Gegenbauer and Other Planar Orthogonal Polynomials on an Ellipse in the Complex Plane

Gernot Akemann¹ · Taro Nagao² · Iván Parra¹ · Graziano Vernizzi³

Received: 24 May 2019 / Revised: 8 April 2020 / Accepted: 7 May 2020 / Published online: 18 September 2020
© The Author(s) 2020

Abstract

We show that several families of classical orthogonal polynomials on the real line are also orthogonal on the interior of an ellipse in the complex plane, subject to a weighted planar Lebesgue measure. In particular these include Gegenbauer polynomials $C_n^{(1+\alpha)}(z)$ for $\alpha > -1$ containing the Legendre polynomials $P_n(z)$ and the subset $P_n^{(\alpha+\frac{1}{2}, \pm\frac{1}{2})}(z)$ of the Jacobi polynomials. These polynomials provide an orthonormal basis and the corresponding weighted Bergman space forms a complete metric space. This leads to a certain family of Selberg integrals in the complex plane. We recover the known orthogonality of Chebyshev polynomials of the first up to fourth kind. The limit $\alpha \rightarrow \infty$ leads back to the known Hermite polynomials orthogonal in the entire complex plane. When the ellipse degenerates to a circle we obtain the weight function and monomials known from the determinantal point process of the ensemble of truncated unitary random matrices.

Keywords Planar orthogonal polynomials · Ellipse · Bergman space · Selberg integral

Communicated by Peter Forrester.

✉ Gernot Akemann
akemann@physik.uni-bielefeld.de

Taro Nagao
nagao@math.nagoya-u.ac.jp

Iván Parra
iparra@physik.uni-bielefeld.de

Graziano Vernizzi
gvernizzi@siena.edu

¹ Faculty of Physics, Bielefeld University, P.O. Box 100131, 33501 Bielefeld, Germany

² Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan

³ Siena College, 515 Loudon Road, Loudonville, NY 12211, USA

Mathematics Subject Classification 30C10 · 33C45 · 42C05 · 15B52

1 Introduction

Orthogonal polynomials in the complex plane play an important role for non-Hermitian random matrix theory. A prominent example is the elliptic Ginibre ensemble with complex normal matrix elements, having different variances for their real and imaginary parts, [23]. Its complex eigenvalues follow a determinantal point process, with its kernel constituted by the Hermite polynomials orthogonal in the complex plane [9]. Likewise, the chiral partner of this ensemble leads to a kernel of generalised Laguerre polynomials orthogonal in the complex plane [2,21] and [13]. The respective kernels allow for a complete characterisation of all complex eigenvalue correlation functions of these ensembles of random matrices. Moreover, in the limit of weak non-Hermiticity introduced in [8], these nontrivial polynomials allow us to study an interpolation between the statistics of real eigenvalues of Hermitian random matrices on the one hand, e.g., of the Gaussian Unitary Ensemble characterised by Hermite polynomials on the real line, and those of complex eigenvalues, e.g., of the Ginibre ensemble, being characterised by monomial polynomials in the complex plane. We refer to [3] for a list of interpolating limiting kernels known to date.

In this paper we ask the question whether further classical orthogonal polynomials on the real line also form a set of orthogonal polynomials on a two dimensional domain in the complex plane. Orthogonal polynomials on the real line or subsets thereof, as well as those on one-dimensional curves on the complex plane—typically the unit circle—are a classical topic in mathematics [25]. Therefore, it is quite surprising that relatively few works have addressed this question. The orthogonality of Chebyshev polynomials of the second kind on the interior of an unweighted ellipse probably goes back to [11]. The fact that Hermite polynomials are also orthogonal with respect to a Gaussian weight in the complex plane was first shown in 1990 [6], see [5] for an independent proof. Generalised Laguerre polynomials in the complex plane were found in the context of applications to quantum field theory in [21], see [2] for a concise orthogonality proof. The orthogonality of all Chebyshev polynomials of the first to fourth kind on an ellipse can be found in [17].

While the Gram–Schmidt construction of orthogonal polynomials on any subset of the real line and in the complex plane is completely analogous, given that all moments exist, see [26], the fact that the former always satisfy a three-step recursion relation is special. While Lempert [16] showed that we cannot expect any finite term recurrence for orthogonal polynomials in the complex plane in general, it was shown much more recently that the existence of a finite term recurrence relation on an unweighted bounded domain with sufficiently regular boundary implies that the domain is an ellipse and the recursion depth is three [14,22]. This suggests searching for elliptic domains as our polynomials originating from the real line do have a three step recurrence. We note, however, that the aforementioned results [14,22] only apply to unweighted domains. It is an open question whether orthogonal polynomials with finite-term recurrences exist on weighted, bounded domains other than the ellipse. For

the Chebyshev polynomials of first, third and fourth kind, the weight function on the ellipse is no longer flat [17].

In this work we obtain the following results. We show that the classical Gegenbauer or ultraspherical polynomials $C_n^{(1+\alpha)}(z)$, for $\alpha > -1$, provide a family of planar orthogonal polynomials on the interior of an ellipse parametrised by $h(z) := (\operatorname{Re} z)^2/a^2 + (\operatorname{Im} z)^2/b^2$, with $a > b > 0$ and weight function $(1 - h(z))^\alpha$. They generalise the monomials that appear in the determinantal point process on the unit disc, obtained from the ensemble of truncated unitary random matrices [27]. Furthermore, we find a subset of the Jacobi polynomials to be orthogonal on a weighted ellipse. These findings allow us to recover the orthogonality of all four Chebyshev polynomials from [17]. All these planar orthogonal polynomials lead to examples for Selberg- (or Mehta-) type integrals in the complex plane containing a Vandermonde determinant modulus squared, when determining the normalisation of the corresponding determinantal point processes; see [7] for a review of Selberg integrals.

As an application we use the Gegenbauer polynomials to construct further (non-classical) planar orthogonal polynomials on a weighted ellipse that do not satisfy a recursion relation of finite depth. In that sense, the ellipse is not a special domain in the complex plane once nontrivial weight functions are allowed. At present we do not know a random matrix model that leads to a determinantal point process with a kernel of Gegenbauer or a subset of Jacobi polynomials, apart from a trivial normal matrix representation. In this work we restrict ourselves to polynomials of finite degree. The asymptotics of the Bergman kernel in the limit of weak non-Hermiticity, both in the bulk and at the edge of the ellipse, will be presented elsewhere [19].

The remainder of this article is organised as follows. To prepare the ground, in Sect. 2 we show that the weighted Bergman space on the ellipse is a complete metric space. To that aim, in Sect. 3 we prove the orthogonality of the Gegenbauer polynomials $C_n^{(1+\alpha)}(z)$ of even degree, for $\alpha > -1$, with respect to the inner product on the weighted ellipse. The case with an odd degree is very similar and presented in “Appendix A”. This immediately implies the orthogonality of Legendre polynomials $P_n(z)$ as well, and we recover the orthogonality of Chebyshev polynomials of the second kind $U_n(z)$. In “Appendix C” an alternative orthogonality proof for Gegenbauer polynomials independent of the degree is given which in contrast relies on the known orthogonality of the Chebyshev polynomials of the second kind on the unweighted ellipse. The proof of the latter from [12] is collected in “Appendix B” for completeness.

In Sect. 4 we prove that two families of particular Jacobi polynomials $P_n^{(\alpha+\frac{1}{2}, \pm\frac{1}{2})}(z)$, for $\alpha > -1$, are orthogonal on weighted ellipses. The known orthogonality of the Chebyshev polynomials of the third, fourth and first kind, $V_n(z)$, $W_n(z)$ and $T_n(z)$ respectively, follow as a consequence. In Sect. 5 we construct an explicit example for orthogonal polynomials on a weighted ellipse that do not satisfy a recursion relation of finite depth. The construction is based on Gegenbauer polynomials and the Heine-formula for planar orthogonal polynomials. Here, we also present the Selberg integral based on the family of Gegenbauer polynomials $C_n^{(1+\alpha)}(z)$ as an example that can be analytically continued in α .

2 Weighted Bergman Space on the Interior of an Ellipse

To begin let us fix some notation for the quantities to be considered. For $a > b > 0$ the function

$$h(z) := (\operatorname{Re} z)^2/a^2 + (\operatorname{Im} z)^2/b^2 \quad (2.1)$$

provides an explicit parametrisation of the interior of an ellipse E :

$$E = \{z \in \mathbb{C} : h(z) < 1\}. \quad (2.2)$$

For $0 < p < \infty$ and $-1 < \alpha < \infty$, we will denote by $A_\alpha^p := A_\alpha^p(E) \subseteq L^p(E, dA_\alpha)$ the (weighted) Bergman space of the ellipse E ; i.e., the subspace of analytic functions in $L^p(E, dA_\alpha)$ with finite p -norm. The area measure

$$dA_\alpha(z) = (1 + \alpha)(1 - h(z))^\alpha dA(z) \quad (2.3)$$

is defined in terms of the normalised area measure on the ellipse $dA(z) = dx dy / (\pi ab)$, with $z = x + iy$, together with $h(z)$ defined in the parametrisation of the ellipse (2.1). It is not difficult to see that it is normalised $\forall \alpha > -1$:

$$\int_E dA_\alpha(z) = \frac{1 + \alpha}{\pi} \int_0^1 dr r \int_0^{2\pi} d\theta (1 - r^2)^\alpha = 1, \quad (2.4)$$

after changing variables to

$$x = ar \cos(\theta), \quad y = br \sin(\theta), \quad (2.5)$$

with $r \in [0, 1)$, $\theta \in [0, 2\pi]$ and Jacobian $J(r, \theta) = abr$. For $1 \leq p < \infty$ the associated L^p -norm is defined by

$$\|f\|_{p,\alpha} = \left(\int_E |f(z)|^p dA_\alpha(z) \right)^{1/p}, \quad (2.6)$$

and for $0 < p < 1$ the corresponding metric is given by

$$d(f, g) = \int_E |f(z) - g(z)|^p dA_\alpha(z). \quad (2.7)$$

In this section we show that the Bergman space A_α^p is a Banach space when $1 \leq p < \infty$, and a complete metric space when $0 < p < 1$. The proof is quite standard and follows the lines of Corollary 1.12 and Proposition 1.13 in [4].

Proposition 2.1 *Let $0 < p < \infty$ and $-1 < \alpha < \infty$, and K be a compact subset of E , with positive minimum distance to ∂E . Then there is a positive constant C such that*

$$\sup_K |f(z)|^p \leq C \|f\|_{p,\alpha}^p,$$

for all $f \in A_\alpha^p$.

Proof Let $t \in E$ and $0 < \varrho < \text{dist}(t, \partial E) =: d$ be arbitrary. We define the smaller ellipse

$$E_\varrho = \{z \in \mathbb{C} : h_\varrho(z) := (\text{Re } z)^2/(a - \varrho/2)^2 + (\text{Im } z)^2/(b - \varrho/2)^2 \leq 1\}, \quad (2.8)$$

and suppose that there is a point $z_0 \in B(t, \varrho/2) \setminus E_\varrho$ such that

$$\{|z_0 - w| : w \in \partial E\} \subseteq \{|z - w| : z \in B(t, \varrho/2); w \in \partial E\}. \quad (2.9)$$

Taking the infimum on both sides of (2.9), we obtain

$$\text{dist}(B(t, \varrho/2), \partial E) \leq \text{dist}(z_0, \partial E). \quad (2.10)$$

But (2.10) implies that $d - \varrho/2 \leq \varrho/2$, therefore $B(t, \varrho/2) \subseteq E_\varrho$. In consequence we obtain

$$\sup_{z \in B(t, \varrho/2)} h(z) \leq \sup_{z \in E_\varrho} h(z) \leq h(z_*) =: c(\varrho), \quad z_* \in \partial E_\varrho. \quad (2.11)$$

It is easy to see that $0 < c(\varrho) < 1$, and it can be computed explicitly by introducing a Lagrange multiplier, for example.

Thus, given $f \in A_\alpha^p$, $B(t, r) \subseteq E$ with positive minimum distance to the boundary ∂E , i.e., $0 < \varrho < \text{dist}(B(t, r), \partial E)$, we can find another positive constant $C > 0$ such that

$$\begin{aligned} |f(z)|^p &\leq \frac{4}{\pi \varrho^2} \int_{B(z, \varrho/2)} |f(w)|^p \, dA(w) \\ &\leq C \int_{B(z, \varrho/2)} |f(w)|^p \, dA_\alpha(w) \\ &\leq C \int_E |f(w)|^p \, dA_\alpha(w) \\ &= C \|f\|_{p,\alpha}^p \quad \text{for } z \in B(t, r). \end{aligned} \quad (2.12)$$

In the first step we have used the subharmonicity of $|f|^p$. In the second step the upper bound is trivial for negative $-1 < \alpha < 0$, due to $0 \leq h(z)$, whereas for positive $\alpha > 0$ we have used the estimate from (2.11). $\square$

One immediate consequence of Proposition 2.1 is that any Cauchy sequence $\{f_n\} \in A_\alpha^p$ is locally bounded, and so by Montel's Theorem it constitutes a normal family. Thus, some subsequence converges locally uniformly in E , to a function in A_α^p , and we have the following.

Corollary 2.2 *For every $0 < p < \infty$, $-1 < \alpha < \infty$, the weighted Bergman space A_α^p is closed in $L^p(E, dA_\alpha)$.*

Proof Let $\{f_n\}$ be a Cauchy sequence in A_α^p and $f \in L^p(E, dA_\alpha)$, such that $\int |f_n - f|^p dA_\alpha \rightarrow 0$ as $n \rightarrow \infty$. By Montel's Theorem $\{f_n\}$ converges locally uniformly to a function g that is analytic in E . Since $\|f_n - f\|_{p,\alpha}^p \rightarrow 0$, this implies that f_n converges in measure to f . By Riesz' Theorem there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(z) \rightarrow f(z)$ almost everywhere. Thus $f = g$ almost everywhere and so $f \in A_\alpha^p$. $\square$

For $p \geq 1$ it follows from Corollary 2.2 that the Bergman space is a Banach space, and in particular for $p = 2$ a Hilbert space. In the next section we will consider the Bergman space for $p = 2$, A_α^2 , as a Hilbert space, with the notion for the inner product defined as

$$\langle f, g \rangle_\alpha := \int_E f(z) \overline{g(z)} dA_\alpha(z), \quad (2.13)$$

for two integrable functions $f, g \in A_\alpha^2$.

For analyticity it is of course important that we consider the interior of the ellipse E (2.2), being an open set. Because the boundary of the ellipse ∂E is one-dimensional and of measure zero in the complex plane, all integrals over E and $\overline{E}$ agree, i.e.,

$$\int_E f(z) \overline{g(z)} dA_\alpha(z) = \int_{\overline{E}} f(z) \overline{g(z)} dA_\alpha(z), \quad (2.14)$$

for $-1 < \alpha$. We will come back to this point when relating to the ensemble of truncated unitary matrices in Remark 3.6.

3 Orthogonality of Gegenbauer and Legendre Polynomials

For any non-negative integer n and real parameter $\alpha > -1$ let us define the polynomials

$$p_n^{(\alpha)}(z) := \frac{1}{\sqrt{h_n}} C_n^{(1+\alpha)}\left(\frac{z}{c}\right), \quad (3.1)$$

where $C_n^{(1+\alpha)}(x)$ are the standard Gegenbauer polynomials on the real line having real coefficients, now taken with a complex argument. We recall that the ellipse E in (2.2) defining the inner product (2.13) is parametrised by the real numbers $a > b > 0$. The constant

$$c := \sqrt{a^2 - b^2} > 0 \quad (3.2)$$

then gives the location of the right focus of the ellipse E , and we define by

$$h_n := h_n(a, b) = \frac{1 + \alpha}{1 + \alpha + n} C_n^{(1+\alpha)}\left(\frac{a^2 + b^2}{a^2 - b^2}\right) > 0, \quad (3.3)$$

the norms of the Gegenbauer polynomials in the complex plane. Their positivity follows from (3.21), and a short argument goes as follows. Because the Gegenbauer polynomials (3.1) have all their zeros in $(-1, 1)$, the fact that the argument $(a^2 + b^2)/(a^2 - b^2) > 1$ in (3.3) is to the right of this interval, together with the

positivity of the leading coefficient of the $C_n^{(1+\alpha)}(x)$ there [20], leads to the positivity of h_n for all integers $n \geq 0$. We claim the following

Theorem 3.1 *The set of polynomials $\{p_n^{(\alpha)}\}_{n \in \mathbb{N}}$ defined in (3.1) forms an orthonormal basis for A_α^2 for any $\alpha > -1$.*

In view of the previous subsection we need to prove the orthonormality and completeness of the basis. The former is shown in the following lemma, whereas the completeness is deferred to the very end of this section.

Lemma 3.2 *For the sequence of Gegenbauer polynomials $\{C_n^{(1+\alpha)}\}_{n \in \mathbb{N}}$, with $-1 < \alpha$, on the domain (2.2) with weight (2.3), the following orthogonality relation holds*

$$\int_E C_m^{(1+\alpha)}\left(\frac{z}{c}\right) C_n^{(1+\alpha)}\left(\frac{\bar{z}}{c}\right) dA_\alpha(z) = \frac{1+\alpha}{1+\alpha+n} C_n^{(1+\alpha)}\left(\frac{a^2+b^2}{a^2-b^2}\right) \delta_{nm} \quad (3.4)$$

where $a > b > 0$ and $c = \sqrt{a^2 - b^2}$.

Remark 3.3 In this section we will present an elementary proof of the orthogonality relation (3.4). Due to the reflection symmetry of the weight function, domain and parity of the polynomials, the proof can be split into even and odd polynomials separately. Because these two cases are very similar we only present the one for the even polynomials in the main body of the paper here. For completeness we have put the proof for the odd polynomials into the “Appendix A”. A second independent proof valid for the orthogonality of the even and odd polynomials alike is presented in “Appendix C”. It starts by assuming the known orthogonality of Chebyshev polynomials of the second kind U_n on the unweighted ellipse, which can be found in [11] and that is reproduced for completeness in “Appendix B”. While the orthogonality proof in “Appendix C” is more elegant, there the determination of the norms h_n is much more cumbersome and therefore will not be presented.

In addition to the proof presented below, the orthogonality of the Chebyshev polynomials of the first up to fourth kind follows as a corollary, as demonstrated in Sect. 4. This establishes an independent proof of [17], where the orthogonality of all four kind was shown previously.

Proof It is sufficient to show that for all $m \in \mathbb{N}$

$$\int_E C_m^{(1+\alpha)}\left(\frac{z}{c}\right) \left(\frac{\bar{z}}{c}\right)^j dA_\alpha(z) = 0, \quad \text{for } j = 0, 1, \dots, m-1. \quad (3.5)$$

Since both the weight function (2.3) and domain (2.2) are invariant under the reflection $z \rightarrow -z$, and the polynomials have parity, $C_n^{(1+\alpha)}(-z) = (-1)^n C_n^{(1+\alpha)}(z)$, without restriction we assume that either $m = 2n$ and $j = 2l$ are both even, or $m = 2n + 1$ and $j = 2l + 1$ are both odd, and $l < n$. In the following we will only present the even-even case. The odd-odd case follows from the same line of arguments and is collected in “Appendix A” for the reader’s convenience.

We rewrite the integral (3.5) with $z = x + iy$ in terms of elliptic coordinates. With this aim we change variables as follows:

$$x = ar \cos(\theta), \quad y = br \sin(\theta), \quad \text{with } r \in [0, 1), \quad \theta \in [0, 2\pi]. \quad (3.6)$$

The Jacobian for this transformation reads $J(r, \theta) = abr$, and we obtain for the complex arguments

$$\frac{z(r, \theta)}{c} = \frac{r}{2}(Re^{i\theta} + R^{-1}e^{-i\theta}), \quad \text{with } R := \frac{a+b}{c} = \sqrt{\frac{a+b}{a-b}}. \quad (3.7)$$

We also obtain $h(z) = r^2$ from (2.1). This leads to the following expression

$$\begin{aligned} \int_E C_{2n}^{(1+\alpha)}\left(\frac{z}{c}\right)\left(\frac{\bar{z}}{c}\right)^{2l} dA_\alpha(z) &= \\ &= \frac{1+\alpha}{\pi} \int_0^1 dr r \int_0^{2\pi} d\theta C_{2n}^{(1+\alpha)}\left(\frac{z(r, \theta)}{c}\right)\left(\frac{\overline{z(r, \theta)}}{c}\right)^{2l} (1-r^2)^\alpha. \end{aligned} \quad (3.8)$$

The even Gegenbauer polynomials can be written in terms of Gauß' hypergeometric function in the following way, see, e.g., [10, 8.932.2].

$$\begin{aligned} C_{2n}^{(1+\alpha)}\left(\frac{z(r, \theta)}{c}\right) &= \frac{(-1)^n \Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(1+\alpha)} F\left(-n, n+\alpha+1; \frac{1}{2}; \frac{z(r, \theta)^2}{c^2}\right) \\ &= \frac{(-1)^n}{2\Gamma(1+\alpha)n!} \sum_{p=0}^n \sum_{k=0}^{2p} (-1)^p \binom{n}{p} \binom{2p}{k} \frac{\Gamma(1+\alpha+n+p)\Gamma(p)}{\Gamma(2p)} r^{2p} R^{2(k-p)} e^{2i\theta(k-p)} \\ &= \frac{(-1)^n}{2\Gamma(1+\alpha)n!} \sum_{p=0}^n \sum_{k=0}^{2p} (-1)^p \binom{n}{p} \binom{2p}{k} \frac{\Gamma(1+\alpha+n+p)\Gamma(p)}{\Gamma(2p)} r^{2p} R^{2(p-k)} e^{2i\theta(p-k)}. \end{aligned} \quad (3.9)$$

Here, we introduced two representations, both to be used below, using the binomial theorem for (3.7) in two equivalent ways. In order to prepare the integration in (3.8), we spell out the complex conjugated variable to the power $2l$:

$$\begin{aligned} \left(\frac{\overline{z(r, \theta)}}{c}\right)^{2l} &= \left(\frac{r}{2}\right)^{2l} (Re^{-i\theta} + R^{-1}e^{i\theta})^{2l} \\ &= \left(\frac{r}{2}\right)^{2l} \left[\sum_{k=1}^l \binom{2l}{k+l} R^{-2k} e^{2i\theta k} + \binom{2l}{l} + \sum_{k=1}^l \binom{2l}{k+l} R^{2k} e^{-2i\theta k} \right]. \end{aligned} \quad (3.10)$$

From the radial integral in (3.8) we obtain, including all prefactors,

$$\frac{1+\alpha}{\pi} \int_0^1 dr r^{2p+1} \frac{r^{2l}}{2^{2l}} (1-r^2)^\alpha = \frac{\Gamma(2+\alpha)\Gamma(1+p+l)}{2^{2l+1}\pi\Gamma(2+\alpha+p+l)}. \quad (3.11)$$

For the remaining angular integration we thus have

$$\begin{aligned} & \int_E C_{2n}^{(1+\alpha)} \left(\frac{z}{c} \right) \left(\frac{\bar{z}}{c} \right)^{2l} dA_\alpha(z) \\ &= \frac{(1+\alpha)(-1)^n}{2^{2l+1}\pi} \sum_{k'=1}^l \sum_{p=0}^n \sum_{k=0}^{2p} \binom{2l}{k'+l} \frac{(-1)^p \Gamma(1+\alpha+n+p)\Gamma(1+l+p)}{(n-p)!k!(2p-k)!\Gamma(2+\alpha+l+p)} \\ & \quad \times R^{2(p-k-k')} \int_0^{2\pi} d\theta e^{2i\theta(p-k+k')} \\ & \quad + \frac{(1+\alpha)(-1)^n}{2^{2l+1}\pi} \sum_{p=0}^n \sum_{k=0}^{2p} \binom{2l}{l} \frac{(-1)^p \Gamma(1+\alpha+n+p)\Gamma(1+l+p)}{(n-p)!k!(2p-k)!\Gamma(2+\alpha+l+p)} \\ & \quad \times R^{2(p-k)} \int_0^{2\pi} d\theta e^{2i\theta(p-k)} \\ & \quad + \frac{(1+\alpha)(-1)^n}{2^{2l+1}\pi} \sum_{k'=1}^l \sum_{p=0}^n \sum_{k=0}^{2p} \binom{2l}{k'+l} \frac{(-1)^p \Gamma(1+\alpha+n+p)\Gamma(1+l+p)}{(n-p)!k!(2p-k)!\Gamma(2+\alpha+l+p)} \\ & \quad \times R^{2(k-p+k')} \int_0^{2\pi} d\theta e^{2i\theta(k-p-k')}. \end{aligned} \quad (3.12)$$

In the first step we have already simplified the binomial factors and Gamma-functions from (3.9). Notice that in the first two terms, obtained from integrating over the first two contributions on the right-hand side of (3.10), we have used the second identity in (3.9), whereas for the last sum from (3.10) we have used the first form of identity in (3.9). We now evaluate each of the multiple sums in (3.12) individually. In the last triple sum we have $k = p + k'$ due to the angular integration, and because $k \leq 2p$ and thus $k' \leq p$ we obtain for it

$$\begin{aligned} & \frac{(1+\alpha)(-1)^n}{2^{2l}} \sum_{k'=1}^l \binom{2l}{k'+l} \sum_{p=k'}^n \frac{(-1)^p \Gamma(1+\alpha+n+p)\Gamma(1+l+p)}{(n-p)!(k'+p)!(p-k')!\Gamma(2+\alpha+l+p)} R^{4k'} \\ &:= \frac{(1+\alpha)(-1)^n}{2^{2l}} \sum_{k'=1}^l \binom{2l}{k'+l} a_{k'} R^{4k'}. \end{aligned} \quad (3.13)$$

It is a polynomial in R of degree $4l$. We have to show that all its coefficients $a_{k'} = a_{k'}(n, l)$ vanish for $l < n$. Before we do that let us compute the other sums in (3.12). From the second term in (3.12), the double sum, we obtain from $p = k$

$$\frac{(1+\alpha)(-1)^n}{2^{2l}} \binom{2l}{l} \sum_{p=0}^n \frac{(-1)^p \Gamma(1+\alpha+n+p) \Gamma(1+l+p)}{(n-p)!(p!)^2 \Gamma(2+\alpha+l+p)} = \frac{(1+\alpha)(-1)^n}{2^{2l}} \binom{2l}{l} a_0, \quad (3.14)$$

which is R -independent. It is the same as the contribution in (3.13) for $k' = 0$. For the first triple sum in (3.12) we have again $k = p + k'$ and thus $k' \leq p$:

$$\frac{(1+\alpha)(-1)^n}{2^{2l}} \sum_{k'=1}^l \binom{2l}{k'+l} \sum_{p=k'}^n \frac{(-1)^p \Gamma(1+\alpha+n+p) \Gamma(1+l+p)}{(n-p)!(k'+p)!(p-k')! \Gamma(2+\alpha+l+p)} R^{-4k'}. \quad (3.15)$$

It agrees with (3.13) replacing $R \rightarrow R^{-1}$. So in summary, if we can show that all coefficients $a_{k'}$ vanish for $k' = 0, 1, \dots, l$ when $l < n$, we are done. This can be seen as follows. From the definition (3.13) we have, after a change of variables,

$$\begin{aligned} a_k &= \frac{(-1)^k}{(n-k)!} \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} \frac{\Gamma(1+\alpha+n+k+p) \Gamma(1+l+k+p)}{(2k+p)! \Gamma(2+\alpha+l+k+p)} \\ &= \frac{(-1)^k}{(n-k)! \Gamma(1+\alpha)} \int_0^1 dx x^{l+k} (1-x)^\alpha \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} \frac{\Gamma(1+\alpha+n+k+p)}{\Gamma(1+2k+p)} x^p \\ &= \frac{(-1)^k \Gamma(1+\alpha+n+k)}{(n-k)! \Gamma(1+\alpha)(2k)!} \int_0^1 dx x^{l+k} (1-x)^\alpha F(-n+k, 1+\alpha+n+k; 1+2k; x). \end{aligned} \quad (3.16)$$

This reduces the problem to showing that the integral containing the hypergeometric function vanishes, when $l < n$ and $\alpha > -1$. Let us introduce a regularising parameter $\varepsilon > 0$. We then have

$$\left| x^{l+k+\varepsilon} (1-x)^\alpha F(-n+k, 1+\alpha+n+k, 1+2k, x) \right| \leq C_F x^{l+k} (1-x)^\alpha, \quad x \in [0, 1],$$

for some constant C_F . Since $x^{l+k}(1-x)^\alpha \in L^1([0, 1])$, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} &\int_0^1 dx x^{l+k} (1-x)^\alpha F(-n+k, 1+\alpha+n+k; 1+2k; x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^1 dx x^{l+k+\varepsilon} (1-x)^\alpha F(-n+k, 1+\alpha+n+k; 1+2k; x) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(1+2k) \Gamma(1+l+k+\varepsilon) \Gamma(1+\alpha+n-k) \Gamma(n-l-\varepsilon)}{\Gamma(1+k+n) \Gamma(2+\alpha+n+l+\varepsilon) \Gamma(-\varepsilon-(l-k))} \end{aligned}$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{(-1)^{l-k-1} \Gamma(1+2k) \Gamma(1+l+k+\varepsilon) \Gamma(1+\alpha+n-k) \Gamma(l+1-k+\varepsilon)}{\pi \Gamma(1+k+n) \Gamma(2+\alpha+n+l+\varepsilon)} \times \Gamma(n-l-\varepsilon) \sin(\pi \varepsilon). \quad (3.17)$$

In the second step we have used the following integral, see [10, 7.512.2],

$$\int_0^1 t^{\varrho-1} (1-t)^{\beta-\gamma-m} F(-m, \beta; \gamma; t) dt = \frac{\Gamma(\gamma) \Gamma(\varrho) \Gamma(\beta-\gamma+1) \Gamma(\gamma-\varrho+m)}{\Gamma(\gamma+m) \Gamma(\beta-\gamma+\varrho+1) \Gamma(\gamma-\varrho)}$$

for $m = 0, 1, 2, \dots; \operatorname{Re} \varrho > 0, \operatorname{Re}(\beta-\gamma) > m-1$.

In the next step we use Euler's reflection formula. Finally, the limit

$$\lim_{\varepsilon \rightarrow 0} \Gamma(n-l-\varepsilon) \sin(\pi \varepsilon) = \begin{cases} -\pi & l = n \\ 0 & l < n \end{cases} \quad (3.18)$$

establishes the claimed orthogonality of (3.5) for even indices.

In order to compute the squared norm on the right-hand side of (3.4), we first compute (3.12) for $n = l$. For that purpose we summarise the result for the coefficients a_k in (3.16) that follows from the above:

$$a_k(n, n) = \frac{(-1)^n \Gamma(1+\alpha+n+k) \Gamma(1+\alpha+n-k)}{\Gamma(1+\alpha) \Gamma(2n+\alpha+2)}, \quad (3.19)$$

which we insert into (3.13). We thus obtain for this, as well as for (3.15) at $n = l$,

$$\begin{aligned} & \sum_{k'=1}^n \binom{2n}{k'+n} a_{k'}(n, n) R^{\pm 4k'} \\ &= \frac{(-1)^n (2n)!}{\Gamma(1+\alpha) \Gamma(2n+\alpha+2)} \sum_{k'=1}^n \frac{\Gamma(1+\alpha+n+k') \Gamma(1+\alpha+n-k')}{(n-k')! (n+k')!} R^{\pm 4k'} \\ &= \frac{(-1)^n (2n)!}{\Gamma(1+\alpha) \Gamma(2n+\alpha+2)} \sum_{k=n+1}^{2n} \frac{\Gamma(1+\alpha+k) \Gamma(1+\alpha+2n-k)}{(2n-k)! k!} R^{\mp 4(n-k)} \\ &= \frac{(-1)^n (2n)!}{\Gamma(1+\alpha) \Gamma(2n+\alpha+2)} \sum_{k=0}^{n-1} \frac{\Gamma(1+\alpha+2n-k) \Gamma(1+\alpha+k)}{k! (2n-k)!} R^{\pm 4(n-k)}, \end{aligned} \quad (3.20)$$

after relabelling the sum twice. With this result it is easy to see that we can write the three contributions (3.13)–(3.15) at $n = l$ in (3.12) as a single sum as

$$\begin{aligned} & \int_E C_{2n}^{(1+\alpha)} \left(\frac{z}{c} \right) \left(\frac{\bar{z}}{c} \right)^{2l} dA_\alpha(z) = \\ &= \delta_{n,l} \frac{(1+\alpha)(2n)!}{2^{2n} \Gamma(1+\alpha) \Gamma(2n+\alpha+2)} \sum_{k=0}^{2n} \frac{\Gamma(1+\alpha+k) \Gamma(1+\alpha+2n-k)}{\Gamma(2n-k+1) \Gamma(k+1)} R^{4(n-k)}. \end{aligned} \quad (3.21)$$

The remaining sum can be related to a single Gegenbauer polynomial as follows. Because this sum is invariant under $k \rightarrow 2n - k$, we can write it as

$$\begin{aligned} &= \frac{1}{2} \sum_{k=0}^{2n} \frac{\Gamma(1+\alpha+k)\Gamma(1+\alpha+2n-k)}{\Gamma(2n-k+1)\Gamma(k+1)} \left(R^{4(n-k)} + R^{-4(n-k)} \right) \\ &= \sum_{k=0}^{2n} \frac{\Gamma(1+\alpha+k)\Gamma(1+\alpha+2n-k)}{\Gamma(2n-k+1)\Gamma(k+1)} \cosh[(2n-2k)\ln(R^2)] \\ &= \Gamma(1+\alpha)^2 C_{2n}^{(1+\alpha)} \left(\frac{a^2+b^2}{a^2-b^2} \right). \end{aligned} \quad (3.22)$$

In the last step we have used the (analytically continued) relation [20, 18.5.11]

$$C_j^{(1+\alpha)}(\cos \theta) = \sum_{l=0}^j \frac{(1+\alpha)_l (1+\alpha)_{j-l}}{l!(j-l)!} \cos((j-2l)\theta), \quad (3.23)$$

with $(a)_n = \Gamma(a+n)/\Gamma(n)$ being the Pochhammer symbol, together with

$$\cosh[\ln(R^2)] = \frac{1}{2}(R^2 + R^{-2}) = \frac{a^2+b^2}{a^2-b^2}, \quad (3.24)$$

which follows from (3.7). In order to obtain (3.4) we still need to multiply (3.21) with the leading power of the Gegenbauer polynomial, which is easy to obtain from the first line of (3.9), see [20]

$$C_{2l}^{(1+\alpha)}(x) = \frac{\Gamma(2l+1+\alpha)2^{2l}}{\Gamma(1+\alpha)(2l)!} x^{2l} + O(x^{2l-2}). \quad (3.25)$$

Because the lower powers give zero, combined with (3.22) we finally have

$$\int_E C_{2n}^{(1+\alpha)}\left(\frac{z}{c}\right) C_{2l}^{(1+\alpha)}\left(\frac{\bar{z}}{c}\right) dA_\alpha(z) = \delta_{2n,2l} \frac{(1+\alpha)}{(2n+\alpha+1)} C_{2n}^{(1+\alpha)}\left(\frac{a^2+b^2}{a^2-b^2}\right), \quad (3.26)$$

which agrees with (3.4) for even indices. The proof for the odd polynomials follows exactly in the same way, and for completeness we have collected the necessary steps in “Appendix A”. $\square$

Remark 3.4 In the case $\alpha = 0$ we recover the orthogonality relation for Chebyshev polynomials of the second kind, due to $U_n(x) = C_n^{(1)}(x)$, which goes back to [11]. We will come back to this statement in Sect. 4.

For $\alpha = -1/2$ we obtain as a special case the orthogonality of the Legendre polynomials $P_n(x) = C_n^{(1/2)}(x)$.

Corollary 3.5 *The Legendre polynomials P_n are orthogonal with respect to the weight function dA_α defined in (2.3) at $\alpha = -1/2$:*

$$\int_E P_m\left(\frac{z}{c}\right) P_n\left(\frac{\bar{z}}{c}\right) dA_{-\frac{1}{2}}(z) = \frac{1}{1+2n} P_n\left(\frac{a^2+b^2}{a^2-b^2}\right) \delta_{n,m}. \quad (3.27)$$

We have not been able to find this result in the literature.

Furthermore, we can make contact with Hermite polynomials¹ $H_n(x)$ as polynomials in the full complex plane. Setting $z \rightarrow z/\sqrt{1+\alpha}$ and taking α to infinity in (3.4), we have from [20, 18.7.24]

$$\lim_{\alpha \rightarrow \infty} (1+\alpha)^{-\frac{n}{2}} C_n^{(1+\alpha)}\left((1+\alpha)^{-\frac{1}{2}}x\right) = H_n(x)/n!, \quad (3.28)$$

leading to

$$\int_{\mathbb{C}} H_m(z/c) H_n(\bar{z}/c) e^{-h(z)} d^2z = \pi n! ab \left(2 \frac{a^2+b^2}{a^2-b^2}\right)^n \delta_{n,m}, \quad (3.29)$$

with $h(z)$ defined in (2.1). This reproduces the known orthogonality relation for Hermite polynomials in the complex plane, obtained by van Eijndhoven and Meyers [6, Eq.(0.5)] for $a = \sqrt{\frac{1}{1-A}}$ and $b = \sqrt{\frac{A}{1-A}}$, with $0 < A < 1$, see also [5].

Remark 3.6 In the limit $c \rightarrow 0$, when the ellipse E becomes a disc, we obtain for integer values of α the weight function that results from the complex eigenvalues of the ensemble of truncated unitary random matrices studied in [27], with monomials as orthogonal polynomials. This can be seen as follows: We have from (3.9) that the monic Gegenbauer polynomials occurring in (3.4) read:

$$\tilde{P}_n^{(\alpha)}(z) := \frac{n!c^n}{2^n(1+\alpha)_n} C_n^{(1+\alpha)}(z/c). \quad (3.30)$$

Multiplying (3.4) with the corresponding factors, we can take the limit $b \rightarrow a$, implying $c \rightarrow 0$ in this orthogonality relation, to obtain

$$\int_{x^2+y^2 < a^2} z^m \bar{z}^n (1+\alpha) \left(1 - \frac{|z|^2}{a^2}\right)^\alpha \frac{d^2z}{\pi a^2} = \frac{\Gamma(n+1)\Gamma(1+\alpha)(1+\alpha)}{\Gamma(1+\alpha+n)(1+\alpha+n)} a^{2n} \delta_{n,m}, \quad (3.31)$$

where $z = x + iy$. After rescaling $z \rightarrow az$, and dividing (3.31) by $(1+\alpha)$, we arrive at the weight function and monic polynomials for the complex eigenvalues in the ensemble of truncated unitary random matrices [27] on the unit disc. It is defined starting from the circular unitary ensemble of Haar distributed unitary random matrices of size $N \times N$, and truncating these to the upper left block of size $M \times M$ with $N > M$, by removing $N - M$ rows and columns. The weight function reads $w(z) = (1 - |z|^2)^{N-M-1}$, that is, we have to identify $\alpha = N - M - 1 \geq 0$. In

¹ We denote by this the Hermite polynomials orthogonal with respect to $\exp[-x^2]$ on $\mathbb{R}$.

this case there is no singularity on the boundary of the circle, and we may extend our integration from inside the disc to include the boundary, see (2.14). In this ensemble this is important as for large M and small truncation $N - M$ a substantial fraction of eigenvalues of the truncated unitary matrix may remain on the unit circle. We refer to [27] for a further discussion of the limiting behaviour.

In analogy to the relation between the Ginibre ensemble and its elliptic version, our Gegenbauer polynomials can thus be viewed as the orthogonal polynomials of an elliptic version of the truncated unitary ensemble [27], with an appropriate random matrix realisation yet to be constructed.

Remark 3.7 Finally, we can establish contact with the usual orthogonality relation for the Gegenbauer polynomials on the real interval $[-1, 1]$. The change of variables for the imaginary part $y = \frac{b}{a}\hat{y}$ maps the ellipse to a disc of radius a . Together with $dA_\alpha(z) = (1 + \alpha)(1 - (x/a)^2 - (y/b)^2)^\alpha dx dy / (ab\pi)$, this allows us to take the limit $b \rightarrow 0$ on (3.4):

$$\begin{aligned} & \lim_{b \rightarrow 0} \int_E C_m^{(1+\alpha)}\left(\frac{z}{c}\right) C_n^{(1+\alpha)}\left(\frac{\bar{z}}{c}\right) dA_\alpha(z) \\ &= \int_{-a}^a C_m^{(1+\alpha)}\left(\frac{x}{a}\right) C_n^{(1+\alpha)}\left(\frac{x}{a}\right) \left(1 - \frac{x^2}{a^2}\right)^\alpha \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(1 - \frac{\hat{y}^2}{a^2-x^2}\right)^\alpha \frac{(1+\alpha)d\hat{y}dx}{a^2\pi} \\ &= \int_{-1}^1 C_m^{(1+\alpha)}(x) C_n^{(1+\alpha)}(x) (1-x^2)^{\alpha+\frac{1}{2}} F\left(\frac{1}{2}, -\alpha; \frac{3}{2}; 1\right) \frac{2(1+\alpha)}{\pi} dx \\ &= \frac{1+\alpha}{1+\alpha+n} C_n^{(1+\alpha)}(1) \delta_{n,m}. \end{aligned} \quad (3.32)$$

Identities for the Gegenbauer polynomial [20, Table 18.6.1]

$$C_n^{(1+\alpha)}(1) = \frac{\Gamma(2+2\alpha+n)}{\Gamma(2+2\alpha)\Gamma(n+1)}, \quad (3.33)$$

and for Gauß' hypergeometric function [10, 9.122] at unity

$$F\left(\frac{1}{2}, -\alpha; \frac{3}{2}; 1\right) = \frac{\sqrt{\pi}\Gamma(1+\alpha)}{2\Gamma(\alpha+3/2)} \quad (3.34)$$

yield the standard orthogonality relation

$$\int_{-1}^1 C_n^{(\alpha+1)}(x) C_m^{(\alpha+1)}(x) (1-x^2)^{\alpha+\frac{1}{2}} dx = \frac{2^{1-2(1+\alpha)}\pi\Gamma(2+2\alpha+n)}{(1+\alpha+n)\Gamma^2(1+\alpha)n!} \delta_{n,m}. \quad (3.35)$$

We can now finish the proof of Theorem 3.1 by showing the completeness of the system of orthogonal polynomials.

Proof Let $f \in A_\alpha^2$ with $\langle f, p_n \rangle_\alpha = 0$ for all $n = 0, 1, 2, \dots$. Then

$$0 = \lim_{b \rightarrow 0} \langle f, p_n \rangle_\alpha = \int_{-1}^1 dx f(ax) C_n^{(1+\alpha)}(x) (1-x^2)^{\alpha+\frac{1}{2}}. \quad (3.36)$$

Hence $f(ax) = 0$ for all $x \in (-1, 1)$, see [25] for the completeness of the Jacobi polynomials on the real line. Since f is regular in E , it follows that $f \equiv 0$, i.e. $\{p_n^\alpha\}$ defined above form an orthonormal basis for A_α^2 . $\square$

4 Orthogonality of Certain Jacobi and all Chebyshev Polynomials

In this section we will first deduce Corollary 4.1 from our Lemma 3.2, by mapping the Gegenbauer polynomials to a certain sub-family of Jacobi polynomials $P_n^{(\alpha+\frac{1}{2}, \pm\frac{1}{2})}$ orthogonal on an ellipse, with a different weight function. We will not use the standard, symmetric representation [20, 18.7.1]

$$C_n^{(1+\alpha)}(z) = \frac{(2+2\alpha)_n}{(\alpha+\frac{3}{2})_n} P_n^{(\alpha+\frac{1}{2}, \alpha+\frac{1}{2})}(z), \quad (4.1)$$

which is linear, but rather a quadratic transformation that leads to a non trivial orthogonality relation, as described below. Second, we will use this corollary to show the orthogonality of Chebyshev polynomials of the first, second, third and fourth kind T_n , U_n , V_n and W_n , respectively, that were derived in a different way in [17], see Corollary 4.4 below. We will come back to the polynomials U_n of the second kind, where the orthogonality was already stated in Remark 3.4, following from Lemma 3.2.

Let us summarise our first statement as follows.

Corollary 4.1 *Define the ellipse E as before in (2.2), and the function*

$$j(w) = \frac{a}{b^2}|c+w| - \frac{c}{b^2}\operatorname{Re}(c+w). \quad (4.2)$$

It satisfies $0 < j(w) < 1$ on E . Then, for $\alpha > -1$ the following two sub-families of Jacobi polynomials are orthogonal on E : First,

$$\int_E P_n^{(\alpha+\frac{1}{2}, -\frac{1}{2})}(w/c) P_m^{(\alpha+\frac{1}{2}, -\frac{1}{2})}(\bar{w}/c) dB_\alpha^-(w) = \frac{((1/2)_n)^2}{((\alpha+1)_n)^2} \frac{1+\alpha}{1+\alpha+2n} C_{2n}^{(1+\alpha)}\left(\frac{a}{c}\right) \delta_{n,m}, \quad (4.3)$$

with respect to the weight function

$$dB_\alpha^-(w) := \frac{(1+\alpha)}{2\pi b} \frac{(1-j(w))^\alpha}{|c+w|} d^2w, \quad (4.4)$$

and, second,

$$\int_E P_n^{(\alpha+\frac{1}{2}, \frac{1}{2})}(w/c) P_m^{(\alpha+\frac{1}{2}, \frac{1}{2})}(\bar{w}/c) dB_\alpha^+(w) = \frac{2c(1/2)_{n+1}^2(1+\alpha)(2+\alpha)}{a(\alpha+1)_{n+1}^2(2+\alpha+2n)} C_{2n+1}^{(1+\alpha)}\left(\frac{a}{c}\right) \delta_{n,m}, \quad (4.5)$$

with weight

$$dB_{\alpha}^{+}(w) := \frac{(1+\alpha)(2+\alpha)}{2\pi ab} (1-j(w))^{\alpha} d^2w. \quad (4.6)$$

The measures $dB_{\alpha}^{\pm}(w)$ are chosen to be normalised, $\int_E dB_{\alpha}^{\pm}(w) = 1$.

Proof We begin with the orthogonality relation (4.3). Using the quadratic transformation [20, 18.7.15], we have for all even Gegenbauer polynomials

$$C_{2n}^{(\alpha+1)}(z/c) = \frac{(\alpha+1)_n}{(\frac{1}{2})_n} P_n^{(\alpha+\frac{1}{2}, -\frac{1}{2})} \left(2 \left(\frac{z}{c} \right)^2 - 1 \right), \quad (4.7)$$

which leads us to identify the map to a new coordinate w

$$\frac{w}{c} = 2 \left(\frac{z}{c} \right)^2 - 1. \quad (4.8)$$

In order to specify the orthogonality relation following from Lemma 3.2 for this family of Jacobi polynomials in w/c , we have to determine the domain and weight function resulting from the map (4.8) of the ellipse (2.2) and weight (2.3). In order to make the mapping (4.8), to be inverted for z , unique,

$$\frac{z(w)}{c} = \sqrt{\frac{w+c}{2c}}, \quad (4.9)$$

we subdivide the ellipse $E = E_{+} \cup E_{-}$, with

$$E^{+(-)} = \{z = x + iy \in \mathbb{C} : x^2/a^2 + y^2/b^2 < 1, x > (<) 0\}. \quad (4.10)$$

Because the weight and measure are invariant under the inversion $z \rightarrow -z$, that maps $E^{-} \rightarrow E^{+}$, we have for the even Gegenbauer polynomials before the map (4.9)

$$\begin{aligned} \int_E C_{2n}^{(\alpha+1)}(z/c) C_{2l}^{(\alpha+1)}(\bar{z}/c) dA_{\alpha}(z) &= 2 \int_{E^{+}} C_{2n}^{(\alpha+1)}(z/c) C_{2l}^{(\alpha+1)}(\bar{z}/c) dA_{\alpha}(z) \\ &= \int_{\tilde{E}} P_n^{(\alpha+\frac{1}{2}, -\frac{1}{2})}(w/c) P_m^{(\alpha+\frac{1}{2}, -\frac{1}{2})}(\bar{w}/c) d\tilde{B}_{\alpha}^{-}(w), \end{aligned} \quad (4.11)$$

which leads to the claimed orthogonality in the second line, as we will explain now. The map (4.9) has a square root cut for $\{\operatorname{Re}(w) + c < 0\}$, and for the Jacobian we obtain from

$$\frac{dz(w)}{dw} = \frac{\sqrt{c}}{2\sqrt{2}} \frac{1}{\sqrt{w+c}} \Rightarrow d^2z = \frac{c d^2w}{8|w+c|}. \quad (4.12)$$

In order to determine the domain to be $\tilde{E} \subset \mathbb{C} \setminus \{\operatorname{Re}(w) + c < 0\}$, resulting from (4.9), we introduce two auxiliary quantities $A > B > 0$ in terms of the parameters $a > b > 0$ of the original ellipse E in (2.2):

$$A = \frac{a^2 + b^2}{2a^2b^2}, \quad B = \frac{a^2 - b^2}{2a^2b^2} = \frac{c^2}{2a^2b^2}. \quad (4.13)$$

Here, we have recalled the definition (3.2) of parameter c . These two quantities satisfy

$$A^2 - B^2 = \frac{2B}{c^2} \quad \text{and} \quad \frac{Bc^2}{2} + 1 = \frac{(a^2 + b^2)^2}{4a^2b^2}. \quad (4.14)$$

Furthermore, we can write for $z = x + iy$

$$A|z|^2 - B\operatorname{Re}(z^2) = \frac{(a^2 + b^2)(x^2 + y^2)}{2a^2b^2} - \frac{(a^2 - b^2)(x^2 - y^2)}{2a^2b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \quad (4.15)$$

Therefore, the domain E expressed in terms of the new variable $w = u + iv$ reads

$$\begin{aligned} 1 > A|z|^2 - B\operatorname{Re}(z^2) &= \frac{cA}{2}|w + c| - \frac{cB}{2}\operatorname{Re}(w + c) \\ &= \frac{cA}{2}\sqrt{(u + c)^2 + v^2} - \frac{cB}{2}(u + c), \end{aligned} \quad (4.16)$$

which is the defining equation for the new domain. The claim that it is again given by an ellipse, with new parameters $\tilde{a}$ and $\tilde{b}$ to be determined, can be seen as follows. From (4.16) we have

$$\begin{aligned} 0 &< \frac{cA}{2}\sqrt{(u + c)^2 + v^2} < 1 + \frac{cB}{2}(u + c) \\ \Leftrightarrow \quad \frac{c^2}{4}(A^2 - B^2)u^2 + uc \left(\frac{c^2(A^2 - B^2)}{2} - B \right) + \frac{c^2A^2}{4}v^2 &< 1 - \frac{c^4(A^2 - B^2)}{4} + Bc^2 \\ \Leftrightarrow \quad c^2u^2 + \frac{c^2(a^2 + b^2)^2}{4a^2b^2}v^2 &< (a^2 + b^2)^2, \end{aligned} \quad (4.17)$$

which is obtained after squaring the inequality, using (4.14) and multiplying with $4a^2b^2$. We are thus led to define the new domain as

$$\tilde{E} := \{w = u + iv \in \mathbb{C} : u^2/\tilde{a}^2 + v^2/\tilde{b}^2 < 1\}, \quad \text{with} \quad \tilde{a} = \frac{(a^2 + b^2)}{c}, \quad \tilde{b} = \frac{2ab}{c}. \quad (4.18)$$

We note that $c^2 = \tilde{a}^2 - \tilde{b}^2 = a^2 - b^2$ follows. It remains to show (4.2), which follows from (4.15) and (4.16) as

$$1 - h(z) = 1 - \frac{cA}{2}|w + c| + \frac{cB}{2}\operatorname{Re}(w + c) = 1 - \tilde{j}(w), \quad (4.19)$$

together with $cA/2 = \tilde{a}/\tilde{b}^2$ and $cB/2 = c/\tilde{b}^2$. Inserting all these into the right hand side of (4.11), we arrive at (4.3) with weight (4.4). The fact that the weight (4.4) is normalised to unity immediately follows from setting $n = 0$ in (4.3), as $C_0^{(1+\alpha)}(x) = 1$. Dropping the tilde on all quantities we arrive at the statement in (4.3).

It is important to note here that once we consider the Jacobi polynomials (4.3), with weight (4.4) on the new ellipse (4.18), there is no more square root cut inside, which would lead to a slit domain. To see this we multiply the equation defining E

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} < 1$$

$$\Leftrightarrow a^2 \left((u+c)^2 + v^2 \right) < (b^2 + c(u+c))^2, \quad (4.20)$$

by a^2b^2 , to arrive at the second line. While it is clear that the left hand side is always positive, we can take the square root here without crossing zero, due to the following fact. It holds that $b^2 + c(u+c) = cu + a^2$ inside the square on the right hand side is always positive for $u \in (-a, +a)$. This ends the proof for the first set of polynomials.

For the orthogonality relation (4.5) of the second set of polynomials only few modifications are needed. We start from Lemma 3.2 for the odd Gegenbauer polynomials and use the relation [20, 18.7.16]:

$$C_{2n+1}^{(\alpha+1)}(z/c) = \frac{(\alpha+1)_{n+1}}{\left(\frac{1}{2}\right)_{n+1}} \frac{z}{c} P_n^{(\alpha+\frac{1}{2}, \frac{1}{2})}(2(z/c)^2 - 1). \quad (4.21)$$

The identification of variables (4.8) is identical, and the map to E^+ works in the same way as in (4.11), after cancelling the two minus signs obtained from the reflection of the two odd polynomials. Apart from the additional constant factors, we obtain from (4.21) an additional factor

$$\left| \frac{z}{c} \right|^2 = \frac{|w+c|}{2c} \quad (4.22)$$

which cancels the pole from the Jacobian in (4.12). This leads to the orthogonality (4.5) with weight (4.6), after multiplying with an overall factor $(2+\alpha)$ for the correct normalisation of the area measure. This can be seen using that $C_1^{(1+\alpha)}(x) = 2(1+\alpha)x$ for $n = 0$ on the right hand side of (4.5). $\square$

Remark 4.2 In order to show that the Bergman space with weights dB_α^- (4.4) and dB_α^+ (4.6) is closed in L^p , as in Sect. 2, all one needs to do is to find an estimate as in (2.8), such that we can apply Proposition 2.1 and Corollary 2.2 to these.

From (4.20) it follows that $j(z) = 1$ if and only if $z \in \partial E$. It is easy to see that there are no local extrema for $j(z)$ inside of E_ρ , therefore $0 < \max_{z \in E_\rho} j(z) = j(z_*) < 1$ for some $z_* \in E_\rho$. This shows that the Bergman space $A_\alpha^p(E, dB_\alpha^+)$ is closed in $L^p(E, dB_\alpha^+)$. As a consequence of Hölder's inequality we obtain the same result for the weight dB_α^- and $p \geq 1$.

Remark 4.3 As was done in Remark 3.7, we can make contact to the usual orthogonality relations for Jacobi polynomials $P_n^{(\alpha+\frac{1}{2}, \frac{1}{2})}$ on the real line, by rescaling the imaginary part of z , $\text{Im } z \rightarrow \frac{b}{a} \text{Im } z$. The same steps can be taken for $P_n^{(\alpha+\frac{1}{2}, -\frac{1}{2})}$. Without giving any details, in the limit $b \rightarrow 0$ we obtain in analogy to (3.32)

$$\begin{aligned} & \lim_{b \rightarrow 0} \int_E P_m^{(\frac{1}{2}+\alpha, \frac{1}{2})} \left(\frac{z}{c} \right) P_n^{(\frac{1}{2}+\alpha, \frac{1}{2})} \left(\frac{\bar{z}}{c} \right) dB_\alpha^+(z) \\ &= F \left(\frac{1}{2}, -\alpha; \frac{3}{2}; 1 \right) \frac{(1+\alpha)(2+\alpha)}{2^\alpha \pi} \int_{-1}^1 P_m^{(\frac{1}{2}+\alpha, \frac{1}{2})}(x) P_n^{(\frac{1}{2}+\alpha, \frac{1}{2})}(x) (1-x)^{\alpha+\frac{1}{2}} (1+x)^{\frac{1}{2}} dx \\ &= \frac{2(1/2)_{n+1}^2 (1+\alpha)(2+\alpha)}{(\alpha+1)_{n+1}^2 (2+\alpha+2n)} C_{2n+1}^{(1+\alpha)}(1) \delta_{n,m}, \end{aligned} \quad (4.23)$$

which yields the correct normalisation on $[-1, 1]$, see [20, 18.3.1].

Finally, as in (3.36), it is easy to see that Jacobi polynomials $P_n^{(\alpha+\frac{1}{2}, \pm\frac{1}{2})}$ provide an orthonormal basis for the underlying Hilbert space from Corollary 4.1.

In the remaining part of this section we will prove the orthogonality of the Chebyshev polynomials of first to fourth kind as a direct consequence of Corollary 4.1. The following statement is due to [17], where the notation for the polynomials of third and fourth kind is interchanged compared to ours, $V_n \leftrightarrow W_n$. We follow the notation of [20].

Corollary 4.4 *The Chebyshev polynomials satisfy the following orthogonality relations on the ellipse defined in (2.2), with $r = a + b$ and $c^2 = a^2 - b^2$:*

$$\int_E T_n(z/c) T_m(\bar{z}/c) \frac{d^2 z}{|z^2 - c^2|} = \begin{cases} \frac{\pi}{4n} ((r/c)^{2n} - (c/r)^{2n}) \delta_{n,m} & \text{for } n > 0, m \geq 0, \\ 2\pi \ln(r/c) & \text{for } n = m = 0, \end{cases} \quad (4.24)$$

$$\int_E U_n(z/c) U_m(\bar{z}/c) d^2 z = \frac{\pi c^2}{4(1+n)} ((r/c)^{2n+2} - (c/r)^{2n+2}) \delta_{n,m}, \quad (4.25)$$

$$\int_E V_n(z/c) V_m(\bar{z}/c) \frac{d^2 z}{|c+z|} = \frac{\pi c}{1+2n} ((r/c)^{2n+1} - (c/r)^{2n+1}) \delta_{n,m}, \quad (4.26)$$

$$\int_E W_n(z/c) W_m(\bar{z}/c) \frac{d^2 z}{|c-z|} = \frac{\pi c}{1+2n} ((r/c)^{2n+1} - (c/r)^{2n+1}) \delta_{n,m}. \quad (4.27)$$

Note that for better comparison with [17]² our statements are with respect to the flat measure $d^2 z$, rather than the area measure $dA(z) = d^2 z / (\pi ab)$.

Proof We begin with the Chebyshev polynomials of the second kind U_n . Because of the relation [20, 18.7.4]

$$U_n(z) = C_n^{(1)}(z) = \frac{1+n}{P_n^{(1/2, 1/2)}(1)} P_n^{(1/2, 1/2)}(z), \quad (4.28)$$

² In contrast to the orthogonality of the Chebyshev polynomials on the contour given by the boundary of the ellipse ∂E stated in [17] too, the weight function we find here differs from the classical weight on the real line, continued to the ellipse.

we set $\alpha = 0$ in (4.5), to obtain

$$\int_E P_n^{(1/2,1/2)}(z/c) P_m^{(1/2,1/2)}(\bar{z}/c) \frac{d^2 z}{\pi ab} = \frac{2c((1/2)_{n+1})^2}{a((1)_{n+1})^2(1+n)} C_{2n+1}^{(1)}\left(\frac{a}{c}\right) \delta_{n,m}, \quad (4.29)$$

After using

$$P_n^{(\frac{1}{2}, \frac{1}{2})}(1) = \Gamma(n+3/2)/(\Gamma(3/2)\Gamma(n+1)), \quad (4.30)$$

from [20, Table 18.6.1], we arrive at

$$\int_E U_n(z/c) U_m(\bar{z}/c) d^2 z = \frac{\pi cb}{2(1+n)} C_{2n+1}^{(1)}\left(\frac{a}{c}\right) \delta_{n,m}. \quad (4.31)$$

Recalling $r = a + b$ and $c^2 = a^2 - b^2$, we have

$$\frac{1}{2} \left(\frac{r}{c} + \frac{c}{r} \right) = \frac{a}{c} \quad \text{and} \quad \frac{1}{2} \left(\frac{r}{c} - \frac{c}{r} \right) = \frac{b}{c}. \quad (4.32)$$

With this the Gegenbauer polynomial on the right hand side of (4.31) can be simplified as follows. Applying (3.23) we have

$$\begin{aligned} C_{2n+1}^{(1)}(\cos(i \ln(r/c))) &= \sum_{k=0}^{2n+1} \cos((2n+1-2k)i \ln(r/c)) \\ &= \sum_{k=0}^{2n+1} \frac{1}{2} \left((c/r)^{2n+1-2k} + (r/c)^{2n+1-2k} \right) \\ &= \frac{c}{2b} ((r/c)^{2n+2} - (c/r)^{2n+2}). \end{aligned} \quad (4.33)$$

When replacing $C_{2n+1}^{(1)}(a/c)$ in (4.31) we arrive at the statement (4.24).

Let us recall that the orthogonality of $U_n(z)$ (4.28) also follows by setting $\alpha = 0$ in Lemma 3.2, see Remark 3.4. Comparing this statement

$$\int_E U_n(z/c) U_m(\bar{z}/c) d^2 z = \frac{\pi ab}{(1+n)} C_n^{(1)}\left(\frac{a^2+b^2}{a^2-b^2}\right) \delta_{n,m}, \quad (4.34)$$

with (4.31), we find that the following identity must hold:

$$C_{2n+1}^{(1)}(x) = x C_n^{(1)}(2x^2 - 1). \quad (4.35)$$

Indeed this follows from the quadratic relation (4.21) at $\alpha = 0$, and (4.28). We emphasise, however, that beyond $\alpha = 0$ apparently no such identity (4.35) exists, that would allow to further simplify the right-hand side of Lemma 3.2.

The Chebyshev Polynomials of the third kind V_n are related to Jacobi polynomials following [20, 18.7.5]:

$$V_n(z/c) = \frac{1+2n}{P_n^{(1/2, -1/2)}(1)} P_n^{(1/2, -1/2)}(z/c). \quad (4.36)$$

Setting $\alpha = 0$ in (4.3) we obtain

$$\int_E V_n(w/c) V_m(\bar{w}/c) \frac{d^2 w}{|c+w|} = \frac{2\pi b}{1+2n} C_{2n}^{(1)}\left(\frac{a}{c}\right) \delta_{n,m}. \quad (4.37)$$

Here, we have inserted (4.30). Similar to (4.33) we can simplify the Gegenbauer polynomial on the right hand side of (4.37), using (3.23) for an even index. We have

$$\begin{aligned} C_{2n}^{(1)}(\cos(i \ln(r/c))) &= \sum_{k=0}^{2n} \cos((2n-2k)i \ln(r/c)) \\ &= \sum_{k=0}^{2n} \left(\frac{r}{c}\right)^{2n-k} \left(\frac{c}{r}\right)^k \\ &= \frac{c}{2b} ((r/c)^{2n+1} - (c/r)^{2n+1}), \end{aligned} \quad (4.38)$$

which, upon replacing $C_{2n}^{(1)}(a/c)$ in (4.37), leads to the statement (4.25).

The orthogonality relation for Chebyshev polynomials of the fourth kind, W_n , is simple, due to the relation $V_n(-x) = (-1)^n W_n(x)$ which is true for $\forall n \in \mathbb{N}$. A reflection $z \rightarrow -z$ upon (4.37) leads to

$$\int_E W_n(w/c) W_m(\bar{w}/c) \frac{d^2 w}{|c-w|} = \frac{2\pi b}{1+2n} C_{2n}^{(1)}\left(\frac{a}{c}\right) \delta_{n,m}. \quad (4.39)$$

Here, the signs trivially cancel when $n = m$, together with the simplification (4.38) just described, leading to (4.26).

We turn to the orthogonality for the Chebyshev polynomials of the first kind, T_n . The relation [20, 18.7.18]

$$T_{2n+1}(x) = x W_n(2x^2 - 1) \quad (4.40)$$

allows us to find the corresponding weight function and orthogonality of the odd polynomials, starting from (4.39):

$$\begin{aligned} \int_{\tilde{E}} W_n(z'/c) W_m(\bar{z}'/c) \frac{d^2 z'}{|z'-c|} &= 8c \int_{E^+} \frac{z}{c} W_n(2(z/c)^2 - 1) \frac{\bar{z}}{c} W_m(2(\bar{z}/c)^2 - 1) \frac{d^2 z}{|z^2 - c^2|} \\ &= 4c \int_E T_{2n+1}(z/c) T_{2m+1}(\bar{z}/c) \frac{d^2 z}{|z^2 - c^2|}. \end{aligned} \quad (4.41)$$

Here, we use the inverse transformation of (4.8) applied in the proof of Corollary 4.1, for the definition of E^+ and $\tilde{E}$ see (4.10) and (4.18). Thus the polynomials $\{T_n\}$ are orthogonal with respect to $\frac{1}{|z^2 - c^2|} d^2z$. The following well known relation [17] holds for the Joukowski map $z/c = \frac{1}{2}(w/c + c/w)$:

$$T_n(z/c) = \frac{1}{2}((w/c)^n + (c/w)^n) \quad \text{for } n \geq 0, \quad (4.42)$$

which maps the ellipse E to the annulus $A := \{w \in \mathbb{C} : c < |w| < r\}$. Thus we obtain for $n > 0, m \geq 0$

$$\begin{aligned} & \int_E T_n(z/c) T_m(\bar{z}/c) \frac{d^2z}{|z^2 - c^2|} \\ &= \int_A T_n(z(w)/c) T_m(\overline{z(w)}/c) \frac{d^2w}{|w|^2} \\ &= \frac{1}{4} \int_c^r \frac{ds}{s} \int_0^{2\pi} d\theta \left((s/c)^n e^{in\theta} - (c/s)^n e^{-in\theta} \right) \left((s/c)^m e^{-im\theta} - (c/s)^m e^{im\theta} \right) \\ &= \frac{\pi}{4n} ((r/c)^{2n} - (c/r)^{2n}) \delta_{n,m} \\ &= \frac{\pi b}{2nc} C_{2n-1}^{(1)} \left(\frac{a}{c} \right) \delta_{n,m}, \end{aligned} \quad (4.43)$$

by changing to polar coordinates $w = s e^{i\theta}$. When performing the elementary integrations we need to assume that $n > 0$ and $m \geq 0$. Then, the first part of (4.24) follows, while in the last step we have inserted (4.33) in order to compare it to the previous orthogonality relations. Following the same computation for $n = m = 0$, we have with $T_0(x) = 1$,

$$\int_E \frac{1}{|z^2 - c^2|} d^2z = 2\pi \ln(r/c), \quad (4.44)$$

which ends the proof of Corollary 4.4. $\square$

5 Bergman Polynomials, Selberg Integrals and Finite-Term Recurrence

All the orthogonal polynomials on an ellipse we encountered in the previous section satisfy a three-step recursion relation, as they result from classical polynomials on the real line. It was shown by Khavinson and Stylianopoulos in [15], if the planar orthogonal polynomials on a domain D with regular enough boundary satisfy a finite recurrence relation, then the size of the recursion is three and the domain is an ellipse. This result was demonstrated for the unweighted case, also known as Carleman's polynomials, as summarised in Theorem 5.2 below. Using the Gegenbauer polynomials

from Lemma 3.2 that are orthogonal on a weighted ellipse, we will construct an example of a family of orthogonal polynomials with respect to a non-flat weight function, that has no finite-term recursion on such ellipse. Therefore the elliptic domain is not special and weight functions which leads to finite-term recursion are exceptional. In this sense, it remains an open question to characterise the positive Borel measures supported on an ellipse, such that the associated planar orthogonal polynomials do satisfy a three-term recurrence relation.

Because we will work with normalised expectation values to construct such an example, we will state in passing the normalising factor (partition function) for Gegenbauer polynomials, constituting a special case of a Selberg integral in the complex plane.

Consider a bounded simply connected domain D in the complex plane, let $d\mu(z) = w(z)dA(z)$ be a measure on D , where dA is the planar Lebesgue measure, and w a non-negative weight function on D . Given that all moments exist, $\int_D z^k \bar{z}^l w(z) dA(z) < \infty$, a unique sequence of polynomials $p_n(z) = \gamma_n z^n + \dots$, $\gamma_n > 0$, can be constructed using the Gram–Schmidt process, that are orthonormal with respect to $d\mu$, see, e.g., [26]:

$$\int_D p_n(z) \overline{p_m(z)} w(z) dA(z) = \delta_{n,m} . \quad (5.1)$$

In the literature these polynomials are called *Bergman orthonormal polynomials*. For example choosing $D = E$ as an ellipse, and $w(z) = (1 + \alpha)(1 - h(z))^\alpha$, these polynomials p_n are proportional to the Gegenbauer polynomials, see (3.1).

The multiplication operator acting on polynomials can always be represented by expanding $z p_n(z)$ as a series using Bergman polynomials as a basis:

$$z p_n(z) = \sum_{l=0}^{n+1} c_{l,n} p_l(z) \quad n = 0, 1, 2, \dots \quad (5.2)$$

The Fourier coefficients $c_{l,n}$ are then given by

$$c_{l,n} = \int_E z p_n(z) \overline{p_l(z)} w(z) dA(z) . \quad (5.3)$$

These coefficients $c_{l,n}$ constitute the entries of an infinite upper Hessenberg matrix

$$M = \begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} & c_{0,3} & \dots \\ c_{1,0} & c_{1,1} & c_{1,2} & c_{1,3} & \dots \\ 0 & c_{2,1} & c_{2,2} & c_{2,3} & \dots \\ 0 & 0 & c_{3,2} & c_{3,3} & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} .$$

This matrix provides a representation of the Bergman Shift operator, which is linear and defined by $(T_z f)(z) = zf(z)$ with respect to the basis $\{p_n\}_{n \in \mathbb{N}}$.

Definition 5.1 (see [15]) We say that the upper Hessenberg matrix is *banded* or, equivalently, that the orthogonal polynomials p_n satisfy a *finite $(d + 1)$ -term recurrence* if there exists a positive integer d such that

$$c_{l,n} = 0, \text{ for } 0 \leq l < n + 1 - d. \quad (5.4)$$

In [15] Khavinson and Stylianopoulos proved the following:

Theorem 5.2 *If the Bergman polynomials orthogonal with respect to the flat measure, on a bounded simply-connected domain D with regular enough boundary, satisfy a $(d + 1)$ -term recurrence relation with $2 \leq d$, then D is an ellipse and $d = 2$.*

For all orthogonal polynomials supported on the real line, in particular for $I = [-1, 1]$, it is well-known that the associated orthogonal polynomials satisfy a *three-term recurrence* relation ($d = 2$), including a nontrivial weight on I . Because the Gegenbauer polynomials that we found to be orthogonal on the weighted ellipse also satisfy a three-step recurrence, it is a natural question if the above Theorem 5.2 extends to the weighted case, or at least if the weighted ellipse is special. Unfortunately the answer is no and we will construct a family of examples below that has no finite-term recurrence on the weighted ellipse. The fact that in general in the complex plane no finite-term recurrence can be expected was illuminated by [16] in the unweighted case. Notice that the examples of Hermite and Laguerre polynomials are orthogonal on the entire complex plane and thus do not provide a good testing ground to construct polynomials orthogonal on a finite domain.

We will use an alternative representation to Gram–Schmidt that allows us to construct orthogonal polynomials, the Heine formula, see [25]. For a given domain $D \subseteq \mathbb{C}$ in the complex plane, a non-negative weight function $w(z)$, and a normalised area measure dA on D such that all moment exist, we define the following expectation value:

$$\langle \mathcal{O} \rangle_{N,w} = \mathcal{Z}_N^{-1} \int_{D^N} \mathcal{O} |\Delta_N(z)|^2 \prod_{i=1}^N w(z_i) dA(z_i), \quad (5.5)$$

where $\mathcal{O}$ depends on $z_{i=1,\dots,N} \in \mathbb{C}$. Here, $\Delta_N(z) = \prod_{i>j}^N (z_i - z_j)$ is the Vandermonde determinant, and $\mathcal{Z}_N$ is a normalisation constant that ensures $\langle 1 \rangle_{N,w} = 1$. The expectation value can be thought of resulting from the joint density of complex eigenvalues of a complex non-Hermitian random matrix ensemble, such as the elliptic Ginibre ensemble. The Heine formula then states that the orthogonal polynomials of degree N in monic normalisation, $\tilde{p}_N(z) = z^N + \dots$, are given by

$$\tilde{p}_N(z) = \left\langle \prod_{i=1}^N (z - z_i) \right\rangle_{N,w}. \quad (5.6)$$

That is, they are given by the expectation value of a single characteristic polynomial. Denoting the squared norms of the monic polynomials by $\tilde{h}_N$, we have from (5.1)

$$\int_D \tilde{p}_n(z) \overline{\tilde{p}_m(z)} w(z) dA(z) = \delta_{n,m} \tilde{h}_n. \quad (5.7)$$

It is well known (see, e.g., [18]) that the normalisation constant in (5.5) can be expressed in terms of these norms as

$$\mathcal{Z}_N = \int_{D^N} |\Delta_N(z)|^2 \prod_{i=1}^N w(z_i) dA(z_i) = N! \prod_{j=0}^{N-1} \tilde{h}_j. \quad (5.8)$$

Remark 5.3 Selberg integrals. For our Gegenbauer polynomials with weight function $w(z) dA = (1 + \alpha)(1 - h(z))^\alpha dA = dA_\alpha$, we have for the monic polynomials (3.30)

$$\tilde{p}_n^{(\alpha)}(z) = \frac{n! c^n}{2^n (1 + \alpha)_n} C_n^{(1+\alpha)}(z/c), \quad (5.9)$$

with orthogonality relation

$$\int_E \tilde{p}_n^{(\alpha)}(z) \overline{\tilde{p}_m^{(\alpha)}(z)} dA_\alpha(z) = \delta_{n,m} \tilde{h}_n^{(\alpha)}, \quad (5.10)$$

and squared norms

$$\begin{aligned} \tilde{h}_n^{(\alpha)} &= \frac{n!^2 c^{2n}}{2^{2n} (1 + \alpha)_n^2} \frac{1 + \alpha}{1 + \alpha + n} C_n^{(1+\alpha)} \left(\frac{a^2 + b^2}{a^2 - b^2} \right) \delta_{n,m} \\ &= \left(\frac{c}{2} \right)^{2n} \frac{\sqrt{\pi} \Gamma(2 + \alpha) \Gamma(2 + 2\alpha + n) \Gamma(n + 1)}{2^{2\alpha+1} \Gamma(\alpha + \frac{3}{2}) \Gamma(1 + \alpha + n) \Gamma(2 + \alpha + n)} F \left(2 + 2\alpha, -n; \alpha + \frac{3}{2}; \frac{-b^2}{c^2} \right). \end{aligned} \quad (5.11)$$

Here, we have used the representation [10, 8.932.1] of Gegenbauer polynomials in terms of Gauß' hypergeometric function,

$$C_n^{(1+\alpha)}(t) = \frac{\Gamma(2 + 2\alpha + n)}{\Gamma(n + 1) \Gamma(2 + 2\alpha)} F \left(2 + 2\alpha, -n; \alpha + \frac{3}{2}; \frac{1-t}{2} \right). \quad (5.12)$$

Consequently we obtain the following Selberg integral in the complex plane

$$\begin{aligned} & \int_{E^N} |\Delta_N(z)|^\beta \prod_{i=1}^N \left(1 - \frac{1}{a^2} \operatorname{Re}(z_i)^2 - \frac{1}{b^2} \operatorname{Im}(z_i)^2 \right)^\alpha \frac{d^2 z_i}{\pi ab} \Bigg|_{\beta=2} \\ &= N! \frac{\pi^{\frac{N}{2}} N! \Gamma(1+\alpha)^N}{2^{(2\alpha+1)N} \Gamma(\alpha + \frac{3}{2})} \left(\frac{c}{2} \right)^{N(N-1)} \prod_{n=0}^{N-1} \frac{\Gamma(2+2\alpha+n) \Gamma(n+1)}{\Gamma(1+\alpha+n) \Gamma(2+\alpha+n)} \\ & \quad \times F \left(2+2\alpha, -n; \alpha + \frac{3}{2}; \frac{-b^2}{c^2} \right), \end{aligned} \quad (5.13)$$

after using the doubling formula for the Γ -function. This one-parameter family can be analytically continued in α . Of course for general $\alpha \in \mathbb{C}$ it will no longer be positive and can no longer be interpreted as a normalisation constant. It is an open problem how this result could be extended to arbitrary $\beta \in \mathbb{C}$.

Let us return to our example for a set of orthogonal polynomials on the weighted ellipse E , which do not satisfy a finite-term recurrence relation. For this, we will apply the following theorem proved in [1], which generalises Christoffel's Theorem for polynomials on $\mathbb{R}$:

Theorem 5.4 *Let $\{v_i; i = 1, \dots, K\}$ and $\{u_i; i = 1, \dots, L\}$ be two sets of complex numbers which are pairwise distinct among each set. Without loss of generality we assume $K \geq L \geq 0$, where the empty set is permitted. Then, the following statement holds:³*

$$\left\langle \prod_{k=1}^N \left[\prod_{i=1}^K (v_i - z_k) \prod_{j=1}^L (\bar{u}_j - \bar{z}_k) \right] \right\rangle_{N,w} = \frac{\prod_{i=N}^{N+K-1} \tilde{h}_i^{\frac{1}{2}} \prod_{j=N}^{N+L-1} \tilde{h}_j^{\frac{1}{2}}}{\Delta_K(v) \Delta_L(\bar{u})} \det_{1 \leq l, m \leq K} [\mathcal{B}(v_l, \bar{u}_m)], \quad (5.14)$$

with matrix

$$\mathcal{B}(v_l, \bar{u}_m) \equiv \begin{cases} \kappa_{N+L}(v_l, \bar{u}_m) := \sum_{i=0}^{N+L-1} p_i(v_l) \overline{p_i(u_m)} & \text{for } m = 1, \dots, L \\ p_{N+m-1}(v_l) & \text{for } m = L+1, \dots, K \end{cases} \quad (5.15)$$

The monic polynomials $\tilde{p}_n(z)$ are orthogonal with respect to $w(z)$, with squared norms $\tilde{h}_n$ and $p_n(z) = \tilde{p}_n(z)/\sqrt{\tilde{h}_n}$.

The multiplication operation on a sequence of polynomials can be explicitly computed, using the above Theorem 5.4 for $K = 2$ and $L = 1$, as given in [1]. Following the Heine formula (5.6), the polynomials $\{P_n^{(1)}\}_{n \in \mathbb{N}}$ orthogonal with respect to $|v-z|^2 w(z)$

³ The empty products are understood in the following sense: $\Delta_0(x) = \Delta_1(x) = 1$ and $\prod_{i=N}^{M \leq N-1} h_i = 1$.

can be expressed in terms of the polynomials p_n orthogonal with respect to $w(z)$. They are reading in monic normalisation

$$\begin{aligned}\tilde{P}_N^{(1)}(z) &= \left\langle \prod_{i=1}^N (z - z_i) \right\rangle_{N, |v - \cdot|^2 w} = \frac{\left\langle \prod_{i=1}^N (z - z_i) |v - z_i|^2 \right\rangle_{N, w}}{\left\langle \prod_{i=1}^N |v - z_i|^2 \right\rangle_{N, w}} \\ &= h_{N+1}^{\frac{1}{2}} \frac{\kappa_{N+1}(z, \bar{v}) p_{N+1}(v) - \kappa_{N+1}(v, \bar{v}) p_{N+1}(z)}{(v - z) \kappa_{N+1}(v, \bar{v})}.\end{aligned}\quad (5.16)$$

Their respective squared norms $\tilde{h}_N^{(1)}$ are not difficult to compute, using the orthonormality of the underlying polynomials $\tilde{p}_n$ (5.7):

$$\begin{aligned}\tilde{h}_N^{(1)} &= \int \tilde{P}_N^{(1)}(z) \overline{\tilde{P}_N^{(1)}(z)} |v - z|^2 w(z) dA(z) \\ &= \frac{h_{N+1}}{\kappa_{N+1}(v, \bar{v})} \left(\kappa_{N+1}(v, \bar{v}) |\tilde{P}_N^{(1)}(v)|^2 + \kappa_{N+1}(v, \bar{v})^2 \right) \\ &= \frac{h_{N+1} \kappa_{N+2}(v, \bar{v})}{\kappa_{N+1}(v, \bar{v})}.\end{aligned}\quad (5.17)$$

This leads to the orthonormal polynomials

$$P_N^{(1)}(z) = \frac{\kappa_{N+1}(z, \bar{v}) p_{N+1}(v) - \kappa_{N+1}(v, \bar{v}) p_{N+1}(z)}{(v - z) \sqrt{\kappa_{N+1}(v, \bar{v}) \kappa_{N+2}(v, \bar{v})}}.\quad (5.18)$$

The next step is to show that the Fourier coefficients $c_{l,N}$ of

$$z P_N^{(1)}(z) = \sum_{l=0}^{N+1} c_{l,N} P_l^{(1)}(z)\quad (5.19)$$

are (in general) non-zero for $l \leq N - 2$ for our example, when we choose $w(z)$ to be the Gegenbauer weight function, and thus the polynomials to be $\tilde{p}_n^{(\alpha)}$ from (5.10), with squared norms (5.11). Here, we may use the fact that the orthonormalised Gegenbauer polynomials (3.1) in the complex plane also satisfy a *three-term recurrence* relation (C.5), reading

$$z p_n^{(\alpha)}(z) = a_{n+1} p_{n+1}^{(\alpha)}(z) + b_n p_{n-1}^{(\alpha)}(z),\quad (5.20)$$

with

$$a_{n+1} = \frac{c(n+1)}{2(n+\alpha+1)} \sqrt{\frac{h_{n+1}}{h_n}}, \quad b_n = \frac{c(n+2\alpha+1)}{2(n+\alpha+1)} \sqrt{\frac{h_{n-1}}{h_n}}.\quad (5.21)$$

Here, we use the definition from (3.3) for the squared norms h_n of the (un-normalised, non-monic) Gegenbauer polynomials. Notice that in contrast to the recursion for orthonormal Gegenbauer polynomials on the real line, the recurrence (5.20) is not

symmetric, $a_n \neq b_n$. This is due the difference in norm for $[-1, 1]$ and E . From now on we will use the following notation for $\kappa_{i+1}(v, \bar{v}) := \kappa_{i+1}$. A simple calculation implies that the coefficients

$$c_{l,n} = \int_E z P_n^{(1)}(z) \overline{P_l^{(1)}(z)} |v - z|^2 dA_\alpha(z) \quad (5.22)$$

are given by

$$\begin{aligned} c_{l,n} = & \frac{1}{\sqrt{\kappa_{n+1}\kappa_{n+2}\kappa_{l+1}\kappa_{l+2}}} \left[\left(\sum_{k=1}^l a_k p_k^{(\alpha)}(v) p_{k-1}^{(\alpha)}(\bar{v}) - \sum_{k=0}^{\min\{l,n-1\}} b_{k+1} p_k^{(\alpha)}(v) p_{k+1}^{(\alpha)}(\bar{v}) \right) \right. \\ & \times p_{l+1}^{(\alpha)}(\bar{v}) p_{n+1}^{(\alpha)}(v) - \left(a_{l+1} p_l^{(\alpha)}(\bar{v}) \Theta(n-l) - b_{l+2} p_{l+2}^{(\alpha)}(\bar{v}) \Theta(n-l-2) \right) \kappa_{l+1} p_{n+1}^{(\alpha)}(v) \\ & \left. + \kappa_{n+1} \kappa_{n+2} a_{n+2} \delta_{n+1,l} - \kappa_{n+1} b_{n+1} p_n^{(\alpha)}(v) p_{l+1}^{(\alpha)}(\bar{v}) \Theta(l-n) + \kappa_{n+1} \kappa_n a_{n+1} \delta_{n-1,l} \right], \end{aligned} \quad (5.23)$$

where we have used the recursion (5.20) and introduced the step function

$$\Theta(x) := \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases} \quad (5.24)$$

If we only restrict ourselves to those indices $l \leq n-2$ which spoil the three-step recurrence, the remaining terms are simplified considerably and we obtain

$$\begin{aligned} c_{l,n} = & \frac{p_{n+1}^{(\alpha)}(v)}{\sqrt{\kappa_{n+1}\kappa_{n+2}\kappa_{l+1}\kappa_{l+2}}} \left[\left(v \kappa_l + b_l p_{l-1}^{(\alpha)}(v) p_l^{(\alpha)}(\bar{v}) + b_{l+1} p_l^{(\alpha)}(v) p_{l+1}^{(\alpha)}(\bar{v}) \right) p_{l+1}^{(\alpha)}(\bar{v}) \right. \\ & \left. - \left(a_{l+1} p_l^{(\alpha)}(\bar{v}) + b_{l+2} p_{l+2}^{(\alpha)}(\bar{v}) \right) \kappa_{l+1} \right]. \end{aligned} \quad (5.25)$$

In what follows, we will show that these coefficients are non-vanishing for all $n-2 \geq l \geq 0$ and $b > 0$ in general. This means that there can be exceptions, for example $c_{l,n}$ can be zero for some n . This occurs when v is a zero of the Gegenbauer polynomial of degree $n+1$, $p_{n+1}^{(\alpha)}(v)$ in the numerator of (5.25). Notice further that $p_{n+1}^{(\alpha)}$ and $p_{n+2}^{(\alpha)}$ do not have common zeros.

Let us first check that we recover the three-term recurrence in the real limit $b \rightarrow 0$, where we have to show that indeed $c_{l,n} = 0$ in this limit for $l \leq n-2$. When $b = 0$ and the corresponding normalisation constants are understood as $h_n = h_n(a, 0)$, the recursion coefficients (5.21) become symmetric, $a_n = b_n$, as it is known for Gegenbauer polynomials on $[-1, 1]$ [20], see Remark 3.7. We thus obtain for the bracket in (5.25) at $b = 0$

$$p_{l+1}^{(\alpha)}(\bar{v}) \left(v \kappa_l(v, \bar{v}) + b_l p_{l-1}^{(\alpha)}(v) p_l^{(\alpha)}(\bar{v}) + b_{l+1} p_l^{(\alpha)}(v) p_{l+1}^{(\alpha)}(\bar{v}) - \bar{v} \kappa_{l+1}(v, \bar{v}) \right)$$

$$\begin{aligned}
 &= p_{l+1}^{(\alpha)}(\bar{v}) \left(\sum_{i=0}^{l-1} b_{i+1} p_{i+1}^{(\alpha)}(v) p_i^{(\alpha)}(\bar{v}) + \sum_{i=0}^{l+1} b_i p_{i-1}^{(\alpha)}(v) p_i^{(\alpha)}(\bar{v}) \right. \\
 &\quad \left. - \sum_{i=0}^l p_i^{(\alpha)}(v) \left(b_{i+1} p_{i+1}^{(\alpha)}(\bar{v}) + b_i p_{i-1}^{(\alpha)}(\bar{v}) \right) \right) \\
 &= 0.
 \end{aligned} \tag{5.26}$$

Here, we have used the notation $p_{-1}^{(\alpha)} = 0$, and after relabelling the sums, they cancel.

To see that the expression (5.25) is non vanishing in general for $b > 0$, we consider the leading coefficient of (5.25) as a polynomial in $\bar{v}$, which is of degree $2l + 2$. We thus have to focus on

$$\left(b_{l+1} p_{l+1}^{(\alpha)}(\bar{v})^2 - b_{l+2} p_l^{(\alpha)}(\bar{v}) p_{l+2}^{(\alpha)}(\bar{v}) \right) p_l^{(\alpha)}(v). \tag{5.27}$$

Because the polynomials of degree l and $l + 1$ do not have common zeros, it is sufficient to consider the leading coefficients inside the bracket, which read

$$\begin{aligned}
 &\frac{c(l+3+2\alpha)}{2(l+2+\alpha)} \sqrt{\frac{h_l}{h_{l+1}}} \frac{1}{h_{l+1}} \left(\frac{2^{l+1} \Gamma(2+l+\alpha)}{\Gamma(1+\alpha)(l+1)! c^{l+1}} \right)^2 \\
 &- \frac{c(l+3+2\alpha)}{2(l+3+\alpha)} \frac{\sqrt{h_{l+1}}}{h_{l+2}} \frac{1}{\sqrt{h_l}} \frac{2^{2l+2} \Gamma(1+\alpha+l) \Gamma(3+\alpha+l)}{\Gamma(1+\alpha)^2 l! (l+2)! c^{2l+2}},
 \end{aligned} \tag{5.28}$$

upon using (5.21) and (C.3). Inserting (3.3) and recalling [20]

$$C_l^{(1+\alpha)}(1) = \frac{\Gamma(2+2\alpha+l)}{\Gamma(2+2\alpha)l!}, \tag{5.29}$$

it can be shown that (5.28) vanishes only if the following equality holds:

$$\frac{\left(C_{l+1}^{(1+\alpha)}(x) \right)^2}{\left(C_{l+1}^{(1+\alpha)}(1) \right)^2} - \frac{C_l^{(1+\alpha)}(x) C_{l+2}^{(1+\alpha)}(x)}{C_l^{(1+\alpha)}(1) C_{l+2}^{(1+\alpha)}(1)} = 0, \tag{5.30}$$

where

$$x = \frac{a^2 + b^2}{a^2 - b^2}. \tag{5.31}$$

The expression on the left hand side of (5.30), usually denoted by $\Delta_n(x)$, is known as the Turán determinant. By [24, Theorem 1] $\Delta_n(x) = 0$ if and only if $x = \pm 1$. Thus $c_{l,n} \equiv 0$ for $0 \leq l \leq n - 2$ in the limit $b \rightarrow 0$, that is when $x \rightarrow 1$, which brings us back to the real line with a three-step recursion. For $x > 1$ all Fourier coefficients are non-vanishing, $c_{l,n} \neq 0$ for $0 \leq l \leq n - 2$, in our example of polynomials (5.18), and thus there exists no *finite-term* recurrence relation.

Acknowledgements Open Access funding provided by Projekt DEAL. We acknowledge support by the German research council (DFG) through CRC 1283: “Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications” (GA), by the Japan Society for the Promotion of Science, KAKENHI 25400397 (TN), and by the Grant DAAD-CONICYT/Becas Chile, 2016/91609937 (IP). The hospitality of the MFO Oberwolfach during the workshop on “Free Probability” is also acknowledged (GA), in particular their excellent library service.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Appendix

A Proof of Lemma 3.2 for Odd Polynomials

In this appendix we collect the relevant formulae for the proof that (3.5) holds when $m = 2n + 1$ and $j = 2l + 1$ are both odd. We begin by expressing the odd Gegenbauer polynomials in terms of a Gauß hypergeometric function, see e.g., [10, 8.932.3].

$$\begin{aligned} C_{2n+1}^{(1+\alpha)}\left(\frac{z(r, \theta)}{c}\right) &= \frac{2z(r, \theta)}{c} \frac{(-1)^n \Gamma(n+2+\alpha)}{\Gamma(n+1)\Gamma(1+\alpha)} F\left(-n, n+\alpha+2; \frac{3}{2}; \frac{z(r, \theta)^2}{c^2}\right) \\ &= \frac{(-1)^n}{\Gamma(1+\alpha)n!} \sum_{p=0}^n \sum_{k=0}^{2p+1} (-1)^p \binom{n}{p} \binom{2p+1}{k} \frac{\Gamma(2+\alpha+n+p)\Gamma(p+1)}{\Gamma(2p+2)} \\ &\quad \times r^{2p+1} R^{2(k-p)-1} e^{i\theta(2(k-p)-1)} \\ &= \frac{(-1)^n}{\Gamma(1+\alpha)n!} \sum_{p=0}^n \sum_{k=0}^{2p+1} (-1)^p \binom{n}{p} \binom{2p+1}{k} \frac{\Gamma(2+\alpha+n+p)\Gamma(p+1)}{\Gamma(2p+2)} \\ &\quad \times r^{2p+1} R^{2(p-k)+1} e^{i\theta(2(p-k)+1)}, \end{aligned} \quad (\text{A.1})$$

where we have used again the parametrisation (3.7), giving two equivalent representations to be used. Likewise we obtain for the odd powers of the conjugated variable

$$\left(\frac{\bar{z}(r, \theta)}{c}\right)^{2l+1} = \left(\frac{r}{2}\right)^{2l+1} \left[\sum_{k=1}^{l+1} \binom{2l+1}{k+l} R^{1-2k} e^{i\theta(2k-1)} + \sum_{k=1}^{l+1} \binom{2l+1}{k+l} R^{2k-1} e^{i\theta(1-2k)} \right]. \quad (\text{A.2})$$

The radial integral (3.11) can be readily used, and we obtain for the angular integration

$$\int_E C_{2n+1}^{(1+\alpha)}\left(\frac{z}{c}\right) \left(\frac{\bar{z}}{c}\right)^{2l+1} dA_\alpha(z)$$

$$\begin{aligned}
 &= \frac{(1+\alpha)(-1)^n}{2^{2l+2}\pi} \sum_{k'=1}^{l+1} \sum_{p=0}^n \sum_{k=0}^{2p+1} \binom{2l+1}{k'+l} \frac{(-1)^p \Gamma(2+\alpha+n+p) \Gamma(2+l+p)}{(n-p)! k! (2p-k+1)! \Gamma(3+\alpha+l+p)} \\
 &\quad \times R^{2(k-p+k'-1)} \int_0^{2\pi} d\theta e^{2i\theta(k-p-k')} \\
 &\quad + \frac{(1+\alpha)(-1)^n}{2^{2l+2}\pi} \sum_{k'=1}^{l+1} \sum_{p=0}^n \sum_{k=0}^{2p+1} \binom{2l+1}{k'+l} \frac{(-1)^p \Gamma(2+\alpha+n+p) \Gamma(2+l+p)}{(n-p)! k! (2p-k+1)! \Gamma(3+\alpha+l+p)} \\
 &\quad \times R^{2(p-k-k'+1)} \int_0^{2\pi} d\theta e^{2i\theta(p-k+k')}. \tag{A.3}
 \end{aligned}$$

Let us evaluate the first triple sum, where we have $k = p + k'$ due to the angular integration. Because $k \leq 2p + 1$ this implies $k' \leq p + 1$. We thus obtain for it

$$\begin{aligned}
 &\frac{(1+\alpha)(-1)^n}{2^{2l+1}} \sum_{k'=1}^{l+1} \binom{2l+1}{k'+l} \sum_{p=k'-1}^n \frac{(-1)^p \Gamma(2+\alpha+n+p) \Gamma(2+l+p)}{(n-p)! (k'+p)! (p+1-k')! \Gamma(3+\alpha+l+p)} R^{4k'-2} \\
 &:= \frac{(1+\alpha)(-1)^n}{2^{2l+1}} \sum_{k'=1}^{l+1} \binom{2l+1}{k'+l} b_{k'} R^{4k'-2}. \tag{A.4}
 \end{aligned}$$

It is a polynomial in R of degree $4l + 2$, and we have to show that all its coefficients $b_{k'} = b_{k'}(n, l)$ vanish for $l < n$. The second triple sum in (A.3) agrees with (A.4), with $R \rightarrow R^{-1}$. This is because the angular integration in (A.3) leads to $k = p + k'$ and $k' \leq p + 1$. The coefficients b_k can again be rewritten as an integral. After shifting the summation index we have

$$\begin{aligned}
 b_k &= \frac{(-1)^{k-1}}{(n+1-k)!} \sum_{p=0}^{n+1-k} (-1)^p \binom{n+1-k}{p} \frac{\Gamma(1+\alpha+n+k+p) \Gamma(1+l+k+p)}{\Gamma(2k+p) \Gamma(2+\alpha+l+k+p)} \\
 &= \frac{(-1)^{k-1}}{(n+1-k)!} \int_0^1 dx \frac{x^{l+k} (1-x)^\alpha}{\Gamma(1+\alpha)} \sum_{p=0}^{n+1-k} (-1)^p \binom{n+1-k}{p} \frac{\Gamma(1+\alpha+n+k+p)}{\Gamma(2k+p)} x^p \\
 &= \frac{(-1)^{k-1} \Gamma(1+\alpha+n+k)}{(n+1-k)! \Gamma(1+\alpha) \Gamma(2k)} \int_0^1 dx x^{l+k} (1-x)^\alpha F(-n-1+k, 1+\alpha+n+k; 2k; x). \tag{A.5}
 \end{aligned}$$

The very same steps as in the proof for the even polynomials allow us to manipulate the remaining integral as follows:

$$\begin{aligned}
 &\int_0^1 dx x^{l+k} (1-x)^\alpha F(-(n+1-k), 1+\alpha+n+k; 2k; x) \\
 &= \lim_{\varepsilon \rightarrow 0} \int_0^1 dx x^{l+k+\varepsilon} (1-x)^\alpha F(-(n+1-k), 1+\alpha+n+k; 2k; x) \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{\Gamma(2k) \Gamma(1+l+k+\varepsilon) \Gamma(2+\alpha+n-k) \Gamma(n-l-\varepsilon)}{\Gamma(1+k+n) \Gamma(3+\alpha+n+l+\varepsilon) \Gamma(-\varepsilon-(l+1-k))}
 \end{aligned}$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{(-1)^{l-k} \Gamma(2k) \Gamma(1+l+k+\varepsilon) \Gamma(2+\alpha+n-k) \Gamma(l+2-k+\varepsilon)}{\pi \Gamma(1+k+n) \Gamma(3+\alpha+n+l+\varepsilon)} \times \Gamma(n-l-\varepsilon) \sin(\pi \varepsilon). \quad (\text{A.6})$$

Together with (3.18) this establishes the orthogonality of the odd polynomials (A.3).

In order to compute the norms for the odd polynomials we summarise the above results for the coefficients

$$b_k(n, n) = \frac{(-1)^n \Gamma(1+\alpha+n+k) \Gamma(2+\alpha+n-k)}{\Gamma(1+\alpha) \Gamma(3+\alpha+2n)}, \quad (\text{A.7})$$

which has to be inserted into (A.4), and the corresponding equation with $R \rightarrow R^{-1}$. We obtain at $n = l$

$$\begin{aligned} & \sum_{k'=1}^{n+1} \binom{2n+1}{k'+n} b_{k'}(n, n) R^{\pm(4k'-2)} \\ &= \frac{(-1)^n (2n+1)!}{\Gamma(1+\alpha) \Gamma(2n+\alpha+3)} \sum_{k=n+1}^{2n} \frac{\Gamma(1+\alpha+k) \Gamma(2+\alpha+2n-k)}{(2n+1-k)! k!} R^{\mp(4(n-k)+2)} \\ &= \frac{(-1)^n (2n+1)!}{\Gamma(1+\alpha) \Gamma(2n+\alpha+3)} \sum_{k=0}^n \frac{\Gamma(1+\alpha+2n+1-k) \Gamma(1+\alpha+k)}{k! (2n+1-k)!} R^{\pm(4(n-k)+2)}, \end{aligned} \quad (\text{A.8})$$

relabelling the sum twice. These two sums with $R^{\pm}$ can be inserted into (A.3), to give a single sum

$$\begin{aligned} & \int_E C_{2n+1}^{(1+\alpha)} \left(\frac{z}{c} \right) \left(\frac{\bar{z}}{c} \right)^{2l+1} dA_{\alpha}(z) \\ &= \frac{\delta_{n,l} (1+\alpha) (2n+1)!}{2^{2n+1} \Gamma(1+\alpha) \Gamma(2n+\alpha+3)} \sum_{k=0}^{2n+1} \frac{\Gamma(1+\alpha+k) \Gamma(1+\alpha+2n+1-k)}{\Gamma(2n+1-k+1) \Gamma(k+1)} R^{4n+2-4k}. \end{aligned} \quad (\text{A.9})$$

This sum can be written as a single Gegenbauer polynomial, using its invariance under $k \rightarrow 2n+1-k$:

$$\begin{aligned} &= \frac{1}{2} \sum_{k=0}^{2n+1} \frac{\Gamma(1+\alpha+k) \Gamma(1+\alpha+2n+1-k)}{\Gamma(2n+1-k+1) \Gamma(k+1)} \left(R^{4n+2-4k} + R^{-(4n+2-4k)} \right) \\ &= \sum_{k=0}^{2n} \frac{\Gamma(1+\alpha+k) \Gamma(1+\alpha+2n+1-k)}{\Gamma(2n+1-k+1) \Gamma(k+1)} \cosh[(2n+1-2k) \ln(R^2)] \\ &= \Gamma(1+\alpha)^2 C_{2n+1}^{(1+\alpha)} \left(\frac{a^2 + b^2}{a^2 - b^2} \right), \end{aligned} \quad (\text{A.10})$$

where we used again [20, 18.5.11] and (3.24). The leading power of the odd Gegenbauer polynomials can be read off from the first line of (A.1),

$$C_{2l+1}^{(1+\alpha)}(x) = \frac{\Gamma(2l+2+\alpha)2^{2l+1}}{\Gamma(1+\alpha)(2l+1)!}x^{2l+1} + O(x^{2l-1}). \quad (\text{A.11})$$

Multiplying (A.3) with this factor and using that the lower powers vanish yields

$$\int_E C_{2n+1}^{(1+\alpha)}\left(\frac{z}{c}\right) C_{2l+1}^{(1+\alpha)}\left(\frac{\bar{z}}{c}\right) dA_\alpha(z) = \delta_{2n+1,2l+1} \frac{1+\alpha}{2n+\alpha+2} C_{2n+1}^{(1+\alpha)}\left(\frac{a^2+b^2}{a^2-b^2}\right). \quad (\text{A.12})$$

It agrees with (3.4) for odd indices.

B Orthogonality of Chebyshev Polynomials of the Second Kind

For completeness we present an independent proof for the orthogonality of the Chebyshev polynomials of the second kind U_n on the interior of the ellipse E (2.2),

$$\int_E U_m\left(\frac{z}{c}\right) U_n\left(\frac{\bar{z}}{c}\right) dA(z) = \frac{1}{1+n} U_n\left(\frac{a^2+b^2}{a^2-b^2}\right) \delta_{n,m}. \quad (\text{B.1})$$

The argument of the proof is not new and it can be found in [11, pag. 546]. It uses Stokes' theorem (see e.g. [12]), which we restate for the readers convenience.

Let G to be a bounded open set in $\mathbb{C}$, such that the boundary ∂G consists of a finite number of C^1 Jordan curves. For any $F \in C^1(\bar{G})$ Stokes' theorem relates the integral over G to that over its boundary ∂G :

$$\int_G \bar{\partial} F(z) dA(z) = \frac{1}{2i} \int_{\partial G} F(z) dz, \quad \bar{\partial} := \frac{\partial}{\partial \bar{z}}. \quad (\text{B.2})$$

In particular for $F(z) = f(z)\overline{g(z)}$ with f, g analytic, we have

$$\int_G \bar{\partial} [f(z)\overline{g(z)}] dA(z) = \int_G f(z)\overline{g'(z)} dA(z) = \frac{1}{2i} \int_{\partial G} f(z)\overline{g(z)} dz. \quad (\text{B.3})$$

Proof To show (B.1), we can use the well-known formula [20, 18.9.21] relating Chebyshev polynomials of the first T_n and second kind

$$T'_n(z) = nU_{n-1}(z), \quad (\text{B.4})$$

for $n = 1, 2, \dots$. We can thus rewrite the left hand side of (B.1) for any $n, m = 0, 1, \dots$ to apply Stokes' theorem

$$\int_E U_n\left(\frac{z}{c}\right) U_m\left(\frac{\bar{z}}{c}\right) dA(z) = \frac{c^2}{(n+1)(m+1)} \int_E T'_{n+1}\left(\frac{z}{c}\right) \overline{T'_{m+1}\left(\frac{\bar{z}}{c}\right)} dA(z)$$

$$\begin{aligned}
&= \frac{c^2}{(n+1)(m+1)} \int_E \bar{\partial} \left[T'_{n+1} \left(\frac{z}{c} \right) \overline{T_{m+1} \left(\frac{z}{c} \right)} \right] dA(z) \\
&= \frac{c^2}{(n+1)(m+1)} \frac{1}{2i} \int_{\partial E} T'_{n+1} \left(\frac{z}{c} \right) \overline{T_{m+1} \left(\frac{z}{c} \right)} \frac{dz}{\pi ab} . \quad (\text{B.5})
\end{aligned}$$

Next, we use the Joukowski map

$$z(w) = \frac{1}{2} \left(w + \frac{c^2}{w} \right) , \quad (\text{B.6})$$

which maps the circle $|w| = r$ of radius $r := a + b$ onto the boundary $\partial E \ni z$ of the ellipse E . The chain rule

$$T'_{n+1}(z) = \frac{d}{dz} T_{n+1}(z) = \frac{d}{dw} T_{n+1}(z(w)) \frac{dw}{dz} , \quad (\text{B.7})$$

allows us to rewrite

$$\int_{\partial E} T'_{n+1} \left(\frac{z}{c} \right) \overline{T_{m+1} \left(\frac{z}{c} \right)} dz = \int_{|w|=r} \frac{d}{dw} T_{n+1}(z(w)/c) \overline{T_{m+1}(z(w)/c)} dw . \quad (\text{B.8})$$

Note that the contribution from the Jacobian of the transformation (B.6) just cancels the extra factor $\frac{dw}{dz}$ stemming from (B.7). Furthermore, as we have stated already in (4.42), it is well known [17] that

$$T_{n+1}(z(w)/c) = \frac{1}{2} ((w/c)^{n+1} + (c/w)^{n+1}) \quad (\text{B.9})$$

holds for the Joukowski map (B.6). Therefore

$$\frac{d}{dw} T_{n+1}(z(w)/c) = \frac{n+1}{2w} \left[\left(\frac{w}{c} \right)^{n+1} - \left(\frac{c}{w} \right)^{n+1} \right] \quad (\text{B.10})$$

allows us to exploit the orthogonality on the circle

$$\int_{|w|=r} w^a \bar{w}^b \frac{dw}{w} = ir^{a+b} \int_0^{2\pi} e^{i\theta(a-b)} d\theta = 2\pi ir^{2a} \delta_{a,b} , \quad (\text{B.11})$$

as follows:

$$\begin{aligned}
&\int_{\partial E} T'_{n+1} \left(\frac{z}{c} \right) \overline{T_{m+1} \left(\frac{z}{c} \right)} dz \\
&= \frac{n+1}{4} \int_{|w|=r} \left[\left(\frac{w}{c} \right)^{n+1} - \left(\frac{c}{w} \right)^{n+1} \right] \overline{\left[\left(\frac{w}{c} \right)^{m+1} + \left(\frac{c}{w} \right)^{m+1} \right]} \frac{dw}{w} \\
&= \frac{i\pi(n+1)}{2} \left[\left(\frac{r}{c} \right)^{2n+2} - \left(\frac{c}{r} \right)^{2n+2} \right] \delta_{n,m} . \quad (\text{B.12})
\end{aligned}$$

In this form the orthogonality is stated in [11]. To arrive at the right hand side of (B.1) we use (4.33) and (4.38), together with $U_n = C_n^{(1)}$. $\square$

C Alternative Proof of Lemma 3.2

In the proof presented here we do not need to distinguish between Gegenbauer polynomials with even and odd parity. The main assumption to be made here is that the orthogonality of the Chebyshev polynomials of the second kind $U_n(x)$ holds on the unweighted ellipse, as shown in (B.1) in the previous Appendix B. Using $U_n(x) = C_n^{(1)}(x)$, we only need to show that, from (B.1), follows that

$$\int_E C_m^{(1+\alpha)}\left(\frac{z}{c}\right) \bar{z}^j dA_{\alpha(z)} = 0, \quad j = 0, 1, \dots, m-1, \quad (\text{C.1})$$

holds for $\alpha > -1$. The computation of the norms however, is much more involved in this approach and we will not present it here. It leads to the same result as given in Lemma 3.2.

The general Gegenbauer polynomials can be explicitly written in the following form, as can be seen, for example, from the representations in terms of a hypergeometric function (see (3.9) and (A.1)),

$$C_n^{(1+\alpha)}(z) = \sum_{j=0}^n \kappa_j^n(\alpha) z^j, \quad (\text{C.2})$$

with the coefficients reading

$$\kappa_j^n(\alpha) = \begin{cases} \frac{(-1)^{(n-j)/2} 2^j \Gamma(\alpha+1+(n+j)/2)}{\Gamma(\alpha+1)\Gamma(j+1)\Gamma(1+(n-j)/2)}, & \text{for } n-j \text{ even,} \\ 0, & \text{for } n-j \text{ odd.} \end{cases} \quad (\text{C.3})$$

This immediately implies the following relation

$$\kappa_j^n(\alpha) = \frac{\Gamma(\alpha+1+(n+j)/2)}{\Gamma(\alpha+1)\Gamma(1+(n+j)/2)} \kappa_j^n(0), \quad (\text{C.4})$$

between the expansion coefficients for general Gegenbauer and Chebyshev polynomials of the second kind ($\alpha = 0$). The former satisfy the following three-term recurrence relation

$$z C_n^{(1+\alpha)}(z) = \frac{n+1}{2(n+\alpha+1)} C_{n+1}^{(1+\alpha)}(z) + \frac{n+2\alpha+1}{2(n+\alpha+1)} C_{n-1}^{(1+\alpha)}(z), \quad n = 1, 2, 3, \dots \quad (\text{C.5})$$

Let us recall our notation for the inner product (2.13),

$$\langle f, g \rangle_{\alpha} := \int_E f(z) \overline{g(z)} dA_{\alpha}(z). \quad (\text{C.6})$$

From (B.1) we immediately have for arbitrary integers m and j , satisfying $m > j \geq 0$, that

$$\langle C_m^{(1)}(z/c), z^j \rangle_0 = 0. \quad (\text{C.7})$$

Using the three-term recurrence relation (C.5) for $\alpha = 0$, we see that $z^l C_m^{(1)}(z/c)$ can be expanded in terms of $C_k^{(1)}(z/c)$ with $m - l \leq k \leq m + l$. It thus follows that

$$\langle C_m^{(1)}(z/c), z^j z^l \rangle_0 = \langle z^l C_m^{(1)}(z/c), z^j \rangle_0 = 0 \quad (\text{C.8})$$

holds, for $j \geq 0$, $l \geq 0$ and $j + l < m$. Our goal is to use this relation and to prove the orthogonality (C.1) by relating inner products, with general $\alpha > -1$, to those with $\alpha = 0$. In particular we can generalise the above statement (C.8) to the following

Lemma 1 For an arbitrary positive integer m

$$\langle C_m^{(1+\alpha)}(z), z^j z^l \rangle_\alpha = 0 \quad (\text{C.9})$$

holds for $\alpha > -1$, given that $j \geq 0$, $l \geq 0$ and $j + l < m$.

Obviously (C.1) then follows by choosing $l = 0$.

Proof Due to the previously noted invariance of the weight and domain under the reflection $z \rightarrow -z$, we have that

$$\langle z^p, z^q \rangle_\alpha \neq 0 \text{ if and only if } p + q \text{ is even,} \quad (\text{C.10})$$

that is when p and q have the same parity. Furthermore, we can relate the inner products of such monomials with general $\alpha > -1$ and with $\alpha = 0$ as follows. The change of variables (3.7) decouples radial and angular integration and leads to

$$\begin{aligned} \langle z^p, z^q \rangle_\alpha &= \frac{1+\alpha}{\pi} \int_0^1 dr r \left(\frac{r}{2}\right)^{p+q} (1-r^2)^\alpha \int_0^{2\pi} d\theta (Re^{i\theta} + R^{-1}e^{-i\theta})^p (R^{-1}e^{i\theta} + Re^{-i\theta})^q \\ &= \frac{\Gamma(2+\alpha)\Gamma(2+(p+q)/2)}{\Gamma(2+\alpha+(p+q)/2)} \langle z^p, z^q \rangle_0, \end{aligned} \quad (\text{C.11})$$

due to standard integrals

$$\int_0^1 dr r^{p+q+1} (1-r^2)^\alpha = \frac{\Gamma(1+\alpha)\Gamma(1+(p+q)/2)}{\Gamma(2+\alpha+(p+q)/2)}.$$

We proceed to prove (C.9) by induction. For $m = 1$, we can readily find

$$\langle C_1^{(1+\alpha)}(z/c), 1 \rangle_\alpha = \kappa_1^1(\alpha) \langle z/c, 1 \rangle_\alpha = 0, \quad (\text{C.12})$$

which holds due to parity, see (C.10). Now, suppose that the claim (C.9) holds for $m = 1, 2, \dots, k$. We will show

$$\langle C_{k+1}^{(1+\alpha)}(z/c), z^j z^l \rangle_\alpha = 0, \quad (\text{C.13})$$

separately for (i) $j + l \leq k - 2$, (ii) $j + l = k - 1$ and (iii) $j + l = k$.

(i) If $j, l \geq 0$ and $j + l + 1 \leq k - 1$, the induction assumption guarantees that

$$\begin{aligned} 0 &= c^{-1} \langle C_k^{(1+\alpha)}(z/c), z^j \bar{z}^{l+1} \rangle_\alpha = \langle (z/c) C_k^{(1+\alpha)}(z/c), z^j \bar{z}^l \rangle_\alpha \\ &= \frac{k+1}{2(k+\alpha+1)} \langle C_{k+1}^{(1+\alpha)}(z/c), z^j \bar{z}^l \rangle_\alpha + \frac{k+2\alpha+1}{2(k+\alpha+1)} \langle C_{k-1}^{(1+\alpha)}(z/c), z^j \bar{z}^l \rangle_\alpha \\ &= \frac{k+1}{2(k+\alpha+1)} \langle C_{k+1}^{1+\alpha}(z/c), z^j \bar{z}^l \rangle_\alpha. \end{aligned} \quad (\text{C.14})$$

Here, we have used the recursion relation (C.5) and in the second line again the induction assumption, to arrive at the claimed statement.

(ii) If $j, l \geq 0$ and $j + l = k - 1$, we may directly use the expansion (C.2) to obtain

$$\begin{aligned} \frac{1}{1+\alpha} \langle C_{k+1}^{(1+\alpha)}(z/c), z^j \bar{z}^l \rangle_\alpha &= \frac{1}{1+\alpha} \sum_{p=0}^{k+1} \kappa_p^{k+1}(\alpha) \langle (z/c)^p, z^j \bar{z}^l \rangle_\alpha \\ &= \sum_{p=0}^{k+1} \kappa_p^{k+1}(\alpha) \frac{\Gamma(\alpha+1) \Gamma((k+p+3)/2)}{\Gamma((k+2\alpha+p+3)/2)} \langle (z/c)^p, z^j \bar{z}^l \rangle_0 \\ &= \sum_{p=0}^{k+1} \kappa_p^{k+1}(0) \langle (z/c)^p, z^j \bar{z}^l \rangle_0 \\ &= \langle C_{k+1}^{(1)}(z/c), z^j \bar{z}^l \rangle_0 = 0. \end{aligned} \quad (\text{C.15})$$

In the second step we have used the relation (C.11), to be able to relate to the known orthogonality (C.8) via (C.2).

(iii) If $j, l \geq 0$ and $j + l = k$, we see from (C.2) that the expectation value

$$\langle C_{k+1}^{(1+\alpha)}(z/c), z^j \bar{z}^l \rangle_\alpha = \sum_{p=0}^{k+1} \kappa_p^{k+1}(\alpha) \langle (z/c)^p, z^j \bar{z}^l \rangle_\alpha = 0 \quad (\text{C.16})$$

vanishes due to parity: because of (C.3) $k+1-p$ is even, implying that $p+j-l = p+k-2l$ is odd.

□

References

1. Akemann, G., Vernizzi, G.: Characteristic polynomials of complex random matrix models. Nucl. Phys. B **660**(3), 532–556 (2003). [arXiv:hep-th/0212051](https://arxiv.org/abs/hep-th/0212051)
2. Akemann, A., Bender, M.: Interpolation between Airy and Poisson statistics for unitary chiral non-Hermitian random matrix ensembles. J. Math. Phys. **51**, 103524 (2010). [arXiv:1003.4222](https://arxiv.org/abs/1003.4222) [math-ph]
3. Akemann, G., Phillips, M.J.: Universality conjecture for all airy, sine and bessel kernels in the complex plane in random matrix theory. In: Deift, P., Forrester, P. (eds.) Interacting Particle Systems, and Integrable Systems, MSRI Publications, vol. 65, pp. 1–24. Cambridge University Press, Cambridge (2014). ISBN-13: 978-1-107-07992-2 [arXiv:1204.2740](https://arxiv.org/abs/1204.2740) [math-ph]

4. Conway, J.B.: A Course in Functional Analysis, 2nd edn. Springer, New York (1990)
5. Di Francesco, P., Gaudin, M., Itzykson, C., Lesage, F.: Laughlin's wave functions, Coulomb gases and expansions of the discriminant. *Int. J. Mod. Phys. A* **9**, 4257 (1994). (hep-th/9401163)
6. van Eijndhoven, S.J.L., Meyers, J.L.H.: New orthogonality relation for the Hermite polynomials and related Hilbert space. *J. Math. Anal. Appl.* **146**, 89–98 (1990)
7. Forrester, P.J., Warnaar, S.O.: The importance of the Selberg integral. *Bull. Am. Math. Soc. (N.S.)* **45**, 489–534 (2008). [arXiv:0710.3981](https://arxiv.org/abs/0710.3981)
8. Fyodorov, Y.V., Khoruzhenko, B.A., Sommers, H.-J.: Almost-Hermitian random matrices: eigenvalue density in the complex plane. *Phys. Lett. A* **226**, 46–52 (1997). [arXiv:cond-mat/9606173](https://arxiv.org/abs/cond-mat/9606173); Almost-Hermitian Random Matrices: Crossover from Wigner-Dyson to Ginibre eigenvalue statistics. *Phys. Rev. Lett.* **79**, 557–560 (1997). [arXiv:cond-mat/9703152](https://arxiv.org/abs/cond-mat/9703152)
9. Fyodorov, Y.V., Khoruzhenko, B.A., Sommers, H.-J.: Universality in the random matrix spectra in the regime of weak non-Hermiticity. *Ann. Inst. Henri Poincaré* **68**, 449–489 (1998). [arXiv:chao-dyn/9802025](https://arxiv.org/abs/chao-dyn/9802025)
10. Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series, and products. In: Jeffrey, A., Zwillinger, D. (eds.) Table of Integrals, Series, and Products, 7th edn. Academic Press, San Diego (2007)
11. Henrici, P.: Applied and Computational Complex Analysis. Wiley-Interscience, Volume in Pure and Applied Mathematics, vol. 3. Wiley, Hoboken (1993)
12. Hörmander, L.: An Introduction to Complex Analysis in Several Variables, 2nd edn. North-Holland, Amsterdam (1979)
13. Karp, D.: Square summability with geometric weight for classical orthogonal expansions. In: Advances in Analysis, Proceedings of 4th International ISAAC Conference, H. G. W. Begehr et al. (eds.), World Scientific, Singapore, pp. 407–421 (2005). [arXiv:math/0604028](https://arxiv.org/abs/math/0604028) [math.CA]
14. Khavinson, D., Shapiro, H.S.: Dirichlet's problem when the data is an entire function. *Bull. Lond. Math. Soc.* **24**(5), 456–468 (1992)
15. Khavinson, D., Stylianopoulos, N.: Recurrence relations for orthogonal polynomials and algebraicity of solution of the Dirichlet problem. *Int. Math. Series* **12**, 219–228 (2010)
16. Lempert, L.: Recursion for orthogonal polynomials on complex domains. In: Fourier Analysis and Approximation Theory (Proc. Colloq., Budapest, 1976), vol. II, pp. 481–494. North-Holland, Amsterdam (1978)
17. Mason, J., Handscomb, D.: Chebyshev Polynomials. CRC Press, London (2002)
18. Mehta, M.L.: Random Matrices, 3rd edn. Academic Press, London (2004)
19. Nagao, T., Akemann, G., Kieburg, M., Parra, I.: Families of two-dimensional Coulomb gases on an ellipse: correlation functions and universality. *J. Phys. A: Math. Theor.* **53**(7), 075201 (2020). [arXiv:1905.07977](https://arxiv.org/abs/1905.07977) [math-ph]
20. Olver, F.W.J., et al. (eds.): NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge (2010)
21. Osborn, J.C.: Universal results from an alternate random matrix model for QCD with a baryon chemical potential. *Phys. Rev. Lett.* **93**, 222001 (2004). [arXiv:hep-th/0403131](https://arxiv.org/abs/hep-th/0403131)
22. Putinar, M., Stylianopoulos, N.S.: Finite-term relations for planar orthogonal polynomials. *Compl. Anal. Oper. Theory* **1**(3), 447–456 (2007)
23. Sommers, H.J., Crisanti, A., Sompolinsky, H., Stein, Y.: Spectrum of large random asymmetric matrices. *Phys. Rev. Lett.* **60**(19), 1895–1898 (1988)
24. Szwarz, R.: Positivity of Turán determinants for orthogonal polynomials. In: Ross, K.A., Singh, A.I., Anderson, J.M., Sunder, V.S., Litvinov, G.L., Wildberger, N.J. (eds.) Harmonic Analysis and Hypergroups. Trends in Mathematics, pp. 165–182. Birkhäuser, Boston, MA (1998). https://doi.org/10.1007/978-0-8176-4348-5_11
25. Szegő, G.: Orthogonal Polynomials, vol. XXIII, 4th edn. American Mathematical Society, Providence (1975)
26. Van Assche, W.: Orthogonal polynomials in the complex plane and on the real line. In: Ismail, M.E.H., et al. (eds.) Special Functions, Q-Series and Related Topics. Fields Institute Communications, vol. 14, pp. 211–245. American Mathematical Society, Providence (1997)
27. Życzkowski, K., Sommers, H.-J.: Truncations of random unitary matrices. *J. Phys. A: Math. Gen.* **33**, 2045–2057 (2000). [arXiv:chao-dyn/9910032](https://arxiv.org/abs/chao-dyn/9910032)