Asymptotic Behaviour of Christoffel–Darboux Kernel Via Three-Term Recurrence Relation I

Grzegorz Świdorski^{1,2} · Bartosz Trojan³

Received: 5 October 2019 / Revised: 9 March 2020 / Accepted: 12 May 2020 /

Published online: 23 September 2020

© The Author(s) 2020

Abstract

For Jacobi parameters belonging to one of three classes: asymptotically periodic, periodically modulated, and the blend of these two, we study the asymptotic behavior of the Christoffel functions and the scaling limits of the Christoffel–Darboux kernel. We assume regularity of Jacobi parameters in terms of the Stolz class. We emphasize that the first class only gives rise to measures with compact supports.

Keywords Orthogonal polynomials · Asymptotics · Christoffel functions · Scaling limits

Mathematics Subject Classification Primary: 42C05 · 47B36

1 Introduction

Let μ be a probability measure on the real line with infinite support such that for every $n \in \mathbb{N}_0$,

the moments $\int_{\mathbb{R}} x^n d\mu(x)$ are finite.

Communicated by Percy Deift.

✉ Grzegorz Świdorski
grzegorz.swidorski@kuleuven.be

Bartosz Trojan
btrojan@impan.pl

¹ Department of Mathematics, KU Leuven, Celestijnenlaan 200B box 2400, 3001 Leuven, Belgium

² Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

³ Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-696 Warsaw, Poland

Let $L^2(\mathbb{R}, \mu)$ be the Hilbert space of square-integrable functions equipped with the scalar product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} \, d\mu(x).$$

By performing the Gram–Schmidt orthogonalization process on the sequence of monomials $(x^n : n \in \mathbb{N}_0)$ one obtains the sequence of polynomials $(p_n : n \in \mathbb{N}_0)$ satisfying

$$\langle p_n, p_m \rangle = \delta_{nm} \quad (1.1)$$

where δ_{nm} is the Kronecker delta. Moreover, $(p_n : n \in \mathbb{N}_0)$ satisfies the following recurrence relation

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = \frac{x - b_0}{a_0}, \\ xp_n(x) &= a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x), \quad n \geq 1 \end{aligned} \quad (1.2)$$

where

$$a_n = \langle xp_n, p_{n+1} \rangle, \quad b_n = \langle xp_n, p_n \rangle, \quad n \geq 0.$$

Notice that for every n , $a_n > 0$ and $b_n \in \mathbb{R}$. The pair (a_n) and (b_n) is called the Jacobi parameter.

Let $\mathbb{P}_n$ be the orthogonal projection in $L^2(\mathbb{R}, \mu)$ on the space of polynomials of degree at most n . Then $\mathbb{P}_n$ is given by the *Christoffel–Darboux* kernel K_n , that is

$$\mathbb{P}_n f(x) = \int_{\mathbb{R}} K_n(x, y) \overline{f(y)} \, d\mu(y)$$

where from (1.1) one can verify that

$$K_n(x, y) = \sum_{j=0}^n p_j(x) p_j(y). \quad (1.3)$$

To motivate the study of Christoffel–Darboux kernels see surveys [12] and [19].

The asymptotic behavior of K_n is well understood in the case when the measure μ has compact support. In this setup one of the most general results has been proven in [29]. Namely, if I is an open interval contained in $\text{supp}(\mu)$ such that μ is absolutely continuous on I with continuous positive density μ' , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} K_n \left(x + \frac{u}{n}, x + \frac{v}{n} \right) = \frac{\omega'(x)}{\mu'(x)} \frac{\sin((u-v)\pi\omega'(x))}{(u-v)\pi\omega'(x)} \quad (1.4)$$

locally uniformly with respect to $x \in I$ and $u, v \in \mathbb{R}$, provided that μ is *regular* (see [21, Definition 3.1.2]). In formula (1.4), ω' denotes the density of the equilibrium

measure corresponding to the support of μ , see (2.4) for details. In the case when $\text{supp}(\mu)$ is a finite union of compact intervals, μ is regular provided that $\mu' > 0$ almost everywhere in the interior of $\text{supp}(\mu)$. To give some historical perspective, let us also mention three earlier results. The case $\text{supp}(\mu) = [-1, 1]$ and $u = v = 0$ has been examined in [14, Theorem 8], and its extension to a general compact $\text{supp}(\mu)$ has been obtained in [28, Theorem 1]. The extension to all $u, v \in \mathbb{R}$ has been proven in [11].

The best understood class of measures with unbounded support is the class of *exponential weights* (see the monograph [8]). In [10] and [9, Theorem 7.4] under a number of regularity conditions on the function $Q(x) = -\log \mu'(x)$, the following analogue of (1.4) was shown

$$\lim_{n \rightarrow \infty} \frac{1}{\tilde{K}_n(x, x)} \tilde{K}_n\left(x + \frac{u}{\tilde{K}_n(x, x)}, x + \frac{v}{\tilde{K}_n(x, x)}\right) = \frac{\sin(u - v)}{u - v} \quad (1.5)$$

locally uniformly with respect to $x, u, v \in \mathbb{R}$ where $\tilde{K}_n(x, y) = \sqrt{\mu'(x)\mu'(y)} K_n(x, y)$. Unlike (1.4), the formula (1.5) does not give any information if $u = v = 0$. It was recently proved in [4] that under some additional regularity on Q

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n} K_n(x, x) = \frac{1}{2\pi\mu'(x)} \quad (1.6)$$

locally uniformly with respect to $x \in \mathbb{R}$ where

$$\rho_n = \sum_{j=0}^n \frac{1}{a_j}.$$

Let us comment that ρ_n is comparable to n if the sequences (a_n) and (a_n^{-1}) are bounded. By combining (1.5) with (1.6) one obtains

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n} K_n\left(x + \frac{u}{\rho_n}, x + \frac{v}{\rho_n}\right) = \frac{\omega'(0)}{\mu'(x)} \frac{\sin((u - v)\pi\omega'(0))}{(u - v)\pi\omega'(0)} \quad (1.7)$$

where $\omega'(x) = (\pi\sqrt{4 - x^2})^{-1}$ is the density of the equilibrium measure for the interval $[-2, 2]$.

Instead of taking the measure μ as the starting point one can consider polynomials $(p_n : n \in \mathbb{N}_0)$ satisfying the three-term recurrence relation (1.2) with $a_n > 0$ and $b_n \in \mathbb{R}$ for any $n \in \mathbb{N}_0$. Then the Favard's theorem (see, e.g., [18, Theorem 5.10]) states that there is a probability measure μ such that (p_n) is orthonormal in $L^2(\mathbb{R}, \mu)$. The measure μ is unique, if and only if there is exactly one measure with the same moments as μ . It is always the case when the Carleman condition

$$\sum_{n=0}^{\infty} \frac{1}{a_n} = \infty \quad (1.8)$$

is satisfied (see, e.g. [18, Corollary 6.19]). Moreover, the measure μ has compact support, if and only if the Jacobi parameters are bounded.

In this article our starting point is the three-term recurrence relation. We study analogues of (1.6) and (1.7) for three different classes of Jacobi parameters: asymptotically periodic, periodically modulated and a blend of these two; for the definitions, see Sects. 3.1, 3.2 and 3.3, respectively. The first class only gives rise to measures with compact supports. The second class introduced in [5] has the Jacobi parameters uniformly unbounded in the sense that $\liminf a_n = \infty$. The third class has been studied in [1] as an example of unbounded Jacobi parameters corresponding to measures with absolutely continuous parts having supports equal a finite union of closed intervals.

To simplify the exposition in the introduction, we shall focus on the periodic modulations only. Before we formulate our results, let us state some definitions. Let N be a positive integer. We say that sequences $(a_n), (b_n)$ are N -periodically modulated if there are two N -periodic sequences $(\alpha_n : n \in \mathbb{Z})$ and $(\beta_n : n \in \mathbb{Z})$ of positive and real numbers, respectively, such that

- (a) $\lim_{n \rightarrow \infty} a_n = \infty$,
- (b) $\lim_{n \rightarrow \infty} \left| \frac{a_{n-1}}{a_n} - \frac{\alpha_{n-1}}{\alpha_n} \right| = 0$,
- (c) $\lim_{n \rightarrow \infty} \left| \frac{b_n}{a_n} - \frac{\beta_n}{\alpha_n} \right| = 0$.

The crucial rôle is played by the N -step transfer matrices defined by

$$X_n(x) = B_{n+N-1}(x)B_{n+N-2}(x) \cdots B_n(x) \quad \text{where} \quad B_j(x) = \begin{pmatrix} 0 & 1 \\ -\frac{a_{j-1}}{a_j} & \frac{x-b_j}{a_j} \end{pmatrix},$$

and

$$\mathfrak{X}_n(x) = \mathfrak{B}_{n+N-1}(x)\mathfrak{B}_{n+N-2} \cdots \mathfrak{B}_n(x) \quad \text{where} \quad \mathfrak{B}_j(x) = \begin{pmatrix} 0 & 1 \\ -\frac{\alpha_{j-1}}{\alpha_j} & \frac{x-\beta_j}{\alpha_j} \end{pmatrix}.$$

The name is justified by the following property

$$\begin{pmatrix} p_{n+N-1}(x) \\ p_{n+N}(x) \end{pmatrix} = X_n(x) \begin{pmatrix} p_{n-1}(x) \\ p_n(x) \end{pmatrix}.$$

We are interested in the class of Jacobi matrices associated with slowly oscillating sequences introduced in [22]. Let r be a positive integer. We say that the sequence $(x_n : n \in \mathbb{N})$ of vectors from a normed space V belongs to $\mathcal{D}_r(V)$, if it is bounded and for each $j \in \{1, \dots, r\}$,

$$\sum_{n=1}^{\infty} \|\Delta^j x_n\|^{\frac{r}{j}} < \infty$$

where

$$\begin{aligned}\Delta^0 x_n &= x_n, \\ \Delta^j x_n &= \Delta^{j-1} x_{n+1} - \Delta^{j-1} x_n, \quad j \geq 1.\end{aligned}$$

If X is the real line with a Euclidean norm we abbreviate $\mathcal{D}_r = \mathcal{D}_r(X)$. Given a compact set $K \subset \mathbb{C}$ and a normed space R , by $\mathcal{D}_r(K, R)$ we denote the case when X is the space of all continuous mappings from K to R equipped with the supremum norm.

Our first result is the following theorem, see Theorem 4.4. By $GL(2, \mathbb{R})$ we denote 2×2 real invertible matrices equipped with the spectral norm. For a matrix

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

we set $[X]_{i,j} = x_{i,j}$. Lastly, a discriminant of X is $\text{discr } X = (\text{tr } X)^2 - 4 \det X$.

Theorem 1.1 *Let N and r be positive integers and $i \in \{0, 1, \dots, N-1\}$. Suppose that K is a compact interval with non-empty interior contained in*

$$\Lambda = \left\{ x \in \mathbb{R} : \lim_{j \rightarrow \infty} \text{discr } X_{jN+i}(x) \text{ exists and is negative} \right\}.$$

Assume that

$$\lim_{j \rightarrow \infty} \frac{a_{(j+1)N+i-1}}{a_{jN+i-1}} = 1$$

and

$$(X_{jN+i} : j \in \mathbb{N}) \in \mathcal{D}_r(K, GL(2, \mathbb{R})).$$

Suppose that X is the limit of $(X_{jN+i} : j \in \mathbb{N})$. If

$$\sum_{j=1}^{\infty} \frac{1}{a_{jN+i-1}} = \infty,$$

then for $x \in K$

$$\left(\sum_{j=1}^n \frac{1}{a_{jN+i-1}} \right)^{-1} \sum_{j=0}^n p_{jN+i}^2(x) = \frac{|[X(x)]_{2,1}|}{\pi \mu'(x) \sqrt{-\text{discr } X(x)}} + E_n(x)$$

where

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |E_n(x)| = 0.$$

Let us remark that in Theorem 5.4 we have obtained the quantitative bounds on E_n in the $\mathcal{D}_1$ setting.

Theorem 1.1 is an important step in proving the analogues of (1.6) and (1.7). The following theorem (see Theorem 4.6) provides the analogue of (1.6) for periodic modulations. Similar results are obtained also for the remaining classes, that is for asymptotically periodic in Theorem 4.7, and for the blend in Theorem 4.10.

Theorem 1.2 *Let (a_n) and (b_n) be N -periodically modulated Jacobi parameters. Suppose that there is $r \geq 1$ such that for every $i \in \{0, 1, \dots, N-1\}$,*

$$\left(\frac{a_{kN+i-1}}{a_{kN+i}} : k \in \mathbb{N} \right), \left(\frac{b_{kN+i}}{a_{kN+i}} : k \in \mathbb{N} \right), \left(\frac{1}{a_{kN+i}} : k \in \mathbb{N} \right) \in \mathcal{D}_r,$$

and the Carleman condition (1.8) is satisfied. If $|\operatorname{tr} \mathfrak{X}_0(0)| < 2$, then

$$\frac{1}{\rho_n} K_n(x, x) = \frac{\omega'(0)}{\mu'(x)} + E_n(x)$$

where

$$\lim_{n \rightarrow \infty} |E_n(x)| = 0$$

locally uniformly with respect to $x \in \mathbb{R}$, where ω is the equilibrium measure of

$$\{x \in \mathbb{R} : |\operatorname{tr} \mathfrak{X}_0(x)| \leq 2\}$$

and

$$\rho_n = \sum_{j=0}^n \frac{\alpha_j}{a_j}.$$

Again, in $\mathcal{D}_1$ setup, we obtained the quantitative bound on E_n , see Theorem 5.5 (periodic modulations) and Theorem 5.7 (asymptotically periodic).

We emphasize that Theorem 1.2 solves [3, Conjecture 1] for a larger class of Jacobi parameters than it was originally stated, see Sect. 4.3 for details.

Lastly, we provide the analogue of (1.7) for periodic modulations (see Theorem 5.15). In view of [27, Corollary 7], the asymptotically periodic case follows from [29]. For the blend, see Theorem 5.17.

Theorem 1.3 *Suppose that the hypotheses of Theorem 1.2 are satisfied for $r = 1$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n} K_n\left(x + \frac{u}{\rho_n}, x + \frac{v}{\rho_n}\right) = \frac{\omega'(0)}{\mu'(x)} \frac{\sin((u-v)\pi\omega'(0))}{(u-v)\pi\omega'(0)}$$

locally uniformly with respect to $x, u, v \in \mathbb{R}$.

Let us mention that the hypotheses of Theorems 1.2 and 1.3 for $N = 1$ are satisfied by Hermite polynomials, Meixner–Pollaczek polynomials and Freud weights, see [23, Section 5] for detailed proofs.

Let us present some ideas of the proofs. The basic strategy commonly used is to exploit the Christoffel–Darboux formula, that is

$$K_n(x, y) = a_n \begin{cases} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y}, & \text{if } x \neq y, \\ p_n(x)p'_{n+1}(x) - p'_n(x)p_{n+1}(x), & \text{otherwise.} \end{cases}$$

However, it requires the precise asymptotic of the polynomials as well as its derivatives in terms of both n and x . Unfortunately, for the classes of Jacobi parameters we are interested in they are not available. In the recent article [27], we managed to obtain the asymptotic of $(p_n(x) : n \in \mathbb{N}_0)$ locally uniformly with respect to x . Based on it we develop a method to study $K_n(x, y)$. Namely, we use the formula (1.3), which leads to the need of estimation of the oscillatory sums of a form

$$\sum_{k=0}^n \frac{\gamma_k}{\sum_{j=0}^n \gamma_j} \sin \left(\sum_{j=0}^n \theta_j(x_n) + \sigma(x_n) \right) \sin \left(\sum_{j=0}^n \theta_j(y_n) + \sigma(y_n) \right)$$

where

$$x_n = x + \frac{u}{\rho_n}, \quad \text{and} \quad y_n = x + \frac{v}{\rho_n}.$$

To deal with the sums we prove two auxiliary results (see Lemma 4.1 and Theorem 5.11) that are valid for sequences not necessarily belonging to $\mathcal{D}_r$.

The organization of the article is as follows. In Sect. 2 we present basic definitions used in the article. In Sect. 3 we collect the definitions and basic properties of the three classes of sequences. Section 4 is devoted to the general $\mathcal{D}_r$ setting. In particular, we present there the proofs of Theorems 1.1 and 1.2, and we provide a solution of Ignjatović conjecture. In Sect. 5 we study the case $\mathcal{D}_1$ where we derive quantitative bound on the error in the asymptotic of the polynomials. We also provide the quantitative versions of Theorems 1.1 and 1.2. Finally, we prove Theorem 1.3.

Notation

By $\mathbb{N}$ we denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout the whole article, we write $A \lesssim B$ if there is an absolute constant $c > 0$ such that $A \leq cB$. Moreover, c stands for a positive constant whose value may vary from occurrence to occurrence.

2 Definitions

Given two sequences $a = (a_n : n \in \mathbb{N}_0)$ and $b = (b_n : n \in \mathbb{N}_0)$ of positive and real numbers, respectively, and $k \in \mathbb{N}$, we define k th associated *orthonormal* polynomials as

$$p_0^{[k]}(x) = 1, \quad p_1^{[k]}(x) = \frac{x - b_k}{a_k},$$

$$a_{n+k-1}p_{n-1}^{[k]}(x) + b_{n+k}p_n^{[k]}(x) + a_{n+k}p_{n+1}^{[k]}(x) = xp_n^{[k]}(x), \quad (n \geq 1),$$

For $k = 0$ we usually omit the superscript. A sequence $(u_n : n \in \mathbb{N}_0)$ is a generalized eigenvector associated with $x \in \mathbb{C}$, if for all $n \geq 1$,

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = B_n(x) \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}$$

where

$$B_n(x) = \begin{pmatrix} 0 & 1 \\ -\frac{a_{n-1}}{a_n} & \frac{x-b_n}{a_n} \end{pmatrix}.$$

Let A be the closure in ℓ^2 of the operator acting on sequences having finite support by the matrix

$$\begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & \dots \\ 0 & 0 & a_2 & b_3 & \dots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}. \quad (2.1)$$

The operator A is called *Jacobi matrix*. If the Carleman condition

$$\sum_{n=0}^{\infty} \frac{1}{a_n} = \infty \quad (2.2)$$

is satisfied then the operator A has the unique self-adjoint extension (see e.g., [18, Corollary 6.19]). Let us denote by E_A its spectral resolution of the identity. Then for any Borel subset $B \subset \mathbb{R}$, we set

$$\mu(B) = \langle E_A(B)\delta_0, \delta_0 \rangle_{\ell^2}$$

where δ_0 is the sequence having 1 on the 0th position and 0 elsewhere. The polynomials $(p_n : n \in \mathbb{N}_0)$ form an orthonormal basis of $L^2(\mathbb{R}, \mu)$.

In this article the central object is the *Christoffel–Darboux kernel* defined as

$$K_n(x, y) = \sum_{j=0}^n p_j(x)p_j(y). \quad (2.3)$$

Given a compact set $K \subset \mathbb{R}$ with non-empty interior, there is the unique probability measure ω_K , called the *equilibrium measure* corresponding to K , minimizing the energy

$$I(\nu) = - \int_{\mathbb{R}} \int_{\mathbb{R}} \log |x - y| \nu(dx) \nu(dy), \quad (2.4)$$

among all probability measures ν supported on K . The measure ω_K is absolutely continuous in the interior of K with continuous density, see [16, Theorem IV.2.5, pp. 216].

3 Classes of Sequences

In this article we are interested in Jacobi matrices having entries in one of the three classes defined in terms of periodic sequences. Let us start by fixing some notation.

By $(\alpha_n : n \in \mathbb{Z})$ and $(\beta_n : n \in \mathbb{Z})$ we denote N -periodic sequences of real and positive numbers, respectively. For each $k \geq 0$, let us define polynomials $(p_n^{[k]} : n \in \mathbb{N}_0)$ by relations

$$\begin{aligned} p_0^{[k]}(x) &= 1, \quad p_1^{[k]}(x) = \frac{x - \beta_k}{\alpha_k}, \\ \alpha_{n+k-1} p_{n-1}^{[k]}(x) + \beta_{n+k} p_n^{[k]}(x) + \alpha_{n+k} p_{n+1}^{[k]}(x) &= x p_n^{[k]}(x), \quad n \geq 1. \end{aligned}$$

Let

$$\mathfrak{B}_n(x) = \begin{pmatrix} 0 & 1 \\ -\frac{\alpha_{n-1}}{\alpha_n} & \frac{x - \beta_n}{\alpha_n} \end{pmatrix}, \quad \text{and} \quad \mathfrak{X}_n(x) = \prod_{j=n}^{N+n-1} \mathfrak{B}_j(x), \quad n \in \mathbb{Z}$$

where for a sequence of square matrices $(C_n : n_0 \leq n \leq n_1)$ we set

$$\prod_{k=n_0}^{n_1} C_k = C_{n_1} C_{n_1-1} \cdots C_{n_0}.$$

By $\mathfrak{A}$ we denote the Jacobi matrix corresponding to

$$\begin{pmatrix} \beta_0 & \alpha_0 & 0 & 0 & \dots \\ \alpha_0 & \beta_1 & \alpha_1 & 0 & \dots \\ 0 & \alpha_1 & \beta_2 & \alpha_2 & \dots \\ 0 & 0 & \alpha_2 & \beta_3 & \dots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

Let ω be the equilibrium measure corresponding to $\sigma_{\text{ess}}(\mathfrak{A})$. Since $\sigma_{\text{ess}}(\mathfrak{A})$ is a finite union of compact intervals with non-empty interiors (see, e.g., [20, Theorem 5.2.4 and Theorem 5.4.2]), ω is absolutely continuous. Moreover, for x in the interior of $\sigma_{\text{ess}}(\mathfrak{A})$ we obtain

$$\omega'(x) = \frac{1}{N} \sum_{i=0}^{N-1} \frac{|[\mathfrak{X}_i(x)]_{2,1}|}{\pi \sqrt{-\text{discr } \mathfrak{X}_i(x)}} \cdot \frac{1}{\alpha_{i-1}}. \quad (3.1)$$

Indeed, by [24, Proposition 3] (see also Lemma 3.2 below),

$$[\mathfrak{X}_i(x)]_{2,1} = -\frac{\alpha_{i-1}}{\alpha_i} \mathfrak{p}_{N-1}^{[i+1]}(x).$$

Since

$$\mathfrak{X}_{i+1}(x) = (\mathfrak{B}_i(x))(\mathfrak{X}_i(x))(\mathfrak{B}_i(x))^{-1},$$

we have

$$\text{discr } \mathfrak{X}_i(x) = \text{discr } \mathfrak{X}_0(x).$$

Therefore,

$$\frac{1}{N} \sum_{i=0}^{N-1} \frac{|[\mathfrak{X}_i(x)]_{2,1}|}{\pi \sqrt{-\text{discr } \mathfrak{X}_i(x)}} \cdot \frac{1}{\alpha_{i-1}} = \frac{1}{\pi \sqrt{-\text{discr } \mathfrak{X}_0(x)}} \frac{1}{N} \sum_{i=0}^{N-1} \frac{|\mathfrak{p}_{N-1}^{[i+1]}(x)|}{\alpha_i}.$$

Let us recall that the first formula on page 214 of [30] reads

$$\sum_{i=0}^{N-1} \frac{|\mathfrak{p}_{N-1}^{[i+1]}(x)|}{\alpha_i} = |\text{tr } \mathfrak{X}'_0(x)|,$$

hence

$$\frac{1}{N} \sum_{i=0}^{N-1} \frac{|[\mathfrak{X}_i(x)]_{2,1}|}{\pi \sqrt{-\text{discr } \mathfrak{X}_i(x)}} \cdot \frac{1}{\alpha_{i-1}} = \frac{|\text{tr } \mathfrak{X}'_0(x)|}{\pi N \sqrt{-\text{discr } \mathfrak{X}_0(x)}}. \quad (3.2)$$

Now, in view of [30, the formula (3.2)] we obtain (3.1).

3.1 Asymptotically Periodic

Definition 3.1 The Jacobi matrix A associated with $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ has *asymptotically N -periodic entries*, if there are two N -periodic sequences $(\alpha_n : n \in \mathbb{Z})$ and $(\beta_n : n \in \mathbb{Z})$ of positive and real numbers, respectively, such that

- (a) $\lim_{n \rightarrow \infty} |a_n - \alpha_n| = 0$,
- (b) $\lim_{n \rightarrow \infty} |b_n - \beta_n| = 0$.

Let us recall the following lemma.

Lemma 3.2 ([24, Proposition 3]) *Let $(p_n : n \in \mathbb{N}_0)$ be a sequence of orthonormal polynomials associated with $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$. Then for all $n \geq 1$ and $k \geq 1$,*

$$\prod_{j=k}^{n+k-1} B_j(x) = \begin{pmatrix} -\frac{a_{k-1}}{a_k} p_{n-2}^{[k+1]}(x) & p_{n-1}^{[k]}(x) \\ -\frac{a_{k-1}}{a_k} p_{n-1}^{[k+1]}(x) & p_n^{[k]}(x) \end{pmatrix}.$$

Proposition 3.3 *Suppose that A has asymptotically N -periodic entries. Then for each $i \in \{0, 1, \dots, N-1\}$ and $n \geq 0$,*

$$\lim_{k \rightarrow \infty} \left\| \prod_{j=kN+i}^{kN+i+n} B_j(x) - \prod_{j=i}^{i+n} \mathfrak{B}_j(x) \right\| = 0,$$

locally uniformly with respect to $x \in \mathbb{C}$.

Proof Since for each $i \in \{0, 1, \dots, N-1\}$, we have

$$\lim_{k \rightarrow \infty} \|B_{kN+i}(x) - \mathfrak{B}_i(x)\| = 0$$

uniformly on compact subsets of $\mathbb{C}$, the conclusion follows by the continuity of B_n . $\square$

For a Jacobi matrix having asymptotically N -periodic entries, we set

$$X_n(x) = \prod_{j=n}^{N+n-1} B_j(x).$$

Let us denote by $\mathcal{X}_i$ the limit of $(X_{jN+i} : j \in \mathbb{N})$. Then, by Proposition 3.3, we conclude that $\mathcal{X}_i = \mathfrak{X}_i$ for all $i \in \{0, 1, \dots, N-1\}$.

Proposition 3.4 *Suppose that a Jacobi matrix A has asymptotically N -periodic entries. Then for each $i \in \{0, 1, \dots, N-1\}$, and $n \geq 0$,*

$$\lim_{k \rightarrow \infty} p_n^{[kN+i]}(x) = p_n^{[i]}(x), \quad (3.3a)$$

$$\lim_{k \rightarrow \infty} \frac{a_{kN+i}}{\alpha_i} (p_n^{[kN+i]})'(x) = (p_n^{[i]})'(x), \quad (3.3b)$$

$$\lim_{k \rightarrow \infty} \left(\frac{a_{kN+i}}{\alpha_i} \right)^2 (p_n^{[kN+i]})''(x) = (p_n^{[i]})''(x), \quad (3.3c)$$

locally uniformly with respect to $x \in \mathbb{C}$.

Proof Lemma 3.2 together with Proposition 3.3 easily gives (3.3a). Since

$$\lim_{k \rightarrow \infty} \frac{a_{kN+i}}{\alpha_i} = 1,$$

the uniform convergence in (3.3a) entails (3.3b) and (3.3c). $\square$

Corollary 3.5 *Suppose that a Jacobi matrix A has asymptotically N -periodic entries. Then for each $i \in \{0, 1, \dots, N-1\}$,*

$$\lim_{k \rightarrow \infty} \operatorname{tr} (X_{kN+i}(x)) = \operatorname{tr} (\mathfrak{X}_0(x)), \quad (3.4a)$$

$$\lim_{k \rightarrow \infty} \frac{a_{kN+i}}{\alpha_i} \operatorname{tr} (X'_{kN+i}(x)) = \operatorname{tr} (\mathfrak{X}'_0(x)), \quad (3.4b)$$

$$\lim_{k \rightarrow \infty} \left(\frac{a_{kN+i}}{\alpha_i} \right)^2 \operatorname{tr} (X''_{kN+i}(x)) = \operatorname{tr} (\mathfrak{X}''_0(x)), \quad (3.4c)$$

locally uniformly with respect to $x \in \mathbb{C}$.

Proof Since for $x \in \mathbb{C}$,

$$\mathfrak{X}_{i+1}(x) = (\mathfrak{B}_i(x))(\mathfrak{X}_i(x))(\mathfrak{B}_i(x))^{-1},$$

we have

$$\operatorname{tr} (\mathfrak{X}_i^{(s)}(x)) = \operatorname{tr} (\mathfrak{X}_0^{(s)}(x)), \quad \text{for all } s \geq 0. \quad (3.5)$$

By Lemma 3.2,

$$\operatorname{tr} (X_{kN+i}(x)) = p_N^{[kN+i]}(x) - \frac{a_{kN+i-1}}{a_{kN+i}} p_{N-2}^{[kN+i+1]}(x),$$

which together with Proposition 3.4, implies that

$$\lim_{k \rightarrow \infty} \operatorname{tr} (X_{kN+i}(x)) = \operatorname{tr} (\mathfrak{X}_i(x))$$

uniformly on compact subsets of $\mathbb{C}$. Hence, by (3.5) we easily obtain (3.4a).

For $s \in \{1, 2\}$ we write

$$\begin{aligned} & \frac{a_{kN+i-1}}{a_{kN+i}} \left(\frac{a_{kN+i}}{\alpha_i} \right)^s (p_{N-2}^{[kN+i+1]})^{(s)}(x) \\ &= \frac{a_{kN+i-1}}{a_{kN+i}} \left(\frac{a_{kN+i}}{\alpha_i} \right)^s \left(\frac{\alpha_{i+1}}{a_{kN+i+1}} \right)^s \left(\frac{a_{kN+i+1}}{\alpha_{i+1}} \right)^s (p_{N-2}^{[kN+i+1]})^{(s)}(x), \end{aligned}$$

hence, by Proposition 3.4, we obtain

$$\lim_{k \rightarrow \infty} \frac{a_{kN+i-1}}{a_{kN+i}} \left(\frac{a_{kN+i}}{\alpha_i} \right)^s (p_{N-2}^{[kN+i+1]})^{(s)}(x) = \frac{\alpha_{i-1}}{\alpha_i} (p_{N-2}^{[i+1]})^{(s)}(x).$$

Therefore, by Lemma 3.2,

$$\begin{aligned} \lim_{k \rightarrow \infty} \left(\frac{a_{kN+i}}{\alpha_i} \right)^s \operatorname{tr} (X_{kN+i}^{(s)}(x)) &= (p_N^{[i]})^{(s)}(x) - \frac{\alpha_{i-1}}{\alpha_i} (p_{N-2}^{[i+1]})^{(s)}(x) \\ &= \operatorname{tr} (\mathfrak{X}_i^{(s)}(x)), \end{aligned}$$

which finishes the proof. $\square$

3.2 Periodic Modulations

Definition 3.6 We say that the Jacobi matrix A associated with $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ has *N -periodically modulated entries*, if there are two N -periodic sequences $(\alpha_n : n \in \mathbb{Z})$ and $(\beta_n : n \in \mathbb{Z})$ of positive and real numbers, respectively, such that

- (a) $\lim_{n \rightarrow \infty} a_n = \infty$,
- (b) $\lim_{n \rightarrow \infty} \left| \frac{a_{n-1}}{a_n} - \frac{\alpha_{n-1}}{\alpha_n} \right| = 0$,
- (c) $\lim_{n \rightarrow \infty} \left| \frac{b_n}{a_n} - \frac{\beta_n}{\alpha_n} \right| = 0$.

Suppose that A is a Jacobi matrix with N -periodically modulated entries. Observe that, by setting

$$\tilde{a}_n = \frac{a_n}{\alpha_n}, \quad \text{and} \quad \tilde{b}_n = \frac{b_n}{\alpha_n},$$

we obtain

$$\lim_{n \rightarrow \infty} \frac{\tilde{a}_{n-1}}{\tilde{a}_n} = 1, \quad \text{and} \quad \frac{\tilde{b}_n}{\tilde{a}_n} = \frac{b_n}{a_n}.$$

Hence, A is an N -periodic modulation of the Jacobi matrix corresponding to the sequences $(\tilde{a}_n : n \in \mathbb{N}_0)$ $(\tilde{b}_n : n \in \mathbb{N}_0)$ in the usual sense.

Proposition 3.7 *If a Jacobi matrix A has N -periodically modulated entries and $i \in \mathbb{N}$, then*

$$\lim_{n \rightarrow \infty} \frac{\alpha_{n+i}}{\alpha_n} \frac{a_n}{a_{n+i}} = 1.$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+N}} = 1.$$

Proof Since

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} - \frac{\alpha_n}{\alpha_{n+1}} \right| = 0,$$

one has

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+i}} - \frac{\alpha_n}{\alpha_{n+i}} \right| = \lim_{n \rightarrow \infty} \left| \prod_{k=0}^{i-1} \frac{a_{n+k}}{a_{n+k+1}} - \prod_{k=0}^{i-1} \frac{\alpha_{n+k}}{\alpha_{n+k+1}} \right| = 0.$$

Hence, for some $c > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+i}} \frac{\alpha_{n+i}}{\alpha_n} - 1 \right| &= \lim_{n \rightarrow \infty} \frac{\alpha_{n+i}}{\alpha_n} \left| \frac{a_n}{a_{n+i}} - \frac{\alpha_n}{\alpha_{n+i}} \right| \\ &\leq c \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+i}} - \frac{\alpha_n}{\alpha_{n+i}} \right| = 0, \end{aligned}$$

and the proposition follows. $\square$

Proposition 3.8 *Suppose that A has N -periodically modulated entries. Then for each $i \in \{0, 1, \dots, N-1\}$ and $n \geq 0$,*

$$\lim_{k \rightarrow \infty} \left\| \prod_{j=kN+i}^{kN+i+n} B_j(x) - \prod_{j=i}^{i+n} \mathfrak{B}_j(0) \right\| = 0,$$

locally uniformly with respect to $x \in \mathbb{C}$.

Proof Since for each $i \in \{0, 1, \dots, N-1\}$, we have

$$\lim_{k \rightarrow \infty} \|B_{kN+i}(x) - \mathfrak{B}_i(0)\| = 0$$

uniformly on compact subsets of $\mathbb{C}$, the conclusion follows by the continuity of B_n . $\square$

For a Jacobi matrix with N -periodically modulated entries, we set

$$X_n(x) = \prod_{j=n}^{N+n-1} B_j(x).$$

Let us denote by X_i the limit of $(X_{jN+i} : j \in \mathbb{N})$. Then, by Proposition 3.8, we have $X_i(x) = \mathfrak{X}_i(0)$ for all $i \in \{0, 1, \dots, N-1\}$ and $x \in \mathbb{C}$.

Proposition 3.9 *Suppose that a Jacobi matrix A has N -periodically modulated entries. Then for each $i \in \{0, 1, \dots, N-1\}$, and $n \geq 0$,*

$$\lim_{k \rightarrow \infty} p_n^{[kN+i]}(x) = p_n^{[i]}(0), \quad (3.6a)$$

$$\lim_{k \rightarrow \infty} \frac{a_{kN+i}}{\alpha_i} (p_n^{[kN+i]})'(x) = (p_n^{[i]})'(0), \quad (3.6b)$$

$$\lim_{k \rightarrow \infty} \left(\frac{a_{kN+i}}{\alpha_i} \right)^2 (p_n^{[kN+i]})''(x) = (p_n^{[i]})''(0), \quad (3.6c)$$

locally uniformly with respect to $x \in \mathbb{C}$.

Proof By Lemma 3.2 and Proposition 3.8 we obtain that for every $i \geq 0$ and $n \geq 0$,

$$\lim_{k \rightarrow \infty} p_n^{[kN+i]}(x) = p_n^{[i]}(0) \quad (3.7)$$

uniformly on compact subsets of $\mathbb{C}$, which is (3.6a). Next, let us recall that (see e.g. [24, Proposition 2])

$$p'_n(x) = \frac{1}{a_0} \sum_{m=0}^{n-1} \left(p_m(x) p_{n-1}^{[1]}(x) - p_n(x) p_{m-1}^{[1]}(x) \right) p_m(x), \quad (3.8)$$

therefore for every $n \in \mathbb{N}$,

$$\begin{aligned} & (p_n^{[kN+i]})'(x) \\ &= \frac{1}{a_{kN+i}} \sum_{m=0}^{n-1} \left(p_m^{[kN+i]}(x) p_{n-1}^{[kN+i+1]}(x) - p_n^{[kN+i]}(x) p_{m-1}^{[kN+i+1]}(x) \right) p_m^{[kN+i]}(x). \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{a_{kN+i}}{\alpha_i} (p_n^{[kN+i]})'(x) \\ &= \frac{1}{\alpha_i} \sum_{m=0}^{n-1} \left(p_m^{[kN+i]}(x) p_{n-1}^{[kN+i+1]}(x) - p_n^{[kN+i]}(x) p_{m-1}^{[kN+i+1]}(x) \right) p_m^{[kN+i]}(x). \end{aligned} \quad (3.9)$$

By (3.7) the right-hand side of (3.9) tends to

$$\frac{1}{\alpha_i} \sum_{m=0}^{n-1} \left(\mathfrak{p}_m^{[i]}(0) \mathfrak{p}_{n-1}^{[i+1]}(0) - \mathfrak{p}_n^{[i]}(0) \mathfrak{p}_{m-1}^{[i+1]}(0) \right) \mathfrak{p}_m^{[i]}(0).$$

Therefore, by (3.8) we conclude (3.6b). For the proof of (3.6c), let us observe that from (3.9) we get

$$\begin{aligned} & \left(\frac{a_{kN+i}}{\alpha_i} \right)^2 (p_n^{[kN+i]})''(x) \\ &= \frac{1}{\alpha_i} \sum_{m=0}^{n-1} \left(p_m^{[kN+i]}(x) p_{n-1}^{[kN+i+1]}(x) - p_n^{[kN+i]}(x) p_{m-1}^{[kN+i+1]}(x) \right) \frac{a_{kN+i}}{\alpha_i} (p_m^{[kN+i]})'(x) \\ &+ \frac{1}{\alpha_i} \sum_{m=0}^{n-1} \frac{a_{kN+i}}{\alpha_i} \left(p_m^{[kN+i]} p_{n-1}^{[kN+i+1]} - p_n^{[kN+i]} p_{m-1}^{[kN+i+1]} \right)'(x) p_m^{[kN+i]}(x). \end{aligned} \quad (3.10)$$

Therefore, by (3.6b) and (3.7), the first sum in (3.10) approaches to

$$\frac{1}{\alpha_i} \sum_{m=0}^{n-1} \left(\mathfrak{p}_m^{[i]}(0) \mathfrak{p}_{n-1}^{[i+1]}(0) - \mathfrak{p}_n^{[i]}(0) \mathfrak{p}_{m-1}^{[i+1]}(0) \right) (\mathfrak{p}_m^{[i]})'(0). \quad (3.11)$$

In view of (3.6b), for each $m \in \mathbb{N}_0$,

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_{kN+i}}{\alpha_i} (p_m^{[kN+i+1]})'(x) &= \lim_{k \rightarrow \infty} \frac{a_{kN+i}}{\alpha_i} \frac{\alpha_{i+1}}{a_{kN+i+1}} \frac{a_{kN+i+1}}{\alpha_{i+1}} (p_m^{[kN+i+1]})'(x) \\ &= (\mathfrak{p}_m^{[i+1]})'(0). \end{aligned} \quad (3.12)$$

Hence, by (3.6b), (3.12), and (3.7), we obtain

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{a_{kN+i}}{\alpha_i} \left(p_m^{[kN+i]} p_{n-1}^{[kN+i+1]} - p_n^{[kN+i]} p_{m-1}^{[kN+i+1]} \right)'(x) \\ &= \left(\mathfrak{p}_m^{[kN+i]} \mathfrak{p}_{n-1}^{[kN+i+1]} - \mathfrak{p}_n^{[kN+i]} \mathfrak{p}_{m-1}^{[kN+i+1]} \right)'(0). \end{aligned}$$

Consequently, the second sum in (3.10) tends to

$$\frac{1}{\alpha_i} \sum_{m=0}^{n-1} \left(\mathfrak{p}_m^{[i]} \mathfrak{p}_{n-1}^{[i+1]} - \mathfrak{p}_n^{[i]} \mathfrak{p}_{m-1}^{[i+1]} \right)'(0) \mathfrak{p}_m^{[i]}(0). \quad (3.13)$$

Finally, putting (3.11) and (3.13) into (3.10), by (3.8), we obtain (3.6c). This completes the proof. $\square$

By reasoning analogous to the proof of Corollary 3.5 we obtain the following corollary.

Corollary 3.10 Suppose that a Jacobi matrix A has N -periodically modulated entries. Then for each $i \in \{0, 1, \dots, N-1\}$,

$$\lim_{k \rightarrow \infty} \operatorname{tr} (X_{kN+i}(x)) = \operatorname{tr} (\mathfrak{X}_0(0)), \quad (3.14a)$$

$$\lim_{k \rightarrow \infty} \frac{a_{kN+i}}{\alpha_i} \operatorname{tr} (X'_{kN+i}(x)) = \operatorname{tr} (\mathfrak{X}'_0(0)), \quad (3.14b)$$

$$\lim_{k \rightarrow \infty} \left(\frac{a_{kN+i}}{\alpha_i} \right)^2 \operatorname{tr} (X''_{kN+i}(x)) = \operatorname{tr} (\mathfrak{X}''_0(0)), \quad (3.14c)$$

locally uniformly with respect to $x \in \mathbb{C}$.

3.3 A Blend of Bounded and Unbounded Parameters

Definition 3.11 The Jacobi matrix A associated with sequences $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ is an N -periodic blend if there are an asymptotically N -periodic Jacobi matrix $\tilde{A}$ associated with sequences $(\tilde{a}_n : n \in \mathbb{N}_0)$ and $(\tilde{b}_n : n \in \mathbb{N}_0)$, and a sequence of positive numbers $(\tilde{c}_n : n \in \mathbb{N}_0)$, such that

- (i) $\lim_{n \rightarrow \infty} \tilde{c}_n = \infty$, and $\lim_{m \rightarrow \infty} \frac{\tilde{c}_{2m+1}}{\tilde{c}_{2m}} = 1$,
- (ii) $a_{k(N+2)+i} = \begin{cases} \tilde{a}_{kN+i} & \text{if } i \in \{0, 1, \dots, N-1\}, \\ \tilde{c}_{2k} & \text{if } i = N, \\ \tilde{c}_{2k+1} & \text{if } i = N+1, \end{cases}$
- (iii) $b_{k(N+2)+i} = \begin{cases} \tilde{b}_{kN+i} & \text{if } i \in \{0, 1, \dots, N-1\}, \\ 0 & \text{if } i \in \{N, N+1\}. \end{cases}$

Proposition 3.12 For $i \in \{1, 2, \dots, N-1\}$,

$$\lim_{j \rightarrow \infty} B_{j(N+2)+i}(x) = \mathfrak{B}_i(x)$$

locally uniformly with respect to $x \in \mathbb{R}$. Moreover,

$$\lim_{j \rightarrow \infty} B_{((j+1)(N+2))}(x) B_{j(N+2)+N+1}(x) B_{(j(N+2)+N)}(x) = C(x)$$

locally uniformly with respect to $x \in \mathbb{R}$, where

$$C(x) = \begin{pmatrix} 0 & -1 \\ \frac{\alpha_{N-1}}{\alpha_0} & -\frac{2x-\beta_0}{\alpha_0} \end{pmatrix}.$$

Proof The argument is contained in the proof of [27, Corollary 9]. □

For a Jacobi matrix that is an N -periodic blend, we set

$$X_n(x) = \prod_{j=n}^{N+n+1} B_j(x).$$

In view of Proposition 3.12, for $i \in \{1, 2, \dots, N\}$, the sequence $(X_{j(N+2)+i} : j \in \mathbb{N})$ converges to $\mathcal{X}_i$ locally uniformly on $\mathbb{C}$ where

$$\mathcal{X}_i(x) = \left(\prod_{j=1}^{i-1} \mathfrak{B}_j(x) \right) C(x) \left(\prod_{j=i}^{N-1} \mathfrak{B}_j(x) \right). \quad (3.15)$$

We set

$$\Lambda = \{x \in \mathbb{R} : |\operatorname{tr} \mathcal{X}_1(x)| < 2\}.$$

Theorem 3.13 *There are non-empty open and disjoint intervals $(I_j : 1 \leq j \leq N)$ such that*

$$\Lambda = \bigcup_{j=1}^N I_j.$$

Moreover, for $x \in \Lambda$,

$$\omega'(x) = \frac{1}{N\pi\sqrt{-\operatorname{discr} \mathcal{X}_1(x)}} \left(\sum_{i=1}^{N-1} \frac{|[\mathcal{X}_i(x)]_{2,1}|}{\alpha_{i-1}} + 2 \frac{|[\mathcal{X}_N(x)]_{2,1}|}{\alpha_{N-1}} \right)$$

where ω is the equilibrium measure corresponding to $\overline{\Lambda}$.

Proof Let us begin with the case $N = 1$. Then

$$\mathcal{X}_1(x) = \begin{pmatrix} 0 & 1 \\ -1 & \frac{x-\beta_0/2}{\alpha_0/2} \end{pmatrix}.$$

Therefore, by (3.1) one obtains

$$\omega'(x) = \frac{1}{\pi\sqrt{-\operatorname{discr} \mathcal{X}_1(x)}} \frac{|[\mathcal{X}_1(x)]_{2,1}|}{\alpha_0/2},$$

which ends the proof for $N = 1$.

Suppose that $N \geq 2$. For $k \in \mathbb{Z}$ and $i \in \{0, 1, \dots, N-1\}$, we set

$$\tilde{\alpha}_{kN+i} = \alpha_i \cdot \begin{cases} \frac{1}{\sqrt{2}} & \text{if } i \in \{0, N-1\}, \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{\beta}_{kN+i} = \beta_i \cdot \begin{cases} \frac{1}{2} & \text{if } i = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Let

$$\tilde{\mathfrak{B}}_j(x) = \begin{pmatrix} 0 & 1 \\ -\frac{\tilde{\alpha}_{j-1}}{\tilde{\alpha}_j} & \frac{x-\tilde{\beta}_j}{\tilde{\alpha}_j} \end{pmatrix} \quad \text{and} \quad \tilde{\mathfrak{X}}_n(x) = \prod_{j=n}^{N+n-1} \tilde{\mathfrak{B}}_j(x).$$

By (3.1),

$$\tilde{\omega}'(x) = \frac{1}{N\pi\sqrt{-\operatorname{discr} \tilde{\mathfrak{X}}_1(x)}} \sum_{i=0}^{N-1} \frac{|[\tilde{\mathfrak{X}}_i(x)]_{2,1}|}{\tilde{\alpha}_{i-1}} \quad (3.16)$$

where $\tilde{\omega}$ is the equilibrium measure corresponding to the closure of

$$\tilde{\Lambda} = \{x \in \mathbb{R} : \operatorname{discr} \tilde{\mathfrak{X}}_1(x) < 0\}.$$

In particular, $\tilde{\Lambda}$ is the union of N non-empty open and disjoint intervals, see [20, Theorem 5.4.2]. Notice that

$$-\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \tilde{\mathfrak{B}}_0 \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} = C.$$

We next show the following claim. □

Claim 3.14 *If $N \geq 2$, then*

$$\mathcal{X}_1 = -\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \tilde{\mathfrak{X}}_1 \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.17)$$

$$\mathcal{X}_j = -\tilde{\mathfrak{X}}_j, \quad \text{for } i = 2, 3, \dots, N-1, \quad (3.18)$$

$$\mathcal{X}_N = -\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \tilde{\mathfrak{X}}_0 \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.19)$$

For $N = 2$, the identities (3.17) and (3.19) can be checked by direct computation. For $N \geq 3$, we first observe that

$$\mathfrak{B}_1 = \tilde{\mathfrak{B}}_1 \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.20)$$

$$\mathfrak{B}_j = \tilde{\mathfrak{B}}_j, \quad \text{for } i = 2, 3, \dots, N-2, \quad (3.21)$$

$$\mathfrak{B}_{N-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \tilde{\mathfrak{B}}_{N-1}. \quad (3.22)$$

Consequently,

$$\mathcal{X}_1 = C \prod_{j=1}^{N-1} \mathfrak{B}_j = -\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \tilde{\mathfrak{X}}_1 \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly, one can show (3.18) and (3.19).

Now, using (3.17), we easily get

$$\operatorname{discr} \mathcal{X}_1 = \operatorname{discr} \tilde{\mathcal{X}}_1,$$

which implies that $\Lambda = \tilde{\Lambda}$. Hence, $\tilde{\omega} = \omega$. Moreover, by Claim 3.14, we obtain

$$\begin{aligned} |[\tilde{\mathcal{X}}_0(x)]_{2,1}| &= \sqrt{2} |[\mathcal{X}_N(x)]_{2,1}|, \\ |[\tilde{\mathcal{X}}_1(x)]_{2,1}| &= \frac{1}{\sqrt{2}} |[\mathcal{X}_1(x)]_{2,1}|, \\ |[\tilde{\mathcal{X}}_i(x)]_{2,1}| &= |[\mathcal{X}_i(x)]_{2,1}| \quad \text{for } i = 2, 3, \dots, N-1. \end{aligned} \quad (3.23)$$

Therefore, for $x \in \Lambda$, formula (3.16) gives

$$\begin{aligned} \omega'(x) &= \frac{1}{N\pi \sqrt{-\operatorname{discr} \tilde{\mathcal{X}}_1(x)}} \sum_{i=0}^{N-1} \frac{|[\tilde{\mathcal{X}}_i(x)]_{2,1}|}{\tilde{\alpha}_{i-1}} \\ &= \frac{1}{N\pi \sqrt{-\operatorname{discr} \mathcal{X}_1(x)}} \left(2 \frac{|[\mathcal{X}_N(x)]_{2,1}|}{\alpha_{N-1}} + \sum_{i=1}^{N-1} \frac{|[\mathcal{X}_i(x)]_{2,1}|}{\alpha_{i-1}} \right) \end{aligned}$$

which finishes the proof. $\square$

Corollary 3.15

$$\sum_{i=1}^{N-1} \frac{|[\mathcal{X}_i(x)]_{2,1}|}{\alpha_{i-1}} + 2 \frac{|[\mathcal{X}_N(x)]_{2,1}|}{\alpha_{N-1}} = |\operatorname{tr} \mathcal{X}'_1(x)|.$$

Proof In view of [30, formula (3.2)],

$$\sum_{i=0}^{N-1} \frac{|[\tilde{\mathcal{X}}_i(x)]_{2,1}|}{\tilde{\alpha}_{i-1}} = |\operatorname{tr} \tilde{\mathcal{X}}'_1(x)|.$$

By Claim 3.14,

$$\operatorname{tr} \tilde{\mathcal{X}}'_1(x) = \operatorname{tr} \mathcal{X}'_1(x),$$

which together with (3.23) concludes the proof. $\square$

Remark 3.16 We want to emphasize that Theorem 3.13 says that Λ is the disjoint union of *exactly* N non-empty open intervals. This should be compared with the discussion at the beginning of Section 5 in [1].

Proposition 3.17 *Suppose that a Jacobi matrix A is an N -periodic blend. Then for each $i \in \{1, 2, \dots, N\}$ and $n \geq 0$,*

$$\lim_{k \rightarrow \infty} p_n^{[k(N+2)+i]}(x) = [\mathcal{X}_i(x)]_{2,2}, \quad (3.24a)$$

$$\lim_{k \rightarrow \infty} \frac{a_{k(N+2)+i}}{\alpha_i} (p_n^{[k(N+2)+i]})'(x) = [\mathcal{X}'_i(x)]_{2,2}, \quad (3.24b)$$

$$\lim_{k \rightarrow \infty} \left(\frac{a_{k(N+2)+i}}{\alpha_i} \right)^2 (p_n^{[k(N+2)+i]})''(x) = [\mathcal{X}''_i(x)]_{2,2}, \quad (3.24c)$$

locally uniformly with respect to $x \in \mathbb{C}$.

Proof Lemma 3.2 together with Proposition 3.3 easily gives (3.24a). Since

$$\lim_{k \rightarrow \infty} \frac{a_{k(N+2)+i}}{\alpha_i} = 1,$$

the uniform convergence in (3.24a) entails (3.24b) and (3.24c). $\square$

Corollary 3.18 *Suppose that a Jacobi matrix A is an N -periodic blend. Then for each $i \in \{1, 2, \dots, N\}$,*

$$\lim_{k \rightarrow \infty} \operatorname{tr} (X_{k(N+2)+i}(x)) = \operatorname{tr} (\mathcal{X}_1(x)), \quad (3.25a)$$

$$\lim_{k \rightarrow \infty} \frac{a_{k(N+2)+i}}{\alpha_i} \operatorname{tr} (X'_{k(N+2)+i}(x)) = \operatorname{tr} (\mathcal{X}'_1(x)), \quad (3.25b)$$

$$\lim_{k \rightarrow \infty} \left(\frac{a_{k(N+2)+i}}{\alpha_i} \right)^2 \operatorname{tr} (X''_{k(N+2)+i}(x)) = \operatorname{tr} (\mathcal{X}''_1(x)), \quad (3.25c)$$

locally uniformly with respect to $x \in \mathbb{C}$.

4 Christoffel Functions for $\mathcal{D}_r, r \geq 1$

4.1 General Case

In this section we determine the asymptotic behavior of the Christoffel–Darboux kernel (2.3) on the diagonal. We start by showing the following lemma. In its proof we use the following well-known fact, called the Stolz–Cesàro theorem: if $(c_n : n \in \mathbb{N})$ is a sequence of real numbers strictly increasing and approaching infinity, and $(f_n : n \in \mathbb{N})$ is a sequence of real functions on a compact set K , such that

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}(x) - f_n(x)}{c_{n+1} - c_n} = L(x)$$

uniformly with respect to $x \in K$, then

$$\lim_{n \rightarrow \infty} \frac{f_n(x)}{c_n} = L(x)$$

uniformly with respect to $x \in K$. The classical scalar version (see, e.g. [15, Section 3.1.7]) quickly generalizes to the form above.

Lemma 4.1 *Let $(\gamma_n : n \geq 0)$ be a sequence of positive numbers and $(\theta_n : n \geq 0)$ be a sequence of continuous functions on some compact set $K \subset \mathbb{R}^d$ with values in $(0, 2\pi)$. Suppose that there is a function $\theta : K \rightarrow (0, 2\pi)$ such that*

$$\lim_{n \rightarrow \infty} \theta_n(x) = \theta(x)$$

uniformly with respect to $x \in K$. Then there is $c > 0$ such that for all $x \in K$ and $n \in \mathbb{N}$,

$$\left| \sum_{k=0}^n \gamma_k \exp \left(i \sum_{j=0}^k \theta_j(x) \right) \right| \leq c \left(1 + \sum_{k=0}^{n-1} \left(|\gamma_{k+1} - \gamma_k| + \gamma_{k+1} |\theta_{k+1}(x) - \theta(x)| \right) \right). \quad (4.1)$$

In particular, if

$$\sum_{k=0}^{\infty} \gamma_k = \infty, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\gamma_{n-1}}{\gamma_n} = 1,$$

then

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\gamma_k}{\sum_{j=0}^n \gamma_j} \exp \left(i \sum_{j=0}^k \theta_j(x) \right) = 0 \quad (4.2)$$

uniformly with respect to $x \in K$.

Proof Let us observe that

$$\sum_{k=0}^n \gamma_k \exp \left(i(k+1)\theta(x) + i\eta_k(x) \right) = \sum_{k=0}^n \gamma_k e^{i\eta_k(x)} (s_{k+1}(x) - s_k(x))$$

where

$$s_k(x) = \sum_{j=0}^k e^{ij\theta(x)} = e^{ik\theta(x)/2} \frac{\sin((k+1)\theta(x)/2)}{\sin(\theta(x)/2)},$$

and

$$\eta_k(x) = \sum_{j=0}^k (\theta_j(x) - \theta(x)).$$

Hence, by the summation by parts we get

$$\begin{aligned} & \sum_{k=0}^n \gamma_k \exp \left(i(k+1)\theta(x) + i\eta_k(x) \right) \\ &= \gamma_n e^{i\eta_n(x)} s_{n+1}(x) - \gamma_0 e^{i\eta_0(x)} s_0(x) - \sum_{k=0}^{n-1} \left(\gamma_{k+1} e^{i\eta_{k+1}(x)} - \gamma_k e^{i\eta_k(x)} \right) s_{k+1}(x). \end{aligned} \quad (4.3)$$

Since

$$\sup_{k \in \mathbb{N}} \sup_{x \in K} |s_k(x)| < \infty,$$

the first term in (4.3) is bounded by a constant multiple of γ_n . Moreover,

$$|\gamma_n e^{i\eta_n(x)} s_{n+1}(x)| \leq c \left(\sum_{k=0}^{n-1} |\gamma_{k+1} - \gamma_k| + 1 \right)$$

because

$$\gamma_n \leq \sum_{k=0}^{n-1} |\gamma_{k+1} - \gamma_k| + \gamma_0.$$

We treat the second term in (4.3) similarly. Finally, the third term in (4.3) can be bounded by a constant multiple of

$$\sum_{k=0}^{n-1} |\gamma_{k+1} - \gamma_k| + \sum_{k=0}^{n-1} \gamma_{k+1} |\eta_{k+1}(x) - \eta_k(x)|$$

which together with the identity

$$\eta_{k+1}(x) - \eta_k(x) = \theta_{k+1}(x) - \theta(x),$$

entails that (4.3) is bounded by a constant multiple of

$$1 + \sum_{k=0}^{n-1} |\gamma_{k+1} - \gamma_k| + \sum_{k=0}^{n-1} \gamma_{k+1} |\theta_{k+1}(x) - \theta(x)|$$

proving (4.1).

Lastly, we observe that by the Stolz–Cesàro theorem,

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} |\gamma_{k+1} - \gamma_k|}{\sum_{k=0}^n \gamma_k} = \lim_{n \rightarrow \infty} \frac{|\gamma_n - \gamma_{n-1}|}{\gamma_n} = 0.$$

Similarly, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} \frac{\gamma_{j+1}}{\sum_{k=0}^n \gamma_k} \sup_{x \in K} |\theta_{j+1}(x) - \theta(x)| &= \lim_{n \rightarrow \infty} \frac{\gamma_n \cdot \sup_{x \in K} |\theta_n(x) - \theta(x)|}{\gamma_n} \\ &= \lim_{n \rightarrow \infty} \sup_{x \in K} |\theta_n(x) - \theta(x)| = 0, \end{aligned}$$

thus

$$\lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} \frac{\gamma_{j+1}}{\sum_{k=0}^n \gamma_k} |\theta_{j+1}(x) - \theta(x)| = 0$$

uniformly with respect to $x \in K$ proving (4.2), and the lemma follows. $\square$

The following theorem has been proved in [27].

Theorem 4.2 [27, Theorem C] *Let N and r be positive integers and $i \in \{0, 1, \dots, N-1\}$. Suppose that K is a compact interval contained in*

$$\Lambda = \left\{ x \in \mathbb{R} : \lim_{j \rightarrow \infty} \text{discr } X_{jN+i}(x) \text{ exists and is negative} \right\}.$$

Assume that

$$\lim_{j \rightarrow \infty} \frac{a_{(j+1)N+i-1}}{a_{jN+i-1}} = 1$$

and

$$(X_{jN+i} : j \in \mathbb{N}) \in \mathcal{D}_r(K, \text{GL}(2, \mathbb{R})).$$

Suppose that X is the limit of $(X_{jN+i} : j \in \mathbb{N})$. Then there is a probability measure ν such that $(p_n : n \in \mathbb{N}_0)$ are orthonormal in $L^2(\mathbb{R}, \nu)$, which is purely absolutely continuous with continuous and positive density ν' on K . Moreover, there are $M > 0$ and a real continuous function $\eta : K \rightarrow \mathbb{R}$ such that for all $k \geq M$,

$$\lim_{k \rightarrow \infty} \sup_{x \in K} \left| \sqrt{a_{(k+1)N+i-1}} p_{kN+i}(x) - \sqrt{\frac{2|[X(x)]_{2,1}|}{\pi \nu'(x) \sqrt{-\text{discr } X(x)}}} \sin \left(\sum_{j=M+1}^k \theta_{jN+i}(x) + \eta(x) \right) \right| = 0$$

where

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |\theta_n(x) - \arccos(\tfrac{1}{2} \text{tr } \mathfrak{X}(x))| = 0. \quad (4.4)$$

The proof of the following proposition is inspired by [2].

Proposition 4.3 *Under the hypotheses of Theorem 4.2, we have*

$$\inf_{n \in \mathbb{N}} a_n > 0. \quad (4.5)$$

Proof First, let us consider the case when the moment problem for ν is indeterminate, that is

$$\sum_{n=0}^{\infty} \frac{1}{a_n} < \infty. \quad (4.6)$$

But (4.6) implies that

$$\lim_{n \rightarrow \infty} a_n = \infty,$$

which easily gives (4.5).

Assume now that the moment problem for ν is determinate. By Theorem 4.2, the measure ν has non-trivial absolutely continuous part. To obtain a contradiction, suppose that there is a strictly increasing sequence $(L_j : j \in \mathbb{N}_0)$ such that

$$\lim_{j \rightarrow \infty} a_{L_j} = 0.$$

Without loss of generality we may assume that

$$\sum_{j=0}^{\infty} a_{L_j} < \infty. \quad (4.7)$$

The Jacobi matrix (2.1) can be written in the following block-form

$$\mathcal{A} = \begin{pmatrix} J_0 & a_{L_0} P_0 & 0 & 0 & \dots \\ a_{L_0} P_0^t & J_1 & a_{L_1} P_1 & 0 & \dots \\ 0 & a_{L_1} P_1^t & J_2 & a_{L_2} P_2 & \dots \\ 0 & 0 & a_{L_2} P_2^t & J_3 & \dots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

where each J_i is a finite Jacobi matrix with dimension

$$\begin{cases} L_0 & \text{if } i = 0, \\ L_i - L_{i-1} & \text{otherwise,} \end{cases}$$

and each P_i is a rectangular matrix

$$P_i = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

Let

$$\mathcal{B} = \text{diag}(J_0, J_1, J_2, \dots).$$

By A and B we denote the restrictions of $\mathcal{A}$ and $\mathcal{B}$ to the maximal domains, respectively, that is

$$\text{Dom}(A) = \{x \in \ell^2 : \mathcal{A}x \in \ell^2\},$$

and

$$\text{Dom}(B) = \{x \in \ell^2 : \mathcal{B}x \in \ell^2\}.$$

The determinacy of the moment problem for ν is equivalent to A being self-adjoint. Moreover,

$$\nu(\cdot) = \langle E_A(\cdot)\delta_0, \delta_0 \rangle_{\ell^2} \quad (4.8)$$

where E_A is the spectral resolution of A and δ_0 is the sequence having 1 on the zero position and zero elsewhere. In view of (4.7), the operator $A - B$ is self-adjoint and belongs to the trace class. Hence, by the Kato–Rosenblum theorem (see, e.g., [17, Theorem 9.29])

$$\sigma_{\text{ac}}(A) = \sigma_{\text{ac}}(B).$$

Since B is unitary equivalent to an operator acting by the multiplication by a real-valued sequence, B has only discrete spectrum. Therefore,

$$\sigma_{\text{ac}}(A) = \emptyset,$$

and consequently, by (4.8), the measure ν has no non-trivial absolutely continuous part, which leads to the contradiction. $\square$

For $i \in \{0, 1, \dots, N-1\}$ and $n \in \mathbb{N}$ we set

$$K_{i;n}(x, y) = \sum_{j=0}^n p_{jN+i}(x) p_{jN+i}(y), \quad x, y \in \mathbb{R},$$

and

$$\rho_{i;n} = \sum_{j=0}^n \frac{1}{a_{jN+i}}.$$

Let us recall that the Carleman condition (2.2) implies that there is a unique probability measure μ such that $(p_n : n \in \mathbb{N}_0)$ are orthonormal in $L^2(\mathbb{R}, \mu)$.

Theorem 4.4 *Let N and r be positive integers and $i \in \{0, 1, \dots, N-1\}$. Suppose that K is a compact interval with non-empty interior contained in*

$$\Lambda = \left\{ x \in \mathbb{R} : \lim_{j \rightarrow \infty} \text{discr } X_{jN+i}(x) \text{ exists and is negative} \right\}.$$

Assume that

$$\lim_{j \rightarrow \infty} \frac{a_{(j+1)N+i-1}}{a_{jN+i-1}} = 1$$

and

$$(X_{jN+i} : j \in \mathbb{N}) \in \mathcal{D}_r(K, \text{GL}(2, \mathbb{R})).$$

Suppose that X is the limit of $(X_{jN+i} : j \in \mathbb{N})$. If

$$\sum_{j=0}^{\infty} \frac{1}{a_{jN+i-1}} = \infty,$$

then

$$K_{i;n}(x, x) = \frac{|[X(x)]_{2,1}|}{\pi \mu'(x) \sqrt{-\text{discr } X(x)}} \rho_{i-1;n+1} + E_{i;n}(x) \quad (4.9)$$

where

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{i-1;n+1}} \sup_{x \in K} |E_{i;n}(x)| = 0.$$

Proof Fix a compact interval K with non-empty interior contained in Λ . In view of Theorem 4.2, there is $M > 0$ such that for all $k \geq M$,

$$a_{(k+1)N+i-1} p_{kN+i}^2(x) = \frac{2|[X(x)]_{2,1}|}{\pi \mu'(x) \sqrt{-\text{discr } X(x)}} \sin^2 \left(\eta(x) + \sum_{j=M+1}^k \theta_{jN+i}(x) \right) + E_{kN+i}(x)$$

where

$$\lim_{k \rightarrow \infty} \sup_{x \in K} |E_{kN+i}(x)| = 0.$$

Since $2 \sin^2(x) = 1 - \cos(2x)$, we have

$$\begin{aligned} \sum_{k=M}^n p_{kN+i}^2(x) &= \frac{|[\mathcal{X}(x)]_{2,1}|}{\pi \mu'(x) \sqrt{-\operatorname{discr} \mathcal{X}(x)}} \sum_{k=M}^n \frac{1}{a_{(k+1)N+i-1}} \left(1 - \cos \left(2\eta(x) + 2 \sum_{j=M+1}^k \theta_{jN+i}(x) \right) \right) \\ &\quad + \sum_{k=M}^n \frac{1}{a_{(k+1)N+i-1}} E_{kN+i}(x). \end{aligned}$$

Notice that the Stolz–Cesàro theorem gives

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{i-1;n+1}} \sum_{j=M}^n \frac{1}{a_{(j+1)N+i-1}} E_{jN+i}(x) = \lim_{n \rightarrow \infty} E_{nN+i}(x) = 0$$

uniformly with respect to $x \in K$. Moreover, by Lemma 4.1

$$\frac{1}{\rho_{i-1;n+1}} \sum_{k=M}^N \frac{1}{a_{(k+1)N+i-1}} \cos \left(2\eta(x) + 2 \sum_{j=M+1}^k \theta_{jN+i}(x) \right) = 0.$$

Finally, since there is $c > 0$ such that

$$\sup_{x \in K} \sum_{k=0}^{M-1} p_{kN+i}^2(x) \leq c,$$

we conclude that

$$\lim_{n \rightarrow \infty} \left| \frac{1}{\rho_{i-1;n+1}} K_{i;n}(x, x) - \frac{|[\mathcal{X}(x)]_{2,1}|}{\pi \mu'(x) \sqrt{-\operatorname{discr} \mathcal{X}(x)}} \right| = 0, \quad (4.10)$$

which completes the proof. $\square$

Remark 4.5 In view of [27, Proposition 7], for each compact set $K \subset \mathbb{R}$ with non-empty interior there is $c > 0$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned} &\sum_{j=n}^{\infty} \sup_{x \in K} \|X_{(j+1)N+i}(x) - X_{jN+i}(x)\| \\ &\leq c \sum_{j=n}^{\infty} \left(\left| \frac{a_{(j+1)N+i-1}}{a_{(j+1)N+i}} - \frac{a_{jN+i-1}}{a_{jN+i}} \right| + \left| \frac{b_{(j+1)N+i}}{a_{(j+1)N+i}} - \frac{b_{jN+i}}{a_{jN+i}} \right| + \left| \frac{1}{a_{(j+1)N+i}} - \frac{1}{a_{jN+i}} \right| \right). \end{aligned}$$

Moreover, the right-hand side is comparable to a constant multiple of

$$\sum_{j=n}^{\infty} \sup_{x \in K} \|B_{(j+1)N+i}(x) - B_{jN+i}(x)\|.$$

4.2 Application to the Classes

We are now ready to prove the main theorem of this section.

Theorem 4.6 *Let A be a Jacobi matrix with N -periodically modulated entries. Suppose that there is $r \geq 1$ such that for every $i \in \{0, 1, \dots, N-1\}$,*

$$\left(\frac{a_{kN+i-1}}{a_{kN+i}} : k \in \mathbb{N} \right), \left(\frac{b_{kN+i}}{a_{kN+i}} : k \in \mathbb{N} \right), \left(\frac{1}{a_{kN+i}} : k \in \mathbb{N} \right) \in \mathcal{D}_r,$$

and

$$\sum_{n=0}^{\infty} \frac{1}{a_n} = \infty. \quad (4.11)$$

If $|\operatorname{tr} \mathfrak{X}_0(0)| < 2$, then

$$K_n(x, x) = \frac{\omega'(0)}{\mu'(x)} \rho_n + E_n(x)$$

where

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n} E_n(x) = 0$$

locally uniformly with respect to $x \in \mathbb{R}$, where ω is the equilibrium measure corresponding to $\sigma_{\text{ess}}(\mathfrak{A})$, with $\mathfrak{A}$ being the Jacobi matrix associated with $(\alpha_n : n \in \mathbb{N}_0)$ and $(\beta_n : n \in \mathbb{N}_0)$, and

$$\rho_n = \sum_{j=0}^n \frac{\alpha_j}{a_j}.$$

Proof Let K be a compact interval in $\mathbb{R}$ with non-empty interior. Observe that, by Remark 4.5, for each $i \in \{0, 1, \dots, N-1\}$ the sequence $(X_{jN+i} : j \geq 0)$ belongs to $\mathcal{D}_r(K, \operatorname{GL}(2, \mathbb{R}))$. Moreover, by Proposition 3.8 we have

$$\lim_{j \rightarrow \infty} X_{jN+i}(x) = \mathfrak{X}_i(0),$$

which together with $\operatorname{discr} \mathfrak{X}_i(0) < 0$ implies that $\Lambda = \mathbb{R}$. Since for each $n, n' \in \mathbb{N}_0$, by Proposition 3.7, we have

$$\lim_{j \rightarrow \infty} \frac{a_{jN+n'}}{a_{jN+n}} = \frac{\alpha_{n'}}{\alpha_n},$$

by the Stolz–Cesàro theorem

$$\begin{aligned}\lim_{j \rightarrow \infty} \frac{\rho_{i';j}}{\rho_{jN+i}} &= \lim_{j \rightarrow \infty} \frac{\frac{1}{a_{jN+i'}}}{\sum_{k=1}^N \frac{\alpha_{i+k}}{a_{jN+i+k}}} \\ &= \frac{1}{N\alpha_{i'}}.\end{aligned}\quad (4.12)$$

Consequently, the Carleman condition (4.11) implies that

$$\lim_{k \rightarrow \infty} \sum_{j=0}^k \frac{1}{a_{jN+i}} = \infty,$$

for each $i \in \{0, 1, \dots, N-1\}$. Thus, by Theorem 4.4, we obtain

$$\lim_{n \rightarrow \infty} \sup_{x \in K} \left| \frac{1}{\rho_{i;n}} K_{i;n}(x, x) - \frac{|[\mathfrak{X}_i(0)]_{2,1}|}{\pi \mu'(x) \sqrt{-\operatorname{discr} \mathfrak{X}_i(0)}} \cdot \frac{\alpha_i}{\alpha_{i-1}} \right| = 0. \quad (4.13)$$

Fix $i \in \{0, 1, \dots, N-1\}$ and consider $n = kN + i$ for $k \in \mathbb{N}_0$. We write

$$K_{kN+i}(x, x) = \sum_{i'=0}^{N-1} K_{i';k}(x, x) + \sum_{i'=i+1}^{N-1} (K_{i';k-1}(x, x) - K_{i';k}(x, x)).$$

Observe that

$$\sup_{x \in K} |K_{i';k}(x, x) - K_{i';k-1}(x, x)| = \sup_{x \in K} p_{kN+i'}^2(x) \leq c,$$

hence, by (4.13),

$$\lim_{k \rightarrow \infty} \frac{1}{\rho_{kN+i}} K_{kN+i}(x, x) = \sum_{i'=0}^{N-1} \frac{|[\mathfrak{X}_{i'}(0)]_{2,1}|}{\pi \mu'(x) \sqrt{-\operatorname{discr} \mathfrak{X}_{i'}(0)}} \cdot \frac{\alpha_{i'}}{\alpha_{i'-1}} \cdot \lim_{k \rightarrow \infty} \frac{\rho_{i';k}}{\rho_{kN+i}}. \quad (4.14)$$

Since

$$\mathfrak{X}_{i+1}(x) = (\mathfrak{B}_i(x))(\mathfrak{X}_i(x))(\mathfrak{B}_i(x))^{-1},$$

we immediately get

$$\operatorname{discr} \mathfrak{X}_{i'}(x) = \operatorname{discr} \mathfrak{X}_0(x). \quad (4.15)$$

Finally, putting (4.12) and (4.15) into (4.14), we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{\rho_{kN+i}} K_{kN+i}(x, x) &= \frac{1}{\pi \mu'(x)} \sum_{i'=0}^{N-1} \frac{|[\mathfrak{X}_{i'}(0)]_{2,1}|}{\sqrt{-\operatorname{discr} \mathfrak{X}_{i'}(0)}} \frac{1}{N \alpha_{i'-1}} \\ &= \frac{1}{N \pi \mu'(x) \sqrt{-\operatorname{discr} \mathfrak{X}_0(0)}} \sum_{i'=0}^{N-1} \frac{|[\mathfrak{X}_{i'}(0)]_{2,1}|}{\alpha_{i'-1}} \end{aligned}$$

which together with (3.1) completes the proof. $\square$

The following theorem has essentially the same proof as Theorem 4.6.

Theorem 4.7 *Let A be a Jacobi matrix with asymptotically N -periodic entries. Suppose that there is $r \geq 1$ such that for every $i \in \{0, 1, \dots, N-1\}$,*

$$\left(\frac{a_{kN+i-1}}{a_{kN+i}} : k \in \mathbb{N} \right), \left(\frac{b_{kN+i}}{a_{kN+i}} : k \in \mathbb{N} \right), \left(\frac{1}{a_{kN+i}} : k \in \mathbb{N} \right) \in \mathcal{D}_r. \quad (4.16)$$

Let K be a compact interval with non-empty interior contained in

$$\Lambda = \{x \in \mathbb{R} : |\operatorname{tr} \mathfrak{X}_0(x)| < 2\}.$$

Then

$$K_n(x, x) = \frac{\omega'(x)}{\mu'(x)} \rho_n + E_n(x)$$

with

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n} \sup_{x \in K} |E_n(x)| = 0$$

where ω is the equilibrium measure corresponding to $\sigma_{\text{ess}}(\mathfrak{A})$, with $\mathfrak{A}$ being the Jacobi matrix associated with $(\alpha_n : n \in \mathbb{N}_0)$ and $(\beta_n : n \in \mathbb{N}_0)$, and

$$\rho_n = \sum_{j=0}^n \frac{\alpha_j}{a_j}.$$

Remark 4.8 If the Jacobi matrix A has asymptotically N -periodic entries, then the condition (4.16) is equivalent to

$$(a_n : n \in \mathbb{N}_0), (b_n : n \in \mathbb{N}_0) \in \mathcal{D}_r.$$

To see this, let us observe that

$$\inf_{n \in \mathbb{N}_0} a_n > 0.$$

Hence, by [27, Corollary 2]

$$\left(\frac{1}{a_n} : n \in \mathbb{N}_0\right) \in \mathcal{D}_r,$$

thus by [27, Corollary 1]

$$\left(\frac{a_{n-1}}{a_n} : n \in \mathbb{N}\right), \left(\frac{b_n}{a_n} : n \in \mathbb{N}_0\right) \in \mathcal{D}_r.$$

Analogously, one can prove the opposite implication.

Remark 4.9 If the Jacobi matrix A has asymptotically N -periodic entries, then

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{n} = 1. \quad (4.17)$$

Indeed, by the Stolz–Cesàro theorem, we have

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{n} = \lim_{n \rightarrow \infty} \frac{\alpha_n}{a_n} = 1.$$

Let us recall that in this case $\text{supp}(\mu)$ is compact and μ' is continuous and positive in the interior of $\text{supp}(\mu)$. Thus, in view of (4.17), Theorem 4.7 follows from [28, Theorem 1]. We want to point out that our approach is completely different.

Theorem 4.10 *Let A be a Jacobi matrix that is an N -periodic blend. Suppose that there is $r \geq 1$ such that for every $i \in \{0, 1, \dots, N-1\}$,*

$$\left(\frac{1}{a_{j(N+2)+i}} : j \in \mathbb{N}_0\right), \left(\frac{b_{j(N+2)+i}}{a_{j(N+2)+i}} : j \in \mathbb{N}_0\right) \in \mathcal{D}_r,$$

and

$$\left(\frac{1}{a_{j(N+2)+N}} : j \in \mathbb{N}_0\right), \left(\frac{1}{a_{j(N+2)+N+1}} : j \in \mathbb{N}_0\right), \left(\frac{a_{j(N+2)+N}}{a_{j(N+2)+N+1}} : j \in \mathbb{N}_0\right) \in \mathcal{D}_r.$$

Let K be a compact subset with non-empty interior contained in

$$\Lambda = \{x \in \mathbb{R} : |\text{tr } X_1(x)| < 2\}$$

where X_1 is the limit of $(X_{j(N+2)+1} : j \in \mathbb{N}_0)$. Then

$$K_n(x, x) = \frac{\omega'(x)}{\mu'(x)} \rho_n + E_n(x)$$

where ω is the equilibrium measure corresponding to $\overline{\Lambda}$, and

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n} \sup_{x \in K} |E_n(x)| = 0,$$

with

$$\rho_n = \sum_{i=0}^{N-1} \sum_{\substack{1 \leq m \leq n \\ m \equiv i \pmod{N+2}}} \frac{\alpha_i}{a_m}.$$

Proof Let K be a compact set with non-empty interior contained in Λ . Fix $i \in \{1, 2, \dots, N\}$. Let us recall that the sequence $(X_{j(N+2)+i} : j \in \mathbb{N})$ converges to X_i . Moreover, we claim the following holds.

Claim 4.11 *The sequence $(X_{j(N+2)+i} : j \in \mathbb{N})$ belongs to $\mathcal{D}_r(K, \text{GL}(2, \mathbb{R}))$.*

For the proof, let us observe that if $i \in \{0, 1, \dots, N-1\}$,

$$\inf_{j \in \mathbb{N}} \frac{1}{a_{j(N+2)+i}} > 0,$$

thus, by [27, Corollary 2],

$$(a_{j(N+2)+i} : j \in \mathbb{N}_0) \in \mathcal{D}_r,$$

and consequently, by [27, Corollary 1(i)],

$$(b_{j(N+2)+i} : j \in \mathbb{N}_0) \in \mathcal{D}_r.$$

Moreover,

$$\left(\frac{1}{a_{j(N+2)+N} \cdot a_{j(N+2)+N+1}} : j \in \mathbb{N}_0 \right) \in \mathcal{D}_r.$$

Since

$$\lim_{j \rightarrow \infty} \frac{a_{j(N+2)+N}}{a_{j(N+2)+N+1}} = 1,$$

by [27, Corollary 1], we get

$$\left(\frac{a_{j(N+2)+N+1}}{a_{j(N+2)+N}} : j \in \mathbb{N}_0 \right) \in \mathcal{D}_r.$$

Now, the conclusion follows from the proof of [27, Corollary 9].

Since

$$\lim_{k \rightarrow \infty} \sum_{j=0}^k \frac{1}{a_{j(N+2)+i}} = \infty,$$

and, in view of (3.15),

$$\text{discr } \mathcal{X}_i = \text{discr } \mathcal{X}_1,$$

therefore, by Theorem 4.4, we obtain

$$K_{i;k}(x, x) = \frac{|[\mathcal{X}_i(x)]_{2,1}|}{\pi \mu'(x) \sqrt{-\text{discr } \mathcal{X}_i(x)}} \rho_{i-1;k} + E_{i;k}(x) \quad (4.18)$$

where

$$\lim_{k \rightarrow \infty} \frac{1}{\rho_{i-1;k}} \sup_{x \in K} |E_{i;k}(x)| = 0.$$

Next, we show

$$\lim_{k \rightarrow \infty} \sup_{x \in K} \frac{1}{\rho_{N-1;k}} |K_{0;k}(x, x) - K_{N;k}(x, x)| = 0. \quad (4.19)$$

and

$$\lim_{k \rightarrow \infty} \sup_{x \in K} \frac{1}{\rho_{N-1;k}} |K_{N+1;k}(x, x)| = 0. \quad (4.20)$$

First, we prove the following claim.

Claim 4.12 *There is $c > 0$ such that for all $k \in \mathbb{N}_0$,*

$$\sup_{x \in K} |p_{k(N+2)+N+1}(x)| \leq c \frac{1}{a_{k(N+2)+N+1}}, \quad (4.21)$$

and

$$\sup_{x \in K} |p_{k(N+2)+N}(x) + p_{(k+1)(N+2)}(x)| \leq c \left(\left| 1 - \frac{a_{k(N+2)+N}}{a_{k(N+2)+N+1}} \right| + \frac{1}{a_{k(N+2)+N+1}^2} \right). \quad (4.22)$$

For the proof, let us notice that [27, Theorem A] with $i = 1$, implies

$$\sup_{x \in K} a_{k(N+2)} (p_{k(N+2)}^2(x) + p_{k(N+2)+1}^2(x)) \leq c.$$

Since $(a_{k(N+2)} : k \in \mathbb{N}_0)$ is bounded, by [27, Corollary 9] with $i = 1$,

$$\sup_{k \in \mathbb{N}} \sup_{x \in K} |p_{k(N+2)+1}(x)| \leq c, \quad (4.23)$$

consequently we get

$$\sup_{k \in \mathbb{N}} \sup_{x \in K} |p_{k(N+2)}(x)| \leq c. \quad (4.24)$$

Analogous reasoning for $i = N$ shows that

$$\sup_{k \in \mathbb{N}} \sup_{x \in K} |p_{k(N+2)+N}(x)| \leq c. \quad (4.25)$$

Recall that we have the following recurrence relation

$$p_{k(N+2)+N+1}(x) = \frac{x - b_{(k+1)(N+2)}}{a_{k(N+2)+N+1}} p_{(k+1)(N+2)}(x) - \frac{a_{(k+1)(N+2)}}{a_{k(N+2)+N+1}} p_{(k+1)(N+2)+1}(x).$$

Therefore, by (4.23) and (4.24), we easily get (4.21). Moreover, we have

$$p_{(k+1)(N+2)}(x) = \frac{x}{a_{k(N+2)+N+1}} p_{k(N+2)+N+1}(x) - \frac{a_{k(N+2)+N}}{a_{k(N+2)+N+1}} p_{k(N+2)+N}(x),$$

thus

$$\begin{aligned} |p_{(k+1)(N+2)}(x) + p_{k(N+2)+N}(x)| &\leq \frac{|x|}{a_{k(N+2)+N+1}} |p_{k(N+2)+N+1}(x)| \\ &\quad + \left| 1 - \frac{a_{k(N+2)+N}}{a_{k(N+2)+N+1}} \right| |p_{k(N+2)+N}(x)|, \end{aligned}$$

which together with (4.21) and (4.25) entails (4.22).

Now using Claim 4.12 together with (4.24) and (4.25), we easily see that

$$\begin{aligned} |K_{0;k}(x, x) - K_{N;k}(x, x)| &= \left| \sum_{j=0}^k p_{j(N+2)}^2(x) - \sum_{j=0}^k p_{j(N+2)+N}^2(x) \right| \\ &\leq c \sum_{j=0}^k \frac{1}{a_{j(N+2)+N+1}^2} + \left| 1 - \frac{a_{j(N+2)+N}}{a_{j(N+2)+N+1}} \right|. \end{aligned}$$

Since $(a_{k(N+2)+N-1} : k \in \mathbb{N}_0)$ is bounded, by the Stolz–Cesàro theorem, we obtain

$$\lim_{k \rightarrow \infty} \frac{1}{\rho_{N-1;k}} \sum_{j=0}^k \frac{1}{a_{j(N+2)+N+1}^2} = \lim_{k \rightarrow \infty} \frac{a_{k(N+2)+N-1}}{a_{k(N+2)+N+1}^2} = 0,$$

and

$$\lim_{k \rightarrow \infty} \frac{1}{\rho_{N-1;k}} \sum_{j=0}^k \left| 1 - \frac{a_{j(N+2)+N}}{a_{j(N+2)+N+1}} \right| = \lim_{k \rightarrow \infty} a_{k(N+2)+N-1} \left| 1 - \frac{a_{k(N+2)+N}}{a_{k(N+2)+N+1}} \right| = 0,$$

which gives (4.19). To prove (4.20), we reason analogously. Namely, by Claim 4.12, we have

$$|K_{N+1;k}(x, x)| \leq c \sum_{j=0}^k \frac{1}{a_{j(N+2)+N+1}^2},$$

thus, by the Stolz–Cesàro theorem, we get

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup_{x \in K} \frac{1}{\rho_{N-1;k}} |K_{N+1;k}(x, x)| &\leq c \lim_{k \rightarrow \infty} \frac{1}{\rho_{N-1;k}} \sum_{j=0}^k \frac{1}{a_{j(N+2)+N+1}^2} \\ &= \lim_{k \rightarrow \infty} \frac{a_{k(N+2)+N-1}}{a_{k(N+2)+N+1}^2} = 0. \end{aligned}$$

Notice that, by [27, Corollary 9] and (4.21), we have

$$\sup_{n \in \mathbb{N}_0} \sup_{x \in K} |p_n(x)| \leq c. \quad (4.26)$$

Next we show the following statement.

Claim 4.13 *For each $i \in \{0, 1, \dots, N+1\}$ and $i' \in \{0, 1, \dots, N-1\}$,*

$$\lim_{k \rightarrow \infty} \frac{\rho_{i';k}}{\rho_{k(N+2)+i}} = \frac{1}{N\alpha_{i'}}. \quad (4.27)$$

For the proof, let us consider $i \in \{0, 1, \dots, N-2\}$. By the Stolz–Cesàro theorem, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\rho_{i';k}}{\rho_{k(N+2)+i}} &= \lim_{k \rightarrow \infty} \frac{\frac{1}{a_{k(N+2)+i'}}}{\frac{\alpha_{i+1}}{a_{k(N+2)+i+1}} + \dots + \frac{\alpha_{N-1}}{a_{k(N+2)+N-1}} + \frac{\alpha_0}{a_{(k+1)(N+2)}} + \dots + \frac{\alpha_i}{a_{(k+1)(N+2)+i}}} \\ &= \frac{1}{N\alpha_{i'}}. \end{aligned}$$

For $i = N-1$ we obtain

$$\lim_{k \rightarrow \infty} \frac{\rho_{i';k}}{\rho_{k(N+2)+N-1}} = \lim_{k \rightarrow \infty} \frac{\frac{1}{a_{k(N+2)+i'}}}{\frac{\alpha_0}{a_{(k+1)(N+2)}} + \dots + \frac{\alpha_{N-1}}{a_{(k+1)(N+2)+N-1}}} = \frac{1}{N\alpha_{i'}}.$$

Finally, we observe that

$$\begin{aligned}\rho_{(k+1)(N+2)+N-1} - \rho_{k(N+2)+N-1} &= \rho_{(k+1)(N+2)+N} - \rho_{k(N+2)+N} \\ &= \rho_{(k+1)(N+2)+N+1} - \rho_{k(N+2)+N+1},\end{aligned}$$

which entails (4.27) for $i \in \{N, N+1\}$.

Now, writing

$$K_{k(N+2)+i}(x, x) = \sum_{i'=0}^{N+1} K_{i';k}(x, x) + \sum_{i'=i+1}^{N+1} (K_{i';k-1}(x, x) - K_{i';k}(x, x)),$$

by (4.19), (4.20) and (4.26), we obtain

$$K_{k(N+2)+i}(x, x) = \sum_{i'=1}^{N-1} K_{i';k}(x, x) + 2K_{N;k}(x, x) + o(\rho_{k(N+2)+i})$$

uniformly with respect to $x \in K$. Using (4.18) together with (4.27), we arrive at

$$K_n(x, x) = \frac{1}{N\pi\mu'(x)\sqrt{-\operatorname{discr}\mathcal{X}_1(x)}} \left(\sum_{i=1}^{N-1} \frac{|[\mathcal{X}_i(x)]_{2,1}|}{\alpha_{i-1}} + 2 \frac{|[\mathcal{X}_N(x)]_{2,1}|}{\alpha_{N-1}} \right) \rho_n + o(\rho_n)$$

uniformly with respect to $x \in K$. To finish the proof, it is sufficient to invoke Theorem 3.13. $\square$

Remark 4.14 If A is a Jacobi matrix that is an N -periodic blend then

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{n} = \frac{N}{N+2}. \quad (4.28)$$

Indeed, for each $i \in \{0, 1, \dots, N+1\}$, by Claim 4.13, we have

$$\lim_{k \rightarrow \infty} \frac{\rho_{k(N+2)+i}}{k(N+2)+i} = \lim_{k \rightarrow \infty} \frac{\rho_{k(N+2)+i}}{\rho_{0;k}} \cdot \frac{\rho_{0;k}}{k(N+2)+i} = N\alpha_0 \cdot \lim_{k \rightarrow \infty} \frac{\rho_{0;k}}{k(N+2)+i}.$$

Now, by the Stolz–Cesàro theorem, we obtain

$$\lim_{k \rightarrow \infty} \frac{\rho_{0;k}}{k(N+2)+i} = \lim_{k \rightarrow \infty} \frac{1}{(N+2)a_{kN}} = \frac{1}{(N+2)\alpha_0},$$

and the formula (4.28) follows.

Let us recall that $\operatorname{supp}(\mu)$ is *not* compact. In [1, Theorem 5] examples were provided of Jacobi parameters from this class such that the set of the accumulation points of $\operatorname{supp}(\mu)$ is equal to the compact set $\overline{\Lambda}$. From [27, Corollary 9] the density μ' is continuous and positive in Λ . Hence, in view of (4.28), the hypothesis on the compactness of $\operatorname{supp}(\mu)$ from [28, Theorem 1] cannot be replaced by compactness of the set of its accumulation points.

4.3 Ignjatović Conjecture

In this section we show the relation between Theorem 4.6 and the conjecture due to Ignjatović [3, Conjecture 1].

Conjecture 4.15 (Ignjatović, 2016) *Suppose that*

- (C₁) $\lim_{n \rightarrow \infty} a_n = \infty$;
- (C₂) $\lim_{n \rightarrow \infty} \Delta a_n = 0$;
- (C₃) *There exist n_0, m_0 such that $a_{n+m} > a_n$ holds for all $n \geq n_0$ and all $m \geq m_0$;*
- (C₄) $\sum_{n=0}^{\infty} \frac{1}{a_n} = \infty$;
- (C₅) *There exists $\kappa > 1$ such that $\sum_{n=0}^{\infty} \frac{1}{a_n^\kappa} < \infty$;*
- (C₆) $\sum_{n=0}^{\infty} \frac{|\Delta a_n|}{a_n^2} < \infty$;
- (C₇) $\sum_{n=0}^{\infty} \frac{|\Delta^2 a_n|}{a_n} < \infty$.

If

$$-2 < \lim_{n \rightarrow \infty} \frac{b_n}{a_n} < 2,$$

then for any $x \in \mathbb{R}$, the limit

$$\lim_{n \rightarrow \infty} \left(\sum_{j=0}^n \frac{1}{a_j} \right)^{-1} \sum_{j=0}^n p_j^2(x),$$

exists and is positive.

In [3, Corollary 3] the conclusion was shown to hold in the case $b_n \equiv 0$. Later, in [23, Corollary 3], the result was extended to a more general class of sequences with

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = 0,$$

and it was shown that

$$\lim_{n \rightarrow \infty} \left(\sum_{j=0}^n \frac{1}{a_j} \right)^{-1} \sum_{j=0}^n p_j^2(x) = \frac{1}{2\pi\mu'(x)}.$$

Our results imply the following corollary.

Corollary 4.16 *Let N be a positive integer and let $r \geq 1$. Suppose that for each $i \in \{0, 1, \dots, N-1\}$,*

$$\left(\frac{a_{jN+i-1}}{a_{jN+i}} : j \in \mathbb{N} \right), \left(\frac{b_{jN+i}}{a_{jN+i}} : j \in \mathbb{N}_0 \right), \left(\frac{1}{a_{jN+i}} : j \in \mathbb{N}_0 \right) \in \mathcal{D}_r,$$

and

$$\lim_{n \rightarrow \infty} \frac{a_{n-1}}{a_n} = 1, \quad \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = q, \quad \lim_{n \rightarrow \infty} a_n = \infty.$$

If $|q| < 2$ with

$$q \notin \left\{ 2 \cos\left(j \frac{\pi}{N}\right) : j = 1, 2, \dots, N-1 \right\}, \quad (4.29)$$

and the Carleman condition is satisfied, then

$$\lim_{n \rightarrow \infty} \left(\sum_{j=0}^n \frac{1}{a_j} \right)^{-1} \sum_{j=0}^n p_j^2(x) = \frac{1}{\pi \sqrt{4 - q^2 \mu'(x)}}, \quad (4.30)$$

locally uniformly with respect to $x \in \mathbb{R}$.

Proof Let

$$\alpha_n \equiv 1, \quad \beta_n \equiv q.$$

By (3.1),

$$\omega'(x) = \frac{1}{\pi \sqrt{4 - (x - q)^2}}, \quad x \in (-2 + q, 2 + q).$$

Hence, the conclusion follows from Theorem 4.6 provided that we can show

$$|\operatorname{tr} \mathfrak{X}_0(0)| < 2. \quad (4.31)$$

To do so, let us observe that

$$\mathfrak{X}_0(0) = \begin{pmatrix} 0 & 1 \\ -1 & -q \end{pmatrix}^N = \begin{pmatrix} 0 & 1 \\ -1 & \frac{-q/2-0}{1/2} \end{pmatrix}^N.$$

Hence, by Lemma 3.2

$$\mathfrak{X}_0(0) = \begin{pmatrix} -U_{N-2}(-\frac{q}{2}) & U_{N-1}(-\frac{q}{2}) \\ -U_{N-1}(-\frac{q}{2}) & U_N(-\frac{q}{2}) \end{pmatrix},$$

where $(U_n : n \in \mathbb{N}_0)$ is the sequence of the Chebyshev polynomials of the second kind (see [13, formula (1.6)]). By [13, formula (1.7)]

$$U_N(-\frac{q}{2}) - U_{N-2}(-\frac{q}{2}) = 2 \cos\left(N \arccos(-\frac{q}{2})\right).$$

Hence, (4.29) implies (4.31). The proof is complete. $\square$

Remark 4.17 The case when (4.29) is violated is more complicated and demands stronger hypotheses. We refer to [26] for more details.

As it was shown in [23, Section 4] conditions (C_1) – (C_7) imply the hypotheses of Corollary 4.16 with $b_n \equiv 0$, $N = 1$ and $r = 1$. On the other hand, in Conjecture 4.15 no regularity assumptions on (b_n) were imposed, whereas in Corollary 4.16 we asked for

$$\left(\frac{b_{jN+i}}{a_{jN+i}} : j \in \mathbb{N}_0 \right) \in \mathcal{D}_r,$$

for each $i \in \{0, 1, \dots, N-1\}$. The following example illustrates that some regularity assumption on (b_n) is necessary.

Example 4.18 Let

$$a_n = \sqrt{n+1}, \quad \text{and} \quad b_n = \begin{cases} 1 & n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Then the measure μ is absolutely continuous on $\mathbb{R} \setminus [0, 1]$ with continuous and positive density, 0 is not a mass point of μ , and

$$\lim_{n \rightarrow \infty} \left(\sum_{j=0}^n \frac{1}{a_j} \right)^{-1} \sum_{j=0}^n p_j^2(0) = \infty. \quad (4.32)$$

For the proof, we set

$$X_n(x) = B_{n+1}(x)B_n(x),$$

that is

$$X_n(x) = \left(\begin{array}{cc} -\frac{a_{n-1}}{a_n} & \frac{x-b_n}{a_n} \\ -\frac{a_{n-1}}{a_n} \frac{x-b_{n+1}}{a_{n+1}} & -\frac{a_n}{a_{n+1}} + \frac{a_n}{a_{n+1}} \frac{x-b_n}{a_n} \end{array} \right). \quad (4.33)$$

Since

$$a_{n+1}(X_n(x) + \text{Id}) = \left(\begin{array}{cc} \frac{a_{n+1}}{a_n}(a_n - a_{n-1}) & \frac{a_{n+1}}{a_n}(x - b_n) \\ -\frac{a_{n-1}}{a_n}(x - b_{n+1}) & (a_{n+1} - a_n) + (x - b_{n+1})\frac{x-b_n}{a_n} \end{array} \right),$$

we obtain

$$\mathcal{R}_0 = \lim_{n \rightarrow \infty} a_{2n+1}(X_{2n}(x) + \text{Id}) = \begin{pmatrix} 0 & x-1 \\ -x & 0 \end{pmatrix}.$$

Consequently,

$$\Lambda = \{x \in \mathbb{R} : \text{discr } \mathcal{R}_0(x) < 0\} = \mathbb{R} \setminus [0, 1].$$

Therefore, by [24, Theorem D] and [24, Proposition 11], the measure μ is purely absolutely continuous on Λ with positive continuous density proving the first assertion.

Next, let us observe that, by (4.33),

$$\begin{aligned} \begin{pmatrix} p_{2n}(0) \\ p_{2n+1}(0) \end{pmatrix} &= X_{2n-1}(0) \begin{pmatrix} p_{2n-2}(0) \\ p_{2n-1}(0) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{a_{2n-2}}{a_{2n-1}} & 0 \\ \frac{a_{2n-2}}{a_{2n-1}} \frac{1}{a_{2n}} & -\frac{a_{2n-1}}{a_{2n}} \end{pmatrix} \begin{pmatrix} p_{2n-2}(0) \\ p_{2n-1}(0) \end{pmatrix}. \end{aligned}$$

Thus

$$p_{2n}(0) = (-1)^n \frac{a_0 a_2 \dots a_{2n-2}}{a_1 a_3 \dots a_{2n-1}} = (-1)^n \frac{\sqrt{(2n)!}}{2^n n!}, \quad (4.34)$$

and

$$p_{2n+1}(0) = -\frac{1}{a_{2n}} p_{2n}(0) - \frac{a_{2n-1}}{a_{2n}} p_{2n-1}(0). \quad (4.35)$$

By (4.34) and (4.35), $(p_{2n+1}(0) : n \in \mathbb{N}_0)$ satisfies

$$\begin{cases} x_n = -\frac{1}{\sqrt{2n+1}} (-1)^n \frac{\sqrt{(2n)!}}{2^n n!} - \sqrt{\frac{2n}{2n+1}} x_{n-1}, & n \geq 1. \\ x_0 = -1. \end{cases} \quad (4.36)$$

It can be verified that

$$x_n = (-1)^{n+1} \frac{\sqrt{(n+1)(2n+2)!}}{(n+1)! 2^{n+1/2}}$$

is the only solution of (4.36). Hence,

$$p_{2n+1}^2(0) = \frac{(n+1)(2n+2)!}{((n+1)!)^2 2^{2n+1}}.$$

Using Stirling's formula, we can find that

$$\lim_{n \rightarrow \infty} \frac{p_{2n+1}^2(0)}{\sqrt{n+1}} = \frac{\sqrt{\pi}}{2}. \quad (4.37)$$

Now, by the Stolz–Cesàro theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sum_{j=0}^{2n} \frac{1}{\sqrt{j+1}} \right)^{-1} \sum_{j=0}^{n-1} p_{2j+1}^2(0) &= \lim_{n \rightarrow \infty} \frac{p_{2n-1}^2(x)}{\frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{2n+1}}} \\ &= \frac{2}{\sqrt{\pi}} \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{2n+1}}} = \infty \end{aligned}$$

proving (4.32). Moreover, by (4.37), the sequence $(p_n(0) : n \in \mathbb{N}_0)$ is not square summable. Hence, 0 is not a mass point of μ .

Remark 4.19 It turns out that in the previous example the measure μ on $(0, 1)$ can have only mass points without any accumulation points on this set (see [25]). Moreover, we have recently shown [26] that in this case

$$\lim_{n \rightarrow \infty} \left(\sum_{j=0}^n \frac{1}{a_j} \right)^{-1} \sum_{j=0}^n p_j^2(x) = \frac{|2x-1|}{4\pi\sqrt{x^2-x}} \frac{1}{\mu'(x)},$$

locally uniformly with respect to $x \in \mathbb{R} \setminus [0, 1]$. So the value of the limit is different than in (4.30).

5 Christoffel–Darboux Kernel for $\mathcal{D}_1$

For $\mathcal{D}_1$ class, we can describe the speed of convergence in Theorem 4.2 and Theorem 4.4. First, let us show a refined asymptotic of polynomials.

5.1 Asymptotics of Polynomials

Theorem 5.1 *Let N be a positive integer and $i \in \{0, 1, \dots, N-1\}$. Suppose that K is a compact interval with non-empty interior contained in*

$$\Lambda = \left\{ x \in \mathbb{R} : \lim_{j \rightarrow \infty} \text{discr } X_{jN+i}(x) \text{ exists and is negative} \right\}.$$

Assume that

$$\lim_{j \rightarrow \infty} \frac{a_{(j+1)N+i-1}}{a_{jN+i-1}} = 1$$

and

$$(X_{jN+i} : j \in \mathbb{N}) \in \mathcal{D}_1(K, \text{GL}(2, \mathbb{R})).$$

Let X denote the limit of $(X_{jN+i} : j \in \mathbb{N})$. Then there is a probability measure ν such that $(p_n : n \in \mathbb{N}_0)$ are orthonormal in $L^2(\mathbb{R}, \nu)$ which is purely absolutely continuous

with continuous and positive density v' on K . Moreover, there are $M > 0$ and a real continuous function $\eta : K \rightarrow \mathbb{R}$, such that for all $k \geq M$,

$$\sqrt{a_{(k+1)N+i-1}} p_{kN+i}(x) = \sqrt{\frac{2|[X(x)]_{2,1}|}{\pi v'(x) \sqrt{-\operatorname{discr} X(x)}}} \sin \left(\sum_{j=M+1}^k \theta_{jN+i}(x) + \eta(x) \right) + E_{kN+i}(x)$$

where

$$\sup_{x \in K} |E_{kN+i}(x)| \leq c \sum_{j=k}^{\infty} \sup_{x \in K} \|X_{(j+1)N+i}(x) - X_{jN+i}(x)\|,$$

and

$$\theta_n(x) = \arccos \left(\frac{\operatorname{tr} X_n(x)}{2\sqrt{\det X_n(x)}} \right). \quad (5.1)$$

Proof Let us fix a compact interval K with non-empty interior. Since X is the uniform limit of $(X_{jN+i} : j \in \mathbb{N})$, there are $\delta > 0$ and $M > 0$ such that for all $x \in K$ and $k \geq M$,

$$\operatorname{discr} X_{kN+i}(x) \leq -\delta < 0.$$

Therefore, the matrix $X_{kN+i}(x)$ has two eigenvalues λ_{kN+i} and $\overline{\lambda_{kN+i}}$ where

$$\lambda_n(x) = \frac{\operatorname{tr} X_n(x)}{2} + \frac{i}{2} \sqrt{-\operatorname{discr} X_n(x)}. \quad (5.2)$$

Let us next observe that for $k \geq M$,

$$\Im \lambda_{kN+i}(x) = \frac{1}{2} \sqrt{-\operatorname{discr} X_{kN+i}(x)} \geq \frac{1}{2} \sqrt{\delta}.$$

Moreover,

$$X_{kN+i}(x) = C_k(x) D_k(x) C_k^{-1}(x)$$

where

$$C_k = \begin{pmatrix} 1 & 1 \\ \frac{\lambda_{kN+i} - [X_{kN+i}]_{1,1}}{[X_{kN+i}]_{1,2}} & \frac{\overline{\lambda_{kN+i}} - [X_{kN+i}]_{1,1}}{[X_{kN+i}]_{1,2}} \end{pmatrix}, \quad D_k(x) = \begin{pmatrix} \lambda_{kN+i} & 0 \\ 0 & \overline{\lambda_{kN+i}} \end{pmatrix},$$

which is well-defined since $[X_{kN+i}(x)]_{1,2} \neq 0$ for any $x \in K$. Let

$$\phi_{kN+i} = \frac{p_{(k+1)N+i} - \overline{\lambda_{kN+i}} \cdot p_{kN+i}}{\prod_{j=M+1}^k \lambda_{jN+i}}.$$

We claim the following holds true.

Claim 5.2 *There is $c > 0$ such that for all $m \geq n \geq M$, and $x \in K$,*

$$|\phi_{mN+i}(x) - \phi_{nN+i}(x)| \leq c \sum_{j=n}^{\infty} |\lambda_{(j+1)N+i}(x) - \lambda_{jN+i}(x)|.$$

We start by writing

$$p_{mN+i} = \left\langle C_{m-1} \left(\prod_{j=n}^{m-1} D_j C_j^{-1} C_{j-1} \right) C_{n-1}^{-1} \begin{pmatrix} p_{nN+i-1} \\ p_{nN+i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle.$$

Let us now introduce two auxiliary functions

$$q_m = \left\langle C_{\infty} \left(\prod_{j=n}^{m-1} D_j \right) C_{n-1}^{-1} \begin{pmatrix} p_{nN+i-1} \\ p_{nN+i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle,$$

and

$$\psi_m = \frac{q_{m+1} - \overline{\lambda_{mN+i}} \cdot q_m}{\prod_{j=M+1}^m \lambda_{jN+i}}.$$

Notice that

$$p_{mN+i} - q_m = \left\langle Y_m \begin{pmatrix} p_{nN+i-1} \\ p_{nN+i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

where

$$Y_m = C_{\infty} \left(\prod_{j=n}^{m-1} D_j C_j C_{j-1}^{-1} - \prod_{j=n}^{m-1} D_j \right) C_{n-1}^{-1} + (C_{\infty} - C_{m-1}) \left(\prod_{j=n}^{m-1} D_j \right) C_{n-1}^{-1}.$$

In view of [27, Proposition 1], we have

$$\begin{aligned} \|Y_m\| &\lesssim \left(\prod_{j=n}^{m-1} \|D_j\| \right) \left(\sum_{j=n-1}^{\infty} \|\Delta C_j\| + \|C_{\infty} - C_{m-1}\| \right) \\ &\lesssim \left(\prod_{j=n}^{m-1} \|D_j\| \right) \sum_{j=n-1}^{\infty} \|\Delta C_j\|, \end{aligned}$$

thus

$$\|Y_m\| \lesssim \prod_{j=n}^{m-1} |\lambda_{jN+i}| \cdot \sum_{j=n-1}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|.$$

Next, by [27, Claim 2], there is $c > 0$ such that for all $n \geq M$ and $x \in K$,

$$\frac{\sqrt{p_{nN+i}^2(x) + p_{nN+i-1}^2(x)}}{\prod_{j=M+1}^{n-1} |\lambda_{jN+i}(x)|} \leq c,$$

and consequently, for all $m \geq n \geq M$,

$$\left| \frac{p_{mN+i} - q_m}{\prod_{j=M+1}^{m-1} \lambda_{jN+i}} \right| \lesssim \sum_{j=n-1}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|.$$

In particular, we obtain

$$|\phi_{mN+i} - \psi_m| \lesssim \sum_{j=n-1}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|. \quad (5.3)$$

Next, we notice that

$$q_{m+1} - \overline{\lambda_{mN+i}} \cdot q_m = \left\langle C_{\infty} \left(D_m - \overline{\lambda_{mN+i}} \text{Id} \right) \left(\prod_{j=n}^{m-1} D_j \right) C_{n-1}^{-1} \begin{pmatrix} p_{nN+i-1} \\ p_{nN+i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle.$$

Since

$$\frac{1}{\prod_{j=n}^m \lambda_{jN+i}} \left(D_m - \overline{\lambda_{mN+i}} \text{Id} \right) \left(\prod_{j=n}^{m-1} D_j \right) = \frac{1}{\lambda_{mN+i}} \begin{pmatrix} \lambda_{mN+i} - \overline{\lambda_{mN+i}} & 0 \\ 0 & 0 \end{pmatrix},$$

we obtain

$$\begin{aligned} |\psi_m - \psi_n| &\lesssim \left| \frac{\overline{\lambda_{mN+i}}}{\lambda_{mN+i}} - \frac{\overline{\lambda_{nN+i}}}{\lambda_{nN+i}} \right| \\ &\lesssim \sum_{j=n}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|, \end{aligned}$$

which together with (5.3) implies that for all $m \geq n > M$ and $x \in K$,

$$|\phi_{mN+i}(x) - \phi_{nN+i}(x)| \lesssim \sum_{j=n}^{\infty} |\lambda_{(j+1)N+i}(x) - \lambda_{jN+i}(x)|.$$

In particular, the sequence $(\phi_{mN+i} : m \in \mathbb{N})$ converges. Let us denote by φ its limit. Since polynomials p_n have real coefficients, by taking the imaginary part we arrive at

$$\left| \frac{1}{2} \sqrt{-\operatorname{discr} X_{nN+i}(x)} \frac{p_{nN+i}(x)}{\prod_{j=M+1}^n |\lambda_{jN+i}(x)|} - |\varphi(x)| \sin \left(\sum_{j=M+1}^n \arg \lambda_{jN+i}(x) + \arg \varphi(x) \right) \right| \\ \lesssim \sum_{j=n}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|.$$

Because

$$\det X_{jN+i} = \prod_{k=jN+i}^{(j+1)N+i-1} \det B_k = \frac{a_{jN+i-1}}{a_{(j+1)N+i-1}},$$

we obtain

$$\prod_{j=M+1}^n |\lambda_{jN+i}|^2 = \prod_{j=M+1}^n \det X_{jN+i} \\ = \frac{a_{(M+1)N+i-1}}{a_{(n+1)N+i-1}}.$$

Therefore, by [27, Theorem 6],

$$\left| \sqrt{a_{(n+1)N+i-1}} \sqrt{-\operatorname{discr} X_{nN+i}(x)} p_{nN+i}(x) \right. \\ \left. - \sqrt[4]{-\operatorname{discr} X(x)} \sqrt{\frac{2\|X(x)\|_{2,1}}{\pi v'(x)}} \sin \left(\sum_{j=M+1}^n \arg \lambda_{jN+i}(x) + \varphi(x) \right) \right| \\ \lesssim \sum_{j=n}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|.$$

Observe that, by (5.2),

$$\left| \frac{1}{\sqrt{-\operatorname{discr} X_{nN+i}(x)}} - \frac{1}{\sqrt{-\operatorname{discr} X(x)}} \right| \lesssim \sum_{j=n}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|,$$

thus

$$\left| \sqrt{a_{(n+1)N+i-1}} p_{nN+i}(x) - \sqrt{\frac{2\|X(x)\|_{2,1}}{\pi v'(x)}} \sqrt{-\operatorname{discr} X(x)} \sin \left(\sum_{j=M+1}^n \arg \lambda_{jN+i}(x) + \varphi(x) \right) \right| \\ \lesssim \sum_{j=n}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|.$$

Since

$$|\lambda_{(j+1)N+i} - \lambda_{jN+i}| \lesssim \|X_{(j+1)N+i} - X_{jN+i}\|,$$

we finish the proof. $\square$

Remark 5.3 Under the assumption of Theorem 5.1, we have

$$\theta_{kN+i}(x) = \arccos\left(\frac{1}{2} \operatorname{tr} X(x)\right) + E_{kN+i}(x)$$

where

$$\sup_{x \in K} |E_{kN+i}(x)| \leq c \sum_{j=k}^{\infty} \sup_{x \in K} \|X_{(j+1)N+i}(x) - X_{jN+i}(x)\|.$$

Indeed, since for $m \geq n \geq M$,

$$\left| \frac{\operatorname{tr} X_{nN+i}}{2\sqrt{\det X_{nN+i}}} - \frac{\operatorname{tr} X_{mN+i}}{2\sqrt{\det X_{mN+i}}} \right| = \left| \frac{\lambda_{nN+i}}{|\lambda_{nN+i}|} - \frac{\lambda_{mN+i}}{|\lambda_{mN+i}|} \right| \lesssim \sum_{j=n}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|,$$

we get

$$\left| \theta_n(x) - \arccos\left(\frac{1}{2} \operatorname{tr} X(x)\right) \right| \lesssim \sum_{j=n}^{\infty} |\lambda_{(j+1)N+i} - \lambda_{jN+i}|,$$

which proves our statement.

5.2 Christoffel Functions

We are now in the position to prove the main theorem of this section.

Theorem 5.4 *Let N be a positive integer and $i \in \{0, 1, \dots, N-1\}$. Suppose that K is a compact interval with non-empty interior and contained in*

$$\Lambda = \left\{ x \in \mathbb{R} : \lim_{j \rightarrow \infty} \operatorname{discr} X_{jN+i}(x) \text{ exists and is negative} \right\}.$$

Assume that

$$\lim_{j \rightarrow \infty} \frac{a_{(j+1)N+i-1}}{a_{jN+i-1}} = 1$$

and

$$(X_{jN+i} : j \in \mathbb{N}) \in \mathcal{D}_1(K, GL(2, \mathbb{R})).$$

If

$$\sum_{j=1}^{\infty} \frac{1}{a_{jN+i-1}} = \infty,$$

then

$$K_{i;n}(x, x) = \frac{||\mathcal{X}(x)||_{2,1}}{\pi \mu'(x) \sqrt{-\operatorname{discr} \mathcal{X}(x)}} \rho_{i-1;n+1} + E_{i;n}(x),$$

where $\mathcal{X}$ is the limit of $(X_{jN+i} : j \in \mathbb{N})$, and

$$\begin{aligned} \sup_{x \in K} |E_{i;n}(x)| &\leq c \sum_{k=0}^n \left(\left| \frac{1}{a_{(k+1)N+i-1}} - \frac{1}{a_{kN+i-1}} \right| \right. \\ &\quad \left. + \frac{1}{a_{(k+1)N+i-1}} \sum_{j \geq k} \sup_{x \in K} \|X_{(j+1)N+i}(x) - X_{jN+i}(x)\| \right). \end{aligned}$$

Proof Let $K \subset \Lambda$ be a compact interval with non-empty interior. By Theorem 5.1, there are $c > 0$ and $M \in \mathbb{N}$ such that for all $k \geq M$,

$$a_{(k+1)N+i-1} p_{kN+i}^2(x) = \frac{2||\mathcal{X}(x)||_{2,1}}{\pi \mu'(x) \sqrt{-\operatorname{discr} \mathcal{X}(x)}} \sin^2 \left(\eta(x) + \sum_{j=M}^k \theta_{jN+i}(x) \right) + E_{kN+i}(x)$$

where

$$\sup_{x \in K} |E_{kN+i}(x)| \leq c \sum_{j \geq k} \sup_{x \in K} \|X_{(j+1)N+i}(x) - X_{jN+i}(x)\|.$$

In view of the identity $2 \sin^2(x) = 1 - \cos(2x)$, we get

$$\begin{aligned} \sum_{k=M}^n p_{kN+i}^2(x) &= \frac{||\mathcal{X}(x)||_{2,1}}{\pi \mu'(x) \sqrt{-\operatorname{discr} \mathcal{X}(x)}} \sum_{k=M}^n \frac{1}{a_{(k+1)N+i-1}} \left(1 - \cos \left(2\eta(x) + 2 \sum_{j=M}^k \theta_{jN+i}(x) \right) \right) \\ &\quad + \sum_{k=M}^n \frac{1}{a_{(k+1)N+i-1}} E_{kN+i}(x). \end{aligned}$$

Since there is $c > 0$ such that

$$\sup_{x \in K} \sum_{k=0}^{M-1} p_{kN+i}^2(x) \leq c,$$

by Lemma 4.1 and Remark 5.3, we obtain

$$\left| K_{i;n}(x, x) - \frac{|[\mathcal{X}(x)]_{2,1}|}{\pi \mu'(x) \sqrt{-\operatorname{discr} \mathcal{X}(x)}} \rho_{i-1;n+1} \right| \leq c \sum_{k=0}^n \left(\left| \frac{1}{a_{(k+1)N+i-1}} - \frac{1}{a_{kN+i-1}} \right| + \frac{1}{a_{(k+1)N+i-1}} \sum_{j \geq k} \sup_{x \in K} \|X_{(j+1)N+i}(x) - X_{jN+i}(x)\| \right)$$

which completes the proof. $\square$

Theorem 5.5 *Let A be a Jacobi matrix with N -periodically modulated entries. Suppose that for each $i \in \{0, 1, \dots, N-1\}$,*

$$\left(\frac{a_{jN+i-1}}{a_{jN+i}} : j \in \mathbb{N} \right), \left(\frac{b_{jN+i}}{a_{jN+i}} : j \in \mathbb{N}_0 \right), \left(\frac{1}{a_{jN+i}} : j \in \mathbb{N}_0 \right) \in \mathcal{D}_1,$$

and

$$\sum_{n=0}^{\infty} \frac{1}{a_n} = \infty.$$

If $|\operatorname{tr} \mathfrak{X}_0(0)| < 2$ then

$$K_n(x, x) = \frac{\omega'(0)}{\mu'(x)} \rho_n + E_n(x)$$

where ω is the equilibrium measure corresponding to $\sigma_{\operatorname{ess}}(\mathfrak{A})$ with $\mathfrak{A}$ being the Jacobi matrix associated with $(\alpha_n : n \in \mathbb{N}_0)$ and $(\beta_n : n \in \mathbb{N}_0)$,

$$\rho_n = \sum_{j=0}^n \frac{\alpha_j}{a_j},$$

and for each compact interval $K \subset \mathbb{R}$ with non-empty interior

$$\sup_{x \in K} |E_n(x)| \leq c \sum_{m=0}^{n+N} \frac{1}{a_m} \sum_{j \geq 0} \sup_{x \in K} \|B_{m+(j+1)N}(x) - B_{m+jN}(x)\|.$$

Proof Let K be a compact interval with non-empty interior and contained in $\mathbb{R}$. By Remark 4.5, for each $i \in \{0, 1, \dots, N-1\}$, the sequence $(X_{jN+i} : j \in \mathbb{N})$ belongs to $\mathcal{D}_1(K, \operatorname{GL}(2, \mathbb{R}))$, thus, by Proposition 3.8, we have

$$\lim_{j \rightarrow \infty} X_{jN+i}(x) = \mathfrak{X}_i(0)$$

uniformly with respect to $x \in K$. Since

$$\mathfrak{X}_{i+1}(x) = (\mathfrak{B}_i(x))(\mathfrak{X}_i(x))(\mathfrak{B}_i(x))^{-1},$$

we have

$$\text{discr } \mathfrak{X}_i(x) = \text{discr } \mathfrak{X}_0(x).$$

By Theorem 5.4,

$$K_{i;n}(x, x) = \frac{|\mathfrak{X}_i(0)|_{2,1}}{\pi \mu'(x) \sqrt{-\text{discr } \mathfrak{X}_0(0)}} \rho_{i-1;n+1} + E_{i;n}(x)$$

where

$$\sup_{x \in K} |E_{i;n}(x)| \leq c \sum_{k=0}^n \frac{1}{a_{(k+1)N+i-1}} \sum_{j \geq k} \sup_{x \in K} \|B_{(j+1)N+i}(x) - B_{jN+i}(x)\|.$$

For $k \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, N-1\}$ we write

$$K_{kN+i}(x, x) = \sum_{i'=0}^{N-1} K_{i';k}(x, x) + \sum_{i'=i+1}^{N-1} (K_{i';k-1}(x, x) - K_{i';k}(x, x)).$$

Since

$$|K_{i';k-1}(x, x) - K_{i';k}(x, x)| = p_{kN+i'}^2(x) \leq c \frac{1}{a_{kN+i'}},$$

we obtain

$$K_{kN+i}(x, x) = \frac{1}{\pi \mu'(x) \sqrt{-\text{discr } \mathfrak{X}_0(x)}} \sum_{i'=0}^{N-1} |\mathfrak{X}_{i'}(x)|_{2,1} \cdot \rho_{i'-1;k+1} + E_{kN+i}(x), \quad (5.4)$$

where

$$\sup_{x \in K} |E_n(x)| \leq c \sum_{m=0}^n \frac{1}{a_{m+N-1}} \sum_{j \geq 0} \sup_{x \in K} \|B_{m+(j+1)N}(x) - B_{m+jN}(x)\|.$$

We next claim the following holds true. □

Claim 5.6 For each $i', i'' \in \{0, 1, \dots, N-1\}$,

$$|\alpha_{i'} \rho_{i';k} - \alpha_{i''} \rho_{i'';k}| \leq c \sum_{m=0}^{(k+1)N} \frac{1}{a_m} \sum_{j \geq 0} \sup_{x \in K} \|B_{m+jN}(x) - B_{m+(j+1)N}(x)\|. \quad (5.5)$$

For the proof let us observe that, by Proposition 3.7, we have

$$\begin{aligned} \left| \frac{\alpha_{i-1}}{\alpha_i} - \frac{a_{kN+i-1}}{a_{kN+i}} \right| &\leq \sum_{j \geq k} \left| \frac{a_{jN+i-1}}{a_{jN+i}} - \frac{a_{(j+1)N+i-1}}{a_{(j+1)N+i}} \right| \\ &\leq c \sum_{j \geq k} \sup_{x \in K} \|B_{jN+i}(x) - B_{(j+1)N+i}(x)\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \alpha_{i'} \rho_{i';k} - \alpha_{i''} \rho_{i'';k} \right| &\leq \sum_{i=1}^{N-1} \sum_{j=0}^k \left| \frac{\alpha_{i-1}}{a_{jN+i-1}} - \frac{\alpha_i}{a_{jN+i}} \right| \\ &\leq \sum_{i=1}^{N-1} \sum_{j=0}^k \frac{\alpha_i}{a_{jN+i-1}} \left| \frac{\alpha_{i-1}}{\alpha_i} - \frac{a_{jN+i-1}}{a_{jN+i}} \right|, \end{aligned}$$

which implies (5.5).

Now, using Claim 5.6, we can write

$$\begin{aligned} |N\alpha_i \rho_{i;k+1} - \rho_{(k+1)N}| &\leq \sum_{i'=0}^{N-1} |\alpha_i \rho_{i;k+1} - \alpha_{i'} \rho_{i';k+1}| \\ &\leq c \sum_{m=0}^{(k+1)N} \frac{1}{a_m} \sum_{j \geq 0} \sup_{x \in K} \|B_{m+jN}(x) - B_{m+(j+1)N}(x)\|. \end{aligned}$$

Hence, by (5.4), we obtain

$$\begin{aligned} \left| K_{kN+i}(x, x) - \frac{1}{N\pi\mu'(x)\sqrt{-\operatorname{discr} \mathfrak{X}_0(0)}} \sum_{i'=0}^{N-1} \frac{|[\mathfrak{X}_{i'}(0)]_{2,1}|}{\alpha_{i'-1}} \rho_{kN+i} \right| \\ \leq c \sum_{m=0}^{(k+1)N} \frac{1}{a_m} \sum_{j \geq 0} \sup_{x \in K} \|B_{m+jN}(x) - B_{m+(j+1)N}(x)\|, \end{aligned}$$

which together with (3.1), concludes the proof. $\square$

The following theorem has a proof analogous to Theorem 5.5.

Theorem 5.7 *Let A be a Jacobi matrix with asymptotically N -periodic entries. Suppose that for each $i \in \{0, 1, \dots, N-1\}$,*

$$\left(\frac{a_{jN+i-1}}{a_{jN+i}} : j \in \mathbb{N} \right), \left(\frac{b_{jN+i}}{a_{jN+i}} : j \in \mathbb{N}_0 \right), \left(\frac{1}{a_{jN+i}} : j \in \mathbb{N}_0 \right) \in \mathcal{D}_1,$$

Let K be a compact interval with non-empty interior contained in

$$\Lambda = \{x \in \mathbb{R} : |\operatorname{tr} \mathfrak{X}_0(x)| < 2\}.$$

Then

$$K_n(x, x) = \frac{\omega'(x)}{\mu'(x)} \rho_n + E_n(x)$$

where ω is the equilibrium measure corresponding to $\sigma_{ess}(\mathfrak{A})$ with $\mathfrak{A}$ being the Jacobi matrix associated to $(\alpha_n : n \in \mathbb{N}_0)$ and $(\beta_n : n \in \mathbb{N}_0)$,

$$\rho_n = \sum_{j=0}^n \frac{\alpha_j}{a_j},$$

and

$$\sup_{x \in K} |E_n(x)| \leq c \sum_{m=0}^{n+N} \frac{1}{a_m} \sum_{j \geq 0} \sup_{x \in K} \|B_{m+(j+1)N}(x) - B_{m+jN}(x)\|. \quad (5.6)$$

In the following two examples we want to compare the estimate (5.6) with some known results.

Example 5.8 (*Generalized Jacobi*) Let h be a real-analytic positive function on the neighborhood of $[-1, 1]$. Let μ be a probability measure supported on $[-1, 1]$ with the density

$$\mu'(x) = c \cdot h(x)(x+1)^{\gamma_1}(x-1)^{\gamma_2}, \quad x \in (-1, 1),$$

where $\gamma_1, \gamma_2 > -1$, and c is the normalizing constant. Then (see [6, Theorem 1.10])

$$a_n = \frac{1}{2} + \frac{c_1}{n^2} + O(n^{-3}), \quad b_n = \frac{c_2}{n^2} + O(n^{-3}).$$

Therefore, by Theorem 5.5 we obtain

$$\frac{1}{\rho_n} K_n(x, x) = \frac{1}{\pi \sqrt{1-x^2}} \frac{1}{\mu'(x)} + O(n^{-1}).$$

Hence, we obtain the same rate as in [7, Theorem 1.1(a)].

Example 5.9 (*Pollaczek-type*) Let μ be a probability measure supported on $[-1, 1]$ with the density

$$\mu'(x) = c \cdot \exp(-(1-x^2)^{-\gamma}), \quad x \in (-1, 1),$$

where $\gamma \in (0, \frac{1}{2})$, and c is the normalizing constant. Then (see, [32, Corollary 4])

$$a_n = \frac{1}{2} + \frac{c_1}{2} n^{-2/(1+2\gamma)} + O(n^{-2}), \quad b_n = 0.$$

Hence, Theorem 5.5 implies

$$\frac{1}{\rho_n} K_n(x, x) = \frac{1}{\pi \sqrt{1-x^2}} \frac{1}{\mu'(x)} + O(n^{-1}).$$

It should be compared with [31, Theorem 1(i)].

5.3 Auxiliary Results

Lemma 5.10 *Let $(\gamma_k : k \geq 0)$ be a sequence of positive numbers such that*

$$\sum_{k=0}^{\infty} \gamma_k = \infty, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\gamma_{n-1}}{\gamma_n} = 1.$$

Assume that $(\theta_n : n \geq 0)$ is a sequence of continuous functions on some open set $U \subset \mathbb{R}^d$ with values in $(0, 2\pi)$. Suppose that there is $\theta : U \rightarrow (0, 2\pi)$ such that

$$\lim_{n \rightarrow \infty} \theta_n(x) = \theta(x)$$

locally uniformly with respect to $x \in U$. Let $(r_n : n \in \mathbb{N})$ be a sequence of positive numbers such that

$$\lim_{n \rightarrow \infty} r_n = \infty.$$

For $x \in U$, and $a, b \in \mathbb{R}$, we set

$$x_n = x + \frac{a}{r_n}, \quad \text{and} \quad y_n = x + \frac{b}{r_n}.$$

Then for each compact subset $K \subset U$, $L > 0$, and any function $\sigma : U \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\gamma_k}{\sum_{j=0}^n \gamma_j} \cos \left(\sigma(x_n) + \sigma(y_n) + \sum_{j=0}^k (\theta_j(x_n) + \theta_j(y_n)) \right) = 0 \quad (5.7)$$

uniformly with respect to $x \in K$, and $a, b \in [-L, L]$.

Proof Let us fix K a compact subset of U and $L > 0$. Select $R > 0$ so that

$$\left(x - \frac{2L}{R}, x + \frac{2L}{R} \right) \subset U$$

for all $x \in K$, and let $N \in \mathbb{N}$ be such that $r_n \geq R$ for all $n \geq N$. For $(x, a, b) \in U \times (-2L, 2L)^2$, we set

$$\begin{aligned}\tilde{\theta}_j(x, a, b) &= \theta_j\left(x + \frac{a}{R}\right) + \theta_j\left(x + \frac{b}{R}\right), \text{ and} \\ \tilde{\theta}(x, a, b) &= \theta\left(x + \frac{a}{R}\right) + \theta\left(x + \frac{b}{R}\right).\end{aligned}$$

Then

$$\lim_{j \rightarrow \infty} \tilde{\theta}_j(x, a, b) = \tilde{\theta}(x, a, b)$$

uniformly with respect to $(x, a, b) \in K \times [-L, L]^2$. By Lemma 4.1, there is $c > 0$ such that

$$\left| \sum_{k=0}^n \gamma_k \exp\left(i \sum_{j=0}^k \tilde{\theta}_j(x, a, b)\right) \right| \leq c \sum_{k=0}^{n-1} |\gamma_{k+1} - \gamma_k| + \gamma_{k+1} |\tilde{\theta}_{k+1}(x, a, b) - \tilde{\theta}(x, a, b)| \quad (5.8)$$

for all $x \in K$, $a, b \in [-L, L]$, and $n \in \mathbb{N}$. Since

$$\tilde{\theta}_j\left(x, \frac{Ra}{r_n}, \frac{Rb}{r_n}\right) = \theta_j(x_n) + \theta_j(y_n),$$

by (5.8),

$$\begin{aligned}& \left| \sum_{k=0}^n \gamma_k \exp\left(i \sum_{j=0}^k \theta_j(x_n) + \theta_j(y_n)\right) \right| \\ & \leq c \sum_{k=0}^{n-1} |\gamma_{k+1} - \gamma_k| + \gamma_{k+1} \left(|\theta_{k+1}(x_n) - \theta(x_n)| + |\theta_{k+1}(y_n) - \theta(y_n)| \right) \quad (5.9)\end{aligned}$$

for all $x \in K$, $a, b \in [-L, L]$, and $n \geq N$. Finally,

$$\begin{aligned}& \cos\left(\sum_{j=0}^k (\theta_j(x_n) + \theta_j(y_n)) + \sigma(x_n) + \sigma(y_n)\right) \\ & = \cos\left(\sum_{j=0}^k (\theta_j(x_n) + \theta_j(y_n))\right) \cos(\sigma(x_n) + \sigma(y_n)) \\ & \quad - \sin\left(\sum_{j=0}^k (\theta_j(x_n) + \theta_j(y_n))\right) \sin(\sigma(x_n) + \sigma(y_n)),\end{aligned}$$

which together with (5.9) implies that there are $c > 0$ and $N \in \mathbb{N}$, such that for any function $\sigma : U \rightarrow \mathbb{R}$ and all $x \in K$, $a, b \in [-L, L]$ and $n \geq N$,

$$\left| \sum_{k=0}^n \gamma_k \cos \left(\sigma(x_n) + \sigma(y_n) + \sum_{j=0}^k (\theta_j(x_n) + \theta_j(y_n)) \right) \right| \leq c \sum_{k=0}^{n-1} |\gamma_{k+1} - \gamma_k| + \gamma_{k+1} (|\theta_{k+1}(x_n) - \theta(x_n)| + |\theta_{k+1}(y_n) - \theta(y_n)|). \quad (5.10)$$

Finally, (5.7) follows from (5.10) by the Stolz–Cesàro theorem, since

$$\lim_{n \rightarrow \infty} \frac{1}{\sum_{k=0}^n \gamma_k} \sum_{k=0}^{n-1} |\gamma_{k+1} - \gamma_k| = \lim_{n \rightarrow \infty} \frac{|\gamma_n - \gamma_{n-1}|}{\gamma_n} = 0,$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{\sum_{k=0}^n \gamma_k} \sum_{k=0}^n \gamma_{k+1} (|\theta_{k+1}(x_n) - \theta(x_n)| + |\theta_{k+1}(y_n) - \theta(y_n)|) \\ &= \lim_{n \rightarrow \infty} |\theta_n(x_n) - \theta(x_n)| + |\theta_n(y_n) - \theta(y_n)| = 0. \end{aligned}$$

□

Theorem 5.11 *Let U be an open subset of $\mathbb{R}$. Let $(\gamma_k : k \geq 0)$ be a sequence of positive numbers such that*

$$\sum_{k=0}^{\infty} \gamma_k = \infty, \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\gamma_{k-1}}{\gamma_k} = 1. \quad (5.11)$$

Assume that $(\theta_k : k \geq 0)$ is a sequence of $C^2(U)$ functions with values in $(0, 2\pi)$ such that for each compact set $K \subset U$ there are functions $\theta : K \rightarrow (0, 2\pi)$ and $\psi : K \rightarrow (0, \infty)$, and $c > 0$ such that

- (a) $\lim_{n \rightarrow \infty} \sup_{x \in K} |\theta_n(x) - \theta(x)| = 0$,
- (b) $\lim_{n \rightarrow \infty} \sup_{x \in K} |\gamma_n^{-1} \cdot \theta'_n(x) - \psi(x)| = 0$,
- (c) $\sup_{n \in \mathbb{N}} \sup_{x \in K} |\gamma_n^{-2} \cdot \theta''_n(x)| \leq c$.

For $x \in U$ and $a, b \in \mathbb{R}$ we set

$$x_n = x + \frac{a}{\sum_{k=0}^n \gamma_k}, \quad \text{and} \quad y_n = x + \frac{b}{\sum_{k=0}^n \gamma_k}.$$

Then for any continuous function $\sigma : U \rightarrow \mathbb{R}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\gamma_k}{\sum_{j=0}^n \gamma_j} \sin \left(\sum_{j=0}^n \theta_j(x_n) + \sigma(x_n) \right) \sin \left(\sum_{j=0}^n \theta_j(y_n) + \sigma(y_n) \right) \\ = \frac{\sin((b-a)\psi(x))}{2(b-a)\psi(x)} \end{aligned} \quad (5.12)$$

locally uniformly with respect to $x \in U$, and $a, b \in \mathbb{R}$.

Proof Let us fix a compact set $K \subset U$ and $L > 0$. We write

$$\begin{aligned} 2 \cdot \sin \left(\sum_{j=0}^n \theta_j(x_n) + \sigma(x_n) \right) \sin \left(\sum_{j=0}^n \theta_j(y_n) + \sigma(y_n) \right) \\ = \cos \left(\sum_{j=0}^n (\theta_j(x_n) - \theta_j(y_n)) + \sigma(x_n) - \sigma(y_n) \right) \\ - \cos \left(\sum_{j=0}^n (\theta_j(x_n) + \theta_j(y_n)) + \sigma(x_n) + \sigma(y_n) \right). \end{aligned} \quad (5.13)$$

In view of Corollary 5.10, the second term has no contribution to the limit (5.12). To deal with the first term in (5.13), we write

$$\begin{aligned} \cos \left(\sum_{j=0}^n (\theta_j(x_n) - \theta_j(y_n)) + \sigma(x_n) - \sigma(y_n) \right) \\ = \cos \left(\sum_{j=0}^n \theta_j(x_n) - \theta_j(y_n) \right) \cos \left(\sigma(x_n) - \sigma(y_n) \right) \\ - \sin \left(\sum_{j=0}^n \theta_j(x_n) - \theta_j(y_n) \right) \sin \left(\sigma(x_n) - \sigma(y_n) \right). \end{aligned} \quad (5.14)$$

Now, by the continuity of σ , the second term in (5.14) has no contribution to the limit (5.12). Hence, it is enough to show that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\gamma_k}{\sum_{j=0}^n \gamma_j} \cos \left(\sum_{j=0}^n (\theta_j(x_n) - \theta_j(y_n)) \right) = \frac{\sin((b-a)\psi(x))}{(b-a)\psi(x)}$$

uniformly with respect to $x \in K$, and $a, b \in [-L, L]$. We first prove the following claim. $\square$

Claim 5.12 *There is $c > 0$ such that for all $j \in \mathbb{N}$ and $n \in \mathbb{N}$,*

$$\left| \theta_j(y_n) - \theta_j(x_n) - (b-a)\theta'_j(x) \left(\sum_{\ell=0}^n \gamma_\ell \right)^{-1} \right| \leq c \left(\sum_{\ell=0}^n \gamma_\ell \right)^{-2} \sup_{x \in K} |\theta''_j(x)|. \quad (5.15)$$

For the proof, let us write Taylor's polynomial for θ_j centered at x , that is,

$$\theta_j(y) = \theta_j(x) + \theta'_j(x)(y-x) + E_j(x; y)$$

where

$$|E_j(x; y)| \leq \frac{1}{2} |y-x|^2 \sup_{w \in [x, y]} |\theta''_j(w)|.$$

Therefore,

$$\theta_j(y_n) - \theta_j(x_n) = \theta'_j(x)(y_n - x_n) + E_j(x; y_n) - E_j(x; x_n),$$

which leads to (5.15).

Let us now observe that, by the mean value theorem and Claim 5.12, we have

$$\begin{aligned} & \left| \cos \left(\sum_{j=0}^k \theta_j(y_n) - \theta_j(x_n) \right) - \cos \left((b-a) \left(\sum_{\ell=0}^n \gamma_\ell \right)^{-1} \sum_{j=0}^k \theta'_j(x) \right) \right| \\ & \leq \sum_{j=0}^k \left| \theta_j(y_n) - \theta_j(x_n) - (b-a) \left(\sum_{\ell=0}^n \gamma_\ell \right)^{-1} \theta'_j(x) \right| \\ & \leq c \left(\sum_{\ell=0}^n \gamma_\ell \right)^{-2} \sum_{j=0}^k \sup_{x \in K} |\theta''_j(x)|. \end{aligned}$$

Hence,

$$\begin{aligned} & \left| \sum_{k=0}^n \frac{\gamma_k}{\sum_{j=0}^n \gamma_j} \left(\cos \left(\sum_{j=0}^k \theta_j(x_n) - \theta_j(y_n) \right) - \cos \left((b-a) \left(\sum_{\ell=0}^n \gamma_\ell \right)^{-1} \sum_{j=0}^k \theta'_j(x) \right) \right) \right| \\ & \leq c \left(\sum_{\ell=0}^n \gamma_\ell \right)^{-3} \sum_{k=0}^n \gamma_k \sum_{j=0}^k \sup_{x \in K} |\theta''_j(x)|. \end{aligned}$$

Now, by the Stolz–Cesàro theorem, we have

$$\lim_{n \rightarrow \infty} \frac{\gamma_n}{\sum_{j=0}^n \gamma_j} = \lim_{n \rightarrow \infty} \frac{\gamma_n - \gamma_{n-1}}{\gamma_n} = 0,$$

thus, by repeated application of the Stolz–Cesàro theorem we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sum_{\ell=0}^n \gamma_{\ell} \right)^{-3} \sum_{k=0}^n \gamma_k \sum_{j=0}^k \sup_{x \in K} |\theta_j''(x)| &= \frac{1}{3} \lim_{n \rightarrow \infty} \left(\sum_{\ell=0}^n \gamma_{\ell} \right)^{-2} \sum_{j=0}^n \sup_{x \in K} |\theta_j''(x)| \\ &= \frac{1}{6} \lim_{n \rightarrow \infty} \gamma_n^{-1} \left(\sum_{\ell=0}^n \gamma_{\ell} \right)^{-1} \sup_{x \in K} |\theta_n''(x)|. \end{aligned}$$

In view of (c), it is enough to show that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{\gamma_k}{\sum_{j=0}^n \gamma_j} \cos \left((b-a) \left(\sum_{\ell=0}^n \gamma_{\ell} \right)^{-1} \sum_{j=0}^k \theta_j'(x) \right) = \frac{\sin((b-a)\psi(x))}{(b-a)\psi(x)}.$$

For the proof, we write

$$\begin{aligned} &\sum_{k=0}^n \frac{\gamma_k}{\sum_{j=0}^n \gamma_j} \cos \left((b-a) \left(\sum_{\ell=0}^n \gamma_{\ell} \right)^{-1} \sum_{j=0}^k \theta_j'(x) \right) \\ &= \sum_{k=0}^n \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} (b-a)^{2m} \gamma_k \left(\sum_{\ell=0}^n \gamma_{\ell} \right)^{-2m-1} \left(\sum_{j=0}^k \theta_j'(x) \right)^{2m} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} (b-a)^{2m} \left(\sum_{\ell=0}^n \gamma_{\ell} \right)^{-2m-1} \sum_{k=0}^n \gamma_k \left(\sum_{j=0}^k \theta_j'(x) \right)^{2m}. \end{aligned}$$

We now claim the following.

Claim 5.13 *For each $m \in \mathbb{N}$,*

$$\lim_{n \rightarrow \infty} \left(\sum_{\ell=0}^n \gamma_{\ell} \right)^{-2m-1} \sum_{k=0}^n \gamma_k \left(\sum_{j=0}^k \theta_j'(x) \right)^{2m} = \frac{1}{2m+1} (\psi(x))^{2m}. \quad (5.16)$$

By (b) and the Stolz–Cesàro theorem, we get

$$\lim_{n \rightarrow \infty} \frac{\sum_{j=0}^n \theta_j'(x)}{\sum_{\ell=0}^n \gamma_{\ell}} = \psi(x).$$

Since

$$\begin{aligned} \frac{1}{2m+1} \left(\frac{\sum_{j=0}^n \theta_j'(x)}{\sum_{\ell=0}^n \gamma_{\ell}} \right)^{2m} &\leq \frac{\gamma_n \left(\sum_{j=0}^n \theta_j'(x) \right)^{2m}}{\left(\sum_{\ell=0}^n \gamma_{\ell} \right)^{2m+1} - \left(\sum_{\ell=0}^{n-1} \gamma_{\ell} \right)^{2m+1}} \\ &\leq \frac{1}{2m+1} \left(\frac{\sum_{j=0}^n \theta_j'(x)}{\sum_{\ell=0}^{n-1} \gamma_{\ell}} \right)^{2m}, \end{aligned}$$

we get

$$\lim_{n \rightarrow \infty} \frac{\gamma_n \left(\sum_{j=0}^n \theta'_j(x) \right)^{2m}}{\left(\sum_{\ell=0}^n \gamma_\ell \right)^{2m+1} - \left(\sum_{\ell=0}^{n-1} \gamma_\ell \right)^{2m+1}} = \frac{1}{2m+1} (\psi(x))^{2m}.$$

Therefore, another application of the Stolz–Cesàro theorem leads to (5.16).

Let us notice that for some $c > 0$,

$$\sum_{k=0}^n \gamma_k \left(\sum_{j=0}^k \theta'_j(x) \right)^{2m} \leq c^{2m} \left(\sum_{k=0}^n \gamma_k \right)^{2m+1},$$

thus, we have the estimate

$$\left| \frac{(-1)^m}{(2m)!} (b-a)^{2m} \left(\sum_{\ell=0}^n \gamma_\ell \right)^{-2m-1} \sum_{k=0}^n \gamma_k \left(\sum_{j=0}^k \theta'_j(x) \right)^{2m} \right| \leq \frac{1}{(2m)!} (b-a)^{2m} c^{2m}.$$

Hence, by the dominated convergence theorem and Claim 5.13, we can compute

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} (b-a)^{2m} \left(\sum_{\ell=0}^n \gamma_\ell \right)^{-2m-1} \sum_{k=0}^n \gamma_k \left(\sum_{j=0}^k \theta'_j(x) \right)^{2m} \\ &= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m+1)!} ((b-a)\psi(x))^{2m} \\ &= \frac{\sin((b-a)\psi(x))}{(b-a)\psi(x)}, \end{aligned}$$

which finishes the proof of the theorem. $\square$

5.4 Christoffel–Darboux Kernel

Let us recall the definition

$$\theta_n(x) = \arccos \left(\frac{\operatorname{tr} X_n(x)}{2\sqrt{\det X_n(x)}} \right).$$

Proposition 5.14 *Let A be a Jacobi matrix with N -periodically modulated entries. Then for every compact subset $K \subset \mathbb{R}$, we have*

$$\lim_{n \rightarrow \infty} \sup_{x \in K} \left| \frac{a_n}{\alpha_n} \theta'_n(x) + \frac{\operatorname{tr} \mathfrak{X}'_0(0)}{N\sqrt{-\operatorname{discr} \mathfrak{X}_0(0)}} \right| = 0, \quad (5.17)$$

and

$$\sup_{x \in K} |\theta_n''(x)| \leq c \left(\frac{\alpha_n}{a_n} \right)^2 \quad (5.18)$$

for some $c > 0$.

Proof Let us fix $i \in \{0, 1, \dots, N-1\}$. Since

$$\det X_{kN+i}(x) = \frac{a_{kN+i-1}}{a_{(k+1)N+i-1}}, \quad (5.19)$$

by Proposition 3.7, we conclude that

$$\lim_{k \rightarrow \infty} \det X_{kN+i}(x) = 1. \quad (5.20)$$

The chain rule applied to (5.1) leads to

$$\begin{aligned} \theta'_{kN+i}(x) &= - \left(4 - \left(\frac{\operatorname{tr} X_{kN+i}(x)}{\sqrt{\det X_{kN+i}(x)}} \right)^2 \right)^{-1/2} \frac{\operatorname{tr} X'_{kN+i}(x)}{\sqrt{\det X_{kN+i}(x)}} \\ &= - \frac{\operatorname{tr} X'_{kN+i}(x)}{\sqrt{-\operatorname{discr} X_{kN+i}(x)}}, \end{aligned}$$

thus, by (5.20) and Corollary 3.10, we obtain (5.17). Consequently, in view of (5.19),

$$\theta''_{kN+i}(x) = - \frac{\operatorname{tr} X''_{kN+i}(x)}{\sqrt{-\operatorname{discr} X_{kN+i}(x)}} - \frac{(\operatorname{tr} X'_{kN+i}(x))^2 \operatorname{tr} X_{kN+i}(x)}{(-\operatorname{discr} X_{kN+i}(x))^{3/2}}.$$

Therefore, the estimate (5.18) is a consequence of Corollary 3.10. $\square$

Theorem 5.15 *Let A be a Jacobi matrix with N -periodically modulated entries. Suppose that for each $i \in \{0, 1, \dots, N-1\}$,*

$$\left(\frac{a_{jN+i-1}}{a_{jN+i}} : j \in \mathbb{N} \right), \left(\frac{b_{jN+i}}{a_{jN+i}} : j \in \mathbb{N}_0 \right), \left(\frac{1}{a_{jN+i}} : j \in \mathbb{N}_0 \right) \in \mathcal{D}_1 \quad (5.21)$$

and

$$\sum_{j=0}^{\infty} \frac{1}{a_j} = \infty. \quad (5.22)$$

If $|\operatorname{tr} \mathfrak{X}_0(0)| < 2$, then

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n} K_n \left(x + \frac{u}{\rho_n}, x + \frac{v}{\rho_n} \right) = \frac{\omega'(0)}{\mu'(x)} \cdot \frac{\sin((u-v)\pi\omega'(0))}{(u-v)\pi\omega'(0)}$$

locally uniformly with respect to $x, u, v \in \mathbb{R}$, where

$$\rho_n = \sum_{j=0}^n \frac{\alpha_j}{a_j},$$

and ω is the equilibrium measure corresponding to $\sigma_{ess}(\mathfrak{A})$ with $\mathfrak{A}$ being the Jacobi matrix associated with $(\alpha_n : n \in \mathbb{N}_0)$ and $(\beta_n : n \in \mathbb{N}_0)$.

Proof Let us fix a compact set $K \subset \Lambda$ with non-empty interior and $L > 0$. Let $\tilde{K} \subset \Lambda$ be a compact set containing K in its interior. There is $n_0 > 0$ such that for all $x \in K$, $n \geq n_0$, $u \in [-L, L]$, and $i \in \{0, 1, \dots, N-1\}$,

$$x + \frac{u}{N\alpha_i \rho_{i;n}}, x + \frac{u}{\rho_{nN+i}} \in \tilde{K}.$$

Given $u, v \in [-L, L]$, we set

$$\begin{aligned} x_{i;n} &= x + \frac{u}{N\alpha_i \rho_{i;n}}, & x_{nN+i} &= x + \frac{u}{\rho_{nN+i}}, \\ y_{i;n} &= x + \frac{v}{N\alpha_i \rho_{i;n}}, & y_{nN+i} &= x + \frac{v}{\rho_{nN+i}}. \end{aligned}$$

Remark 4.5 together with (5.21) entails that $(X_{jN+i} : j \in \mathbb{N})$ belongs to $\mathcal{D}_1$. Moreover,

$$\lim_{k \rightarrow \infty} X_{kN+i}(x) = \mathfrak{X}_i(0)$$

uniformly with respect to $x \in K$. Let us recall that

$$\text{discr } \mathfrak{X}_i(x) = \text{discr } \mathfrak{X}_0(x).$$

In view of (4.12), the Carleman condition (5.22) implies that

$$\lim_{k \rightarrow \infty} \rho_{i;k} = \infty.$$

Hence, by Theorem 5.1, there are $c > 0$ and $M \in \mathbb{N}$ such that for all $x, y \in \tilde{K}$ and $k \geq M$,

$$\begin{aligned} p_{kN+i}(x)p_{kN+i}(y) &= \frac{|[\mathfrak{X}_i(0)]_{2,1}|}{\pi \sqrt{-\text{discr } \mathfrak{X}_0(y)}} \cdot \frac{2}{\sqrt{\mu'(x)\mu'(y)}} \\ &\times \frac{1}{a_{(k+1)N+i-1}} \sin \left(\sum_{j=M+1}^k \theta_{jN+i}(x) + \eta(x) \right) \sin \left(\sum_{j=M+1}^k \theta_{jN+i}(y) + \eta(y) \right) \\ &+ E_{kN+i}(x, y) \end{aligned} \quad (5.23)$$

where

$$\sup_{x, y \in \tilde{K}} |E_{kN+i}(x, y)| \leq c \sum_{j \geq k} \sup_{x \in \tilde{K}} \|X_{(j+1)N+i}(x) - X_{jN+i}(x)\|.$$

Hence, we obtain

$$\begin{aligned} \sum_{k=M}^n p_{kN+i}(x) p_{kN+i}(y) &= \frac{|[\mathfrak{X}_i(0)]_{2,1}|}{\pi \sqrt{-\operatorname{discr} \mathfrak{X}_0(x)}} \cdot \frac{2}{\sqrt{\mu'(x)\mu'(y)}} \\ &\times \sum_{k=M}^n \frac{1}{a_{(k+1)N+i-1}} \sin \left(\sum_{j=M+1}^k \theta_{jN+i}(x) + \eta(x) \right) \sin \left(\sum_{j=M+1}^k \theta_{jN+i}(y) + \eta(y) \right) \\ &+ \sum_{k=M}^n \frac{1}{a_{(k+1)N+i-1}} E_{kN+i}(x, y). \end{aligned}$$

Let

$$\gamma_k = \frac{N\alpha_{i-1}}{a_{kN+i-1}}, \quad \text{and} \quad \psi(x) = -\frac{\operatorname{tr} \mathfrak{X}'_0(0)}{N\sqrt{-\operatorname{discr} \mathfrak{X}_0(0)}}.$$

By Proposition 5.14

$$\lim_{n \rightarrow \infty} \sup_{x \in \tilde{K}} \left| \frac{a_{kN+i-1}}{\alpha_{i-1}} \theta'_{kN+i}(x) - \psi(x) \right| = 0.$$

Moreover, by (3.1) and (3.2)

$$|\psi(x)| = \pi \omega'(0).$$

Therefore, by Theorem 5.11 we get

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{N\alpha_i \rho_{i;n}} \sum_{k=M}^n \frac{N\alpha_{i-1}}{a_{kN+i-1}} \sin \left(\sum_{j=M+1}^k \theta_{jN+i}(x_{i-1;n}) + \eta(x_{i-1;n}) \right) \\ &\quad \sin \left(\sum_{j=M+1}^k \theta_{jN+i}(y_{i-1;n}) + \eta(y_{i-1;n}) \right) \\ &= \frac{\sin((u-v)\pi\omega'(0))}{2(u-v)\pi\omega'(0)}. \end{aligned}$$

Now, by uniformness and (4.12), for any $i' \in \{0, 1, \dots, N-1\}$, we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{N\alpha_{i-1}}{\rho_{nN+i}} K_{i';n}(x_{nN+i'}, y_{nN+i'}) \\ &= \frac{|[\mathfrak{X}_i(0)]_{2,1}|}{\pi \mu'(x) \sqrt{-\text{discr} \mathfrak{X}_0(0)}} \cdot \lim_{n \rightarrow \infty} \frac{\sin \left((u-v) \frac{\rho_{nN+i'}}{N\alpha_{i-1}\rho_{i-1;n}} \pi \omega'(0) \right)}{(uv) \frac{\rho_{nN+i'}}{N\alpha_{i-1}\rho_{i-1;n}} \pi \omega'(0)} \\ &= \frac{|[\mathfrak{X}_i(0)]_{2,1}|}{\pi \mu'(x) \sqrt{-\text{discr} \mathfrak{X}_0(0)}} \cdot \frac{\sin((u-v)\pi \omega'(0))}{(u-v)\pi \omega'(0)} \end{aligned} \quad (5.24)$$

uniformly with respect to $x \in K$ and $u, v \in [-L, L]$. Here, we have also used that

$$\sup_{m \in \mathbb{N}} \sup_{x \in \tilde{K}} |p_m(x)| \leq c. \quad (5.25)$$

Since

$$K_{nN+i}(x, y) = \sum_{i'=0}^{N-1} K_{i';n}(x, y) + \sum_{i'=i+1}^{N-1} (K_{i';n-1}(x, y) - K_{i';n}(x, y)),$$

by (5.25) and (5.24), we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{\rho_{nN+i}} K_{nN+i}(x_{nN+i}, y_{nN+i}) \\ &= \frac{1}{N} \sum_{i'=0}^{N-1} \frac{N\alpha_{i'-1}}{\rho_{nN+i}} K_{i';n}(x_{nN+i}, y_{nN+i}) \frac{1}{\alpha_{i'-1}} \\ &= \frac{\sin((u-v)\pi \omega'(0))}{(u-v)\pi \omega'(0)} \cdot \frac{1}{N\pi \mu'(x) \sqrt{-\text{discr} \mathfrak{X}_0(0)}} \sum_{i'=0}^{N-1} \frac{|[\mathfrak{X}_{i'}(0)]_{2,1}|}{\alpha_{i'-1}} \end{aligned}$$

which together with (3.1) concludes the proof. $\square$

Proposition 5.16 *Let A be a Jacobi matrix that is N -periodic blend. Let K be a non-empty compact interval contained in*

$$\Lambda = \{x \in \mathbb{R} : |\text{tr} \mathcal{X}_1(x)| < 2\}$$

where $\mathcal{X}_1$ is the limit of $(X_{j(N+2)+1} : j \in \mathbb{N}_0)$. Then for each $i \in \{1, 2, \dots, N\}$,

$$\lim_{n \rightarrow \infty} \sup_{x \in K} \left| \frac{a_{n(N+2)+i}}{\alpha_i} \theta'_{n(N+2)+i}(x) + \frac{\text{tr} \mathcal{X}'_1(x)}{N \sqrt{-\text{discr} \mathcal{X}_1(x)}} \right| = 0, \quad (5.26)$$

and

$$\sup_{x \in K} |\theta''_{n(N+2)+i}(x)| \leq c \left(\frac{\alpha_i}{a_{n(N+2)+i}} \right)^2 \quad (5.27)$$

for some $c > 0$.

Proof Let us fix $i \in \{1, 2, \dots, N\}$. Since

$$\det X_{k(N+2)+i}(x) = \frac{a_{k(N+2)+i-1}}{a_{(k+1)(N+2)+i-1}}, \quad (5.28)$$

we conclude that

$$\lim_{k \rightarrow \infty} \det X_{k(N+2)+i}(x) = 1. \quad (5.29)$$

The chain rule applied to (5.1) leads to

$$\begin{aligned} \theta'_{k(N+2)+i}(x) &= - \left(4 - \left(\frac{\operatorname{tr} X_{k(N+2)+i}(x)}{\sqrt{\det X_{k(N+2)+i}(x)}} \right)^2 \right)^{-1/2} \frac{\operatorname{tr} X'_{k(N+2)+i}(x)}{\sqrt{\det X_{k(N+2)+i}(x)}} \\ &= - \frac{\operatorname{tr} X'_{k(N+2)+i}(x)}{\sqrt{-\operatorname{discr} X_{k(N+2)+i}(x)}} \end{aligned}$$

thus, by (5.29) and Corollary 3.18, we obtain (5.26). Consequently, in view of (5.28),

$$\theta''_{k(N+2)+i}(x) = - \frac{\operatorname{tr} X''_{k(N+2)+i}(x)}{\sqrt{-\operatorname{discr} X_{k(N+2)+i}(x)}} - \frac{(\operatorname{tr} X'_{k(N+2)+i}(x))^2 \operatorname{tr} X_{k(N+2)+i}(x)}{(-\operatorname{discr} X_{k(N+2)+i}(x))^{3/2}}.$$

Therefore, the estimate (5.27) is a consequence of Corollary 3.18. $\square$

Theorem 5.17 Let A be a Jacobi matrix that is N -periodic blend. Suppose that for each $i \in \{0, 1, \dots, N-1\}$,

$$\left(\frac{1}{a_{j(N+2)+i}} : j \in \mathbb{N}_0 \right), \left(\frac{b_{j(N+2)+i}}{a_{j(N+2)+i}} : j \in \mathbb{N}_0 \right) \in \mathcal{D}_1$$

and

$$\left(\frac{1}{a_{j(N+2)+N}} : j \in \mathbb{N}_0 \right), \left(\frac{1}{a_{j(N+2)+N+1}} : j \in \mathbb{N}_0 \right), \left(\frac{a_{j(N+2)+N}}{a_{j(N+2)+N+1}} : j \in \mathbb{N}_0 \right) \in \mathcal{D}_1.$$

Let K be a compact interval with non-empty interior contained in

$$\Lambda = \{x \in \mathbb{R} : |\operatorname{tr} X_1(x)| < 2\}$$

where X_1 is the limit of $(X_{j(N+2)+1} : j \in \mathbb{N}_0)$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_n} K_n \left(x + \frac{u}{\rho_n}, x + \frac{v}{\rho_n} \right) = \frac{\omega'(x)}{\mu'(x)} \cdot \frac{\sin((u-v)\pi\omega'(x))}{(u-v)\pi\omega'(x)}$$

locally uniformly with respect to $x \in K$, and $u, v \in \mathbb{R}$, where ω is the equilibrium measure corresponding to $\bar{\Lambda}$, and

$$\rho_n = \sum_{i=0}^{N-1} \sum_{\substack{m=0 \\ m \equiv i \pmod{N+2}}}^n \frac{\alpha_m}{a_m}.$$

Proof Let $K \subset \Lambda$ be a compact interval with non-empty interior and let $L > 0$. Let $\tilde{K} \subset \Lambda$ be a compact set containing K in its interior. There is $n_0 > 0$ such that for all $x \in K, n \geq n_0, i \in \{0, 1, \dots, N+1\}$, and $u \in [-L, L]$,

$$x + \frac{u}{\rho_{n(N+2)+i}} \in \tilde{K}.$$

Given $x \in K$ and $u, v \in [-L, L]$, we set

$$x_n = x + \frac{u}{\rho_n}, \quad \text{and} \quad y_n = x + \frac{v}{\rho_n}.$$

For each $i \in \{1, 2, \dots, N\}$ and $i' \in \{0, 1, \dots, N+1\}$, by the reasoning analogous to the proof of Theorem 5.15 one can show that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{N\alpha_{i-1}}{\rho_{i;n}} K_{i;n}(x_{n(N+2)+i'}, y_{n(N+2)+i'}) \\ &= \frac{[X_i(x)]_{2,1}}{\pi \mu'(x) \sqrt{-\text{discr } X_i(x)}} \cdot \frac{\sin((u-v)\pi\omega'(x))}{(u-v)\pi\omega'(x)} \end{aligned} \quad (5.30)$$

uniformly with respect to $x \in K$ and $u, v \in [-L, L]$. By Claim 4.11, for each $i \in \{1, 2, \dots, N\}$ the sequence $(X_{j(N+2)+i} : j \in \mathbb{N}_0)$ belongs to $\mathcal{D}_1(K, \text{GL}(2, \mathbb{R}))$ and converges to X_i satisfying

$$\text{discr } X_i = \text{discr } X_1.$$

For $i = 0$, by (4.26) and Claim 4.12,

$$\begin{aligned} & |K_{0;n}(x, y) - K_{N;n}(x, y)| \\ & \leq c \left(\sum_{j=0}^n |p_{j(N+2)}(x) + p_{j(N+2)+N}(x)| + |p_{j(N+2)}(y) + p_{j(N+2)+N}(y)| \right) \\ & \leq c \sum_{j=0}^k \frac{1}{a_{j(N+2)+N+1}^2} + \left| 1 - \frac{a_{j(N+2)+N}}{a_{j(N+2)+N+1}} \right|. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{N-1;n}} \sup_{x, y \in \tilde{K}} |K_{0;n}(x, y) - K_{N;n}(x, y)| = 0. \quad (5.31)$$

Similarly, for $i = N + 1$, one can show

$$\lim_{n \rightarrow \infty} \frac{1}{\rho_{N-1;n}} \sup_{x, y \in \tilde{K}} |K_{N+1;n}(x, y)| = 0. \quad (5.32)$$

Now, let $i \in \{0, 1, \dots, N + 1\}$. We write

$$K_{n(N+2)+i}(x, y) = \sum_{i'=0}^{N+1} K_{i';n}(x, y) + \sum_{i'=i+1}^{N+1} (K_{i';n}(x, y) - K_{i';n-1}(x, y)).$$

By (4.26), (5.31) and (5.32), we obtain

$$K_{n(N+2)+i}(x, y) = \sum_{i'=1}^{N-1} K_{i';n}(x, y) + 2K_{N;n}(x, y) + o(\rho_{N-1;n})$$

uniformly with respect to $x, y \in \tilde{K}$. Therefore, by (5.30) and (4.12),

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{\rho_{n(N+2)+i}} K_{n(N+2)+i}(x_{n(N+2)+i}, y_{n(N+2)+i}) \\ & = \sum_{i'=0}^{N-1} \lim_{n \rightarrow \infty} \frac{N\alpha_{i'-1}}{\rho_{i';n}} K_{i';n}(x_{n(N+2)+i}, y_{n(N+2)+i}) + 2 \lim_{n \rightarrow \infty} \frac{N\alpha_{N-1}}{\rho_{N-1;n}} K_{N-1;n}(x_{n(N+2)+i}, y_{n(N+2)+i}) \\ & = \frac{\sin((u-v)\pi\omega'(x))}{(u-v)\pi\omega'(x)} \cdot \frac{1}{\pi\mu'(x)\sqrt{-\operatorname{discr}\mathcal{X}_1(x)}} \left(\sum_{i'=1}^{N-1} \frac{|[X_{i'}(x)]_{2,1}|}{\alpha_{i'-1}} + 2 \frac{|[X_N(x)]_{2,1}|}{\alpha_{N-1}} \right) \end{aligned}$$

which together with Theorem 3.13 finishes the proof. $\square$

Acknowledgements The first author was partially supported by the Foundation for Polish Science (FNP) and by long term structural funding, Methusalem grant of the Flemish Government.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Boutet de Monvel, A., Janas, J., Naboko, S.: Unbounded Jacobi matrices with a few gaps in the essential spectrum: constructive examples. *Integral Equ. Oper. Theory* **69**(2), 151–170 (2011)
2. Dombrowski, J.: Quasitriangular matrices. *Proc. Am. Math. Soc.* **69**(1), 95–96 (1978)
3. Ignjatović, A.: Asymptotic behaviour of some families of orthonormal polynomials and an associated Hilbert space. *J. Approx. Theory* **210**, 41–79 (2016)
4. Ignjatovic, A., Lubinsky, D.S.: On an asymptotic equality for reproducing kernels and sums of squares of orthonormal polynomials. In: Govil, N., Mohapatra, R., Qazi, M., Schmeisser, G. (eds.) *Progress in Approximation Theory and Applicable Complex Analysis*. Springer Optim. Appl., vol. 117, pp. 129–144. Springer, Cham (2017)
5. Janas, J., Naboko, S.: Spectral analysis of selfadjoint Jacobi matrices with periodically modulated entries. *J. Funct. Anal.* **191**(2), 318–342 (2002)
6. Kuijlaars, A.B.J., McLaughlin, K.T.-R., Van Assche, W., Vanlessen, M.: The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1, 1]$. *Adv. Math.* **188**(2), 337–398 (2004)
7. Kuijlaars, A.B.J., Vanlessen, M.: Universality for eigenvalue correlations from the modified Jacobi unitary ensemble. *Int. Math. Res. Not.* **2002**(30), 1575–1600 (2002)
8. Levin, E., Lubinsky, D.S.: *Orthogonal Polynomials for Exponential Weights*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 4. Springer, New York (2001)
9. Levin, E., Lubinsky, D.S.: Universality limits in the bulk for varying measures. *Adv. Math.* **219**(3), 743–779 (2008)
10. Levin, E., Lubinsky, D.S.: Universality limits for exponential weights. *Constr. Approx.* **29**(2), 247–275 (2009)
11. Lubinsky, D.S.: A new approach to universality limits involving orthogonal polynomials. *Ann. Math.* **170**(2), 915–939 (2009)
12. Lubinsky, D.S.: An update on local universality limits for correlation functions generated by unitary ensembles, *SIGMA Symmetry Integrability Geom. Methods Appl.* **12**, Paper No. 078, 36 (2016)
13. Mason, J.C., Handscomb, D.C.: *Chebyshev Polynomials*. Chapman and Hall/CRC, Boca Raton (2002)
14. Máté, A., Nevai, P., Totik, V.: Szegő's extremum problem on the unit circle. *Ann. Math.* (2) **134**(2), 433–453 (1991)
15. Mureşan, M.: *A Concrete Approach to Classical Analysis*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York (2009)
16. Saff, E.B., Totik, V.: *Logarithmic Potentials with External Fields*, vol. 316. Springer, Berlin (1997)
17. Schmüdgen, K.: *Unbounded Self-Adjoint Operators on Hilbert Space*. Graduate Texts in Mathematics, vol. 265. Springer, Cham (2012)
18. Schmüdgen, K.: *The Moment Problem*. Graduate Texts in Mathematics, vol. 277. Springer, Cham (2017)
19. Simon, B.: The Christoffel–Darboux kernel, perspectives in partial differential equations, harmonic analysis and applications. In: *Proc. Sympos. Pure Math.*, vol. 79. Amer. Math. Soc., Providence, RI, pp. 295–335 (2008)
20. Simon, B.: *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*. Princeton University Press, Princeton (2010)
21. Stahl, H., Totik, V.: *General orthogonal polynomials*, *Encyclopedia of Mathematics and its Applications*, vol. 43. Cambridge University Press, Cambridge (1992)
22. Stolz, G.: Spectral theory for slowly oscillating potentials I. Jacobi matrices. *Manuscr. Math.* **84**, 245–260 (1994)

23. Świdorski, G.: Periodic perturbations of unbounded Jacobi matrices II: formulas for density. *J. Approx. Theory* **216**, 67–85 (2017)
24. Świdorski, G.: Periodic perturbations of unbounded Jacobi matrices III: the soft edge regime. *J. Approx. Theory* **233**, 1–36 (2018)
25. Świdorski, G., Trojan, B.: About essential spectra of unbounded Jacobi matrices. [arXiv:2006.07959](https://arxiv.org/abs/2006.07959) (2020)
26. Świdorski, G., Trojan, B.: Asymptotic behaviour of Christoffel–Darboux kernel via three-term recurrence relation II. [arXiv:2004.07826](https://arxiv.org/abs/2004.07826) (2020)
27. Świdorski, G., Trojan, B.: Asymptotics of orthogonal polynomials with slowly oscillating recurrence coefficients. *J. Funct. Anal.* **278**(3), 108326 (2020)
28. Totik, V.: Asymptotics of Christoffel functions for general measures on the real line. *J. Anal. Math.* **81**, 283–303 (2000)
29. Totik, V.: Universality and fine zero spacing on general sets. *Ark. Mat.* **47**(2), 361–391 (2009)
30. Van Assche, W.: Christoffel functions and Turán determinants on several intervals. *J. Comput. Appl. Math.* **48**(1), 207–223 (1993)
31. Xu, Sh-X, Zhao, Y.-Q., Zhou, J.-R.: Universality for eigenvalue correlations from the unitary ensemble associated with a family of singular weights. *J. Math. Phys.* **52**(9), 093302 (2011)
32. Zhou, J.-R., Xu, Sh-X, Zhao, Y.-Q.: Uniform asymptotics of a system of Szegő class polynomials via the Riemann–Hilbert approach. *Anal. Appl.* **09**(04), 447–480 (2011)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.