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CONSTRUCTIVE APPROXIMATION

ERRATUM

Erratum to: Rate of Convergence in Trotter's Approximation Theorem

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In this note we discuss the validity of Theorem 1.1 stated in [1] and give a correct version of formula [1, (1.4)] (Statement 1).

Moreover, we also give some cases in which formula [1, (1.4)] holds with further assumptions (Statement 2).

Finally, we make some clarifications in order to justify the validity of the results concerning the application to Bernstein operators given in [1, Sect. 2] (Statement 3).

Statement 1 Formula [1, (1.4)] becomes

$$\begin{aligned}\left\|T(t)u - L_n^{k(n)}u\right\| \leq{}& Me^{\omega t e^{\omega/n}} \int_0^t e^{-s\omega e^{\omega/n}} \psi_n\bigl(T(s)u\bigr)\,ds \\ &+ M\exp\bigl(\omega t_n e^{\omega/n}\bigr)\left|\frac{k(n)}{n} - t\right|\varphi_n(u) \\ &+ M\exp\left(2\omega\frac{k(n)}{n}e^{\omega/n}\right)\left(\frac{\omega}{n}\frac{k(n)}{n} + \frac{\sqrt{k(n)}}{n}\right)\varphi_n(u) \qquad (1)\end{aligned}$$

for every $u \in D \cap \{v \in D \mid T(t)v \in D\}$.

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Indeed, the term $\|T(t)u - S_n(t)\|$ at the right-hand side of [1, (1.6)] satisfies

$$T(t)u - S_n(t)u = \int_0^t \frac{d}{dt} S_n(t-s)T(s)u\,ds$$
$$= \int_0^t S_n(t-s)(A - A_n)T(s)u\,ds,$$

and from [1, (1.5)] and the inequality $e^x - 1 \le xe^x$, $x \ge 0$, we have

$$\begin{aligned}\big\|T(t)u - S_n(t)\big\| &\le M \int_0^t e^{n(t-s)(e^{\omega/n}-1)} \big\|(A - A_n)T(s)u\big\|\,ds \\ &\le M \int_0^t e^{\omega(t-s)e^{\omega/n}} \big\|(A - A_n)T(s)u\big\|\,ds \\ &= Me^{\omega t e^{\omega/n}} \int_0^t e^{-s\omega e^{\omega/n}} \big\|(A - A_n)T(s)u\big\|\,ds \\ &\le Me^{\omega t e^{\omega/n}} \int_0^t e^{-s\omega e^{\omega/n}} \psi_n(T(s)u)\,ds. \qquad (2)\end{aligned}$$

Statement 2 Formula [1, (1.4)] is valid if we add one of the following assumptions in [1, Theorem 1.1]:

1. The operators L_n commute each other, i.e., $L_n L_m = L_m L_n$ for every $n, m \ge 1$.
   In this case the proof requires no modification since the estimate of the first term given after (1.6) holds true (see, e.g., [2, p. 215]).
2. We have

$$\big\|(A_n - A)T(t)u\big\| \le \big\|T(t)\big\|\psi_n(u) \qquad (3)$$

for every $u \in D$.

In this case from (2) we have

$$\begin{aligned}\big\|T(t)u - S_n(t)\big\| &\le Me^{\omega t e^{\omega/n}} \int_0^t e^{-s\omega e^{\omega/n}} \big\|T(s)\big\|\psi_n(u)\,ds \\ &\le M^2 \psi_n(u) e^{\omega t e^{\omega/n}} \int_0^t e^{-s\omega e^{\omega/n}} e^{\omega s}\,ds \\ &\le M^2 t \psi_n(u) e^{\omega t e^{\omega/n}}.\end{aligned}$$

We observe that condition (3) holds if:

(i) the operators L_n commute with the limit semigroup $T(t)$, i.e., $L_n T(t) = T(t) L_n$ for every $n \in \mathbb{N}$ and $t \ge 0$;

(ii) $T(t)(D) \subset D$ for every $t \ge 0$, and

$$\big\|\psi_n\big(T(t)u\big)\big\| \le \big\|T(t)\big\|\psi_n(u)$$

for every $u \in D$.

Statement 3 We point out that the application to the Bernstein operators is valid as well, since the above assumption (ii) is satisfied. Indeed, we observe that the semigroup generated by the closure of the differential operator (2.1) maps $C^m(K_d)$ into $C^m(K_d)$ for every $m \geq 1$, and, moreover (see [3]),

$$\left\| T(t)f \right\|_{C^m(K_d)} \leq \|f\|_{C^m(K_d)}.$$

From the classical interpolation theory, it is well known that the existence of a linear extension operator which continuously maps $C^2(K_d)$ into $C^2(\mathbb{R}^d)$, $C^3(K_d)$ into $C^3(\mathbb{R}^d)$, and $C^{2+\alpha}(K_d)$ into $C^{2+\alpha}(\mathbb{R}^d)$ for every $\alpha \in]0,1[$ yields that $C^{2,\alpha}(K_d)$ is an intermediate space between $C^2(K_d)$ and $C^3(K_d)$.

The existence of a continuous linear operator E from $C^m(K_d)$ to $C^m(\mathbb{R}^d)$, from $C^{m+\alpha}(K_d)$ to $C^{m+\alpha}(\mathbb{R}^d)$, and from $C^{m+1}(K_d)$ to $C^{m+1}(\mathbb{R}^d)$ may be easily obtained similarly to the proof of [4, Lemma 6.37].

First of all, we observe that the construction of an extension operator can be reduced to the case where the domain is a quadrant $Q_k = \{x \in \mathbb{R}^d \mid x_1, x_2, \dots, x_k \geq 0\}$. Indeed, it is possible to use a partition of unity and suitable local coordinates near the boundary which maps $U \cap K_d$ into $V \cap Q_k$ where U and V are neighborhoods of ∂K_d and ∂Q_k. For every $x \in \partial K_d$ the existence of these local coordinates is achieved by considering the following cases. If $0 \leq x_1 + \cdots + x_d < 1$ and $x_i = 0$ for $i \in J$ where J is a nonempty subset of $\{1, 2, \dots, d\}$, then we can consider a linear invertible map T which permutes the coordinates with indices in J into the coordinates $\{x_1, \dots, x_{|J|}\}$. Then there exists a neighborhood U of x such that $T(U \cup K_d) = U \cup Q_{|J|}$.

If $x_1 + \cdots + x_d = 1$, there exists at least one $i \in \{1, 2, \dots, d\}$ for which $x_i > 0$, and we can consider the linear map $(x_1, \dots, x_d) \mapsto (x_1' = x_1, \dots, x_i' = 1 - x_1 - x_2 - \cdots - x_d, \dots, x_d' = x_d)$; hence we reduce to the preceding case since $x_1' + x_2' + \cdots + x_d' = 1 - x_i < 1$.

The second step consists in defining the extension operator over Q_k. This is achieved using the Hestenes–Whitney extension method. First we consider the domain $\mathbb{R}^d_{r+} = \{x \in \mathbb{R}^d | x_r \geq 0\}$ and define the extension operator $\Pi_r : C(\mathbb{R}^d_{r+}) \to C(\mathbb{R}^d_{r+})$ by setting

$$\Pi_r f(x) = \begin{cases} f(x_1, \dots, x_n) & \text{if } x_r \geq 0, \\ \sum_{k=1}^{m+1} \lambda_k f(x_1, \dots, -kx_r, \dots, x_n) & \text{if } x_r \leq 0, \end{cases}$$

where λ_k, $k = 1, \dots, m+1$ are constants determined by the system of equation

$$\sum_{k=1}^{m+1} (-k)^i \lambda_k = 1, \quad i = 0, 1, \dots, m. \tag{4}$$

If $f \in C^{m+1}(\mathbb{R}^d_{r+})$, the function $\Pi_r f$ has continuous derivatives up to the order $m+1$, and

$$\|\Pi_r f\|_{C^{m+1}(\mathbb{R}^d)} \leq C \|f\|_{C^{m+1}(\mathbb{R}^d_{r+})};$$

moreover, it is also easy to check that $\Pi_r f \in C^m(\mathbb{R}^d)$ if $f \in C^m(\mathbb{R}^d_{r+})$ and $\Pi_r f \in C^{m+\alpha}(\mathbb{R}^d)$ if $f \in C^{m+\alpha}(\mathbb{R}^d_{r+})$ for every $0 < \alpha < 1$. Now the extension operator for Q_k is obtained by considering the operator $\Pi = \Pi_k \circ \Pi_{k-1} \circ \cdots \circ \Pi_1$.

Returning to our situation, $C^{2,\alpha}(K_d)$ is an intermediate space between $C^2(K_d)$ and $C^3(K_d)$, and therefore $T(t)$ maps $C^{2,\alpha}(K_d)$ into itself and $\|T(t)f\|_{C^{2+\alpha}(K_d)} \leq \|f\|_{C^{2+\alpha}(K_d)}$, $0<\alpha<1$. Then

$$\left\|\big(n(B_n - I) - A\big)T(t)f\right\| \leq \frac{L_{(T(t)f)''}}{n^{\alpha/2}} \leq \frac{L_{f''}}{n^{\alpha/2}},$$

and consequently the seminorm ψ defined after [1, (2.3), p. 340] satisfies $\psi(T(t)u) \leq \psi(u)$, and estimate [1, (1.4)] holds.

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