharp Remez inequality. *Constr. Approx.* **52**, 233–246 (2020)
27. Widom, H.: Extremal polynomials associated with a system of curves in the complex plane, *Advances in Math.*, no. 2, **3**, (1969), 127–232

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