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CONSTRUCTIVE APPROXIMATION

Approximation of Holomorphic Functions on Compact Subsets of $\mathbb{R}^N$

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Abstract We present several results giving some estimates for the error in best polynomial approximation of holomorphic functions on compact subsets of $\mathbb{R}^N$. We base our approach on the Bernstein–Walsh–Siciak theorem, which states in terms of the Siciak extremal function how fast a holomorphic function, defined in an appropriate neighborhood of a compact L-regular set $K \subset \mathbb{C}^N$, can be approximated on K by complex polynomials. Our purpose is among others to state a result for an arbitrary compact subset of $\mathbb{R}^N$ (not necessarily L-regular) and to replace the Siciak extremal function (which can hardly ever be computed, especially if $N > 1$) simply by the distance function to K.

Keywords Polynomial approximation · The Bernstein–Walsh–Siciak theorem · Subanalytic set · Degree of approximation · Extremal function · Pluricomplex Green function · (HCP) property

Mathematics Subject Classification 32E30 · 41A10 · 41A25 · 32B20 · 03C64 · 32U35

1 Introduction

We identify $\mathbb{R}^N$ with the set $\{z \in \mathbb{C}^N : \operatorname{Im}(z_\nu) = 0 \text{ for } \nu = 1, \ldots, N\}$. Throughout the paper, $\mathbb{N} := \{1, 2, 3, \ldots\}$.

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For a nonempty set $A \subset \mathbb{C}^N$ and $h : A \longrightarrow \mathbb{C}^{N'}$, we set $\|h\|_A := \sup_{z \in A} |h(z)|$, where $|\ |$ denotes the Euclidean norm in $\mathbb{C}^{N'}$. If $\emptyset \neq A \subset B \subset \mathbb{C}^N$ and $\xi : B \longrightarrow \mathbb{C}$, then for each $n \in \mathbb{N}$, we write

$$E_n(\xi;\ A) := \inf \left\{ \|\xi - Q\|_A \ :\ Q \in \mathbb{C}[z],\ \deg Q \leq n \right\}.$$

Suppose that $U \subset \mathbb{C}^N$ is a nonempty open set. We will denote by $H^\infty(U)$ the Banach space of all bounded and holomorphic functions in U (with the norm $\|\ \|_U$).

The problem of approximation of holomorphic functions (of several variables) was studied by many authors – see, for example, [4–10,16,32,33,37] and the huge bibliography therein. Our aim is among others to prove Theorems 1.1 and 1.2. Theorem 1.1 is the basis for Theorem 1.2, which directly concerns the polynomial approximation of holomorphic functions.

Theorem 1.1 *There exists a constant $\varepsilon_N > 0$ (depending only on $N \in \mathbb{N}$) such that, for each compact set $K \subset \mathbb{R}^N$ containing at least two distinct points,*

$$\Phi_K(z) \geq 1 + \frac{\varepsilon_N}{\operatorname{diam} K} \operatorname{dist}(z;\ K),$$

for all $z \in \mathbb{C}^N$.[1]

Theorem 1.2 *Let $K \subset \mathbb{R}^N$ be a compact set containing at least two distinct points. For each $\lambda > 0$, set $K_\lambda := \{z \in \mathbb{C}^N : \operatorname{dist}(z;\ K) < \lambda\}$. Assume that $0 < \upsilon < \varsigma(K)$.*[2] *Then, there exists a function $\vartheta : (0, +\infty) \longrightarrow (0, +\infty)$ (depending on K and υ) such that, for each $\lambda \in (0, +\infty)$, each $f \in H^\infty(K_\lambda)$, and each $n \in \mathbb{N}$,*

$$E_n(f;\ K) \leq \frac{\vartheta(\lambda)\|f\|_{K_\lambda}}{(1 + \upsilon\lambda)^n}.$$

The proof of Theorem 1.2 relies on Theorem 3.1, which is usually called the Bernstein–Walsh–Siciak theorem. Theorem 3.1 is a very precise version of the Oka–Weil theorem, but when we want to apply this theorem directly, we encounter a significant inconvenience. Namely, it uses the sets of the form $\{z \in \mathbb{C}^N : \Phi_K(z) < R\}$, where Φ_K is the Siciak extremal function and $R > 1$. The problem is that the function Φ_K can hardly ever be computed (even for very simple sets). The advantage of our approach is that the sets $\{z \in \mathbb{C}^N : \Phi_K(z) < R\}$ are replaced by the natural sets $K_\lambda = \{z \in \mathbb{C}^N : \operatorname{dist}(z;\ K) < \lambda\}$. It should be stressed, however, that there is also a disadvantage of such an approach. This is underlined in Remark 6.6. As we explain in the example following Corollary 5.5, for the set $K := [-1, 1]$ and the family of functions $f_\lambda : K_\lambda \ni w \longmapsto 1/(w - i\lambda) \in \mathbb{C}$ with $\lambda \in (0, +\infty)$, the estimate of Corollary 4.1 (which is very closely connected with Theorem 1.2) is asymptotically exact as $\lambda \to 0$. On the other hand, this is not the case for the family

[1] Φ_K denotes the Siciak extremal function (see Sect. 3) and $\operatorname{diam} K$ stands for the diameter of K.

[2] The constant $\varsigma(K)$ is a certain constant depending on K and is precisely defined in Sect. 4 via Theorem 1.1.

$g_\lambda : K_\lambda \ni w \longmapsto 1/(w - \lambda - 1) \in \mathbb{C}$ (cf. the example following Remark 6.6). We will try to explain this phenomenon now. First of all, note that K_λ is the rectangle in the complex plane with corners $(\pm 1, \pm\lambda)$ with semicircles of radius λ attached to the left and right sides, i.e., the "racetrack". By Corollary 4.1 and the example following Corollary 5.5, for each holomorphic function $f : K_\lambda \longrightarrow \mathbb{C}$,

$$\limsup_{n\to\infty} \sqrt[n]{E_n(f;\ K)} \le \frac{1}{R},$$

where $R := 1 + \lambda$. However, it is well known (the Bernstein theorem) that the region of analyticity sufficient in order to attain this order of approximation is the set

$$D(R) := \{w \in \mathbb{C} : \ \Phi_K(w) < R\} = \{w \in \mathbb{C} : \ |w + \sqrt{w^2 - 1}| < R\},$$

which is an ellipse with the major and minor semiaxes equal to $(R + 1/R)/2$ and $(R - 1/R)/2$, respectively. Consider now two cases.

Case 1: λ is small. Then,

$$\frac{1}{2}\left(R - \frac{1}{R}\right) = \lambda\frac{2+\lambda}{2(1+\lambda)} \approx \lambda.$$

This means that the sets K_λ and $D(R)$ are very close to each other in the vertical direction near the origin (note that the singular points of the functions f_λ defined above lie just on the vertical line $i\mathbb{R}$). On the other hand,

$$\frac{1}{2}\left(R + \frac{1}{R}\right) - 1 = \frac{\lambda^2}{2(1+\lambda)} \approx \frac{\lambda^2}{2}.$$

This means that the sets K_λ and $D(R)$ are not very close to each other in the horizontal direction (note that the singular points of the functions g_λ defined above lie just on the horizontal line $\mathbb{R}$).

Case 2: λ is large. Then, the sets K_λ and $D(R)$ differ significantly. The first one is close to a disc of radius R, while the second is close to a disc of radius $R/2$. Moreover, if f is holomorphic on the disc $\{w \in \mathbb{C} : \ |w| < R\}$, then the estimate $\limsup_{n\to\infty} \sqrt[n]{E_n(f;\ K)} \le 1/R$ is easily obtained, via the Cauchy estimates, by considering the n-th Taylor polynomial for f.

To sum up:

- Theorem 1.2 gives new information if Φ_K is not calculable (or is calculable, but its expression is complicated) and λ's are small.
- If Φ_K is calculable (this is a very rare situation), then of course the Bernstein–Walsh–Siciak theorem gives better bounds for the error in best approximation by polynomials. However, in some situations, (as indicated above) the estimates of Theorem 1.2 are very close to the corresponding estimates obtained via the Bernstein–Walsh–Siciak theorem.

- If λ's are large (compared with the set K), then perhaps Theorem 1.2 is not very interesting, because then the sets K_λ's are very nearly balls and then constructive processes such as Taylor polynomials can be used to obtain similar estimates (see the example above).

Note that the sets K_λ are just the sublevel sets of the distance function. Of course, the idea of the study of the error in best polynomial approximation of holomorphic functions (or the study of convergence of interpolatory processes) in terms of the appropriate sublevel sets is not new. For example, Walsh considered these problems in terms of the sublevel sets $\{z : G_K(z) < \lambda\}$, where G_K is the Green function for K (K is a compact, regular subset of $\mathbb{C}$) – see [16]. In several variables, we refer the reader to the papers [32,33] of Siciak who considered the above mentioned sublevel sets of his extremal function (see also [7], where a problem of convergence of Kergin interpolating polynomials of a holomorphic function is considered).

The sets K_λ appear for example in [11]. Davis gives some results concerning the speed of convergence of some interpolating polynomials of a holomorphic function defined in the lemniscate interior, that is, in the set $\{z : |P(z)| < \lambda\}$, where P is a complex polynomial of one variable. The sets K_λ are mentioned at the end of Chapter IV of [11], in the context of the Hilbert theorem (on approximation by the lemniscates).

Theorem 1.2 gives an upper bound for the error in best polynomial approximation of holomorphic functions on compact subsets of $\mathbb{R}^N$. In Sects. 6, 7, and 8, we also investigate the problem of estimating this error from below (see Theorems 6.3, 7.4, and Corollary 7.5). However, contrary to Theorem 1.2, Theorem 6.3 requires an additional assumption on the set.

2 Preliminaries

Recall that a set $A \subset \mathbb{C}^N$ is said to be locally analytic in $\mathbb{C}^N$ if for each point $a \in A$, there exists an open neighborhood $U \subset \mathbb{C}^N$ and holomorphic functions $\xi_1, \ldots, \xi_q : U \longrightarrow \mathbb{C}$ such that

$$A \cap U = \{z \in U : \xi_1(z) = \ldots = \xi_q(z) = 0\}.$$

This concept will be used in Sect. 7.

Suppose that $\emptyset \neq A \subset B \subset \mathbb{C}^N$, and denote by $\mathcal{B}(B;\, \mathbb{C})$ the Banach space of all bounded functions $\xi : B \longrightarrow \mathbb{C}$ (with the norm $\|\ \|_B$). Let $n \in \mathbb{N}$. It is straightforward to check that:

- For each $\xi \in \mathcal{B}(B;\, \mathbb{C})$ and $\alpha \in \mathbb{C}$,

$$E_n(\alpha\xi;\, A) = |\alpha| E_n(\xi;\, A).$$

- For all $\xi_1,\, \xi_2 \in \mathcal{B}(B;\, \mathbb{C})$,

$$|E_n(\xi_1;\, A) - E_n(\xi_2;\, A)| \leq E_n(\xi_1 - \xi_2;\, A) \leq \|\xi_1 - \xi_2\|_B.$$

In particular, the function $\mathcal{B}(B;\, \mathbb{C}) \ni \xi \longmapsto E_n(\xi;\, A) \in \mathbb{R}$ is continuous.

- If (r_n) is a sequence of real numbers, then the set

$$\{\xi \in \mathcal{B}(B;\ \mathbb{C}) \mid \forall\, n \in \mathbb{N} :\ E_n(\xi;\ A) \leq r_n \|\xi\|_B\}$$

is closed in $\mathcal{B}(B;\ \mathbb{C})$.

Lemma 2.1 ([22]) Let X be a Banach space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$). Suppose that a sequence of sets $V_k \subset X$ ($k \in \mathbb{N}$) satisfies the following conditions:

(1) $\operatorname{Int}\left(\bigcup_{k\in\mathbb{N}} V_k\right) \neq \emptyset$;
(2) For each $k \in \mathbb{N}$, there exist $j_1, j_2 \in \mathbb{N}$ such that $\overline{V}_k \subset V_{j_1}$ and $[0, +\infty) \cdot V_k \subset V_{j_2}$;
(3) For each $j \in \mathbb{N}$, $x_0 \in V_j$, and $r > 0$, there exists $\mu = \mu(j, x_0, r) \in \mathbb{N}$ such that

$$(V_j - x_0) \cap \{x \in X :\ \|x\| = r\} \subset V_\mu.$$

Then, $X = V_{k_0}$ for some $k_0 \in \mathbb{N}$.

Proof See [22], Lemma 2.3. □

3 A Proof of Theorem 1.1

Theorem 1.1 concerns Siciak's extremal function. Recall that the extremal function, associated with a compact set $K \subset \mathbb{C}^N$ and introduced by Siciak in [32], is defined by the formula

$$\Phi_K(z) := \sup\{|P(z)|^{1/\deg P} :\ P \in \mathbb{C}[z] \text{ is nonconstant and } \|P\|_K \leq 1\}$$

for $z \in \mathbb{C}^N$ (cf. [14,30,32,33]). It is a deep result that $\log \Phi_K = V_K$, where

$$V_K(z) := \sup\{u(z) :\ u \in \mathcal{L}(\mathbb{C}^N),\ u \leq 0 \text{ on } K\}$$

and $\mathcal{L}(\mathbb{C}^N)$ denotes the Lelong class of plurisubharmonic functions in $\mathbb{C}^N$ with minimal growth of type 1 (cf. [14,33,39]). Note that the definition of V_K makes sense for any subset of $\mathbb{C}^N$ – not necessarily compact. The extremal function is a very useful tool in real and complex analysis (for example, in the theory of holomorphic functions, in approximation theory, as well as in potential and pluripotential theory). In general, as we noted before, an effective formula for Φ_K is unknown, even for very simple sets. For some very particular sets, it is however computed. For example, $\Phi_{[-1,1]}(w) = |w + \sqrt{w^2 - 1}|$ for $w \in \mathbb{C}$, where the square root is so chosen that $\Phi_{[-1,1]} \geq 1$.

It is particularly important to recognize, given a point $a \in K$, whether Φ_K is continuous at a (if so, then we say that K is L-regular at a). This problem was studied among others in [14,15,18–21,24–27,29,31–33]. A compact set $K \subset \mathbb{C}^N$ is said to be L-regular if Φ_K is continuous in $\mathbb{C}^N$. The continuity of Φ_K in $\mathbb{C}^N$ is equivalent to the continuity of Φ_K on K (cf. [33], Proposition 6.1).

Below, we state two results relevant to the proofs of Theorems 1.1 and 1.2. The first one is due to Siciak (cf. [32,33]), but because of the contributions made in one variable (i.e., for $N = 1$) by Bernstein and Walsh (cf. [1,38]), it is often called the Bernstein–Walsh–Siciak theorem.

Theorem 3.1 (Siciak) *Assume that a nonempty compact set $K \subset \mathbb{C}^N$ is L-regular. Suppose that $R > 1$ and $f : D(R) \longrightarrow \mathbb{C}$ is holomorphic, where $D(R) := \{z \in \mathbb{C}^N : \Phi_K(z) < R\}$. Then,*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(f;\ K)} \le \frac{1}{R}.$$

Proof Cf. [32] (see also [33], Theorem 8.5). □

Lemma 3.2 *Assume that $K \subset \mathbb{R}^N$ is a compact set containing at least two distinct points. Then, for each $z \in \mathbb{C}^N$,*

$$\Phi_K(z) \ge 1 + \varpi(z)\,\mathrm{dist}(z;\ K),$$

where $\varpi(z) := \sqrt{(\mathrm{diam}\,K)^{-1}\,(\mathrm{diam}\,K + 2\,\mathrm{dist}(\mathrm{Re}(z);\ K))^{-1}}$.

Proof Fix $a \in \mathbb{C}^N$. Set $b := \mathrm{Re}(a)$, $\delta := (\mathrm{dist}(b;\ K))^2$, $\delta' := (\mathrm{dist}(a;\ K))^2$. It is straightforward to check that $\delta' = \delta + |\mathrm{Im}(a)|^2$. Let moreover $R := \mathrm{diam}\,K\big(\mathrm{diam}\,K + 2\sqrt{\delta}\big)$. Define

$$K_a := \left\{x \in \mathbb{R}^N : \sqrt{\delta} \le |x - b| \le \sqrt{R + \delta}\right\}.$$

Note that $K \subset K_a$. Consider the polynomial

$$\Upsilon : \mathbb{C}^N \ni z \longmapsto R + \delta - \sum (z_\nu - b_\nu)^2 \in \mathbb{C}.$$

An easy computation shows that

- $\Upsilon(K_a) = [0,\ R]$,
- $\Upsilon(a) = R + \delta'$.

Claim 1 $\Phi_K(a) \ge \sqrt{\Phi_{\Upsilon(K_a)}(\Upsilon(a))}$.
By the definition of Siciak's extremal function, we obtain easily the following estimates (which imply Claim 1):

$$\Phi_K(a) \ge \Phi_{K_a}(a) \ge \sqrt{\Phi_{\Upsilon(K_a)}(\Upsilon(a))}.$$

Claim 2 $\Phi_K(a) \ge 1 + \sqrt{R^{-1}}\,\mathrm{dist}(a;\ K)$.

Since $\Upsilon(K_a) = [0,\ R]$, it follows that

$$\Phi_{\Upsilon(K_a)}(\Upsilon(a)) = \Phi_{[0,\ R]}(\Upsilon(a)) = \Phi_{[-1,\ 1]}\left(\frac{2\Upsilon(a)}{R} - 1\right)$$
$$= \Phi_{[-1,\ 1]}\left(\frac{2(R+\delta')}{R} - 1\right) = \frac{\left(\sqrt{R+\delta'} + \sqrt{\delta'}\right)^2}{R}.$$

According to Claim 1, we obtain therefore

$$\Phi_K(a) \geq \sqrt{\Phi_{\Upsilon(K_a)}(\Upsilon(a))} = \frac{\sqrt{R+\delta'} + \sqrt{\delta'}}{\sqrt{R}} \geq 1 + \sqrt{\frac{\delta'}{R}} = 1 + \frac{1}{\sqrt{R}}\,\mathrm{dist}(a;\ K),$$

which is the desired estimate.

Obviously, Claim 2 completes the proof of our lemma. □

Białas-Cież and Kosek claim in [2] that so far very few examples of sets with so–called Łojasiewicz–Siciak property are known. Their paper is devoted to the problem of delivering some new examples of such sets (which are connected with iterated function systems). Recall that a compact set $K \subset \mathbb{C}^N$ satisfies the Łojasiewicz–Siciak condition if it is polynomially convex,[3] and there exist constants $\sigma > 0$, $\rho > 0$ such that

$$\Phi_K(z) \geq 1 + \rho\,(\mathrm{dist}(z;\ K))^{\sigma} \quad \text{as } \mathrm{dist}(z;\ K) \leq 1 \ \ (z \in \mathbb{C}^N).$$

We note in [23] that a straightforward consequence of Lemma 3.2 is that each compact subset of $\mathbb{R}^N$ satisfies the Łojasiewicz–Siciak condition with the exponent 1. However, this is insufficient for our purpose, and we will need Theorem 1.1 which is a more precise result.

Proof of Theorem 1.1 Take $\epsilon \in (0,\ 1)$ and fix $a \in \mathbb{C}^N$. We will show that

$$\Phi_K(a) \geq 1 + \frac{\varepsilon_N}{\eta}\mathrm{dist}(a;\ K), \tag{1}$$

where

$$\eta := \mathrm{diam} K\ , \qquad \varepsilon_N := \min\left\{\frac{\epsilon}{\sqrt{N}}, \sqrt{\frac{1-\epsilon}{1-\epsilon+2\sqrt{N}}}\right\}.$$

Choose $a' = (a'_1, \ldots, a'_N) \in K$, and let $C := C_1 \times \cdots \times C_N$, where $C_\nu := \{w \in \mathbb{C} :\ |w - a'_\nu| \leq \eta\}$ $(\nu = 1, \ldots, N)$. Clearly, $K \subset C$. We will show first that, for all $z \in \mathbb{C}^N$,

[3] We call a compact set $K \subset \mathbb{C}^N$ polynomially convex if

$$K = \hat{K} := \left\{z \in \mathbb{C}^N :\ |Q(z)| \leq \|Q\|_K \text{ for each } Q \in \mathbb{C}[z_1, \ldots, z_N]\right\}.$$

$$\Phi_C(z) \geq 1 + \frac{1}{\sqrt{N}\eta}\operatorname{dist}(z;\ C). \tag{2}$$

Since $\Phi_C \equiv 1$ in C,[4] it is sufficient to prove (2) for $z \in \mathbb{C}^N \setminus C$. Assume, therefore, that $z \in \mathbb{C}^N \setminus C$. Since $|z_{\nu_0} - a'_{\nu_0}| > \eta$ for some $\nu_0 \leq N$, we get

$$\Phi_C(z) \geq \max_\nu \left\{\frac{1}{\eta}|z_\nu - a'_\nu|\right\} = \max_\nu \left\{1 + \frac{1}{\eta}\operatorname{dist}(z_\nu;\ C_\nu)\right\}$$

(the inequality follows easily from the definition of Φ_C). Consequently,

$$\Phi_C(z) \geq 1 + \max_\nu \left\{\frac{1}{\eta}\operatorname{dist}(z_\nu;\ C_\nu)\right\} \geq 1 + \frac{1}{\sqrt{N}\eta}\operatorname{dist}(z;\ C).$$

The proof of (2) is complete.

Case 1: $\operatorname{dist}(a;\ C) \geq \epsilon \operatorname{dist}(a;\ K)$. Then, on account of (2),

$$\Phi_K(a) \geq \Phi_C(a) \geq 1 + \frac{1}{\sqrt{N}\eta}\operatorname{dist}(a;\ C) \geq 1 + \frac{\epsilon}{\sqrt{N}\eta}\operatorname{dist}(a;\ K) \geq 1 + \frac{\varepsilon_N}{\eta}\operatorname{dist}(a;\ K).$$

Case 2: $\operatorname{dist}(a;\ C) \leq \epsilon \operatorname{dist}(a;\ K)$. Take $y \in C$ such that $|a - y| = \operatorname{dist}(a;\ C)$. We have

$$\operatorname{dist}(a;\ K) \leq |a - a'| \leq |a - y| + |a' - y| \leq \operatorname{dist}(a;\ C) + \sqrt{N}\eta \leq \epsilon \operatorname{dist}(a;\ K) + \sqrt{N}\eta.$$

Therefore, $\operatorname{dist}(a;\ K) \leq \dfrac{\sqrt{N}\eta}{1-\epsilon}$. Combining this with Lemma 3.2, we get

$$\begin{aligned}\Phi_K(a) &\geq 1 + \frac{1}{\sqrt{\eta\,(\eta + 2\operatorname{dist}(a;\ K))}}\operatorname{dist}(a;\ K)\\ &\geq 1 + \frac{1}{\eta}\sqrt{\frac{1-\epsilon}{1-\epsilon+2\sqrt{N}}}\operatorname{dist}(a;\ K)\\ &\geq 1 + \frac{\varepsilon_N}{\eta}\operatorname{dist}(a;\ K).\end{aligned}$$

The proof of (1) is complete. □

Remark 3.3 Assume that $\sigma \neq 1$. In Theorem 1.1, if we replace $\operatorname{dist}(z;\ K)$ by $(\operatorname{dist}(z;\ K))^\sigma$, then in general the inequality under consideration does not hold (even for convex sets). Set $K := [-1,\ 1]^N$. Since $\Phi_K(z) = \max\{\Phi_{[-1,\ 1]}(z_1), \ldots, \Phi_{[-1,\ 1]}(z_N)\}$ (cf. [32]), it follows that, for all $t \geq 0$,

$$\Phi_K(it, \ldots, it) = t + \sqrt{t^2+1} \leq 1 + 2t = 1 + \frac{2}{\sqrt{N}}\operatorname{dist}\big((it, \ldots, it);\ K\big).$$

[4] This is an elementary property of the Siciak extremal function.

4 A Proof of Theorem 1.2

For each $N \in \mathbb{N}$, set

$$\gamma_N := \sup\{\varepsilon_N > 0 : \text{ Theorem 1.1 holds with } \varepsilon_N\}.$$

The following problem seems to be interesting.

Problem Find explicitly the constant γ_N.

Since there are compact sets $E \subset \mathbb{R}^N$ such that $\Phi_E \not\equiv +\infty$ in $\mathbb{C}^N \setminus E$ (for example, $E := [-1, 1]^N$ or in general nonpluripolar sets), it follows that $\gamma_N \in (0, +\infty)$. Clearly, for each compact set $K \subset \mathbb{R}^N$ containing at least two different points,

$$\Phi_K(z) \geq 1 + \frac{\gamma_N}{\operatorname{diam} K} \operatorname{dist}(z;\ K),$$

for all $z \in \mathbb{C}^N$.

Assume that $K \subset \mathbb{R}^N$ is a nonempty compact set such that $\Phi_K \not\equiv +\infty$ in $\mathbb{C}^N \setminus K$. Define

$$\overline{\varsigma}(K) := \sup\{\varsigma > 0 :\ \Phi_K(z) \geq 1 + \varsigma \operatorname{dist}(z;\ K)\ \forall z \in \mathbb{C}^N\}.$$

(Note that the supremum above is taken over a nonempty set, because $\gamma_N/\operatorname{diam} K$ belongs to the set under consideration.) Clearly, $\overline{\varsigma}(K) \in (0, +\infty)$.

To sum up: if $K \subset \mathbb{R}^N$ is a nonempty compact set such that $\Phi_K \not\equiv +\infty$ in $\mathbb{C}^N \setminus K$, then

$$\Phi_K(z) \geq 1 + \overline{\varsigma}(K)\operatorname{dist}(z;\ K) \geq 1 + \frac{\gamma_N}{\operatorname{diam} K} \operatorname{dist}(z;\ K),$$

for all $z \in \mathbb{C}^N$.

Let $K \subset \mathbb{R}^N$ be a compact set containing at least two distinct points. Define

$$\varsigma(K) := \begin{cases} \overline{\varsigma}(K) & \text{if } \Phi_K \not\equiv +\infty \text{ in } \mathbb{C}^N \setminus K, \\ \dfrac{\gamma_N}{\operatorname{diam} K} & \text{otherwise.} \end{cases}$$

In the above expression for $\varsigma(K)$, we can set any positive real number instead of $\gamma_N/\operatorname{diam} K$. We cannot however write simply $\overline{\varsigma}(K)$, because $\overline{\varsigma}(K) = +\infty$ if $\Phi_K \equiv +\infty$ in $\mathbb{C}^N \setminus K$. This is the reason why we consider two cases.

Proof of Theorem 1.2 Fix $\lambda \in (0, +\infty)$. We will show first that, for each holomorphic function $g : K_\lambda \longrightarrow \mathbb{C}$,

$$\limsup_{n\to\infty} \sqrt[n]{E_n(g;\ K)} \leq \frac{1}{1 + \varsigma(K)\lambda}. \tag{3}$$

To this end, fix $\epsilon \in (0,\ \varsigma(K)\lambda)$ and write

$$\alpha = \alpha(\epsilon) := 1 + \varsigma(K)\lambda - \epsilon \in (1, +\infty),$$
$$\beta = \beta(\epsilon) := \sqrt{\frac{\alpha(\epsilon)}{1 + \varsigma(K)\lambda}} \in (0,\ 1).$$

Since ϵ can be made arbitrarily small, it suffices to show that

$$\limsup_{n\to\infty} \sqrt[n]{E_n(g;\ K)} \le \frac{1}{\alpha}. \tag{4}$$

For each $j \in \mathbb{N}$, set

$$K_{\langle j\rangle} := \bigcup_{x\in K} \left(x + \left[-\frac{1}{j},\ \frac{1}{j} \right]^N \right).$$

We easily verify that $K_{\langle j\rangle}$ is a compact L-regular set. Moreover, for each $z \in \mathbb{C}^N$, we have

$$\Phi_{K_{\langle j\rangle}}(z) \ge 1 + \varsigma(K_{\langle j\rangle})\mathrm{dist}(z;\ K_{\langle j\rangle})$$

and therefore

$$\Phi_{K_{\langle j\rangle}}(z) \le \alpha \implies \mathrm{dist}(z;\ K_{\langle j\rangle}) \le \frac{\alpha - 1}{\varsigma(K_{\langle j\rangle})}.$$

In particular,

$$\begin{aligned} T_{\langle j\rangle} &:= \left\{ z \in \mathbb{C}^N : \Phi_{K_{\langle j\rangle}}(z) \le \alpha \right\} \setminus K_\lambda \\ &= \left\{ z \in \mathbb{C}^N : \Phi_{K_{\langle j\rangle}}(z) \le \alpha,\ \ \mathrm{dist}(z;\ K_{\langle j\rangle}) \le \frac{\alpha - 1}{\varsigma(K_{\langle j\rangle})} \right\} \setminus K_\lambda. \end{aligned}$$

Case 1: $T_{\langle j_0\rangle} = \emptyset$ for some $j_0 \in \mathbb{N}$. Then,

$$\left\{ z \in \mathbb{C}^N : \Phi_{K_{\langle j_0\rangle}}(z) \le \alpha \right\} \subset K_\lambda.$$

Since $\alpha > 1$ and $K_{\langle j_0\rangle}$ is L-regular, it follows by Theorem 3.1 that

$$\limsup_{n\to\infty} \sqrt[n]{E_n(g;\ K)} \le \limsup_{n\to\infty} \sqrt[n]{E_n(g;\ K_{\langle j_0\rangle})} \le \frac{1}{\alpha},$$

which yields (4).

CASE 2: $T_{\langle j\rangle} \neq \emptyset$ for each $j \in \mathbb{N}$. Take $a \in \bigcap_{j\in\mathbb{N}} T_{\langle j\rangle}$.[5] In particular, $a \notin K_\lambda$, that is, $\operatorname{dist}(a;\ K) \geq \lambda$. Since

$$\Phi_K(a) \geq 1 + \varsigma(K)\operatorname{dist}(a;\ K) > \beta(1 + \varsigma(K)\lambda),$$

it follows by the definition of the Siciak extremal function that there exists a nonconstant polynomial $P : \mathbb{C}^N \longrightarrow \mathbb{C}$ such that $\|P\|_K \leq 1$ and

$$|P(a)|^{1/\deg P} > \beta(1 + \varsigma(K)\lambda).$$

The set $\left\{z \in \mathbb{C}^N : |P(z)| < \beta^{-\deg P}\right\}$ is an open neighborhood of K. Consequently, if $j \in \mathbb{N}$ is sufficiently large, then

$$K_{\langle j\rangle} \subset \left\{z \in \mathbb{C}^N : |P(z)| \leq \beta^{-\deg P}\right\}$$

and therefore (again by the definition of the Siciak extremal function)

$$\Phi_{K_{\langle j\rangle}}(a) \geq \left(\beta^{\deg P}|P(a)|\right)^{1/\deg P} > \beta^2(1 + \varsigma(K)\lambda) = \alpha.$$

This is however impossible, because $a \in T_{\langle j\rangle}$.

The proof of (3) is complete. Fix now a positive number $\upsilon < \varsigma(K)$. If $f \in H^\infty(K_\lambda)$, then according to (3) there exists a constant $M(f) > 0$ such that

$$E_n(f;\ K) \leq \frac{M(f)\|f\|_{K_\lambda}}{(1 + \upsilon\lambda)^n}$$

for each $n \in \mathbb{N}$. Consequently, $H^\infty(K_\lambda) = \bigcup_{k\in\mathbb{N}} V_k(K, \upsilon, \lambda)$, where

$$V_k(K, \upsilon, \lambda) := \left\{f \in H^\infty(K_\lambda) \,|\, \forall n \in \mathbb{N} :\ E_n(f;\ K) \leq k\|f\|_{K_\lambda}(1 + \upsilon\lambda)^{-n}\right\}.$$

Moreover, for each $k \in \mathbb{N}$, $V_k(K, \upsilon, \lambda)$ is closed in $H^\infty(K_\lambda)$ and $\mathbb{C} \cdot V_k(K, \upsilon, \lambda) = V_k(K, \upsilon, \lambda)$. Therefore, the assumptions (1) and (2) of Lemma 2.1 are satisfied (as the Banach space X we take $H^\infty(K_\lambda)$). We will now check that the assumption (3) is fulfilled as well.

To this end, fix $j \in \mathbb{N}$, $h_0 \in V_j(K, \upsilon, \lambda)$, and $r > 0$. Let $\mu \in \mathbb{N}$ be the smallest integer such that $\mu \geq j(1 + 2r^{-1}\|h_0\|_{K_\lambda})$. It is enough to verify that the following implication

$$\Big(h \in V_j(K, \upsilon, \lambda),\ \|h - h_0\|_{K_\lambda} = r\Big) \quad \Longrightarrow \quad h - h_0 \in V_\mu(K, \upsilon, \lambda)$$

holds true. Assume therefore that $h \in V_j(K, \upsilon, \lambda)$. Note that, for each $n \in \mathbb{N}$,

$$E_n(h_0;\ K) \leq j\|h_0\|_{K_\lambda}(1 + \upsilon\lambda)^{-n}, \quad E_n(h;\ K) \leq j\|h\|_{K_\lambda}(1 + \upsilon\lambda)^{-n}.$$

[5] Note that the sets $T_{\langle j\rangle}$ are compact and $T_{\langle 1\rangle} \supset T_{\langle 2\rangle} \supset T_{\langle 3\rangle} \supset \cdots$, because $K_{\langle 1\rangle} \supset K_{\langle 2\rangle} \supset K_{\langle 3\rangle} \supset \cdots$.

Moreover,

$$\|h_0\|_{K_\lambda} + \|h\|_{K_\lambda} \le 2\|h_0\|_{K_\lambda} + \|h - h_0\|_{K_\lambda} = 2\|h_0\|_{K_\lambda} + r \le \frac{\mu r}{j}.$$

We obtain therefore, for each $n \in \mathbb{N}$,

$$\begin{aligned} E_n(h - h_0;\ K) &\le E_n(h_0;\ K) + E_n(h;\ K) \le \mu r(1 + \upsilon\lambda)^{-n} \\ &= \mu\|h - h_0\|_{K_\lambda}(1 + \upsilon\lambda)^{-n}. \end{aligned}$$

This means that the implication under consideration is true.

We have checked that all the assumptions of Lemma 2.1 are satisfied. Consequently, there exists $k_0 = k_0(K, \upsilon, \lambda) \in \mathbb{N}$ such that $H^\infty(K_\lambda) = V_{k_0}(K, \upsilon, \lambda)$. We set $\vartheta(\lambda) := k_0$, and the proof is complete. □

Corollary 4.1 *Let $K \subset \mathbb{R}^N$ be a compact set containing at least two distinct points. Then, for each $\lambda \in (0, +\infty)$, and each holomorphic function $f : K_\lambda \longrightarrow \mathbb{C}$,*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(f;\ K)} \le \frac{1}{1 + \varsigma(K)\lambda}.$$

Proof The result follows from the proof of Theorem 1.2, namely from the estimate (3). □

5 An Example Concerning Corollary 4.1

Remark 5.1 In the estimate from Corollary 4.1, λ is with the exponent 1. We will show (see the example below) that even for such a simple set as $K := [-1,\ 1] \subset \mathbb{R}$,

- the exponent 1 cannot be replaced by a smaller one,
- the constant $\varsigma(K)$ cannot be replaced by a bigger one.

To provide an example, we will need some auxiliary lemmas stated below.

Lemma 5.2 *For each $w \in \mathbb{C}$ such that $|w| \ge 1$,*

$$\Phi_{[-1,\,1]}(w) \ge |w| + \sqrt{|w|^2 - 1}.$$

Proof Set $u := |w| + \sqrt{|w|^2 - 1}$, $v := w + \sqrt{w^2 - 1}$, where the square root is so chosen that $|v| \ge 1$. Note that

$$|v| + \frac{1}{|v|} \ge \left|v + \frac{1}{v}\right| = 2|w| = u + \frac{1}{u}.$$

Since the function $[1, +\infty) \ni t \longmapsto t + t^{-1} \in \mathbb{R}$ is increasing, it follows that $|v| \ge u$. This completes the proof. □

Lemma 5.3 *For each $w \in \mathbb{C}$ such that* $|\mathrm{Re(w)}| \geq 1$,

$$\Phi_{[-1,\,1]}(w) \geq |w| + |w - c| \geq 1 + |w - c|,$$

where $c := 1$ *if* $\mathrm{Re(w)} \geq 1$ *and* $c := -1$ *if* $\mathrm{Re(w)} \leq -1$.

Proof Apply Lemma 5.2 along with the obvious estimate: $|w|^2 \geq 1 + |w - c|^2$. □

Lemma 5.4 *For each $w \in \mathbb{C}$,*

$$\Phi_{[-1,\,1]}(w) \geq 1 + |\mathrm{Im(w)}|.$$

Proof Set $v := w + \sqrt{w^2 - 1}$, where the square root is as before. Since $v + v^{-1} = 2w$, it follows easily that $2\mathrm{Im}(w) = \mathrm{Im}(v)\big(1 - |v|^{-2}\big)$. Therefore,

$$|v| \geq 1 + \frac{|v|}{2}\left(1 - |v|^{-2}\right) \geq 1 + |\mathrm{Im(w)}|.$$

□

Corollary 5.5 *For each $w \in \mathbb{C}$,*

$$\Phi_{[-1,\,1]}(w) \geq 1 + \mathrm{dist}\,(w;\ [-1,\ 1])\,.$$

Proof Apply Lemmas 5.3 and 5.4. □

Example Set $K := [-1,\ 1] \subset \mathbb{R}$. For each $\lambda \in (0, +\infty)$, let

$$R_\lambda := \Phi_{[-1,\,1]}(i\lambda) = \lambda + \sqrt{\lambda^2 + 1}.$$

Note that

$$\lim_{\lambda \to 0^+} \frac{R_\lambda - 1}{\lambda} = \lim_{\lambda \to 0^+} \frac{\lambda - 1 + \sqrt{\lambda^2 + 1}}{\lambda} = 1. \tag{5}$$

Therefore, $\varsigma(K) \leq 1$. By Corollary 5.5, $\varsigma(K) \geq 1$. Consequently, $\varsigma(K) = 1$. Consider the following function:

$$f_\lambda : K_\lambda \ni w \longmapsto \frac{1}{w - i\lambda} \in \mathbb{C}.$$

For $R > 1$, let $D(R) := \{w \in \mathbb{C} :\ |w + \sqrt{w^2 - 1}| < R\}$, where the square root is chosen as before.[6] Note that $i\lambda \notin D(R_\lambda)$ and $f_\lambda|_K$ has no holomorphic extension to $D(R')$, for any $R' > R_\lambda$. By a classical result in approximation theory (due to Bernstein),

$$\limsup_{n \to \infty} \sqrt[n]{E_n(f_\lambda;\ K)} = \frac{1}{R_\lambda}$$

[6] We easily see that, for example, $D(R_\lambda)$ is an ellipse with the major and minor semiaxes equal to $\sqrt{\lambda^2 + 1}$ and λ, respectively.

(see Theorem 13.4 in [28] or Corollary 6.2 in the next section). This along with (5) proves the two claims in Remark 5.1. □

6 A Lower Bound Counterpart of Corollary 4.1 for Subanalytic Sets

A subset $A \subset \mathbb{R}^N$ is said to be semianalytic if, for each point in $\mathbb{R}^N$, we can find a neighborhood U such that $A \cap U$ is a finite union of sets of the form

$$\{x \in U : \xi(x) = 0,\ \xi_1(x) > 0,\ \ldots,\ \xi_q(x) > 0\},$$

where $\xi,\ \xi_1,\ \ldots,\ \xi_q$ are (real) analytic functions in U (cf. [17]). A set $A \subset \mathbb{R}^N$ is called subanalytic if, for each point in $\mathbb{R}^N$, there exists a neighborhood U such that $A \cap U$ is the projection of some relatively compact semianalytic set in $\mathbb{R}^{N+N'} = \mathbb{R}^N \times \mathbb{R}^{N'}$ (cf. [3,12,13]).

The following is a generalization of the classical result of Bernstein (stated by Bernstein for $K := [-1,\ 1] \subset \mathbb{R}$).

Theorem 6.1 (Siciak) *Assume that a nonempty compact set $K \subset \mathbb{C}^N$ is L-regular. Suppose that $\zeta > 1$ and $h : K \longrightarrow \mathbb{C}$ has no holomorphic extension to $D(\zeta)$, where $D(\zeta) := \{z \in \mathbb{C}^N : \Phi_K(z) < \zeta\}$. Then,*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(h;\ K)} > \frac{1}{\zeta}.$$

Proof Cf. [33], Theorem 8.5, and Corollary 8.6. □

Corollary 6.2 *Assume that a nonempty compact set $K \subset \mathbb{C}^N$ is L-regular. Suppose that $R > 1$ and $f : D(R) \longrightarrow \mathbb{C}$ is a holomorphic function such that $f|_K$ has no holomorphic extension to $D(R')$, for any $R' > R$.*[7] *Then,*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(f;\ K)} = \frac{1}{R}.$$

Proof It follows from Theorems 3.1 and 6.1. □

Theorem 6.3 *Assume that a nonempty compact set $K \subset \mathbb{R}^N$ is fat*[8] *and subanalytic.*[9] *Suppose that $\lambda_0 > 0$. Then, there exists $\kappa > 0$, $\varrho > 0$ such that, for each $t \in (0,\ \lambda_0]$ and each $h : K \longrightarrow \mathbb{C}$ which has no holomorphic extension to K_t,*[10]

$$\limsup_{n\to\infty} \sqrt[n]{E_n(h;\ K)} > \frac{1}{1 + \varrho\, t^{\kappa}}.$$

[7] $D(R)$ is as in the previous theorem.

[8] We say that a set A is fat if $\overline{A} = \overline{\mathrm{Int} A}$.

[9] See Remark 6.5 (the assumption that K is subanalytic can be significantly weakened).

[10] Recall that $K_t := \{z \in \mathbb{C}^N : \mathrm{dist}(z;\ K) < t\}$.

Proof By Theorems 4.1 and 6.4 in [18], there exists $\kappa = \kappa(\lambda_0) > 0$, $\varrho = \varrho(\lambda_0) > 0$ such that

$$\Phi_K(z) \leq 1 + \varrho\,(\operatorname{dist}(z;\ K))^{\kappa}$$

for $z \in K_{\lambda_0}$ (also see Appendix). In particular, K is L-regular. If $t \in (0,\ \lambda_0]$, then $K_t \subset D(1 + \varrho\, t^{\kappa})$. Now, it is enough to apply Theorem 6.1. □

Corollary 6.4 *Assume that a nonempty compact set* $K \subset \mathbb{R}^N$ *is fat and subanalytic.*[11] *Suppose that* $\lambda_0 > 0$*. Then, there exists* $\kappa > 0$, $\varrho > 0$ *such that, for each* $\lambda \in (0,\ \lambda_0)$ *and each holomorphic function* $f : K_\lambda \longrightarrow \mathbb{C}$ *which has no holomorphic extension to* $K_{\lambda'}$ *for any* $\lambda' > \lambda$,

$$\frac{1}{1 + \varrho\,\lambda^{\kappa}} \leq \limsup_{n\to\infty} \sqrt[n]{E_n(f;\ K)} \leq \frac{1}{1 + \varsigma(K)\lambda}.$$

Proof Fix $\lambda \in (0,\ \lambda_0)$ and a holomorphic function $f : K_\lambda \longrightarrow \mathbb{C}$ which has no holomorphic extension to $K_{\lambda'}$, for any $\lambda' > \lambda$. Note first that $f|_K$ has no holomorphic extension to $K_{\lambda'}$, for any $\lambda' > \lambda$. Indeed, assume that this is not the case and take $\lambda' > \lambda$ and $\tilde{f} : K_{\lambda'} \longrightarrow \mathbb{C}$ being a holomorphic extension of $f|_K$. If C is a connected component of K_λ, then

- $C \cap K \neq \emptyset$ (because, if $a \in K_\lambda$, then $[a, b] \subset K_\lambda$, where $|a - b| = \operatorname{dist}(a;\ K)$ and $b \in K$);
- $C \cap K \subset \{z \in C :\ f(z) - \tilde{f}(z) = 0\}$.

It follows that $f = \tilde{f}$ in C. By the arbitrary character of C, $f = \tilde{f}$ in K_λ, which is a contradiction. Now, it is enough to apply Corollary 4.1 and Theorem 6.3. □

Remark 6.5 Theorem 6.3 and Corollary 6.4 hold true in a much more general setting thanks to the main results obtained by the author in [19] and [21] (see Appendix). However, it is not presented above to make the paper as accessible as possible.

Remark 6.6 In the estimate of Corollary 6.4, we have λ^{κ} on the left-hand side and $\lambda = \lambda^1$ on the right-hand side. The example presented in the previous section shows that the exponent 1 (on the right-hand side) is optimal for $K := [-1,\ 1]$. We will show below that, in the same situation, it is impossible to have the exponent $\kappa > 1/2$.

Example Set $K := [-1,\ 1] \subset \mathbb{R}$. For each $\lambda \in (0, +\infty)$, consider the following function

$$g_\lambda : K_\lambda \ni w \longmapsto \frac{1}{w - \lambda - 1} \in \mathbb{C}.$$

It is holomorphic in K_λ and has no holomorphic extension to $K_{\lambda'}$, for any $\lambda' > \lambda$. Set $R_\lambda := \lambda + 1 + \sqrt{\lambda^2 + 2\lambda}$. For $R > 1$, let $D(R) := \{w \in \mathbb{C} :\ |w + \sqrt{w^2 - 1}| < R\}$,

[11] See Remark 6.5 (the assumption that K is subanalytic can be significantly weakened).

where the square root is chosen as in the previous section. Note that $\lambda+1 \notin D(R_\lambda)$ and $g_\lambda|_K$ has no holomorphic extension to $D(R')$, for any $R' > R_\lambda$. By Corollary 6.2,

$$\limsup_{n\to\infty} \sqrt[n]{E_n(g_\lambda;\ K)} = \frac{1}{R_\lambda}.$$

This equality follows independently from the following fact: for each $c > 1$ and $n \in \mathbb{N}$,

$$E_n\left(\frac{1}{c-w};\ [-1,\ 1]\right) = \frac{1}{(c^2-1)\left(c+\sqrt{c^2-1}\right)^n},$$

(see [34], p. 76). Note that

$$\lim_{\lambda\to 0^+} \frac{R_\lambda - 1}{\sqrt{2\lambda}} = \lim_{\lambda\to 0^+} \frac{\lambda + \sqrt{\lambda^2+2\lambda}}{\sqrt{2\lambda}} = 1.$$

Therefore, the estimate in Corollary 6.4 cannot hold, in the case under consideration, with $\kappa > 1/2$. □

7 Addendum to Theorem 6.3

In Theorem 6.3, the assumption that K is fat and subanalytic cannot be replaced by a weaker assumption that K is fat and L-regular. Even if K has a very simple geometry, for example is definable in an o-minimal structure.[12] It is a consequence of Corollary 7.5 stated below and of the following fact: there are L-regular cusps in $\mathbb{R}^N$ (even definable in some o-minimal structures) which do not satisfy the condition (T4) of Theorem 7.4 with any $\lambda_0 > 0$. A simple example is the set

$$E := \{(x, y) \in \mathbb{R}^2 :\ 0 < x \le 1,\ 0 \le y \le \exp(-x^{-1})\} \cup \{(0, 0)\}$$

(cf. [18]).

Recall the following concept:

Definition 7.1 A set $E \subset \mathbb{C}^N$ is said to be pluripolar if for each $a \in E$, there is an open neighborhood U of a and a plurisubharmonic function $\varphi : U \longrightarrow [-\infty, \infty)$ such that $E \cap U \subset \{\varphi = -\infty\}$.

Moreover, it is convenient for us to introduce the following notation:

Definition 7.2 Let $E \subset \mathbb{C}^N$ be nonempty.

- E is called a (npl)-set if for each open set $U \subset \mathbb{C}^N$ such that $E \subset U$, the following implication holds true: W is a connected component of U such that $W \cap E \neq \emptyset \implies W \cap E$ is not pluripolar.

[12] Knowledge of o-minimal structures is not necessary to follow the present section.

- E is called a (nan)-set if for each open set $U \subset \mathbb{C}^N$ such that $E \subset U$, the following implication holds true: W is a connected component of U such that $W \cap E \neq \emptyset$ $\Longrightarrow$ $W \cap E$ is not a subset of a locally analytic set in $\mathbb{C}^N$ with empty interior.

Remark 7.3 Let $E \subset \mathbb{C}^N$ be nonempty. Then:

- each connected component of E is not pluripolar $\Longrightarrow$ E is a (npl)-set $\Longrightarrow$ E is a (nan)-set. Moreover, the set $E := \{0\} \cup \bigcup_{n\in\mathbb{N}} \left[4^{-n},\ 2\cdot 4^{-n}\right] \subset \mathbb{C}$ is a (npl)-set, but it has a pluripolar connected component.
- for each open set $D \subset \mathbb{C}^N$, the set $D \cap E$ is empty or nonpluripolar $\Longrightarrow$ E is a (npl)-set $\Longrightarrow$ E is a (nan)-set. Moreover, note that, for example, the set $E := [-1,\ 1]^2 \cup ([1,\ 2] \times \{0\}) \subset \mathbb{C}^2$ is a (npl)-set, but for each sufficiently small neighborhood D of $(2, 0)$, the set $D \cap E$ is nonempty and pluripolar.

Theorem 7.4 *Assume that a nonempty set $K \subset \mathbb{C}^N$ is compact. Suppose that $\lambda_0 > 0$. Consider the following conditions:*

(T1) *The set K is L-regular and the following condition $(\mathcal{J})$ holds: There exists $\kappa > 0$, $\varrho > 0$ such that, for each $t \in (0,\ \lambda_0]$ and each $h : K \longrightarrow \mathbb{C}$ which has no holomorphic extension to K_t,*

$$\limsup_{n\to\infty} \sqrt[n]{E_n(h;\ K)} > \frac{1}{1+\varrho\, t^{\kappa}}.$$

(T2) *The set K is a (npl)-set and the condition $(\mathcal{J})$ holds.*
(T3) *The set K is a (nan)-set and the condition $(\mathcal{J})$ holds.*
(T4) *There exists $\kappa > 0$, $\varrho > 0$ such that*

$$\Phi_K(z) \le 1 + \varrho\,(\mathrm{dist}(z;\ K))^{\kappa}$$

for $z \in K_{\lambda_0}$.[13]

Then, (T2) $\Longrightarrow$ (T3) $\Longrightarrow$ (T4) $\Longrightarrow$ (T1). If moreover K is polynomially convex, then (T1) $\Longrightarrow$ (T2).

Corollary 7.5 *Assume that a nonempty set $K \subset \mathbb{R}^N$ is compact. Suppose that $\lambda_0 > 0$. Then, the conditions (T1), (T2), (T3), and (T4) are equivalent.*

Proof It follows from Theorem 7.4 and from the fact that compact subsets of $\mathbb{R}^N \subset \mathbb{C}^N$ are polynomially convex in $\mathbb{C}^N$ (cf. [14], Lemma 5.4.1). □

Remark 7.6 In Theorem 7.4, the implication (T1) $\Longrightarrow$ (T2) does not hold without the additional assumption that K is polynomially convex.

Example Set $K := \{|z| = 1\} \cup \{0\} \subset \mathbb{C}$. By the maximum principle,

- $\hat{K} = \{|z| \le 1\}$,
- $\Phi_K(z) = \Phi_{\hat{K}}(z) = \max\{1, |z|\}$ for $z \in \mathbb{C}$.

[13] This condition is called the (HCP) property and was investigated in many papers.

Therefore,

$$\Phi_K(z) = \Phi_{\hat{K}}(z) = 1 + \mathrm{dist}(z;\ \hat{K}) \le 1 + \mathrm{dist}(z;\ K)$$

for $z \in \mathbb{C}$. It follows that (for each $\lambda_0 > 0$) the condition (T4) of Theorem 7.4 holds. This theorem implies that the condition (T1) is also satisfied. However, the conditions (T2), (T3) do not hold, because K is not a (npl)-set and is not a (nan)-set. □

Remark 7.7 For each $\lambda_0 > 0$, consider the condition:

(T2') The set K is not a pluripolar set and the condition ($\mathcal{J}$) of Theorem 7.4 holds.

Clearly, (T2) $\Longrightarrow$ (T2'). One may ask whether in Theorem 7.4 the condition (T2) can be replaced by this simpler condition (T2'). We will show below that the answer is negative. More precisely, there exists a nonempty compact set $K \subset \mathbb{C}$ such that

- K is polynomially convex,
- the condition (T2') is satisfied,
- the conditions (T3), (T4), and (T1) are not satisfied.

Example Set $K := B \cup \{2\}$, where $B := \{|z| \le 1\} \subset \mathbb{C}$. Fix $\lambda_0 \in (0,\ 1/2]$. Note that K is polynomially convex, because $\mathbb{C} \setminus K$ is connected. Moreover, K is not pluripolar.

The set B satisfies the condition (T4) (see the previous example). By Theorem 7.4, B satisfies the condition (T1) as well and let $\kappa > 0$, $\varrho > 0$ be of ($\mathcal{J}$) for B.

Fix $t \in (0,\ \lambda_0]$ and $h : K \longrightarrow \mathbb{C}$ which has no holomorphic extension to K_t. We will show that

$$\limsup_{n\to\infty} \sqrt[n]{E_n(h;\ K)} > \frac{1}{1 + \varrho\, t^{\kappa}}. \tag{6}$$

Note that

- $K_t = B_t \cup \{|z - 2| < t\}$,
- $B_t \cap \{|z - 2| < t\} = \emptyset$,
- $h|_{\{2\}}$ has a trivial holomorphic extension to $\{|z - 2| < t\}$.

It follows that $h|_B$ has no holomorphic extension to B_t. Since B satisfies the condition (T1), we have

$$\limsup_{n\to\infty} \sqrt[n]{E_n(h;\ K)} \ge \limsup_{n\to\infty} \sqrt[n]{E_n(h;\ B)} > \frac{1}{1 + \varrho\, t^{\kappa}},$$

and the proof of (6) is complete. Consequently, K satisfies (T2'). It is clear that it does not satisfy the conditions (T2), (T3). Theorem 7.4 and the polynomial convexity of K imply that the conditions (T1) and (T4) are not satisfied either (for the set K).

The last assertion can also be proved directly as follows. By Proposition 3.11 in [33],

$$\Phi_K^*(2) = \Phi_B^*(2) = \Phi_B(2) = 2 > 1 = \Phi_K(2)$$

(see the previous example).[14] Since $\Phi_K^*(2) > \Phi_K(2)$, it follows that Φ_K is not continuous at the point 2. In particular, the conditions (T1) and (T4) are not satisfied for K. □

8 A Proof of Theorem 7.4

Proof that (T2) $\Longrightarrow$ (T3). It follows from the fact that a locally analytic subset of $\mathbb{C}^N$ with empty interior is pluripolar. □

Proof that (T4) $\Longrightarrow$ (T1). It is enough to apply Theorem 6.1 (see also the proof of Theorem 6.3). □

Proof that (T1) $\Longrightarrow$ (T2) for K being polynomially convex. It suffices to prove the following assertion: Suppose that a nonempty compact set $K \subset \mathbb{C}^N$ is polynomially convex and L-regular. Then, K is a (npl)-set.

Let $U \subset \mathbb{C}^N$ be an open set such that $K \subset U$. Suppose that W is a connected component of U such that $W \cap K \neq \emptyset$. We need to show that $W \cap K$ is not pluripolar. Suppose that, on the contrary, it is pluripolar.

CASE 1: $K \subset W$. Then, $V_K^* \equiv +\infty$ (see Corollary 3.9 and Theorem 3.10 in [33]). This is however impossible, because V_K is continuous.

CASE 2: $K \setminus W \neq \emptyset$. Consider the function $f : U \longrightarrow \mathbb{C}$ defined by

$$f(z) := \begin{cases} 0 & \text{if } z \in U \setminus W, \\ 1 & \text{if } z \in W. \end{cases}$$

Clearly, this is a holomorphic function. By the Oka–Weil theorem, there exists a polynomial $P \in \mathbb{C}[z_1, \dots, z_N]$ such that $\|f - P\|_K < 1/2$. Obviously, P is not constant. Take $c \in W \cap K$. Since $|P| < 1/2$ in $K \setminus W$, it follows by Theorem 3.10 and Proposition 3.11 in [33] that

$$\begin{aligned} 0 = V_K(c) = V_K^*(c) &= V^*_{(K \setminus W) \cup (K \cap W)}(c) = V^*_{K \setminus W}(c) \\ &\geq V_{K \setminus W}(c) \geq \frac{1}{\deg P} \log 2|P(c)| > 0, \end{aligned}$$

which is a contradiction. □

Proof that (T3) $\Longrightarrow$ (T4). Let $a \in K_{\lambda_0}$. Fix a nonconstant polynomial $P \in \mathbb{C}[z_1, \dots, z_N]$ such that $\|P\|_K \leq 1$. We will show that

$$|P(a)|^{1/\deg P} \leq 1 + \varrho\lambda^\kappa,$$

where $\lambda := \operatorname{dist}(a; K)$ and κ, ϱ are of the condition $(\mathcal{J})$ in (T3). By the arbitrary character of P, we will get then $\Phi_K(a) \leq 1 + \varrho\lambda^\kappa$ and so (T4) will be proved. Clearly, it suffices to show that, for each $\epsilon > 0$,

[14] Recall that ϕ^* denotes the upper semicontinuous regularization of ϕ.

$$|P(a)|^{1/\deg P} \le (1+\epsilon)(1+\epsilon+\varrho\lambda^{\kappa}). \tag{7}$$

To this end, fix $\epsilon > 0$. There exists a compact set $K' \subset \mathbb{C}^N$ such that

- $K \subset K' \subset \{z \in \mathbb{C}^N : |P(z)| < (1+\epsilon)^{\deg P}\}$,
- K' is L-regular.

For example, as K' we can take a finite union of compact balls of sufficiently small radii and covering the set K. From the definition of the Siciak extremal function we easily obtain the estimate

$$|P(a)|^{1/\deg P} \le (1+\epsilon)\Phi_{K'}(a). \tag{8}$$

CASE 1: $\Phi_{K'}(a) \le 1+\epsilon$. Then (7) follows from (8).

CASE 2: $\Phi_{K'}(a) > 1+\epsilon$. Fix $\lambda' \in (\lambda, \lambda_0]$. Take a nonconstant polynomial $Q \in \mathbb{C}[z_1,\dots,z_N]$ such that $\|Q\|_{K'} \le 1$ and $|Q(a)| \ge (\Phi_{K'}(a)-\epsilon)^{\deg Q}$. Set

$$\Omega := \{z \in \mathbb{C}^N : \Phi_{K'}(z) < \Phi_{K'}(a) - \epsilon\}.$$

Clearly, Ω is open in $\mathbb{C}^N$ and $K' \subset \Omega$. Moreover, for each $z \in \Omega$, we have

$$|Q(z)| \le \Phi_{K'}(z)^{\deg Q} < (\Phi_{K'}(a)-\epsilon)^{\deg Q} \le |Q(a)|.$$

Consequently, $|Q(z)| < |Q(a)|$ for each $z \in \Omega$, and hence

$$f : \Omega \ni z \longmapsto \frac{1}{Q(z)-Q(a)} \in \mathbb{C}$$

is a well-defined holomorphic function in Ω.

Let W be a connected component of $K_{\lambda'}$ such that $a \in W$. Take $y \in K$ such that $|a-y| = \operatorname{dist}(a;\ K) = \lambda$. Since $a \in W$ and $[a,\ y] \subset K_{\lambda'}$, it follows that $[a,\ y] \subset W$. In particular, $W \cap K \neq \emptyset$. Moreover, $W \cap K$ is not contained in any locally analytic set in $\mathbb{C}^N$ with empty interior, because K is a (nan)-set.

Note that $f|_{K\cap W}$ has no holomorphic extension to W.[15] Therefore $f|_K$ has no holomorphic extension to $K_{\lambda'}$. By the assumption (T3), we have

$$\limsup_{n\to\infty} \sqrt[n]{E_n(f;\ K')} \ge \limsup_{n\to\infty} \sqrt[n]{E_n(f;\ K)} > \frac{1}{1+\varrho\,(\lambda')^{\kappa}}. \tag{9}$$

[15] We use here the following simple observation. Assume that $H : W \longrightarrow \mathbb{C}$ is a holomorphic function, where $W \subset \mathbb{C}^N$ is open and connected. Suppose moreover that $H^{-1}(0) \neq \emptyset$, $E \subset W$, $E \cap H^{-1}(0) = \emptyset$ and E is not contained in any locally analytic subset of $\mathbb{C}^N$ with empty interior. Then, the map

$$g : E \ni z \longmapsto \frac{1}{H(z)} \in \mathbb{C}$$

has no holomorphic extension to W.

By Theorem 3.1,

$$\limsup_{n\to\infty} \sqrt[n]{E_n(f;\ K')} \le \frac{1}{\Phi_{K'}(a) - \epsilon}. \tag{10}$$

Combining (8), (9), and (10), we obtain

$$|P(a)|^{1/\deg P} \le (1+\epsilon)\Phi_{K'}(a) \le (1+\epsilon)(1+\epsilon+\varrho(\lambda')^{\kappa}).$$

The estimate (7) follows by the arbitrary character of $\lambda' \in (\lambda,\ \lambda_0]$. □

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9 Appendix

To make the article accessible to a broad audience we decided not to use the notion of an o-minimal structure. O-minimal structures can be regarded as a far-reaching generalization of (globally) subanalytic sets (see [35,36]). Some results in this paper stated for subanalytic sets are true in a more general setting. For example, in Theorem 6.3 or Corollary 6.4, it is sufficient to assume that $K \subset \mathbb{R}^N$ is compact, fat, and definable in one of the o-minimal structures considered in [19,21]. The reason is that such a set satisfies, by the main results in [19,21], the (HCP) property. That is, there exists $\kappa > 0$, $\varrho > 0$ such that

$$\Phi_K(z) \le 1 + \varrho\,(\mathrm{dist}(z;\ K))^{\kappa} \tag{HCP}$$

as $\mathrm{dist}(z;\ K) \le 1$.[16] The condition of being a set with the (HCP) property is of very delicate nature. While the problem of checking the L-regularity of a set is difficult, the problem of verifying the (HCP) property is incomparably more difficult.

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[16] See also the condition (T4) of Theorem 7.4.

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