



Optimal Polynomial Prediction Measures and Extremal Polynomial Growth

L. Bos¹ · N. Levenberg² · J. Ortega-Cerdà³

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Abstract

We show that the problem of finding the measure supported on a compact set $K \subset \mathbb{C}$ such that the variance of the least squares predictor by polynomials of degree at most n at a point $z_0 \in \mathbb{C}^d \setminus K$ is a minimum is equivalent to the problem of finding the polynomial of degree at most n , bounded by 1 on K , with extremal growth at z_0 . We use this to find the polynomials of extremal growth for $[-1, 1] \subset \mathbb{C}$ at a purely imaginary point. The related problem on the extremal growth of real polynomials was studied by Erdős (Bull Am Math Soc 53:1169–1176, 1947).

Keywords Optimal polynomial prediction measures · Polynomials of extremal growth · Complex polynomials

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1 Introduction

In this work we consider two classical extremum problems for polynomials. The first is very easy to state. Indeed, let us denote the *complex* polynomials of degree at most n in d complex variables by $\mathbb{C}_n[z]$, $z \in \mathbb{C}^d$. Then for $K \subset \mathbb{C}^d$ compact and $z_0 \in \mathbb{C}^d \setminus K$ an *external* point, we say that $P_n(z) \in \mathbb{C}_n[z]$ has *extremal growth relative to K at z_0*

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✉ L. Bos
leonardpeter.bos@univr.it

¹ Department of Computer Science, University of Verona, Verona, Italy

² Department of Mathematics, Indiana University, Bloomington, IN, USA

³ Departament de Matemàtiques i Informàtica, Universitat de Barcelona (UB), BGSMath, Barcelona, Spain

if

$$P_n = \arg \max_{p \in \mathbb{C}_n[z]} \frac{|p(z_0)|}{\|p\|_K} \quad (1)$$

where $\|p\|_K$ denotes the sup-norm of p on K . Alternatively, we may normalize p to be 1 at the external point and use

$$P_n = \arg \max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\|p\|_K}. \quad (2)$$

We note that for this to be well-defined we require that K be polynomial determining, i.e., if $p \in \mathbb{C}[z]$ is such that $p(x) = 0$ for all $x \in K$, then $p = 0$. We refer the interested reader to the survey [2] for more about what is known about this problem.

The second problem is from the field of optimal design for polynomial regression. To describe it we reduce to the real case $K \subset \mathbb{R}^d$, and note that we may write any $p \in \mathbb{R}_n[z]$ in the form

$$p = \sum_{k=1}^N \theta_k p_k$$

where $\mathcal{B}_s := \{p_1, p_2, \dots, p_N\}$ is a basis for $\mathbb{R}_n[z]$ and $N := \binom{n+d}{d}$ its dimension.

Suppose now that we observe the values of a particular $p \in \mathbb{R}_n[z]$ at a set of $m \geq N$ points $X := \{x_j : 1 \leq j \leq m\} \subset K$ with some random errors; i.e., we observe

$$y_j = p(x_j) + \epsilon_j, \quad 1 \leq j \leq m$$

where we assume that the errors $\epsilon_j \sim N(0, \sigma)$ are independent. In matrix form this becomes

$$y = V_n \theta + \epsilon$$

where $\theta \in \mathbb{R}^N$, $y, \epsilon \in \mathbb{R}^m$ and

$$V_n := \begin{bmatrix} p_1(x_1) & p_2(x_1) & \cdots & p_N(x_1) \\ p_1(x_2) & p_2(x_2) & \cdots & p_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(x_m) & p_2(x_m) & \cdots & p_N(x_m) \end{bmatrix} \in \mathbb{R}^{m \times N}$$

is the associated Vandermonde matrix.

Our assumption on the error vector ϵ means that

$$\text{cov}(\epsilon) = \sigma^2 I_m \in \mathbb{R}^{m \times m}.$$

Now, assuming that V_n is of full rank, the least squares estimate of θ is

$$\widehat{\theta} := (V_n^t V_n)^{-1} V_n^t y.$$

Note that the entries of $\frac{1}{m} V_n^t V_n$ are the discrete inner products of the p_i with respect to the measure

$$\mu_X = \frac{1}{m} \sum_{k=1}^m \delta_{x_k}. \quad (3)$$

More specifically,

$$\frac{1}{m} V_n^t V_n = G_n(\mu_X)$$

where

$$G_n(\mu) := \left[\int_K p_i(x) p_j(x) d\mu \right]_{1 \leq i, j \leq N} \in \mathbb{R}^{N \times N} \quad (4)$$

is the moment, or Gram, matrix of the polynomials p_i with respect to the measure μ .

In general we may consider arbitrary probability measures on K , setting

$$\mathcal{M}(K) := \{\mu : \mu \text{ is a probability measure on } K\}.$$

Now if

$$\mathbf{p}(z) = \begin{bmatrix} p_1(z) \\ p_2(z) \\ \vdots \\ p_N(z) \end{bmatrix} \in \mathbb{R}^N \quad (5)$$

then the least squares estimate of the *observed polynomial* is

$$\mathbf{p}^t(z) \widehat{\theta}.$$

We may compute its variance at any point $z \in \mathbb{R}^d$ to be

$$\begin{aligned} \text{var}(\mathbf{p}^t(z) \widehat{\theta}) &= \sigma^2 \mathbf{p}^t(z) (V_n^t V_n)^{-1} \mathbf{p}(z) \\ &= \frac{1}{m} \sigma^2 \mathbf{p}^t(z) (G_n(\mu_X))^{-1} \mathbf{p}(z) \end{aligned} \quad (6)$$

where μ_X is again given by (3). Now, it is easy to verify that for any $\mu \in \mathcal{M}(K)$ with non-singular Gram matrix,

$$\mathbf{p}^t(z) (G_n(\mu))^{-1} \mathbf{p}(z) = K_n^\mu(z, z)$$

where, for $\{q_1, \dots, q_N\} \subset \mathbb{R}_n[z]$, a μ -orthonormal basis for $\mathbb{R}_n[z]$,

$$K_n^\mu(w, z) := \sum_{k=1}^N q_k(w)q_k(z)$$

is the Bergman kernel for $\mathbb{R}_n[z]$. The function $K_n^\mu(z, z)$ is also known as the (reciprocal of) the Christoffel function for $\mathbb{R}_n[z]$. In particular, we see that the variance (6) is proportional to $K_n^{\mu_X}(z, z)$.

We may generalize easily to the complex case, $K \subset \mathbb{C}^d$, where now the p_j form a basis for $\mathbb{C}_n[z]$ and

$$G_n(\mu) := \left[\int_K \overline{p_i(z)} p_j(z) d\mu \right]_{1 \leq i, j \leq N} \in \mathbb{C}^{N \times N}. \quad (7)$$

In the case that $G_n(\mu)$ is *non-singular* then the kernel is

$$K_n^\mu(w, z) := \sum_{k=1}^N \overline{q_k(w)} q_k(z)$$

for $\{q_1, \dots, q_N\} \subset \mathbb{C}_n[z]$, a μ -orthonormal basis for $\mathbb{C}_n[z]$. Then, for an external point $z_0 \in \mathbb{C}^d \setminus K$, a measure $\mu_0 \in \mathcal{M}(K)$ is said to be an *optimal prediction (or extrapolation) measure for z_0 relative to K* (of order n) if it minimizes the complex analogue of the variance (6) of the polynomial predictor at z_0 ; i.e., if

$$K_n^{\mu_0}(z_0, z_0) = \inf_{\mu \in \mathcal{M}(K)} K_n^\mu(z_0, z_0). \quad (8)$$

However, as it turns out (see Example 1.1 below), such optimal prediction measures need not be definite (i.e., the associated Gram matrix need not be non-singular). Hence we need to re-formulate so that indefinite measures are allowed. Indeed, as is well known there is a variational form for $K_n^\mu(z_0, z_0)$:

$$\begin{aligned} K_n^\mu(z_0, z_0) &= \sup_{p \in \mathbb{C}_n[z]} \frac{|p(z_0)|^2}{\int_K |p(z)|^2 d\mu} \\ &= \sup_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\mu}. \end{aligned} \quad (9)$$

Note that in the case of an indefinite measure this value may be $+\infty$. Any polynomial $P_n^{\mu, z_0} \in \mathbb{C}_n[z]$ such that

$$P_n^{\mu, z_0} = \arg \max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\mu} \quad (10)$$

is said to be a *prediction polynomial* for μ and if μ_0 is an optimal prediction measure we call $P_n^{\mu_0, z_0}$ an *optimal prediction polynomial*. In the case that (9) is ∞ we interpret (10) to mean that the polynomial $P_n^{\mu_0, z_0}$ is such that $\int_K |P_n^{\mu_0, z_0}(z)|^2 d\mu = 0$.

Hence, in general, we say that $\mu_0 \in \mathcal{M}(K)$ is an optimal prediction measure for z_0 relative to K if μ_0 satisfies (8) with K_n^μ defined by (9).

We note that if μ is definite then

$$P_n^{\mu, z_0}(z) = \frac{K_n^\mu(z_0, z)}{K_n^\mu(z_0, z_0)}$$

is *unique* and $\int_K |P_n^{\mu, z_0}(z)|^2 d\mu(z) = 1/K_n^\mu(z_0, z_0)$. In the case that μ is indefinite then P_n^{μ, z_0} need not be unique.

Example 1.1 Consider $K = [-1, 1]^2$ considered as a subset of \mathbb{C}^2 , $z_0 = (2, 0)$, $\mu = \frac{1}{4}\delta_{(-1,0)} + \frac{3}{4}\delta_{(1,0)}$, and degree $n = 1$. Then it is easy to check that any polynomial of the form $P_1^{\mu, z_0}(x, y) = x/2 + cy$, $c \in \mathbb{C}$, is a prediction polynomial for μ . We will see in the next section that μ is an optimal prediction measure which also shows that optimal prediction measures may be indefinite.

In the univariate case however, optimal prediction polynomials are always definite.

Lemma 1.2 Suppose that $K \subset \mathbb{C}$ is $\mathbb{C}_n[z]$ determining and that $z_0 \in \mathbb{C} \setminus K$. Then any optimal prediction measure μ is definite; i.e., the Gram matrix $G_n(\mu)$ is non-singular.

Proof If the support of a measure μ has n or fewer distinct points there exists a polynomial $p \in \mathbb{C}_n[z]$ such that $p \equiv 0$ on the support while $p(z_0) = 1$. Hence

$$\frac{1}{\int_K |p(z)|^2 d\mu} = \infty$$

and μ cannot be an optimal prediction measure as taking any $n + 1$ points a_0, \dots, a_n in K and positive numbers w_0, \dots, w_n with $\sum_{j=0}^n w_j = 1$, the measure $\nu := \sum_{j=0}^n w_j \delta_{a_j}$ is definite. Thus $K_n^\nu(z_0, z)$ is a nontrivial polynomial of degree n with

$$K_n^\nu(z_0, z_0) = \sup_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\nu} < \infty.$$

□

Hoel and Levine [5] show that in the univariate case, for $K = [-1, 1]$, and any $z_0 \in \mathbb{R} \setminus K$, a *real* external point, the optimal prediction measure is unique and is a discrete measure supported at the $n + 1$ extremal points $x_k = \cos(k\pi/n)$, $0 \leq k \leq n$, of $T_n(x)$ the classical Chebyshev polynomial of the first kind (see Lemma 3.1 below). In this case it turns out that

$$K_n^{\mu_0}(z_0, z_0) = T_n^2(z_0). \quad (11)$$

Notably, as is well known, $T_n(x)$ is the polynomial of extremal growth for any point $z_0 \in \mathbb{R} \setminus [-1, 1]$ relative to $K = [-1, 1]$. Also, Erdős [3] has shown that the Chebyshev polynomial is also extreme relative to $[-1, 1]$ for *real* polynomials at points $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$; i.e.,

$$\max_{p \in \mathbb{R}_n[x], \|p\|_{[-1,1]} \leq 1} |p(z_0)| = |T_n(z_0)|.$$

The problem for real polynomials and $|z_0| \leq 1$ or for complex polynomials $p \in \mathbb{C}[z]$ has remained unsolved up to now.

We show in Sect. 2 that (11) is not an accident, and that there is a general equivalence of our two extremum problems. In Sect. 3 we give a complete and unique characterization of optimal prediction measures and polynomials of extremal growth for the case of the unit interval $K = [-1, 1] \subset \mathbb{C}$. Finally, in Sect. 4 we will use this to compute the polynomials of extremal growth and the optimal prediction measures for a purely imaginary complex point $z_0 \in \mathbb{C} \setminus [-1, 1]$.

2 A Kiefer–Wolfowitz Type Equivalence Theorem

Kiefer and Wolfowitz [6] have given a remarkable equivalence between what are called D-optimal and G-optimal designs; i.e., probability measures that maximize the determinant of the *design* matrix $G_n(\mu)$ and those that minimize the maximum over x *interior* to K , of the prediction variance; i.e., minimize $\max_{x \in K} K_n^\mu(x, x)$. Here we give an analogous equivalence, for a single *exterior* point $z_0 \in \mathbb{C}^d \setminus K$, with the problem of extremal polynomial growth.

Combining the definition of an optimal prediction measure (8) and the variational form for the kernel (9), the problem of minimal variance is to find

$$\min_{\mu \in \mathcal{M}(K)} \max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\mu}.$$

It turns out that this can be easily analyzed using the classical Minimax theorem (see e.g. Gamelin [4, Thm. 7.1, Ch. II]).

Proposition 2.1 *The minimal variance is the square of the maximal polynomial growth, i.e.,*

$$\min_{\mu \in \mathcal{M}(K)} \max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\mu} = \max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\|p\|_K^2}.$$

Proof First note that we may simplify to

$$\min_{\mu \in \mathcal{M}(K)} \max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\mu} = 1 / \left\{ \max_{\mu \in \mathcal{M}(K)} \min_{p \in \mathbb{C}_n[z], p(z_0)=1} \int_K |p(z)|^2 d\mu \right\}.$$

Now, for $\mu \in \mathcal{M}(K)$ and $p \in \mathbb{C}_n[z]$ such that $p(z_0) = 1$, let

$$f(\mu, p) := \int_K |p(z)|^2 d\mu.$$

It is easy to confirm that f is quasiconcave in μ and quasiconvex in p and hence by the Minimax Theorem

$$\max_{\mu \in \mathcal{M}(K)} \min_{p \in \mathbb{C}_n[z], p(z_0)=1} \int_K |p(z)|^2 d\mu = \min_{p \in \mathbb{C}_n[z], p(z_0)=1} \max_{\mu \in \mathcal{M}(K)} \int_K |p(z)|^2 d\mu.$$

However, as $\mu = \delta_x \in \mathcal{M}(K)$ for every $x \in K$, it follows that

$$\max_{\mu \in \mathcal{M}(K)} \int_K |p(z)|^2 d\mu = \|p\|_K^2.$$

Consequently, the minimum variance is given by

$$\min_{\mu \in \mathcal{M}(K)} \max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\mu} = \max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\|p\|_K^2},$$

as claimed. \square

We remark that the Minimax theorem in a similar context has been used before to get pointwise estimates of solutions to the $\bar{\partial}$ -equation by Berndtsson in [1, p. 206].

It is also possible to give a more precise relation between the extremal polynomials for the two problems (of minimum variance and extremal growth).

Theorem 2.2 *A measure $\mu_0 \in \mathcal{M}(K)$ is an optimal prediction measure for $z_0 \notin K$ relative to K if and only if there is an associated (optimal) prediction polynomial $P_n^{\mu_0, z_0}(z) \in \mathbb{C}_n[z]$, (10), such that $\|P_n^{\mu_0, z_0}\|_K = \|P_n^{\mu_0, z_0}\|_{L_2(\mu_0)}$, i.e.,*

$$\max_{z \in K} |P_n^{\mu_0, z_0}(z)|^2 = \int_K |P_n^{\mu_0, z_0}(z)|^2 d\mu_0,$$

or, equivalently, if and only if there is an associated prediction polynomial that is also a polynomial of extremal growth at z_0 relative to K .

Proof First suppose that $P_n^{\mu_0, z_0}(z) \in \mathbb{C}_n[z]$ is an optimal prediction polynomial associated with μ_0 such that $\|P_n^{\mu_0, z_0}\|_K = \|P_n^{\mu_0, z_0}\|_{L^2(\mu_0)}$. Then for any $\mu \in \mathcal{M}(K)$,

$$\begin{aligned} K_n^\mu(z_0, z_0) &= \max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\mu} \\ &\geq \frac{1}{\int_K |P_n^{\mu_0, z_0}(z)|^2 d\mu} \\ &\geq \frac{1}{\int_K \|P_n^{\mu_0, z_0}(z)\|_K^2 d\mu} \\ &= \frac{1}{\|P_n^{\mu_0, z_0}(z)\|_K^2} \\ &= \frac{1}{\int_K |P_n^{\mu_0, z_0}(z)|^2 d\mu_0} \\ &= K_n^{\mu_0}(z_0, z_0) \end{aligned}$$

and hence μ_0 is optimal.

To see that $P_n^{\mu_0, z_0}$ is also a polynomial of extremal growth, let $p \in \mathbb{C}_n[z]$ be any other polynomial for which $p(z_0) = 1$. Then

$$\begin{aligned} \|P_n^{\mu_0, z_0}\|_K^2 &= \|P_n^{\mu_0, z_0}\|_{L^2(\mu_0)}^2 \\ &= \int_K |P_n^{\mu_0, z_0}(z)|^2 d\mu_0 \\ &\leq \int_K |p(z)|^2 d\mu_0 \quad (\text{as } P_n^{\mu_0, z_0} \text{ is a prediction polynomial}) \\ &\leq \|p\|_K^2. \end{aligned}$$

Hence

$$\|P_n^{\mu_0, z_0}\|_K = \min_{p \in \mathbb{C}_n[z], p(z_0)=1} \|p\|_K$$

and $P_n^{\mu_0, z_0}$ is indeed a polynomial of extremal growth.

Conversely, suppose that μ_0 is optimal and let $P_n^{\mu_0, z_0}(z) \in \mathbb{C}_n[z]$ be a polynomial of extremal growth for z_0 relative to K , i.e., $P_n^{\mu_0, z_0}(z_0) = 1$ and for any other $p \in \mathbb{C}_n[z]$ such that $p(z_0) = 1$,

$$\|P_n^{\mu_0, z_0}\|_K \leq \|p\|_K.$$

We claim that $P_n^{\mu_0, z_0}$ is an optimal prediction polynomial and that $\|P_n^{\mu_0, z_0}\|_K = \|P_n^{\mu_0, z_0}\|_{L^2(\mu_0)}$.

To see this note that by Proposition 2.1

$$\max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\mu_0} = \frac{1}{\|P_n^{\mu_0, z_0}\|_K^2};$$

i.e.,

$$\min_{p \in \mathbb{C}_n[z], p(z_0)=1} \int_K |p(z)|^2 d\mu_0 = \|P_n^{\mu_0, z_0}\|_K^2.$$

Hence

$$\begin{aligned} \|P_n^{\mu_0, z_0}\|_K^2 &\leq \int_K |P_n^{\mu_0, z_0}(z)|^2 d\mu_0 \\ &\leq \int_K \|P_n^{\mu_0, z_0}\|_K^2 d\mu_0 \\ &= \|P_n^{\mu_0, z_0}\|_K^2; \end{aligned}$$

$$\text{i.e., } \int_K |P_n^{\mu_0, z_0}(z)|^2 d\mu_0 = \|P_n^{\mu_0, z_0}\|_K^2.$$

Moreover,

$$\max_{p \in \mathbb{C}_n[z], p(z_0)=1} \frac{1}{\int_K |p(z)|^2 d\mu_0} = \frac{1}{\|P_n^{\mu_0, z_0}\|_K^2} = \frac{1}{\int_K |P_n^{\mu_0, z_0}(z)|^2 d\mu_0}$$

and so $P_n^{\mu_0, z_0}$ is also an optimal prediction polynomial associated with μ_0 . □

In particular, if μ_0 is definite then

$$\int_K |P_n^{\mu_0, z_0}(z)|^2 d\mu_0 = \|P_n^{\mu_0, z_0}\|_K^2 = 1/K_n^{\mu_0}(z_0, z_0).$$

Remark 2.3 It is easily confirmed that $|P_n^{\mu_0, z_0}(z)| \equiv \|P_n^{\mu_0, z_0}\|_K$ on the support of μ_0 . Consequently optimal prediction measures are always supported on a real algebraic subset of K of degree $2n$.

Example 2.4 Recall the situation of Example 1.1: $K = [-1, 1]^2 \subset \mathbb{C}^2$ with the measure $\mu_0 := \frac{1}{4}\delta_{(-1,0)} + \frac{3}{4}\delta_{(1,0)}$. We show that this is an optimal prediction measure for the external point $z_0 = (2, 0)$ and polynomials of degree at most 1. As mentioned in Example 1.1, the prediction polynomials for this measure and point are $p(x, y) = x/2 + cy$, $c \in \mathbb{C}$. For the particular polynomial $P_1^{\mu_0, z_0}(x, y) := x/2$, we have $\|P_1^{\mu_0, z_0}\|_K^2 = 1/4$ and $\int_K |P_1^{\mu_0, z_0}(x, y)|^2 d\mu_0 = 1/8 + 3/8 = 1/4$. Hence by Theorem 2.2, μ_0 is an optimal prediction measure.

We now give an example showing that optimal prediction measures need not be unique, even in the univariate situation. Let

$$K = \mathbf{D} = \{z \in \mathbb{C} : |z| \leq 1\}$$

and fix z_0 with $|z_0| > 1$. Write $z_0 = |z_0|e^{i\phi}$ for a fixed angle ϕ .

Proposition 2.5 *Consider the measure*

$$d\mu(\theta) := \left[\sum_{k=-\infty}^{\infty} |z_0|^{-|k|} e^{ik(\theta+\phi)} \right] \frac{1}{2\pi} d\theta,$$

i.e., $d\mu$, the Poisson kernel for $1/z_0$ times $d\theta/(2\pi)$, supported on the unit circle. Then μ is an optimal prediction measure for $K = \mathbf{D}$ and $z_0 \notin \mathbf{D}$ for any degree n .

Proof For $j = 0, \pm 1, \pm 2, \dots$, let

$$m_j(\mu) := \int_K z^j d\mu = \int_0^{2\pi} e^{ij\theta} d\mu(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{k=-\infty}^{\infty} |z_0|^{-k} e^{ik(\theta+\phi)} \right] e^{ij\theta} d\theta.$$

It follows easily that for any j ,

$$m_j(\mu) = |z_0|^{-j} e^{-ij\phi} = z_0^{-j}. \quad (12)$$

Thus the Gram matrix for μ with respect to the basis $\{1, z, \dots, z^n\}$ for $\mathbb{C}_1[z]$ is

$$G_n(\mu) = G(z_0^{-1}) := \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} & \dots & z_0^{-n} \\ \bar{z}_0^{-1} & 1 & z_0^{-1} & \dots & z_0^{-(n-1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \bar{z}_0^{-n} & \bar{z}_0^{-(n-1)} & \bar{z}_0^{-(n-2)} & \dots & 1 \end{bmatrix}.$$

More generally, we define, for $|z| \neq 1$,

$$G(z) := \begin{bmatrix} 1 & z & z^2 & \dots & z^n \\ \bar{z} & 1 & z & \dots & z^{(n-1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \bar{z}^n & \bar{z}^{(n-1)} & \bar{z}^{(n-2)} & \dots & 1 \end{bmatrix}.$$

One easily verifies that

$$G(z)^{-1} := \frac{1}{|z|^2 - 1} \begin{bmatrix} -1 & z & 0 & \dots & 0 & 0 \\ \bar{z} & -(1 + |z|^2) & z & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \bar{z} & -(1 + |z|^2) & z \\ 0 & 0 & 0 & \dots & \bar{z} & -1 \end{bmatrix}.$$

Next, letting, for $z \neq 0$,

$$\mathbf{P}(z) := \begin{bmatrix} 1 \\ z^{-1} \\ \cdot \\ \cdot \\ z^{-n} \end{bmatrix} \in \mathbb{C}^{n+1}, \quad (13)$$

we easily verify that

$$\mathbf{P}^*(z)G(z)^{-1}\mathbf{P}(z) = |z|^{-2n}.$$

Thus we have

$$K_n^{\mu_0}(z_0, z_0) = \mathbf{P}^*(z_0^{-1})G(z_0^{-1})^{-1}\mathbf{P}(z_0^{-1}) = |z_0^{-1}|^{-2n} = |z_0|^{2n}.$$

But it is well-known that $p_n(z) = z^n$ is a polynomial of degree n of extremal growth at z_0 relative to K (see [2]); thus we know from Proposition 2.1 the optimal value $\inf_{\nu \in \mathcal{M}(K)} K_n^\nu(z_0, z_0)$ is

$$|p_n(z_0)|^2 = |z_0|^{2n}$$

and the proof is complete. \square

For each degree n we can produce additional optimal prediction measures by taking any discrete measure ν that reproduces the moments of μ present in the Gram matrix $G_n(\mu)$. Such discrete measures ν can be constructed, e.g., by Szegő quadrature (see section 7 of [7]). \square

However, as we will see in the next section, for a real interval and a point exterior to this interval, optimal prediction measures are unique.

3 A Complex Point External to $[-1, 1]$

We now consider $K = [-1, 1] \subset \mathbb{C}$ and $z_0 \in \mathbb{C} \setminus K$. We normalize so that the extremal polynomials are $P_n^{\mu_0, z_0}(z) = \frac{K_n^{\mu_0}(z_0, z)}{\sqrt{K_n^{\mu_0}(z_0, z_0)}}$; these have supremum norm 1 on $[-1, 1]$. As mentioned in Remark 2.3 above, the support of an optimal prediction measure in this case is a subset of $[-1, 1]$ where $|P_n^{\mu_0, z_0}(z)| = 1$, its maximum value. It is not possible that $|P_n^{\mu_0, z_0}(z)| \equiv 1$ on all of $[-1, 1]$ and hence the support of μ_0 consists of at most $2n$ points in $[-1, 1]$, counting multiplicities. Any interior point, being a local maximum of $|P_n^{\mu_0, z_0}|$, must be of even multiplicity and hence there can be at most n interior points. However, exactly n interior (double) points would mean that $z = \pm 1$ are not maximum points of $|P_n^{\mu_0, z_0}(z)|$; i.e., $|P_n^{\mu_0, z_0}(\pm 1)| < 1$. But then the fact that $\lim_{z \rightarrow \pm\infty} |P_n^{\mu_0, z_0}(z)| = \infty$ would imply that there are two other points outside $[-1, 1]$ where $|P_n^{\mu_0, z_0}(z)|$ attains the value 1, giving $2n + 2 > 2n$ real points where the value 1 is attained, an impossibility. Hence there are at most $n - 1$ interior points in the support of μ_0 . The fact that $G_n(\mu_0)$ is non-singular requires that there

are at least $n + 1$ support points, and these must therefore consist of $n - 1$ interior points together with the two endpoints ± 1 ; i.e., $x_0 := -1$, $x_n := +1$ and $n - 1$ internal (double) points $-1 < x_1 < \cdots < x_{n-1} < 1$. Consequently

$$\mu_0 = \sum_{i=0}^n w_i \delta_{x_i}$$

with weights $w_i > 0$, $\sum_{i=0}^n w_i = 1$.

Given the support points x_i there is a simple recipe for the optimal weights, given already in [5].

Lemma 3.1 (Hoel–Levine) *Suppose that $-1 = x_0 < x_1 < \cdots < x_n = +1$ are given. Then among all discrete probability measures supported at these points, the measure with*

$$w_i := \frac{|\ell_i(z_0)|}{\sum_{i=0}^n |\ell_i(z_0)|}, \quad 0 \leq i \leq n \quad (14)$$

with $\ell_i(z)$ the i th fundamental Lagrange interpolating polynomial for these points, minimizes $K_n^\mu(z_0, z_0)$.

Proof We first note that for such a discrete measure, $\{\ell_i(z)/\sqrt{w_i}\}_{0 \leq i \leq n}$ form an orthonormal basis. Hence

$$K_n^\mu(z_0, z_0) = \sum_{i=0}^n \frac{|\ell_i(z_0)|^2}{w_i}. \quad (15)$$

In the case of the weights chosen according to (14) we obtain

$$K_n^{\mu_0}(z_0, z_0) = \left(\sum_{i=0}^n |\ell_i(z_0)| \right)^2. \quad (16)$$

We claim that for any choice of weights K_n^μ given by (15) is at least as large as that given by (16). To see this, just note that by the Cauchy–Schwartz inequality,

$$\begin{aligned} \left(\sum_{i=0}^n |\ell_i(z_0)| \right)^2 &= \left(\sum_{i=0}^n \frac{|\ell_i(z_0)|}{\sqrt{w_i}} \cdot \sqrt{w_i} \right)^2 \\ &\leq \left(\sum_{i=0}^n \frac{|\ell_i(z_0)|^2}{w_i} \right) \cdot \left(\sum_{i=0}^n w_i \right) \\ &= \sum_{i=0}^n \frac{|\ell_i(z_0)|^2}{w_i}. \end{aligned}$$

□

Remark 3.2 We note that the optimal $K_n^{\mu_0}(z_0, z_0)$ given by (16) is the Lebesgue function squared. Hence the problem of finding the support of the optimal prediction measure amounts to finding the $n + 1$ interpolation points $-1 = x_0 < x_1 < \cdots < x_n = +1$ for which the Lebesgue function evaluated at the external point z_0 ,

$$\Lambda(z_0) := \sum_{i=0}^n |\ell_i(z_0)|,$$

is as small as possible.

Recall that the optimal prediction polynomials $P_n^{\mu_0, z_0}(z) = \frac{K_n^{\mu_0}(z_0, z)}{\sqrt{K_n^{\mu_0}(z_0, z_0)}}$ have supremum norms 1 on $[-1, 1]$.

Lemma 3.3 Suppose that the measure μ_0 is supported at the points $-1 = x_0 < x_1 < \cdots < x_n = +1$ with optimal weights given by (14). Then

$$P_n^{\mu_0, z_0}(z) = \sum_{i=0}^n \operatorname{sgn}(\ell_i(z_0)) \ell_i(z)$$

where $\operatorname{sgn}(z) := \bar{z}/|z|$ is the complex sign of $z \in \mathbb{C}$.

Proof Using again the fact that $\{\ell_i(z)/\sqrt{w_i}\}_{0 \leq i \leq n}$ form a set of orthonormal polynomials, we have

$$\begin{aligned} P_n^{\mu_0, z_0}(z) &= \frac{1}{\Lambda(z_0)} \sum_{i=0}^n \frac{\overline{\ell_i(z_0)}}{\sqrt{w_i}} \frac{\ell_i(z)}{\sqrt{w_i}} \\ &= \frac{1}{\Lambda(z_0)} \sum_{i=0}^n \left(\Lambda(z_0) \frac{\overline{\ell_i(z_0)}}{|\ell_i(z_0)|} \right) \ell_i(z) \\ &= \sum_{i=0}^n \frac{\overline{\ell_i(z_0)}}{|\ell_i(z_0)|} \cdot \ell_i(z). \end{aligned}$$

□

Remark 3.4 By the equivalence Theorem 2.2 the support of the optimal prediction measure and the polynomial of extremal growth will be given by those points $-1 = x_0 < x_1 < \cdots < x_n = +1$ for which

$$\max_{-1 \leq x \leq 1} \left| \sum_{i=0}^n \frac{\overline{\ell_i(z_0)}}{|\ell_i(z_0)|} \cdot \ell_i(x) \right| = 1.$$

4 A Purely Imaginary Point External to $[-1, 1]$

In the case of $z_0 = ai$, $0 \neq a \in \mathbb{R}$, a purely imaginary point, it turns out that there are remarkable formulas for the polynomial of extremal growth as well as for the support of the optimal prediction measure. Both of these will depend on the point z_0 (as opposed to the real case $z_0 \in \mathbb{R} \setminus [-1, 1]$ where Hoel and Levine [5] showed that the support is always the set of extreme points of the Chebyshev polynomial $T_n(x)$).

To begin we will first analyze the degrees $n = 1$ and $n = 2$ cases.

4.1 Degree $n = 1$

Here the support of the extremal measure is necessarily $x = -1$ and $x_1 = +1$. We will compute $P_1^{\mu_0, z_0}(z)$ using the formula given in Lemma 3.3. Indeed in this case, $\ell_0(z) = (1 - z)/2$ and $\ell_1(z) = (1 + z)/2$ so that

$$\operatorname{sgn}(\ell_0(ia)) = \operatorname{sgn}\left(\frac{1 - ia}{2}\right) = \frac{1 + ia}{\sqrt{a^2 + 1}}$$

and

$$\operatorname{sgn}(\ell_1(ia)) = \operatorname{sgn}\left(\frac{1 + ia}{2}\right) = \frac{1 - ia}{\sqrt{a^2 + 1}}.$$

Hence,

$$\begin{aligned} P_1^{\mu_0, z_0}(z) &= \frac{1 + ia}{\sqrt{a^2 + 1}} \frac{1 - z}{2} + \frac{1 - ia}{\sqrt{a^2 + 1}} \frac{1 + z}{2} \\ &= \frac{1}{\sqrt{a^2 + 1}} \{1 - iaz\}. \end{aligned}$$

Since ± 1 is necessarily the support of the optimal prediction measure it is immediate that $\|P_1^{\mu_0, z_0}\|_{[-1, 1]} = 1$, as is also easily verified by a simple direct calculation.

4.2 Degree $n = 2$

We claim that the support of the optimal prediction measure is $x_0 = -1$, $x_1 = 0$ and $x_2 = +1$. However, this is not automatic and we will have to verify that the norm of $P_2^{\mu_0, z_0}$ is indeed 1. Now, it is easy to see, for this support, that

$$\ell_0(z) = \frac{z(z - 1)}{2}, \quad \ell_1(z) = 1 - z^2, \quad \ell_2(z) = \frac{z(z + 1)}{2}$$

for which

$$\begin{aligned}\operatorname{sgn}(\ell_0(ia)) &= \operatorname{sgn}\left(\frac{ia(ia-1)}{2}\right) \\ &= \frac{-ia}{|a|} \cdot \frac{-ia-1}{\sqrt{a^2+1}} \\ &= i \operatorname{sgn}(a) \frac{1+ia}{\sqrt{a^2+1}},\end{aligned}$$

$$\operatorname{sgn}(\ell_1(ia)) = \operatorname{sgn}(1+a^2) = +1,$$

and, after a simple calculation,

$$\operatorname{sgn}(\ell_2(ia)) = i \operatorname{sgn}(a) \frac{ia-1}{\sqrt{a^2+1}}.$$

From this we may easily conclude that

$$\begin{aligned}P_2^{\mu_0, z_0}(z) &= \sum_{i=0}^2 \operatorname{sgn}(\ell_i(z_0)) \ell_i(z) \\ &= \frac{\operatorname{sgn}(a)}{\sqrt{a^2+1}} \left(-(a + \operatorname{sgn}(a)\sqrt{a^2+1})z^2 - iz + \operatorname{sgn}(a)\sqrt{a^2+1} \right).\end{aligned}$$

The fact that $\|P_2^{\mu_0, z_0}\|_{[-1,1]} = 1$ is an immediate consequence of the following lemma.

Lemma 4.1 *For $x \in \mathbb{R}$ we have*

$$\begin{aligned}|P_2^{\mu_0, z_0}(x)|^2 &= 1 + \frac{(|a| + \sqrt{a^2+1})^2}{a^2+1} x^2(x^2-1) \\ &= 1 + (x^2-1)R_1^2(x), \quad R_1(x) := \frac{|a| + \sqrt{a^2+1}}{\sqrt{a^2+1}} x.\end{aligned}$$

Proof This follows from elementary calculations starting with the formula for $P_2^{\mu_0, z_0}(x)$ given above. \square

We now define a sequence of polynomials, $Q_n(z)$, based on the above degrees $n = 1$ and $n = 2$ cases, for which we will show that $Q_n(z) = c_n P_n^{\mu_0, z_0}(z)$ for certain $c_n \in \mathbb{C}$ with modulus $|c_n| = 1$. We will also define a sequence of polynomials $R_n(x)$ which will play the role of $R_1(x)$ in the Lemma for general degree n .

Now, as the formula for $P_2^{\mu_0, z_0}$ depends on the sign of a , in order to simplify the formulas we will assume that $a > 0$. For $a < 0$, one may use the relation $P_2^{\mu_0, ia}(z) = P_2^{\mu_0, -ia}(-z)$.

Definition 4.2 For $a > 0$ we define the sequences of polynomials $Q_n(z)$ and $R_n(z)$ by

$$\begin{aligned} Q_1(z) &= -\frac{az + i}{\sqrt{a^2 + 1}}, \quad (= (-i)P_1^{\mu_0, z_0}(z)) \\ Q_2(z) &= \frac{1}{\sqrt{a^2 + 1}} \left(-(a + \sqrt{a^2 + 1})z^2 - iz + \sqrt{a^2 + 1} \right), \quad (= P_2^{\mu_0, z_0}(z)) \\ Q_{n+1}(z) &= 2zQ_n(z) - Q_{n-1}(z), \quad n = 2, 3, \dots \end{aligned}$$

and

$$\begin{aligned} R_0(z) &= \frac{a}{\sqrt{a^2 + 1}}, \\ R_1(z) &= \frac{a + \sqrt{a^2 + 1}}{\sqrt{a^2 + 1}} z, \\ R_{n+1}(z) &= 2zR_n(z) - R_{n-1}(z), \quad n = 1, 2, \dots \end{aligned}$$

Since the recursions are both those of the classical Chebyshev polynomials it is not surprising that there are formulas for $Q_n(z)$ and $R_n(z)$ in terms of these.

Lemma 4.3 *We have*

$$Q_n(z) = \frac{1}{\sqrt{a^2 + 1}} \left(-(az + i)T_{n-1}(z) + \sqrt{a^2 + 1}(1 - z^2)U_{n-2}(z) \right)$$

where $T_n(z)$ is Chebyshev polynomial of the first kind and $U_n(z) := \frac{1}{n+1}T'_{n+1}(z)$ that of the second kind.

Proof Let $q_n(z)$ denote the right side of the proposed identity. We proceed by induction. For $n = 1$ we have

$$\begin{aligned} q_1(z) &= \frac{1}{\sqrt{a^2 + 1}} \left(-(az + i)T_{1-1}(z) + \sqrt{a^2 + 1}(1 - z^2)U_{1-2}(z) \right) \\ &= \frac{1}{\sqrt{a^2 + 1}} \left(-(az + i) \times 1 + 0 \right) \\ &= Q_1(z). \end{aligned}$$

Similarly, for $n = 2$ we have

$$\begin{aligned} q_2(z) &= \frac{1}{\sqrt{a^2 + 1}} \left(-(az + i)T_{2-1}(z) + \sqrt{a^2 + 1}(1 - z^2)U_{2-2}(z) \right) \\ &= \frac{1}{\sqrt{a^2 + 1}} \left(-(az + i)z + \sqrt{a^2 + 1}(1 - z^2) \right) \\ &= \frac{1}{\sqrt{a^2 + 1}} \left(-(a + \sqrt{a^2 + 1})z^2 - iz + \sqrt{a^2 + 1} \right) \\ &= Q_2(z). \end{aligned}$$

The result now follows easily from the fact that both kinds of Chebyshev polynomials satisfy the same recursion as used in the definition of $Q_n(z)$. \square

Lemma 4.4 *We have*

$$R_n(z) = \frac{1}{\sqrt{a^2 + 1}} \left(\sqrt{a^2 + 1} z U_{n-1}(z) + a T_n(z) \right).$$

Proof Let $r_n(z)$ denote the right side of the proposed identity. We again proceed by induction. For $n = 0$ we have

$$\begin{aligned} r_0(z) &= \frac{1}{\sqrt{a^2 + 1}} \left(\sqrt{a^2 + 1} z U_{-1}(z) + a T_0(z) \right) \\ &= \frac{a}{\sqrt{a^2 + 1}} \\ &= R_0(z). \end{aligned}$$

Similarly, for $n = 1$ we have

$$\begin{aligned} r_1(z) &= \frac{1}{\sqrt{a^2 + 1}} \left(\sqrt{a^2 + 1} z U_0(z) + a T_1(z) \right) \\ &= \frac{1}{\sqrt{a^2 + 1}} \left(\sqrt{a^2 + 1} z \times 1 + a \times z \right) \\ &= \frac{a + \sqrt{a^2 + 1}}{\sqrt{a^2 + 1}} z \\ &= R_1(z). \end{aligned}$$

The result now follows easily from the fact that both kinds of Chebyshev polynomials satisfy the same recursion as used in the definition of $R_n(z)$. \square

Now, just for the Chebyshev polynomials $T_n(z)$ and $U_{n-1}(z)$ there is the Pell identity

$$T_n^2(z) - (z^2 - 1)U_{n-1}^2(z) \equiv 1. \quad (17)$$

We will show that for real $z \in \mathbb{R}$, the polynomials $Q_n(z)$ and $R_{n-1}(z)$ satisfy a similar Pell identity.

Proposition 4.5 *For $z = x \in \mathbb{R}$, we have*

$$|Q_n(x)|^2 - (x^2 - 1)R_{n-1}^2(x) \equiv 1.$$

Proof By Lemma 4.3, $z = x \in \mathbb{R}$, we may write

$$\begin{aligned} Q_n(x) &= \frac{1}{\sqrt{a^2 + 1}} \left(-(ax + i)T_{n-1}(x) + \sqrt{a^2 + 1}(1 - x^2)U_{n-2}(x) \right) \\ &= \frac{1}{\sqrt{a^2 + 1}} \left(-iT_{n-1}(x) + \left\{ -axT_{n-1}(x) + \sqrt{a^2 + 1}(1 - x^2)U_{n-2}(x) \right\} \right) \end{aligned}$$

so that

$$|Q_n(x)|^2 = \frac{1}{a^2 + 1} \left(T_{n-1}^2(x) + \left(-axT_{n-1}(x) + \sqrt{a^2 + 1}(1 - x^2)U_{n-2}(x) \right)^2 \right).$$

Hence, using the Chebyshev Pell identity (17),

$$\begin{aligned} & (a^2 + 1)(1 - |Q_n(x)|^2) \\ &= (a^2 + 1) - T_{n-1}^2(x) - a^2x^2T_{n-1}^2(x) \\ &\quad - (a^2 + 1)(1 - x^2)^2U_{n-2}^2(x) + 2a\sqrt{a^2 + 1}x(1 - x^2)U_{n-2}(x)T_{n-1}(x) \\ &= (a^2 + 1)(1 - T_{n-1}^2(x)) + a^2(1 - x^2)T_{n-1}^2(x) - (a^2 + 1)(1 - x^2)^2U_{n-2}^2(x) \\ &\quad + 2a\sqrt{a^2 + 1}x(1 - x^2)U_{n-2}(x)T_{n-1}(x) \\ &= (a^2 + 1)(1 - x^2)U_{n-2}^2(x) + a^2(1 - x^2)T_{n-1}^2(x) - (a^2 + 1)(1 - x^2)^2U_{n-2}^2(x) \\ &\quad + 2a\sqrt{a^2 + 1}x(1 - x^2)U_{n-2}(x)T_{n-1}(x) \\ &= (1 - x^2) \left\{ (a^2 + 1)U_{n-2}^2(x) + a^2T_{n-1}^2(x) - (a^2 + 1)(1 - x^2)U_{n-2}^2(x) \right. \\ &\quad \left. + 2a\sqrt{a^2 + 1}xU_{n-2}(x)T_{n-1}(x) \right\} \\ &= (1 - x^2) \left\{ (a^2 + 1)[1 - (1 - x^2)]U_{n-2}^2(x) + a^2T_{n-1}^2(x) \right. \\ &\quad \left. + 2a\sqrt{a^2 + 1}xU_{n-2}(x)T_{n-1}(x) \right\} \\ &= (1 - x^2) \left\{ (a^2 + 1)x^2U_{n-2}^2(x) + a^2T_{n-1}^2(x) + 2a\sqrt{a^2 + 1}xU_{n-2}(x)T_{n-1}(x) \right\} \\ &= (1 - x^2) \left\{ \sqrt{a^2 + 1}xU_{n-2}(x) + aT_{n-1}(x) \right\}^2 \\ &= (1 - x^2)(a^2 + 1)R_{n-1}^2(x). \end{aligned}$$

The last equality follows from Lemma 4.4. \square

From the Pell identity Proposition 4.5, we immediately have

Corollary 4.6 For $x \in [-1, 1]$,

$$|Q_n(x)| \leq 1$$

and its maximum of 1 is attained at the endpoints $x = \pm 1$ and the zeros of $R_{n-1}(x)$.

Indeed, we claim that the endpoints together with the zeros of $R_{n-1}(x)$ form the support of the optimal prediction measure. To this end we first prove that $R_{n-1}(x)$ has $n - 1$ zeros in $(-1, 1)$.

Lemma 4.7 The polynomials $R_n(x)$ have n distinct zeros in $(-1, 1)$ which interlace the extreme points of $T_n(x)$, $\cos(k\pi/n)$, $0 \leq k \leq n$.

Proof Using the fact that $T'_n(x) = nU_{n-1}(x)$, we have that at an interior extremal point of $T_n(x)$, $\cos(k\pi/n)$, $1 \leq k \leq (n-1)$,

$$\begin{aligned} R_n(\cos(k\pi/n)) &= \frac{1}{\sqrt{a^2+1}} \left(\sqrt{a^2+1} U_{n-1}(\cos(k\pi/n)) + a T_n(\cos(k\pi/n)) \right) \\ &= \frac{1}{\sqrt{a^2+1}} \left(0 + a(-1)^k \right) \\ &= \frac{a}{\sqrt{a^2+1}} (-1)^k \end{aligned}$$

so that

$$\operatorname{sgn}(R_n(\cos(k\pi/n))) = (-1)^k, \quad 1 \leq k \leq (n-1).$$

Further, for $k = 0$, $\cos(k\pi/n) = 1$,

$$\begin{aligned} R_n(1) &= \frac{1}{\sqrt{a^2+1}} \left(\sqrt{a^2+1} U_{n-1}(1) + a T_n(1) \right) \\ &= \frac{1}{\sqrt{a^2+1}} \left(n\sqrt{a^2+1} + a \right) \end{aligned}$$

so that

$$\operatorname{sgn}(R_n(\cos(0\pi/n))) = +1 = (-1)^0.$$

Similarly, for $k = n$, $\cos(k\pi/n) = -1$,

$$\begin{aligned} R_n(-1) &= \frac{1}{\sqrt{a^2+1}} \left(\sqrt{a^2+1} (-1) U_{n-1}(-1) + a T_n(-1) \right) \\ &= \frac{1}{\sqrt{a^2+1}} (n\sqrt{a^2+1} + a) (-1)^n \end{aligned}$$

so that also

$$\operatorname{sgn}(R_n(\cos(n\pi/n))) = (-1)^n.$$

The result follows. \square

Suppose now that μ_0 is the discrete measure supported on ± 1 together with the $n-1$ zeros of $R_{n-1}(x)$, with optimal weights given by Lemma 3.1.

Proposition 4.8 *The polynomials $Q_n(z)$ are of extremal growth at $z_0 = ai$ relative to $K = [-1, 1]$. Specifically, $Q_n(z) = -(i)^n P_n^{\mu_0, z_0}(z)$.*

Proof Let $-1 = x_0 < x_1 < \cdots < x_n = +1$ be the support points with corresponding Lagrange polynomials $\ell_k(z)$. We will show that

$$Q_n(x_k) = -(i)^n \operatorname{sgn}(\ell_k(ai)), \quad 0 \leq k \leq n$$

using the formula

$$\ell_k(z) = \frac{\omega_n(z)}{(z - x_k)\omega'_n(x_k)}, \quad \omega_n(z) := (z^2 - 1)R_{n-1}(z).$$

Our calculations will make use of the elementary facts that

$$\begin{aligned} T_n(ai) &= \frac{(i)^n}{2} \left\{ (a + \sqrt{a^2 + 1})^n + (a - \sqrt{a^2 + 1})^n \right\}, \\ U_n(ai) &= \frac{(i)^n}{2\sqrt{a^2 + 1}} \left\{ (a + \sqrt{a^2 + 1})^{n+1} - (a - \sqrt{a^2 + 1})^{n+1} \right\} \end{aligned}$$

so that

$$\begin{aligned} R_{n-1}(ai) &= \frac{1}{\sqrt{a^2 + 1}} \left(\sqrt{a^2 + 1}(ai)U_{n-2}(ai) + aT_{n-1}(ai) \right) \\ &= (i)^{n-1} \frac{a}{\sqrt{a^2 + 1}} (a + \sqrt{a^2 + 1})^{n-1}. \end{aligned}$$

The endpoints are the easiest case and so we will begin with those. Specifically, for $k = 0$, $x_0 = -1$,

$$\begin{aligned} \ell_0(ai) &= \frac{((ai)^2 - 1)R_{n-1}(ai)}{(ai - (-1))\omega'_n(-1)} \\ &= \frac{-(a^2 + 1)R_{n-1}(ai)}{(ai + 1)(-2R_{n-1}(-1))}. \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{sgn}(\ell_0(ai)) &= \operatorname{sgn}(R_{n-1}(ai)) \operatorname{sgn}(R_{n-1}(-1)) \operatorname{sgn}\left(\frac{1}{ai + 1}\right) \\ &= (-i)^{n-1} (-1)^{n-1} \frac{ai + 1}{\sqrt{a^2 + 1}}. \end{aligned}$$

On the other hand

$$\begin{aligned} Q_n(-1) &= \frac{1}{\sqrt{a^2 + 1}} \left(-(a(-1) + i)T_{n-1}(-1) + \sqrt{a^2 + 1}(1 - (-1)^2)U_{n-2}(-1) \right) \\ &= \frac{1}{\sqrt{a^2 + 1}} (a - i)(-1)^{n-1} \\ &= -(i)^n \operatorname{sgn}(\ell_0(ai)), \end{aligned}$$

as is easily verified.

The other endpoint $x_n = +1$ is very similar and so we suppress the details. Consider now, x_k , $1 \leq k \leq (n-1)$, a zero of $R_{n-1}(x)$. Then

$$\begin{aligned}\ell_k(ai) &= \frac{((ai)^2 - 1)R_{n-1}(ai)}{(ai - x_k)(x_k^2 - 1)R'_{n-1}(x_k)} \\ &= \frac{-(a^2 + 1)R_{n-1}(ai)}{(ai - x_k)(x_k^2 - 1)(R'_{n-1}(x_k))}.\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{sgn}(\ell_k(ai)) &= \operatorname{sgn}(R_{n-1}(ai)) \operatorname{sgn}(R'_{n-1}(x_k)) \operatorname{sgn}\left(\frac{1}{ai - x_k}\right) \\ &= (i)^{n-1}(-1)^k \frac{ai - x_k}{\sqrt{a^2 + x_k^2}}\end{aligned}$$

as $\operatorname{sgn}(R'_{n-1}(x_k)) = (-1)^{n-1-k}$, as is easy to see.

On the other hand, from the formula for $R_{n-1}(x)$ given in Lemma 4.4, we see that $R_{n-1}(x_k) = 0$ implies that

$$T_{n-1}(x_k) = -\frac{\sqrt{a^2 + 1}}{a} x_k U_{n-2}(x_k).$$

Substituting this into the formula for Q_n given in Lemma 4.3 we obtain

$$\begin{aligned}Q_n(x_k) &= \left\{ \frac{(ax_k + i)x_k}{a} + (1 - x_k^2) \right\} U_{n-2}(x_k) \\ &= \left(\frac{a + ix_k}{a} \right) U_{n-2}(x_k).\end{aligned}$$

But by the Pell identity of Proposition 4.5, $|Q_n(x_k)| = 1$ and so we must have

$$Q_n(x_k) = \frac{a + ix_k}{\sqrt{a^2 + x_k^2}} \operatorname{sgn}(U_{n-2}(x_k)).$$

But, as the zeros of R_{n-1} interlace the extreme points of T_{n-1} , i.e., the zeros of U_{n-2} , it is easy to check that $\operatorname{sgn}(U_{n-2}(x_k)) = (-1)^{n-1-k}$. In other words,

$$Q_n(x_k) = (-1)^{n-1-k} \frac{a + ix_k}{\sqrt{a^2 + x_k^2}}$$

which is easily verified to equal $-(i)^n \operatorname{sgn}(\ell_k(ai))$, as claimed. \square

From the recursion formula for $Q_n(z)$ it is easy to see that

$$Q_n(ai) = -(i)^n \sqrt{a^2 + 1} (a + \sqrt{a^2 + 1})^{n-1}.$$

Hence we have

Proposition 4.9 For $n = 1, 2, \dots$

$$\max_{p \in \mathbb{C}_n[z], \|p\|_{[-1,1]} \leq 1} |p(ai)| = \sqrt{a^2 + 1} (|a| + \sqrt{a^2 + 1})^{n-1}$$

and this maximum value is attained by $Q_n(z)$ (for $a > 0$).

It is worth noting that the extremal polynomial and optimal prediction measure, unlike the real case, depend on the exterior point z_0 . Moreover, this extreme value is rather larger than $|T_n(ai)|$. Indeed it is easy to show that

$$\sqrt{a^2 + 1} (|a| + \sqrt{a^2 + 1})^{n-1} - |T_n(ai)| = (\sqrt{a^2 + 1} - |a|) |T_{n-1}(ai)|.$$

One may of course wonder if there are similar formulas for general points $z_0 \in \mathbb{C} \setminus [-1, 1]$ (not just $z_0 = ai$). However numerical experiments seem to indicate that in general there is no three-term recurrence for the extremal polynomials.

Note added in proof It has recently come to our attention that the general extremal values problem has been studied by Yuditskii [8].

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