On a Uniqueness Theorem of E. B. Vul

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Abstract

We recall a uniqueness theorem of E. B. Vul pertaining to a version of the cosine transform originating in spectral theory. Then we point out an application to the Bernstein approximation problem with non-symmetric weights: a theorem of Volberg is proved by elementary means.

Keywords Bernstein approximation problem · Completeness of polynomials · Non-symmetric weights · Quasianalytic classes · Cosine transform

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1 Introduction

A Cosine Transform and a Uniqueness Theorem The goal of this note is to draw attention to a uniqueness theorem for an integral transform originating in the spectral theory of Sturm–Liouville operators, and to point out an application in approximation theory.

Let $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ be a function of bounded variation such that

$$\forall x \geq 0 \quad M_\sigma(x) \stackrel{\text{def}}{=} \int_{-\infty}^0 \exp(x\sqrt{|\lambda|}) |d\sigma(\lambda)| < \infty. \quad (1)$$

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Define

$$(\mathfrak{C}\sigma)(x) = \int_{-\infty}^{\infty} \cos(x\sqrt{\lambda}) d\sigma(\lambda), \quad x \in \mathbb{R}, \quad (2)$$

where (for $\lambda < 0$) $\cos(x\sqrt{\lambda}) = \operatorname{ch}(x\sqrt{-\lambda})$. The transform $\mathfrak{C}$ (and its variants) appeared in the works of Povzner [24] and Krein [14], who showed that a continuous even function $f : [-2a, 2a] \rightarrow \mathbb{R}$ defines an Hermitian-positive kernel $K_f(x, y) = f(x - y) + f(x + y)$ on $L_2(0, a)$ if and only if it has a representation $f = \mathfrak{C}\sigma$ with an increasing σ .

These works led to the question whether the representation $f = \mathfrak{C}\sigma$ is unique for an increasing σ ; i.e., for which (not necessarily increasing) σ the equality $\mathfrak{C}\sigma \equiv 0$ implies that $\sigma \equiv \text{const}$. Levitan proved [17] that if $\mathfrak{C}\sigma \equiv 0$ and

$$\forall x \geq 0 \quad M_\sigma(x) \leq C \exp(Cx^\alpha)$$

for some $\alpha < 2$, then $\sigma \equiv \text{const}$. Levitan and Meiman [19] showed that the same conclusion remains valid for $\alpha = 2$. Finally, Vul proved [32] the following definitive result:

Theorem 1.1 (Vul) *Suppose $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-decreasing convex function.*

1. *If p satisfies*

$$\int^{\infty} \frac{p(s)}{s^3} ds = \infty, \quad (3)$$

and σ is a function of bounded variation such that

$$\forall x \geq 0 \quad M_\sigma(x) \leq C \exp(p^*(x)) \stackrel{\text{def}}{=} C \exp(\sup_{s \geq 0} [xs - p(s)]) , \quad (4)$$

and $\mathfrak{C}\sigma \equiv 0$, then $\sigma \equiv \text{const}$.

2. *If (3) fails, there exists σ of bounded variation such that (4) holds and $\mathfrak{C}\sigma \equiv 0$, and yet $\sigma \not\equiv \text{const}$.*

Remark 1.2 In [32], this result is stated under an additional assumption

$$\lim_{x \rightarrow \infty} \frac{x p^{*'}(x)}{p^*(x)} = \gamma > 1,$$

however, this requirement can be omitted with no essential modifications in the proof (which we reproduce in Sect. 4).

The uniqueness theorems of Levitan, Meiman, and Vul have found numerous applications. Already Krein [15] used the result of [19] to provide sufficient conditions for the self-adjointness of one-dimensional second order differential operators in terms of the tails of (some) spectral measure. In [18], Levitan showed that Theorem 1.1 implies

the following sufficient condition, due to Sears [25], for the self-adjointness of the Schrödinger operator $L = -\Delta + q$ on $\mathbb{R}^d$:

$$q(x) \geq -Q(|x|), \quad \text{where } Q > 0 \text{ is monotone increasing and } \int_0^\infty \frac{dr}{\sqrt{Q(r)}} = \infty. \quad (5)$$

The earlier result of [19] recovers the weaker sufficient condition $q(x) \geq -Ax^2 - B$ due to Titchmarsh [29]. Uniqueness results related to Theorem 1.1 have also been used in other branches of spectral theory, for example, in the study of the spectral edges of random band matrices [28].

In this note, we present an application to the Bernstein approximation problem with non-symmetric weights, and also use the opportunity to make the proof of Vul accessible in English.

Bernstein's Approximation Problem The setting is as follows: given a lower semi-continuous function $W : \mathbb{R} \rightarrow [1, \infty]$ such that $1/W(\lambda) = O(\lambda^{-\infty})$, one inquires whether polynomials are dense in the space

$$C_0(1/W) = \left\{ u \in C(\mathbb{R}) \mid \lim_{\lambda \rightarrow \infty} u(\lambda)/W(\lambda) = 0 \right\}, \\ \|u\|_{C_0(1/W)} = \sup_{\lambda} |u(\lambda)|/W(\lambda).$$

Equivalently, consider the space

$$\mathfrak{Q}_W = \left\{ f(x) = \int e^{ix\lambda} d\sigma(\lambda) : \int W(\lambda) |d\sigma(\lambda)| < \infty \right\}.$$

By a Hahn–Banach argument, polynomials are dense in $C_0(1/W)$ if and only if $\mathfrak{Q}_W$ is quasianalytic in the sense of Hadamard; i.e., a function $f \in \mathfrak{Q}_W$ that has a zero of infinite order has to vanish identically.

We recall some of the classical results on the Bernstein approximation problem, and refer to the surveys [1] and [21] and the book [13] for more details. Hall [11] and Izumi and Kawata [12] proved that the condition

$$\int_{-\infty}^{\infty} \frac{\log W(\lambda)}{1 + \lambda^2} d\lambda = +\infty \quad (6)$$

is necessary for the completeness of polynomials in $C_0(1/W)$. Carleson showed [7] that in general, (6) is not sufficient. However [7, 12], if a weight $W(\lambda)$ satisfying (6) is even, and the function $s \mapsto \log W(e^s)$ is convex, then polynomials are dense in $C_0(1/W)$, and consequently also in $C_0(1/\tilde{W})$ for any $\tilde{W} \geq W$. Still, there are weights $\tilde{W}$ for which polynomials are dense and yet no such minorant $W \leq \tilde{W}$ exists. Several necessary and sufficient conditions for completeness were derived, particularly, by Akhiezer–Bernstein, Pollard, Mergelyan (see [1, 21]), de Branges [8], and more recently by Poltoratski [23], yet these are not always easily verifiable.

Now we turn to the case when W has different rates of growth at $\pm\infty$. In the extreme case $W(\lambda) \equiv +\infty$ for $\lambda < 0$, it follows from the result of Hall and Izumi–Kawata (for example, using the map $f \rightarrow \phi_f$ in (12) below) that the condition

$$\int_0^\infty \frac{\log W(\lambda)}{1 + \lambda^{3/2}} d\lambda = +\infty \quad (7)$$

is necessary for the completeness of polynomials in $C_0(1/W)$. Consequently, if W is an arbitrary weight, it is necessary for the completeness of polynomials in $C_0(1/W)$ that the pair of conditions

$$\int_0^\infty \frac{\log W(\lambda)}{1 + \lambda^{3/2}} d\lambda = +\infty, \quad \int_0^\infty \frac{\log W(-\lambda)}{1 + \lambda^2} d\lambda = +\infty \quad (8)$$

be satisfied by either $W(\lambda)$ or $W(-\lambda)$. In [31], Volberg showed, answering a question asked by Mergelyan and by Ehrenpreis (see [10, Problem 13.8]), that the conditions

$$\int_0^\infty \frac{\log W(\lambda)}{\lambda^{3/2}} d\lambda = +\infty \quad (9)$$

$$\int_0^\infty \frac{\log W(-\lambda)}{\lambda^2} d\lambda = +\infty \quad (10)$$

are sufficient for completeness, provided that W is regular in the following sense:

$$\log W(\lambda) = \begin{cases} \sqrt{\lambda} \epsilon_1(1/\lambda), & \lambda \geq 0 \\ |\lambda| \epsilon_2(1/|\lambda|), & \lambda < 0 \end{cases}, \quad \lim_{t \rightarrow +0} \epsilon_i(t) = \lim_{t \rightarrow +0} \frac{t \epsilon'_i(t)}{\epsilon_i(t)} = \lim_{t \rightarrow +0} \frac{t^2 \epsilon''_i(t)}{\epsilon_i(t)} = 0. \quad (11)$$

Another proof of this result was given by Borichev [5].¹ The proof in [31] is based on a construction of an auxiliary analytic function and delicate estimates of the harmonic measure, whereas that of [5] is based on the method of quasianalytic (or almost holomorphic) extension, put forth by Dyn'kin [9].

The result of [31] was significantly generalised by M. Sodin [26], who relied on a theorem of de Branges [8]. It is shown in [26] that if

$$\text{polynomials are normally dense in } C_0(\mathbb{1}_{[0,\infty)}/W) \quad (9')$$

$$\int_0^\infty \frac{\log W(-\lambda)}{\lambda^2} d\lambda = +\infty \quad (10)$$

¹ The results of [5, 31] are stated for $L_2(1/W)$, whereas here we focus on the space $C_0(1/W)$. The difference is not essential for the current discussion, as both the original results and the current Theorem 1.3 are valid in both cases; see Corollary 1.4. We also mention the work of Bakan [3, 4], who found a general connection between completeness in $C_0(1/W)$ and $L_2(1/W)$.

and W satisfies the regularity assumption

$$\log W(\lambda) = |\lambda|\epsilon_2(1/|\lambda|) \text{ for } \lambda < 0, \quad \text{and} \quad \epsilon_2(t) \searrow 0, \quad \frac{t\epsilon_2'(t)}{\epsilon_2(t)} \searrow 0 \quad \text{as } t \rightarrow +0 \quad (11')$$

then polynomials are also dense in $C_0(1/W)$. Here (9') means that polynomials are dense for any weight differing from $W/\mathbb{1}_{[0,\infty)}$ at a finite number of points; this condition is necessary, and thus can not be further relaxed. The regularity condition (11') is somewhat restrictive, however, it is shown in [26] that some regularity has to be imposed: there exists a weight W satisfying (9) and (10) such that the functions $s \mapsto \log W(\pm e^s)$ are convex and still polynomials are not dense in $C_0(1/W)$.

Here we prove

Theorem 1.3 *Let $W : \mathbb{R} \rightarrow [1, \infty]$ be a (lower semicontinuous) function such that $\frac{1}{W(\lambda)} = O(\lambda^{-\infty})$, and (9) and (10) hold. If the functions $s \mapsto \log W(e^s)$ and $s \mapsto \log W(-e^s)$ are convex on $[s_0, \infty)$, then polynomials are dense in $C_0(1/W)$.*

Equivalently, the class Ω_W is quasianalytic; *a fortiori*, a non-zero $f \in \Omega_W$ can not vanish on a set of positive measure.

Our condition on $W|_{\mathbb{R}_+}$ is much more stringent than the optimal condition (9') of [26] (although less stringent than (11)). On the other hand, the regularity assumptions on $W|_{\mathbb{R}_-}$ are weaker than (11').

More importantly, the proof of Theorem 1.3 is relatively elementary. It is based on the following well-known construction, similar to the one used to relate the Stieltjes moment problem to the Hamburger one. Let

$$f(x) = \int_{-\infty}^{\infty} e^{ix\lambda} d\sigma(\lambda),$$

where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a function of bounded variation such that

$$\forall k \geq 0 \quad \int_0^{\infty} |\lambda|^k |d\sigma(\lambda)| < \infty, \quad \forall x \geq 0 \quad \int_{-\infty}^0 e^{x\sqrt{|\lambda|}} d\sigma(\lambda) < \infty.$$

Denote

$$\phi_f(x) = (\mathfrak{C}\sigma)(x) = \int_{-\infty}^{\infty} \cos(x\sqrt{\lambda}) d\sigma(\lambda). \quad (12)$$

Observing that

$$f^{(k)}(0) = i^k \int \lambda^k d\sigma(\lambda), \quad \phi_f^{(2k)}(0) = (-1)^k \int \lambda^k d\sigma(\lambda), \quad \phi_f^{(2k+1)}(0) = 0,$$

we see that if f has a zero of infinite order at $x = 0$, then so does ϕ_f .

In the recent note [27], we presented an application of the map $f \mapsto \phi_f$ to a problem of analytic quasianalyticity, which corresponds to the case when σ is supported on $\mathbb{R}_+$. Here we use this map and the Denjoy–Carleman theorem to reduce Theorem 1.3 to Theorem 1.1. We note that both the Denjoy–Carleman theorem and Theorem 1.1 can be proved (see [16, §14.3] and Sect. 4 below, respectively) using but the Carleman theorem from complex analysis, and the latter is a direct consequence of the formula for the harmonic measure in the half-plane.

Several Corollaries The following corollaries are derived from Theorem 1.3 by relatively standard methods. The proofs are sketched in Sect. 3.

Corollary 1.4 *Let W be as in Theorem 1.3. For any measure $\mu \geq 0$ with $\int W(\lambda) d\mu(\lambda) < \infty$, polynomials are dense in $L_2(\mu)$.*

Corollary 1.5 *Let W be as in Theorem 1.3. Any measure $\mu \geq 0$ with $\int W(\lambda) d\mu(\lambda) < \infty$ is Hamburger determinate, i.e. it shares its moments with no other non-negative measure on $\mathbb{R}$.*

Finally, we deduce a variant of a result of Volberg from [30].

Corollary 1.6 *Let W be as in Theorem 1.3, and let $\nu \geq 0$ be a measure on $\mathbb{R}$ such that*

$$\int W(1/\lambda) d\nu(\lambda) < \infty.$$

Let $(z_j = a_j + ib_j)_{j=1}^\infty$ be a sequence of points in the upper half-plane such that $z_j \rightarrow 0$ and

$$\lim_{j \rightarrow \infty} \frac{\log W(1/a_j)}{\log(1/b_j)} = \infty. \quad (13)$$

Then the linear span of $\left\{ \frac{1}{t - z_j} \right\}_{j \geq 1}$ is dense in $L_2(\nu)$.

Here the regularity assumptions on $W(\lambda)$ are somewhat weaker than in [30] (where a condition of the form (11') and its counterpart at $+\infty$ are imposed), and the assumption of non-tangential convergence $|b_j| \geq \epsilon |a_j|$ is relaxed to (13).

2 Proof of Theorem 1.3

Assume that $f \in \mathfrak{Q}_W$ has a zero of infinite order, say, at $x = 0$: $f^{(k)}(0) = 0$ for all $k \geq 0$. Then ϕ_f of (12) also satisfies $\phi_f^{(k)}(0) = 0$ for all $k \geq 0$, and

$$\sup_{|y| \leq x} |\phi_f^{(k)}(y)| \leq CM_k(x) \stackrel{\text{def}}{=} C \max_{\lambda \geq 0} \lambda^{k/2} \left\{ W(-\lambda)^{-1} e^{x\sqrt{\lambda}} + W(\lambda)^{-1} \right\}. \quad (14)$$

Let us show that

$$\forall x \geq 0 \quad \sum_{k \geq 1} M_k(x)^{-1/k} = \infty. \quad (15)$$

To this end, set

$$B_k = \max_{\lambda \geq 0} \lambda^{k/2} W(\lambda)^{-1} = \exp \left(\max_s \left[\frac{k}{2}s - \log W(e^s) \right] \right) = \exp(q^*(k/2)),$$

where $q(s) = \log W(e^s)$. From the convexity of q , we deduce (following Ostrowski [22]) that the condition (9) implies (and is equivalent to)

$$\sum_{k=1}^{\infty} B_k^{-1/k} = \infty. \quad (16)$$

Now, from (10) we have that $\log W(-\lambda) \geq (x+1)\sqrt{\lambda}$ for sufficiently large $\lambda \geq \lambda_0(x)$, therefore

$$\max_{\lambda \geq 0} \lambda^{k/2} W(-\lambda)^{-1} e^{x\sqrt{\lambda}} \leq \lambda_0^k e^{x\sqrt{\lambda_0}} + \max_{\lambda \geq 0} \lambda^{k/2} e^{-\sqrt{\lambda}} \leq C_x^{k+1} k!$$

Thus

$$\sum_{k=1}^{\infty} M_k(x)^{-1/k} \geq c_x \sum_{k=1}^{\infty} \min(B_k^{-1/k}, 1/k) = \infty,$$

where on the last step we used (16) and the Cauchy condensation test. According to the Denjoy–Carleman theorem [6], [16, §14.3], the class of functions admitting an estimate (14) with M_k satisfying (15) is quasianalytic in the sense of Hadamard, whence $\phi_f \equiv 0$.

Now we appeal to Theorem 1.1. Let $p(s)$ be the largest convex minorant of $p_0(s) = \log W(-s^2)$, $s \geq 0$. The functions p and p_0 coincide for large s , therefore by (10)

$$\int_{-\infty}^{\infty} \frac{p(s)}{s^3} ds = \infty.$$

We have:

$$\int_{-\infty}^0 \exp(x\sqrt{|\lambda|}) |d\sigma(\lambda)| \leq \sup_{\lambda < 0} W(\lambda)^{-1} e^{x\sqrt{|\lambda|}} \int_{-\infty}^0 W(\lambda) |d\sigma(\lambda)| \leq C \exp(p^*(x)).$$

Adjusting the constants, we can assume that p is non-decreasing. Therefore Theorem 1.1 applies and we obtain $\sigma \equiv \text{const}$ and $f \equiv 0$. $\square$

3 Proof of Corollaries

Proof of Corollary 1.4 Observe that $W_1(\lambda) = \sqrt{W(\lambda)}$ also satisfies the conditions of Theorem 1.3, and that for any $u \in C_0(1/W_1)$

$$\begin{aligned} \|u\|_{L_2(\mu)}^2 &= \int |u(\lambda)|^2 d\mu(\lambda) \\ &\leq \left[\sup |u(\lambda)|/W_1(\lambda) \right]^2 \int W(\lambda) d\mu(\lambda) = C \|u\|_{C_0(1/W_1)}^2. \end{aligned}$$

Any function $L_2(\mu)$ can be approximated by functions in $C_0(1/W)$, and, by Theorem 1.3, these in turn can be approximated by polynomials. $\square$

We mention again that a general reduction of the problem of completeness in $L_2(1/W)$ to that in $C_0(1/W)$ was found by Bakan in [3,4].

Proof of Corollary 1.5 Pick a measure $\mu_1 \geq \mu$ that is not discrete and still satisfies $\int W(\lambda) d\mu_1(\lambda) < \infty$ (for example, one can add to μ a small continuous component near the origin). By corollary 1.4, polynomials are dense in $L_2(\mu_1)$. By a theorem of M. Riesz [2, Theorem 2.3.3], μ_1 is N -extreme. Further [2, §3.4.1], an N -extreme measure is either moment-determinate or discrete (or both). Our μ_1 is not discrete, hence it is moment-determinate, and hence so is μ . $\square$

Proof of Corollary 1.6 The condition (13) implies that

$$\forall k \geq 1 \quad c_k = \inf_j \left[W(1/a_j) b_j^k \right] > 0. \quad (17)$$

Consider the domain

$$\Omega = \left\{ z = a + ib : -1 \leq a \leq 1, \quad b > 0, \quad \forall k \geq 1 \quad W(1/a) b^k \geq c_k \right\}. \quad (18)$$

Then

$$\forall k \geq 1 \quad \inf_{z \in \Omega} \inf_{t \in \mathbb{R}} [W(1/t) |t - z|^k] \geq \inf_{z=a+ib \in \Omega} [W(1/a) b^k] = c_k. \quad (19)$$

If $E = \text{span} \left\{ \frac{1}{t - z_j} \right\}_{j \geq 1}$ is not dense in $L_2(\nu)$, let $u(t) \in E^\perp \setminus \{0\}$, and set

$$g(z) = \int \frac{\bar{u}(t) d\nu(t)}{t - z}.$$

Then $g(z_j) = 0$ for $j = 1, 2, \dots$. Further, (19) implies that g and its derivatives are uniformly bounded in Ω :

$$\begin{aligned} |g^{(k)}(z)| &\leq k! \int \frac{|u(t)| dv(t)}{|t-z|^{k+1}} \leq k! \|u\|_{L_2(v)} \sqrt{\int \frac{dv(t)}{|t-z|^{2k+2}}} \\ &\leq k! \|u\|_{L_2(v)} \sqrt{\int W(1/t) dv(t)} \times \sup_t \frac{1}{W(1/t)|t-z|^{2k+2}} \\ &\leq \frac{k!}{\sqrt{c_{2k+2}}} \|u\|_{L_2(v)} \sqrt{\int W(1/t) dv(t)}, \end{aligned}$$

hence g has a zero of infinite order at $z = 0$; i.e.,

$$\forall k \geq 1 \quad \int \lambda^k \bar{u}(1/\lambda) dv(1/\lambda) = \int \frac{\bar{u}(t) dv(t)}{t^k} = 0.$$

Let $W_1(\lambda) = \max(1, \sqrt{W(\lambda)}/(1 + |\lambda|))$. Then W_1 satisfies the assumptions of Theorem 1.3, and the complex measure $d\sigma(\lambda) = \lambda \bar{u}(1/\lambda) dv(1/\lambda)$ satisfies

$$\begin{aligned} \int W_1(\lambda) |d\sigma(\lambda)| &= \int W_1(\lambda) |\lambda \bar{u}(1/\lambda)| dv(1/\lambda) \\ &\leq \|u\|_{L_2(v)} \sqrt{\int W_1(\lambda)^2 \lambda^2 dv(1/\lambda)} < \infty. \end{aligned}$$

Therefore the measures $(\Re \sigma)_{\pm}$ in the Jordan decomposition of $\Re \sigma$ satisfy the assumptions of Corollary 1.5 and share the same moments, which implies that $(\Re \sigma)_+ = (\Re \sigma)_-$; similarly, $(\Im \sigma)_+ = (\Im \sigma)_-$, whence $u \equiv 0$ (in $L_2(v)$). $\square$

4 Proof of Theorem 1.1

We closely follow [32]. Assume that (3) and (4) hold and that $\mathfrak{C}\sigma \equiv 0$. Adjusting constants, we may assume that $p(0) = 0$; in this case the $s \mapsto p(s)/s$ is non-decreasing:

$$\frac{d}{ds} \frac{p(s)}{s} = -\frac{p(s) - sp'(s)}{s^2} \geq -\frac{p(0)}{s^2} = 0$$

(at the differentiability points of p).

From (4) we have for $\lambda_0 < 0$ and $x \geq 0$:

$$\int_{-\infty}^{\lambda_0} |d\sigma(\lambda)| \leq \exp(-x\sqrt{|\lambda_0|}) \times C \exp(p^*(x)), \quad (20)$$

whence

$$\int_{-\infty}^{\lambda_0} |\mathrm{d}\sigma(\lambda)| \leq C \exp(-p(\sqrt{|\lambda_0|})). \quad (21)$$

Consider the Stieltjes transform

$$F(z) = \int \frac{\mathrm{d}\sigma(\lambda)}{\lambda - (z + i)}, \quad \Im z \geq 0.$$

We have: $|F(z)| \leq C$. If we show that

$$\int_{-\infty}^0 \frac{\log |F(x)|}{1+x^2} dx = -\infty, \quad (22)$$

Carleman's theorem [16, §14.2] will imply that $F \equiv 0$ and hence $\sigma \equiv \text{const}$. Therefore we turn to the proof of (22).

For $x < 0$ let

$$F(x) = F_1(x) + F_2(x), \\ F_1(x) = \int_{-\infty}^{x/4} \frac{\mathrm{d}\sigma(\lambda)}{\lambda - (x + i)}, \quad F_2(x) = \int_{x/4}^{\infty} \frac{\mathrm{d}\sigma(\lambda)}{\lambda - (x + i)}.$$

From (21)

$$|F_1(x)| \leq \int_{-\infty}^{x/4} |\mathrm{d}\sigma(\lambda)| \leq C \exp(-p(\sqrt{|x|}/2)). \quad (23)$$

To prove an estimate of the similar form for $F_2(x)$, we start from the identity

$$\frac{1}{\lambda - (x + i)} = \frac{i}{\sqrt{x + i}} \int_0^{\infty} \cos(u\sqrt{\lambda}) \exp(iu\sqrt{x + i}) du, \quad (24)$$

valid for $x < 0$, $\Im\sqrt{\lambda} < \Im\sqrt{x + i}$. From (24) we have (for $x < x_0 < 0$):

$$F_2(x) = \frac{i}{\sqrt{x + i}} \int_{x/4}^{\infty} \mathrm{d}\sigma(\lambda) \int_0^{\infty} \cos(u\sqrt{\lambda}) \exp(iu\sqrt{x + i}) du \\ = \frac{i}{\sqrt{x + i}} \int_{x/4}^{\infty} \mathrm{d}\sigma(\lambda) \left[\int_0^{u^*} + \int_{u^*}^{\infty} \right] = I_1(x) + I_2(x),$$

where we take $u^* = \frac{p(\sqrt{|x|}/2)}{2\sqrt{|x|}}$. From the assumption $\mathfrak{C}\sigma \equiv 0$,

$$I_1(x) = \frac{i}{\sqrt{x + i}} \int_{-\infty}^{x/4} \mathrm{d}\sigma(\lambda) \int_0^{u^*} \cos(u\sqrt{\lambda}) \exp(iu\sqrt{x + i}) du;$$

estimating

$$\left| \frac{i}{\sqrt{x+i}} \int_0^{u^*} \cos(u\sqrt{\lambda}) \exp(iu\sqrt{x+i}) du \right| \leq \int_0^{u^*} \cos(u\sqrt{\lambda}) du \leq \exp(u^* \sqrt{|\lambda|}) ,$$

we obtain, using (21) and (4):

$$\begin{aligned} |I_1(x)| &\leq \int_{-\infty}^{x/4} \exp(u^* \sqrt{|\lambda|}) |d\sigma(\lambda)| \\ &\leq \sqrt{\int_{-\infty}^{x/4} |d\sigma(\lambda)|} \sqrt{\int_{-\infty}^{x/4} \exp(2u^* \sqrt{|\lambda|}) |d\sigma(\lambda)|} \\ &\leq C \exp\left(-\frac{1}{2} p(\sqrt{|x|}/2) + \frac{1}{2} p^*(2u^*)\right). \end{aligned} \quad (25)$$

Let us show that $p^*(2u^*) \leq \frac{1}{2} p(\sqrt{|x|}/2)$. Let s^* be such that

$$2u^* s^* = p^*(2u^*) + p(s^*) ,$$

then

$$p(s^*)/s^* \leq 2u^* \leq 4u^* = p(\sqrt{|x|}/2)/(\sqrt{|x|}/2) ,$$

whence $s^* \leq \sqrt{|x|}/2$ and

$$p^*(2u^*) \leq 2u^* s^* \leq u^* \sqrt{|x|} = \frac{1}{2} p(\sqrt{|x|}/2) ,$$

as claimed. Returning to (25), we obtain

$$|I_1(x)| \leq C \exp\left(-\frac{1}{4} p(\sqrt{|x|}/2)\right). \quad (26)$$

Now we turn to $I_2(x)$. Using that

$$\begin{aligned} &\left| \frac{i}{\sqrt{x+i}} \int_{u^*}^{\infty} \cos(u\sqrt{\lambda}) \exp(iu\sqrt{x+i}) du \right| \\ &\leq \int_{u^*}^{\infty} \exp(-u\sqrt{|x|}/2) du \leq \exp(-u^* \sqrt{|\lambda|}/2) , \end{aligned}$$

we obtain:

$$|I_2(x)| \leq C \exp(-u^* \sqrt{|x|}/2) = C \exp\left(-\frac{1}{4} p(\sqrt{|x|}/2)\right). \quad (27)$$

Combining (23), (26), and (27) we obtain

$$|F(x)| \leq 3C \exp\left(-\frac{1}{4}p(\sqrt{|x|}/2)\right),$$

whence

$$\begin{aligned} \int_{-\infty}^{x_0} \frac{\log |F(x)|}{1+x^2} dx &\leq -C_1 - c_1 \int_{|x_0|}^{\infty} \frac{p(\sqrt{|x|}/2)}{x^2} dx \\ &\leq -C_1 - c_2 \int_{\sqrt{|x_0|}}^{\infty} \frac{p(s)}{s^3} ds = -\infty, \end{aligned}$$

as claimed.

Vice versa, suppose that (3) fails. Then there exists a non-zero entire function $\Phi(z)$ such that

$$\forall z = x + iy \in \mathbb{C} : \quad |\Phi(z)| \leq \exp(|y| - p(\sqrt{|z|}) - \sqrt{|z|})$$

(see [20, Lemma 5], where such a function is constructed as a product of dilated cardinal sine functions). Then $\Phi_1(z) = e^{iz}\Phi(z)$ satisfies

$$\forall z \in \mathbb{C}_+ : \quad |\Phi_1(z)| \leq \exp(-p(\sqrt{|z|}) - \sqrt{|z|}).$$

Let

$$\sigma(\lambda) = \int_{-\infty}^{\lambda} \Re \Phi_1(\lambda') d\lambda' \quad \text{or} \quad \sigma(\lambda) = \int_{-\infty}^{\lambda} \Im \Phi_1(\lambda') d\lambda',$$

so that $\sigma \not\equiv \text{const}$. Shifting the integration contour up, we see that $\Im \sigma \equiv 0$. We also have:

$$\int_{-\infty}^0 \exp(x\sqrt{|\lambda|}) |\mathrm{d}\sigma(\lambda)| \leq \int_0^{\infty} \exp(x\sqrt{\lambda} - p(\sqrt{\lambda}) - \sqrt{\lambda}) d\lambda \leq C \exp(p^*(x)). \quad \square$$

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