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The exponentiated exponentially weighted moving average control chart

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Abstract
Memory-type control charts are widely used for monitoring small to moderate shifts in the process parameter(s). In the present article, we present an exponentiated exponentially weighted moving average (Exp-EWMA) control chart that weights the past observations of a process using an exponentiated function. We evaluated the run-length characteristics of the Exp-EWMA chart via Monte Carlo simulations. A comparison study versus the CUSUM, EWMA and extended EWMA (EEWMA) charts under similar in-control (IC) run-length properties demonstrates that the Exp-EWMA chart is more effective for detecting small and, under certain circumstances, moderate shifts for both the zero-state and steady-state cases. Moreover, the Exp-EWMA chart has better zero-state out-of-control (OOC) performance than an EWMA chart with smoothing parameter equal to the limit to the infinity of the exponentiated function, while the two charts perform similarly for the steady-state case. Finally, it is shown that the Exp-EWMA chart is more IC robust than its competitors under several non-normal distributions. Two examples are provided to explain the implementation of the proposed chart

Keywords Exp-EWMA chart · Monte Carlo simulation · Run-length distribution · Steady-state · Zero-state

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1 Introduction

Control charts are the most important tool of Statistical Process Control (SPC) and are used in many manufacturing and non-manufacturing processes to detect shifts in the quality characteristic of interest. The process variations are classified into two categories; the common-cause of variation and the assignable-cause of variation. A process operating with only common-cause of variation is considered to be in-control (IC), while a process operating with assignable-cause of variation is said to be out-of-control (OOC). There are two types of control charts in SPC literature; the memory-less and the memory-type charts. The memory-less charts were first introduced by Shewhart (1926). These charts are easy to use and effective in detecting large shifts in the process parameter(s). However, they do not have good detection ability for small and moderate shifts because their charting statistics are based only on the current observation. The most popular memory-type control charts are the cumulative sum (CUSUM) and the exponentially weighted moving average (EWMA) charts, proposed by Page (1954) and Roberts (1959) respectively, where their charting statistics are based on both the current and past observations. These charts are very sensitive in detecting small and moderate shifts.

The properties of the EWMA chart have been studied by many authors. Crowder (1987) evaluated the average run-length (ARL) values of an EWMA chart with asymptotic control limits using integral equations, while later, Crowder (1989) investigated the design of the EWMA chart. Saccucci and Lucas (1990) presented a computer program for evaluating the zero-state and steady-state ARL values of the EWMA chart. Lucas and Saccucci (1990) studied the properties of an EWMA chart with asymptotic control limits and they also presented the optimal EWMA schemes for specific sizes of shifts. Moreover, they presented three enhancements of it; those are (i) the EWMA chart with fast initial response (FIR) feature that makes it more effective when the shift occurs at the beginning, (ii) the combined Shewhart-EWMA chart which is very effective for both small and large shifts and (iii) the robust EWMA chart that provides protection against outliers. Chandrasekaran et al. (1995) proposed a stochastic model of the EWMA chart with variance-adjusted control limits. Steiner (1999) evaluated the run-length properties of the EWMA chart with time-varying control limits using non-homogeneous Markov chains and he concluded that using time-varying control limits is closely related to the FIR feature and provides good performance to early process shifts. Knoth (2005) compared the performance of several EWMA schemes with FIR feature. Human et al. (2011) investigated the IC robustness of the EWMA chart with asymptotic control limits and they showed that an EWMA chart is robust to several non-normal distributions for small values of smoothing parameters. Abbas et al. (2014) presented an EWMA chart using auxiliary information and they showed that it performs better than the univariate and bivariate EWMA charts for small and moderate shifts. Knoth (2015) computed the run-length quantiles for several EWMA schemes to compare their zero-state and steady-state performances.

Several extensions and modifications of the EWMA chart have been proposed in the SPC literature to improve the detection ability of the traditional EWMA chart for specific ranges of shifts. Shamma and Shamma (1992) proposed the double EWMA (DEWMA) chart to make the EWMA chart more efficient for small shifts. Capizzi and Masarotto (2003) presented the adaptive EWMA (AEWMA) chart where the past observations are weighted using a function of the current error. Sheu and Lin (2003) introduced the generally weighted moving average (GWMA) chart which is a generalized extension model of the EWMA chart. Patel and Divecha (2011) presented the modified EWMA (MEWMA) chart which is an enhancement of the EWMA chart. Abbas et al. (2013) proposed the mixed EWMA-CUSUM (MEC) chart which is more effective than the EWMA and CUSUM charts for small shifts. Abbas et al. (2013) presented the progressive mean (PM) chart which is a cumulative average of all observations. Khan et al. (2017) studied in more detail the MEWMA chart and they showed that it is more effective than the classical EWMA chart. Naveed et al. (2018) presented an extended EWMA (EEWMA) chart which gives a positive weight to the current observation and a negative weight to the preceding observations. The EEWMA chart was found more sensitive in detecting shifts than the classical EWMA chart. Naveed et al. (2020, 2021) investigated the performance of the EEWMA chart using the multiple dependent state sampling and the repetitive sampling scheme, respectively. Finally, Nawaz et al. (2021) studied the robustness of the EWMA and DEWMA charts using the repetitive sampling scheme.

In this article, we propose and study a new exponentiated EWMA (Exp-EWMA) chart for normally distributed data to enhance the detection ability of the EWMA chart for small and moderate shifts. The main difference from the EWMA chart is that its smoothing parameter is not constant, but an exponentiated function.

The article is organized as follows. In Sect. 2, the existing CUSUM, EWMA and EEWMA charts are briefly reviewed, while in Sect. 3, the structure of the proposed Exp-EWMA chart is given. In Sect. 4, the run-length distribution of the Exp-EWMA chart is evaluated for both the zero-state and steady-state cases. In Sect. 5, we present the Exp-EWMA chart for the case where the process parameters are unknown. In Sect. 6, a comparison study with the CUSUM, EWMA and EEWMA charts is presented. In addition, we investigate the IC robustness of the proposed chart to non-normality and we compare it with those of its competitors in Sect. 7. Two illustrative examples are provided in Sect. 8 and conclusions are summarized in Sect. 8.

2 Existing control charts

In this section, we give a brief description of the structure of the CUSUM, EWMA and EEWMA control charts for monitoring shifts in the process mean.

2.1 The CUSUM control chart

Let X_{tj}, $t = 1, 2, 3, \dots$ and $j = 1, 2, \dots, n$, be the jth observation in the tth sample of size $n \geq 1$ and assume that $X_{tj} \overset{iid}{\sim} N(\mu, \sigma^2)$. The process is considered to be IC if $\mu = \mu_0$ and $\sigma = \sigma_0$. In this article, we focus on monitoring shifts in the process mean assuming that the process variance is IC and remains constant.

The upper and lower charting statistics of the CUSUM chart are given by

$$C_t^+ = \max\left[0, \left(\overline{X}_t - \mu_0\right) - k + C_{t-1}^+\right]$$

and

$$C_t^- = \max\left[0, -\left(\overline{X}_t - \mu_0\right) - k + C_{t-1}^-\right],$$

where $C_0^+ = C_0^- = 0$ and k is the reference value, computed by $k = \frac{\delta\sigma_0}{2\sqrt{n}}$, where δ is the shift (in terms of the process standard deviation σ_0) that we are interested to detect quickly. An OOC signal is given by the CUSUM chart if $C_t^+ \geq H$ or $C_t^- \geq H$, where $H = \frac{h\sigma_0}{\sqrt{n}}$ $(h > 0)$ is the decision interval.

2.2 The EWMA control chart

The charting statistic of the EWMA chart is denoted as

$$E_t = \lambda\overline{X}_t + (1-\lambda)E_{t-1},$$

where $E_0 = \mu_0$ is the starting value and $0 < \lambda \leq 1$ is the smoothing parameter. The time-varying control limits and the centerline of the EWMA chart are given by

$$UCL_t/LCL_t = \mu_0 \pm L\frac{\sigma_0}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}, CL = \mu_0,$$

where $L > 0$ is the width of the control limits. For large values of t, the asymptotic control limits are given by

$$UCL/LCL = \mu_0 \pm L\frac{\sigma_0}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}}, CL = \mu_0,$$

A process is considered to be OOC if any charting statistic falls on or outside the control limits. The EWMA chart reduces to the Shewhart chart for $\lambda = 1$.

According to Steiner (1999), an EWMA chart is designed with time-varying control limits if there is an evidence that the shift occurs at the start (referred to as zero-state case). On the other hand, if the process is IC for several time periods and the shift occurs later (referred to as steady-state case), it is better to design a control chart with asymptotic control limits. The zero-state and steady-state performances of the EWMA chart are approximately similar for medium to large values of λ (Lucas and Saccucci 1990). Small values of λ are appropriate in detecting small shifts in the process mean while medium and large values are efficient in detecting moderate and large shifts, respectively.

2.3 The EEWMA control chart

The charting statistic of the EEWMA chart is denoted as

$$EE_t = \psi_1\overline{X}_t - \psi_2\overline{X}_{t-1} + (1-\psi_1+\psi_2)EE_{t-1},$$

where $0 < \psi_1 \leq 1$, $0 \leq \psi 2 < \psi_1$ and $\overline{X}_0 = EE_0 = \mu_0$. The time-varying control limits and the centerline of the EEWMA chart are given by

$$UCL_t/LCL_t$$
$$= \mu_0 \pm L\frac{\sigma_0}{\sqrt{n}}\sqrt{\frac{\left(\psi_1^2+\psi_2^2\right)\left(1-\alpha^{2t}\right)}{2(\psi_1-\psi_2)-(\psi_1-\psi_2)^2} - \frac{2\alpha\psi_1\psi_2\left(1-\alpha^{2t-2}\right)}{2(\psi_1-\psi_2)-(\psi_1-\psi_2)^2}},$$
$$CL = \mu_0,$$

where $\alpha = 1 - \psi_1 + \psi_2$ and $L > 0$ is the width of control limits. The asymptotic control limits of the EEWMA chart are given by

$$UCL/LCL = \mu_0 \pm L\frac{\sigma_0}{\sqrt{n}}\sqrt{\frac{\psi_1^2+\psi_2^2-2\alpha\psi_1\psi_2}{2(\psi_1-\psi_2)-(\psi_1-\psi_2)^2}}, \; CL = \mu_0,$$

The EEWMA chart reduces to the EWMA chart for $\psi_2 = 0$. A process is declared as OOC if any charting statistic falls on or outside the control limits. Otherwise, the process is said to be IC.

3 Structure of the proposed Exp-EWMA control chart

The proposed Exp-EWMA control chart is a modification of the EWMA chart where the smoothing parameter is not constant, but changes with t. More specifically, the charting statistic of the Exp-EWMA chart is defined as

$$Z_t = \lambda(t)\overline{X}_t + (1-\lambda(t))\, Z_{t-1}, \tag{1}$$

where $\lambda(t) = e^{-f(t)}$ with $f(t) \geq 0$ continuous function of t, so that the sequence $\{f(t), t \geq 0\}$ converges and $Z_0 = \mu_0$. In this article, we consider the following choice of $f(t)$

$$f(t) = -ln\lambda + a^{\left(c+\frac{1}{t}\right)}, \tag{2}$$

where $0 < \lambda \leq 1$ and $a, c \geq 0$ are two additional design parameters. Many functions of $f(t)$ can be used. However, choosing the above function $f(t)$, the smoothing parameter $\lambda(t)$ of the proposed chart changes over the time and lies between 0 and 1. Moreover, the function $\lambda(t)$ converges soon to the value λe^{-a^c}.

Thus, the charting statistic of the Exp-EWMA chart can also be written as

$$Z_t = \lambda e^{-a^{\left(c+\frac{1}{t}\right)}}\overline{X}_t + \left(1 - \lambda e^{-a^{\left(c+\frac{1}{t}\right)}}\right) Z_{t-1}. \tag{3}$$

From (3), we have

$$
\begin{aligned}
Z_t &= \lambda e^{-a\left(c+\frac{1}{t}\right)}\overline{X}_t + \left(1-\lambda e^{-a\left(c+\frac{1}{t}\right)}\right) Z_{t-1} \\
&= \lambda e^{-a\left(c+\frac{1}{t}\right)}\overline{X}_t + \left(1-\lambda e^{-a\left(c+\frac{1}{t}\right)}\right) \\
&\quad \left[\lambda e^{-a\left(c+\frac{1}{t-1}\right)}\overline{X}_{t-1} + \left(1-\lambda e^{-a\left(c+\frac{1}{t-1}\right)}\right) Z_{t-2}\right] \\
&= \lambda e^{-a\left(c+\frac{1}{t}\right)}\overline{X}_t + \lambda e^{-a\left(c+\frac{1}{t-1}\right)} \left(1-\lambda e^{-a\left(c+\frac{1}{t}\right)}\right)\overline{X}_{t-1} + \left(1-\lambda e^{-a\left(c+\frac{1}{t}\right)}\right) \times \\
&\quad \left(1-\lambda e^{-a\left(c+\frac{1}{t-1}\right)}\right) Z_{t-2} \\
&= \lambda e^{-a\left(c+\frac{1}{t}\right)}\overline{X}_t + \lambda e^{-a\left(c+\frac{1}{t-1}\right)} \left(1-\lambda e^{-a\left(c+\frac{1}{t}\right)}\right)\overline{X}_{t-1} + \left(1-\lambda e^{-a\left(c+\frac{1}{t}\right)}\right) \times \\
&\quad \left(1-\lambda e^{-a\left(c+\frac{1}{t-1}\right)}\right)\left[\left(\lambda e^{-a\left(c+\frac{1}{t-2}\right)}\right)\overline{X}_{t-2} + \left(1-\lambda e^{-a\left(c+\frac{1}{t-2}\right)}\right) Z_{t-3}\right] \\
&= \lambda e^{-a\left(c+\frac{1}{t}\right)}\overline{X}_t + \lambda e^{-a\left(c+\frac{1}{t-1}\right)} \left(1-\lambda e^{-a\left(c+\frac{1}{t}\right)}\right)\overline{X}_{t-1} + \lambda e^{-a\left(c+\frac{1}{t-2}\right)} \\
&\quad \left(1-\lambda e^{-a\left(c+\frac{1}{t}\right)}\right) \times \left(1-\lambda e^{-a\left(c+\frac{1}{t-1}\right)}\right)\overline{X}_{t-2} + \left(1-\lambda e^{-a\left(c+\frac{1}{t}\right)}\right) \\
&\quad \left(1-\lambda e^{-a\left(c+\frac{1}{t-1}\right)}\right)\left(1-\lambda e^{-a\left(c+\frac{1}{t-2}\right)}\right) Z_{t-3}.
\end{aligned}
$$

Proceeding in this way and since $Z_0 = \mu_0$, we get

$$
\begin{aligned}
Z_t &= \lambda e^{-a\left(c+\frac{1}{t}\right)}\overline{X}_t + \sum_{i=1}^{t-1} \lambda e^{-a\left(c+\frac{1}{i}\right)} \left(\prod_{j=i}^{t-1}\left(1-\lambda e^{-a\left(c+\frac{1}{j+1}\right)}\right)\right)\overline{X}_i \\
&\quad + \prod_{j=1}^{t}\left(1-\lambda e^{-a\left(c+\frac{1}{j}\right)}\right)\mu_0.
\end{aligned}
$$

The IC mean value and variance of the statistic Z_t are respectively given by

$$E(Z_t|IC) = \mu_0. \tag{4}$$

and

$$Var(Z_t|IC) = \left[\left(\lambda e^{-a\left(c+\frac{1}{t}\right)}\right)^2 + \sum_{i=1}^{t-1}\left\{\lambda e^{-a\left(c+\frac{1}{i}\right)}\left(\prod_{j=i}^{t-1}\left(1-\lambda e^{-a\left(c+\frac{1}{j+1}\right)}\right)\right)\right\}^2\right]\frac{\sigma_0^2}{n} \tag{5}$$

The proof for the IC mean value is given in Appendix A. The time-varying control limits of the Exp-EWMA chart are denoted as

$$UCL_t/LCL_t = \mu_0 \pm L\sqrt{Var(Z_t|IC)}, CL = \mu_0, \tag{6}$$

where $Var(Z_t|IC)$ is given by (5) and $L > 0$ is the width of the control limits and it is chosen so that the IC performance of the control chart is equal to a pre-specified value.

The asymptotic variance of the charting statistic Z_t is given by $\lim_{t\to+\infty} Var(Z_t|IC) = Var(Z_\infty|IC)$ which converges as $t \to +\infty$. For more details, see the Appendix B. Thus, the asymptotic control limits of the Exp-EWMA chart are denoted as

$$UCL/LCL = \mu_0 \pm L\sqrt{Var(Z_\infty|IC)}, CL = \mu_0,$$

or equivalently

$$UCL/LCL = \mu_0 \pm H\frac{\sigma_0}{\sqrt{n}}, CL = \mu_0, \tag{7}$$

where $H = L\frac{\sqrt{n}}{\sigma_0}\sqrt{Var(Z_\infty|IC)}$.

The Exp-EWMA chart is constructed by plotting the charting statistic Z_t against the sample number t. A process is considered to be OOC if any charting statistic plots on or outside the control limits. We note that the proposed control chart reduces to the EWMA chart for $a = 0$, while for $f(t) = \ln(t)$, we get the PM chart (see Appendix C). We also note that the proposed chart with $a = 1$ reduces to the EWMA chart with smoothing parameter λ/e.

4 The run-length distribution of the Exp-EWMA control chart

The performance of a control chart is usually evaluated using the run-length distribution and its associated characteristics. The run-length is denoted as the number of statistics plotted on a chart before the chart triggers an OOC signal. The most common performance measure is the ARL. When the process is IC, the ARL is denoted as ARL_0 and it should be large to avoid any false alarm. On the other hand, when the process is OOC, the ARL is denoted as ARL_1 and it should be small to detect the shift quickly. Except of the ARL, other characteristics of the run-length distribution, such as the median, the standard deviation (regarded as MRL and SDRL, respectively) and several percentile points, are also evaluated to obtain more information about the run-length distribution. As we mentioned in the subsection 2.2, there are two types of ARL; the zero-state and the steady-state ARL. The first one is evaluated for the case where the shift occurs at the start, while the second one is evaluated for the case where the process is IC for several periods and the shift occurs later in the process.

The run-length characteristics of a control chart can be computed using the Markov chain approach, integral equations and Monte Carlo simulations. Due to the complexity of the charting statistic and the expression of time-varying control limits, in this article, we perform the latter method with 50.000 repetitions using the asymptotic control

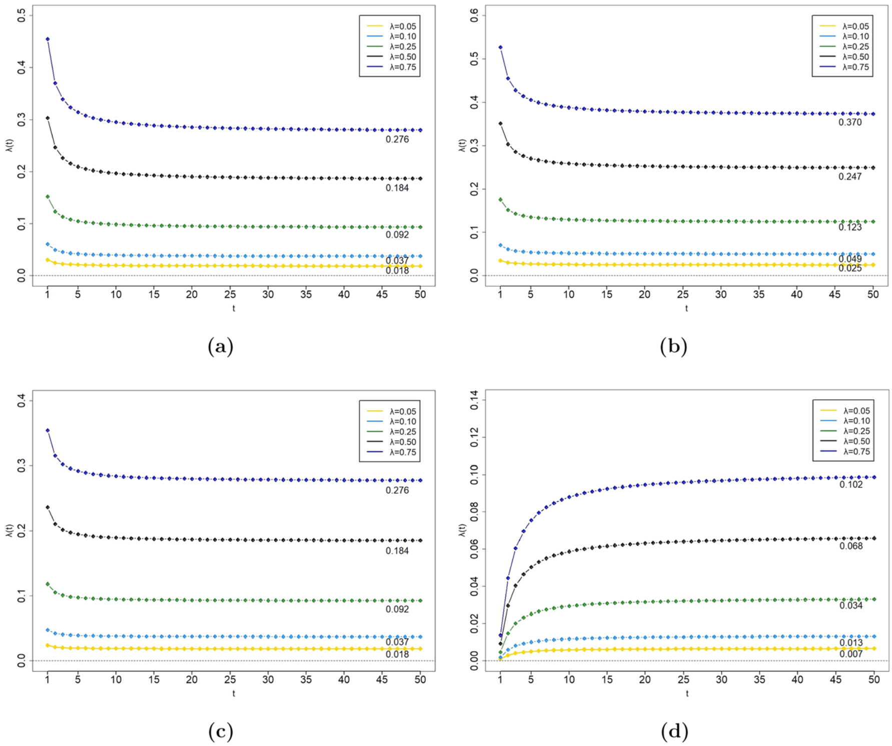

Fig. 1 Profiles $\lambda(t)$ versus t for **a** (a, c)=(0.5,0), **b** (0.5,0.5), **c** (0.75,0) and **d** (2,1)

limits, given by (7). Without loss of generality, we consider individual measurements with $\mu_0 = 0$ and $\sigma_0 = 1$. Furthermore, the results are presented only for positive shifts as they are similar to those for negative shifts.

The performance of the Exp-EWMA chart is investigated using different values of the design parameters (a, c, λ). More specifically, we set (a, c)=(0.5,0), (0.5,0.5), (0.75,0), (2,1) and $\lambda = 0.05, 0.10, 0.25, 0.50, 0.75$. The profiles $\lambda(t)$ for $t = 1, 2, \ldots, 50$ are presented in Fig. 1 from which we observe that $\lambda(t)$ is a decreasing (increasing) function for $a < 1$ $(a > 1)$ and converges soon to the value $\lambda_\infty = \lambda e^{-a^c}$. Thus, it is expected that the steady-state performance of the Exp-EWMA chart to be equal to that of the EWMA chart with λ_∞. This topic will be investigated later.

Tables 1-4 present the zero-state run-length characteristics of the Exp-EWMA chart given an $ARL_0 \approx 500$ for the above values of design parameters (a, c, λ) and for shifts $\delta \in \{0.05, 0.10, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 2.00, 3.00\}$. More specifically, the first row of each cell presents the ARL and SDRL (given in parenthesis) while the second row shows the 5th, 25th, 50th (MRL), 75th and 95th percentile points of the run-length. From these tables, we observe that for specified values (a, c), the IC SDRL value (referred to as $SDRL_0$) and the 75th and 95th percentile points of the IC run-length distribution increase as the value of λ also increases. On the other hand, the 5th, 25th and IC MRL (referred to as MRL_0) decrease as the value of λ increases. It can be

Table 1 Run-length characteristics for the Exp-EWMA control chart with $a = 0.5$ and $c = 0$ for the zero-state case

δ	$(\lambda, H) = (0.05, 0.2172)$	(0.10, 0.3452)	(0.25, 0.6161)	(0.50, 0.9431)	(0.75, 1.2132)
0.00	500.39 (482.68)	500.27 (493.74)	500.13 (506.85)	500.85 (517.76)	500.18 (526.04)
	(45,158,351,686,1460)	(35,149,348,692,1484)	(24,140,345,695,1506)	(17,132,339,698,1539)	(12,127,336,698,1549)
0.05	380.99 (356.61)	400.33 (390.25)	432.95 (435.88)	456.46 (472.67)	469.09 (492.87)
	(41,127,272,520,1092)	(31,123,281,550,1176)	(21,121,299,602,1299)	(15,121,309,634,1397)	(11,119,317,653,1445)
0.10	226.63 (200.71)	253.20 (239.48)	309.32 (308.65)	361.43 (372.05)	394.17 (412.07)
	(32,85,167,303,622)	(26,83,180,346,730)	(18,89,215,428,917)	(13,95,247,505,1104)	(10,100,267,551,1215)
0.25	71.46 (48.07)	75.46 (59.69)	98.22 (92.49)	136.80 (139.06)	173.32 (182.13)
	(19,37,59,93,165)	(15,33,58,100,193)	(10,33,70,134,283)	(7,37,93,190,415)	(6,43,116,243,539)
0.50	28.42 (13.93)	26.37 (15.43)	27.74 (21.54)	35.58 (33.67)	46.29 (47.50)
	(11,18,26,35,55)	(8,15,23,33,56)	(6,13,22,36,71)	(4,12,25,48,103)	(3,13,31,64,142)
0.75	17.05 (6.89)	14.89 (7.12)	13.53 (8.59)	14.82 (12.16)	17.74 (16.98)
	(8,12,16,21,30)	(6,10,14,18,28)	(4,7,12,17,30)	(3,6,11,20,39)	(2,6,12,24,52)
1.00	12.01 (4.27)	10.19 (4.25)	8.56 (4.66)	8.38 (5.96)	9.10 (7.75)
	(6,9,11,14,20)	(5,7,9,12,18)	(3,5,8,11,18)	(2,4,7,11,20)	(2,4,7,12,25)
1.25	9.20 (2.97)	7.67 (2.89)	6.16 (2.98)	5.59 (3.50)	5.66 (4.29)
	(5,7,9,11,15)	(4,6,7,9,13)	(2,4,6,8,12)	(2,3,5,7,12)	(1,3,5,7,14)
1.50	7.44 (2.21)	6.13 (2.12)	4.78 (2.09)	4.15 (2.33)	3.99 (2.69)
	(4,6,7,9,11)	(3,5,6,7,10)	(2,3,4,6,9)	(1,2,4,5,9)	(1,2,3,5,9)
2.00	5.34 (1.41)	4.35 (1.32)	3.30 (1.23)	2.70 (1.27)	2.45 (1.37)
	(3,4,5,6,8)	(3,3,4,5,7)	(2,2,3,4,6)	(1,2,2,3,5)	(1,1,2,3,5)
3.00	3.42 (0.77)	2.77 (0.71)	2.07 (0.63)	1.62 (0.64)	1.42 (0.59)
	(2,3,3,4,5)	(2,2,3,3,4)	(1,2,2,2,3)	(1,1,2,2,3)	(1,1,1,2,2)

Table 2 Run-length characteristics for the Exp-EWMA control chart with $a = 0.5$ and $c = 0.5$ for the zero-state case

δ	$(\lambda, H) = (0.05, 0.2646)$	(0.10, 0.4160)	(0.25, 0.7363)	(0.50, 1.1285)	(0.75, 1.4581)
0.00	500.64 (483.74)	499.56 (493.83)	500.57 (502.97)	500.43 (512.10)	500.92 (516.99)
	(43,156,352,687,1464)	(34,148,348,691,1488)	(25,143,346,694,1499)	(19,136,342,697,1523)	(16,133,342,697,1531)
0.05	389.84 (369.01)	410.16 (397.64)	443.43 (447.15)	469.92 (482.10)	478.56 (492.95)
	(39,127,276,533,1129)	(31,125,289,565,1208)	(23,126,307,614,1322)	(18,128,322,650,1426)	(15,127,326,670,1469)
0.10	237.16 (214.63)	270.64 (259.57)	332.23 (331.43)	388.44 (396.56)	419.16 (431.54)
	(31,85,172,321,662)	(25,86,191,371,784)	(19,95,230,461,992)	(16,106,267,540,1173)	(14,111,286,587,1279)
0.25	73.25 (51.96)	81.07 (67.05)	113.50 (108.97)	165.04 (167.16)	209.66 (216.43)
	(18,36,59,95,175)	(14,34,61,108,214)	(11,36,80,156,329)	(9,46,113,229,500)	(8,55,142,294,643)
0.50	28.14 (14.29)	26.93 (16.55)	31.03 (25.37)	44.52 (43.16)	61.54 (63.06)
	(11,18,25,35,56)	(8,15,23,34,59)	(6,13,24,41,82)	(4,14,31,61,131)	(4,17,41,85,188)
0.75	16.73 (6.87)	14.89 (7.35)	14.44 (9.72)	17.78 (15.49)	23.44 (23.19)
	(8,12,16,20,30)	(6,10,13,18,29)	(4,8,12,19,33)	(3,7,13,24,49)	(2,7,16,32,70)
1.00	11.79 (4.20)	10.12 (4.27)	8.91 (5.05)	9.55 (7.20)	11.38 (10.24)
	(6,9,11,14,20)	(5,7,9,12,18)	(3,5,8,11,19)	(2,4,8,12,24)	(2,4,8,15,32)
1.25	9.06 (2.89)	7.62 (2.86)	6.34 (3.14)	6.15 (4.06)	6.75 (5.49)
	(5,7,9,11,14)	(4,6,7,9,13)	(3,4,6,8,12)	(2,3,5,8,14)	(1,3,5,9,18)
1.50	7.36 (2.15)	6.10 (2.08)	4.88 (2.16)	4.46 (2.59)	4.56 (3.29)
	(4,6,7,9,11)	(3,5,6,7,10)	(2,3,4,6,9)	(2,3,4,6,9)	(1,2,4,6,11)
2.00	5.32 (1.36)	4.36 (1.29)	3.36 (1.25)	2.84 (1.36)	2.66 (1.56)
	(3,4,5,6,8)	(3,3,4,5,7)	(2,2,3,4,6)	(1,2,3,3,5)	(1,2,2,3,6)
3.00	3.47 (0.74)	2.81 (0.69)	2.12 (0.62)	1.68 (0.66)	1.48 (0.64)
	(2,3,3,4,5)	(2,2,3,3,4)	(1,2,2,2,3)	(1,1,2,2,3)	(1,1,1,2,3)

Table 3 Run-length characteristics for the Exp-EWMA control chart with $a = 0.75$ and $c = 0$ for the zero-state case

δ	$(\lambda, H) = (0.05, 0.2165)$	(0.10, 0.3441)	(0.25, 0.6141)	(0.50, 0.9395)	(0.75, 1.2081)
0.00	500.32 (477.13)	500.28 (485.39)	500.38 (495.97)	499.84 (502.62)	500.87 (509.37)
	(49,162,353,685,1449)	(18,154,351,691,1468)	(11,148,349,694,1481)	(7,142,344,694,1506)	(5,141,345,694,1509)
0.05	381.89 (352.91)	401.24 (382.96)	435.08 (428.09)	457.16 (458.85)	470.00 (475.93)
	(24,131,275,520,1084)	(17,128,284,551,1168)	(11,129,305,602,1287)	(7,132,316,632,1366)	(5,133,325,649,1411)
0.10	228.34 (198.44)	254.44 (235.93)	311.66 (303.74)	362.65 (362.08)	396.42 (400.07)
	(21,88,169,305,620)	(15,86,183,347,723)	(10,95,219,429,911)	(6,105,253,503,1086)	(5,112,274,550,1192)
0.25	73.22 (47.71)	77.44 (59.41)	101.01 (92.06)	140.23 (136.90)	177.75 (177.89)
	(14,39,61,94,165)	(11,36,61,102,195)	(7,35,73,138,284)	(5,42,98,193,414)	(4,51,123,247,535)
0.50	29.88 (13.82)	27.85 (15.33)	29.51 (21.56)	38.03 (33.76)	49.65 (47.50)
	(10,20,27,37,56)	(7,17,24,35,58)	(5,14,24,38,72)	(3,14,28,51,105)	(2,16,35,68,145)
0.75	18.28 (6.79)	16.09 (7.03)	14.81 (8.64)	16.36 (12.33)	19.80 (17.22)
	(7,13,17,22,31)	(5,11,15,20,29)	(4,9,13,19,31)	(2,8,13,21,41)	(2,8,15,26,54)
1.00	13.10 (4.20)	11.23 (4.20)	9.58 (4.66)	9.53 (6.07)	10.48 (7.93)
	(6,10,13,15,21)	(4,8,11,13,19)	(3,6,9,12,18)	(2,5,8,12,21)	(2,5,8,14,26)
1.25	10.20 (2.92)	8.60 (2.85)	7.04 (2.99)	6.50 (3.57)	6.70 (4.41)
	(5,8,10,12,16)	(4,7,8,10,14)	(3,5,6,9,13)	(2,4,6,8,13)	(1,4,6,9,15)
1.50	8.35 (2.18)	6.97 (2.10)	5.55 (2.11)	4.90 (2.36)	4.80 (2.77)
	(4,7,8,10,12)	(3,5,7,8,11)	(2,4,5,7,9)	(2,3,4,6,9)	(1,3,4,6,10)
2.00	6.13 (1.40)	5.06 (1.32)	3.91 (1.27)	3.29 (1.31)	3.04 (1.43)
	(4,5,6,7,9)	(3,4,5,6,7)	(2,3,4,5,6)	(1,2,3,4,6)	(1,2,3,4,6)
3.00	4.04 (0.77)	3.31 (0.72)	2.50 (0.65)	2.04 (0.64)	1.79 (0.68)
	(3,4,4,4,5)	(2,3,3,4,4)	(1,2,2,3,4)	(1,2,2,2,3)	(1,1,2,2,3)

Table 4 Run-length characteristics for the Exp-EWMA control chart with $a = 2$ and $c = 1$ for the zero-state case

δ	$(\lambda, H) = (0.05, 0.1030)$	(0.10, 0.1726)	(0.25, 0.3226)	(0.50, 0.5031)	(0.75, 0.6466)
0.00	500.88 (440.64)	500.66 (453.12)	500.81 (463.68)	500.28 (466.71)	500.14 (469.46)
	(77,188,368,674,1368)	(66,178,362,678,1397)	(53,169,361,684,1426)	(48,167,359,686,1434)	(45,165,359,687,1432)
0.05	365.27 (302.67)	376.87 (326.63)	401.48 (364.98)	429.13 (399.40)	440.55 (411.67)
	(68,151,276,484,960)	(59,144,279,505,1026)	(48,141,291,545,1136)	(43,145,310,585,1219)	(42,147,318,603,1258)
0.10	218.81 (161.26)	227.94 (182.26)	257.17 (225.09)	298.65 (272.23)	325.79 (299.63)
	(56,105,173,283,535)	(48,99,174,299,586)	(40,97,191,346,704)	(35,105,218,405,831)	(35,112,237,444,917)
0.25	82.28 (40.25)	79.81 (44.12)	83.77 (57.24)	98.31 (77.66)	114.56 (95.79)
	(36,53,73,101,160)	(31,48,69,100,165)	(25,43,68,107,196)	(22,43,76,130,251)	(20,46,86,153,304)
0.50	40.17 (12.88)	36.81 (13.17)	33.63 (14.88)	34.15 (18.82)	36.63 (23.02)
	(23,31,38,47,65)	(20,27,34,44,62)	(16,23,30,41,63)	(14,21,29,42,71)	(13,20,30,46,82)
0.75	27.29 (6.69)	24.53 (6.61)	21.33 (6.86)	20.00 (7.88)	20.00 (9.16)
	(18,22,26,31,40)	(16,20,24,28,37)	(13,16,20,25,34)	(11,14,18,24,35)	(10,14,18,24,38)
1.00	21.05 (4.23)	18.79 (4.12)	15.96 (4.11)	14.44 (4.42)	13.92 (4.84)
	(15,18,21,24,29)	(13,16,18,21,26)	(11,13,15,18,24)	(9,11,14,17,23)	(8,10,13,16,23)
1.25	17.35 (2.99)	15.43 (2.88)	12.99 (2.80)	11.53 (2.89)	10.91 (3.07)
	(13,15,17,19,23)	(11,13,15,17,21)	(9,11,13,15,18)	(8,9,11,13,17)	(7,9,10,12,17)
1.50	14.89 (2.26)	13.22 (2.16)	11.08 (2.07)	9.73 (2.08)	9.11 (2.14)
	(12,13,15,16,19)	(10,12,13,15,17)	(8,10,11,12,15)	(7,8,9,11,14)	(6,8,9,10,13)
2.00	11.82 (1.47)	10.50 (1.40)	8.76 (1.31)	7.63 (1.28)	7.07 (1.28)
	(10,11,12,13,14)	(8,10,10,11,13)	(7,8,9,10,11)	(6,7,7,8,10)	(5,6,7,8,9)
3.00	8.70 (0.83)	7.76 (0.79)	6.50 (0.73)	5.64 (0.70)	5.21 (0.69)
	(8,8,9,9,10)	(7,7,8,8,9)	(5,6,6,7,8)	(5,5,6,6,7)	(4,5,5,6,6)

also seen that for (a, c)=(0.5,0), (0.5,0.5), (0.75,0) and for medium to large values of λ, the $SDRL_0$ value is larger than 500. This phenomenon may lead to a probability of false OOC signal (see Chan and Zhang 2000). Moreover, the run-length distribution is positively-skewed for the IC state and small to moderate shifts ($\delta \leq 1$) as ARL>MRL, while it is less positively-skewed for larger shifts as in such cases ARL≈MRL.

With regards to the OOC performance of the proposed chart for the zero-state case, we observe that a small value of λ is preferred over larger values to detect small shifts of the process mean, while medium to large values of λ are more effective for moderate to large shifts. For example, for (a, c)=(0.75,0), the Exp-EWMA chart with $\lambda = 0.05$ outperforms the other charts for $\delta \leq 0.25$ whereas the Exp-EWMA charts with $\lambda = 0.10, 0.25, 0.50, 0.75$ have very good detection ability for $\delta = 0.50$, $\delta = 0.75$, $1.00 \leq \delta \leq 1.25$ and $\delta \geq 1.50$, respectively. Finally, it should be noted that for specified values of λ and δ, the OOC performance of the proposed chart depends on the values of the design parameters (a, c). For instance, when $\lambda = 0.05$ and $\delta = 0.25$, the (ARL,MRL) values for (a, c)=(0.5,0), (0.5,0.5), (0.75,0) and (2,1) are (71.46,59), (73.25,59), (73.22,61) and (82.28,73).

Tables 5-8 present the steady-state run-length characteristics of the Exp-EWMA chart for the design parameters (a, c, λ, H) and shifts used to the zero-state case. Comparing with the results of Tables 1-4, we observe that there are differences between the zero-state and steady-state run-length characteristics. More specifically, for $a < 1$, the steady-state ARL_0 and the percentile points of the IC run-length distribution are smaller than the corresponding zero-state values for small values of λ and vice versa for medium to large values. Furthermore, there is no a significant difference between the zero-state and steady-state $SDRL_0$ values. It should be also mentioned that for $a < 1$ and small values of λ, the steady-state ARL_1 are larger than the corresponding zero-state ARL_1 for $\delta \geq 0.25$ or 0.50. For example, the zero-state ARL_1 values of the Exp-EWMA ($a = 0.5, c = 0, \lambda = 0.10, H = 0.3452$) chart are 400.33, 253.20, 75.46, 26.37, 14.89, 10.19, 7.67, 6.13, 4.35 and 2.77 for $\delta =$ 0.05, 0.10, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 2.00 and 3.00, respectively (see Table 1), while the corresponding steady-state ARL_1 values are 389.37, 246.29, 76.01, 27.94, 16.65, 11.85, 9.22, 7.57, 5.63 and 3.80 (see Table 5). On the other hand, for $a < 1$ and medium to large values of λ, the steady-state ARL_1 values are larger than the corresponding zero-state ARL_1 values over the entire range of shifts. For instance, the zero-state ARL_1 values of the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.50, H =$ 0.9395) chart are 457.16, 362.65, 140.23, 38.03, 16.36, 9.53, 6.50, 4.90, 3.29 and 2.04 for $\delta = 0.05, 0.10, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 2.00$ and 3.00, respectively (see Table 3), while the corresponding steady-state ARL_1 values are 460.96, 367.75, 142.82, 39.30, 17.28, 10.26, 7.13, 5.47, 3.76 and 2.41 (see Table 7). With regards to the Exp-EWMA ($a = 2, c = 1$) chart, the steady-state ARL values are smaller than the corresponding zero-state ARL values due to the large difference between the zero-state and steady-state ARL_0 values.

Table 5 Run-length characteristics for the Exp-EWMA control chart with $a = 0.5$ and $c = 0$ for the steady-state case

δ	$(\lambda, H) = (0.05, 0.2172)$	(0.10, 0.3452)	(0.25, 0.6161)	(0.50, 0.9431)	(0.75, 1.2132)
0.00	464.24 (482.46)	487.45 (495.06)	509.90 (514.10)	521.15 (520.88)	528.17 (528.58)
	(24,144,354,707,1522)	(18,136,337,677,1471)	(22,142,348,696,1502)	(26,148,361,727,1561)	(27,152,366,730,1585)
0.05	351.92 (355.23)	389.37 (391.84)	443.13 (443.65)	475.30 (476.35)	494.44 (494.60)
	(9,100,244,488,1060)	(17,112,269,540,1171)	(22,129,308,615,1323)	(24,136,329,660,1417)	(25,142,343,685,1479)
0.10	212.57 (201.20)	246.29 (238.89)	315.77 (311.74)	378.12 (376.12)	417.04 (418.25)
	(9,72,154,291,615)	(15,78,173,339,727)	(19,94,221,436,937)	(20,110,265,525,1123)	(22,120,288,577,1258)
0.25	69.81 (50.63)	76.01 (60.95)	102.38 (93.65)	146.00 (141.68)	187.18 (185.13)
	(7,34,59,94,166)	(10,34,60,101,196)	(11,36,74,138,292)	(11,45,102,200,431)	(12,56,130,257,554)
0.50	29.56 (16.79)	27.94 (16.67)	30.10 (21.95)	39.88 (34.59)	53.15 (49.26)
	(5,18,28,39,60)	(6,16,25,36,60)	(6,15,24,39,735)	(6,15,30,54,108)	(6,18,38,73.151)
0.75	18.62 (9.47)	16.65 (8.44)	15.52 (9.14)	17.44 (12.75)	21.50 (18.00)
	(4,12,18,25,35)	(5,11,16,21,32)	(4,9,14,20,33)	(4,9,14,23,42)	(4,9,16,29,57)
1.00	13.64 (6.51)	11.85 (5.46)	10.24 (5.15)	10.33 (6.39)	11.56 (8.34)
	(3,9,13,18,25)	(4,8,11,15,22)	(3,7,9,13,20)	(3,6,9,13,23)	(3,6,9,15,28)
1.25	10.79 (4.95)	9.22 (4.00)	7.64 (3.45)	7.16 (3.77)	7.51 (4.64)
	(3,7,11,14,19)	(3,6,9,12,16)	(3,5,7,10,14)	(2,5,6,9,14)	(2,4,6,10,16)
1.50	8.94 (3.98)	7.57 (3.15)	6.12 (2.57)	5.49 (2.58)	5.47 (2.95)
	(2,6,9,12,16)	(3,5,7,10,13)	(2,4,6,8,11)	(2,4,5,7,10)	(2,3,5,7,11)
2.00	6.72 (2.87)	5.63 (2.22)	4.42 (1.68)	3.77 (1.52)	3.55 (1.58)
	(2,5,7,9,12)	(2,4,6,7,9)	(2,3,4,5,7)	(2,3,4,5,6)	(2,2,3,4,6)
3.00	4.57 (1.85)	3.80 (1.39)	2.93 (1.00)	2.42 (0.83)	2.18 (0.77)
	(2,3,5,6,8)	(2,3,4,5,6)	(1,2,3,4,5)	(1,2,2,3,4)	(1,2,2,3,3)

Table 6 Run-length characteristics for the Exp-EWMA control chart with $a = 0.5$ and $c = 0.5$ for the steady-state case

δ	$(\lambda, H) = (0.05, 0.2646)$	(0.10, 0.4160)	(0.25, 0.7363)	(0.50, 1.1285)	(0.75, 1.4581)
0.00	471.63 (484.90)	489.92 (493.61)	505.86 (505.52)	514.14 (512.70)	518.70 (518.79)
	(13,128,324,658,1426)	(20,138,340,682,1471)	(24,144,352,702,1512)	(26,148,357,713,1533)	(27,150,361,716,1555)
0.05	365.48 (368.45)	401.49 (399.84)	449.94 (450.68)	481.60 (480.30)	494.44 (492.32)
	(13,105,253,506,1098)	(19,116,280,557,1205)	(22,130,312,625,1347)	(24,139,334,667,1438)	(26,143,344,685,1473)
0.10	224.67 (215.14)	264.45 (259.31)	337.51 (333.65)	398.94 (400.21)	435.39 (432.81)
	(12,74,161,308,656)	(16,81,185,364,786)	(19,99,236,468,1000)	(21,115,277,552,1195)	(23,127,302,604,1303)
0.25	71.40 (53.75)	80.94 (68.23)	116.18 (109.76)	172.30 (169.62)	219.97 (217.78)
	(9,34,59,95,175)	(10,34,62,108,216)	(11,38,83,158,336)	(12,51,120,237,510)	(13,65,153,304,652)
0.50	28.64 (16.44)	27.80 (17.56)	32.76 (25.95)	48.28 (44.14)	67.74 (65.09)
	(6,17,26,38,59)	(6,16,24,36,62)	(6,15,26,43,84)	(6,17,35,66,137)	(6,22,48,93,198)
0.75	17.68 (8.89)	16.00 (8.34)	15.79 (10.13)	19.96 (16.13)	26.72 (23.88)
	(4,11,17,23,33)	(5,10,15,21,31)	(4,9,13,20,35)	(4,9,15,26,52)	(4,10,20,36,74)
1.00	12.83 (5.99)	11.18 (5.19)	10.04 (5.43)	11.02 (7.58)	13.54 (10.88)
	(4,9,12,17,23)	(4,8,11,14,21)	(3,6,9,13,20)	(3,6,9,14,26)	(3,6,10,18,35)
1.25	10.09 (4.50)	8.62 (3.73)	7.31 (3.47)	7.32 (4.31)	8.27 (5.84)
	(3,7,10,13,18)	(3,6,8,11,15)	(3,5,7,9,14)	(2,4,6,9,16)	(2,4,7,11,20)
1.50	8.34 (3.59)	7.04 (2.90)	5.78 (2.51)	5.42 (2.80)	5.72 (3.52)
	(3,6,8,11,14)	(3,5,7,9,12)	(2,4,5,7,10)	(2,3,5,7,11)	(2,3,5,7,13)
2.00	6.24 (2.56)	5.20 (2.01)	4.11 (1.58)	3.59 (1.54)	3.49 (1.72)
	(2,5,6,8,11)	(2,4,5,6,9)	(2,3,4,5,7)	(2,3,3,4,6)	(1,2,3,4,7)
3.00	4.23 (1.64)	3.49 (1.24)	2.69 (0.91)	2.24 (0.78)	2.04 (0.77)
	(2,3,4,5,7)	(2,3,3,4,6)	(1,2,3,3,4)	(1,2,2,3,4)	(1,2,2,2,3)

Table 7 Run-length characteristics for the Exp-EWMA control chart with $a = 0.75$ and $c = 0$ for the steady-state case

δ	$(\lambda, H) = (0.05, 0.2165)$	(0.10, 0.3441)	(0.25, 0.6141)	(0.50, 0.9395)	(0.75, 1.2081)
0.00	457.17 (474.99)	478.68 (485.98)	497.68 (499.11)	504.80 (503.55)	508.26 (508.02)
	(9,120,312,640,1393)	(17,133,332,666,1442)	(23,141,346,690,1488)	(25,143,349,704,1510)	(26,147,353,702,1522)
0.05	347.59 (351.24)	382.92 (385.13)	433.19 (432.29)	460.96 (461.28)	476.26 (476.56)
	(9,98,241,482,1048)	(16,110,265,531,1153)	(22,126,302,602,1288)	(23,132,319,639,1374)	(24,137,331,660,1417)
0.10	210.79 (199.51)	243.20 (235.77)	310.03 (305.41)	367.75 (366.25)	402.77 (403.57)
	(9,71,153,288,611)	(14,77,171,335,715)	(18,92,217,429,923)	(20.107,257,511,1092)	(21,116,279,557,1208)
0.25	69.46 (50.47)	75.47 (60.50)	101.00 (92.33)	142.82 (138.41)	181.64 (178.70)
	(7,34,59,94,165)	(10,33,60,101,194)	(11,36,73,136,288)	(11,44,100,196,420)	(12,54,127,250,537)
0.50	29.46 (16.77)	27.82 (16.61)	29.91 (21.83)	39.30 (34.04)	52.06 (48.23)
	(5,17,28,39,60)	(6,16,25,36,59)	(6,15,24,39,73)	(6,15,29,53,107)	(6,18,37,71,148)
0.75	18.56 (9.46)	16.59 (8.42)	15.45 (9.11)	17.28 (12.60)	21.18 (17.71)
	(4,12,18,24,35)	(5,11,16,21,32)	(4,9,14,20,33)	(4,8,14,23,42)	(4,9,16,28,56)
1.00	13.59 (6.50)	11.81 (5.45)	10.20 (5.14)	10.26 (6.33)	11.44 (8.23)
	(3,9,13,18,25)	(4,8,11,15,22)	(3,7,9,13,20)	(3,6,9,13,23)	(3,6,9,15,28)
1.25	9.66 (4.25)	9.19 (3.99)	7.61 (3.44)	7.13 (3.75)	7.45 (4.60)
	(3,7,9,12,17)	(3,6,9,12,16)	(3,5,7,10,14)	(2,4,6,9,14)	(2,4,6,9,16)
1.50	7.97 (3.38)	7.55 (3.15)	6.10 (2.56)	5.47 (2.57)	5.43 (2.93)
	(3,6,8,10,14)	(3,5,7,10,13)	(2,4,6,8,11)	(2,4,5,7,10)	(2,3,5,7,11)
2.00	5.95 (2.40)	5.62 (2.22)	4.41 (1.68)	3.76 (1.51)	3.54 (1.57)
	(2,4,6,8,10)	(2,4,6,7,9)	(2,3,4,5,7)	(2,3,4,5,6)	(1,2,3,4,6)
3.00	4.03 (1.52)	3.79 (1.39)	2.92 (1.00)	2.41 (0.83)	2.17 (0.77)
	(2,3,4,5,7)	(2,3,4,5,6)	(1,2,3,4,5)	(1,2,2,3,4)	(1,2,2,3,3)

Table 8 Run-length characteristics for the Exp-EWMA control chart with $a = 2$ and $c = 1$ for the steady-state case

δ	$(\lambda, H) = (0.05, 0.1030)$	(0.10, 0.1726)	(0.25, 0.3226)	(0.50, 0.5031)	(0.75, 0.6466)
0.00	384.94 (431.35)	425.31 (448.79)	452.60 (458.65)	466.00 (463.85)	468.61 (468.35)
	(1,71,248,548,1246)	(4,104,288,598,1317)	(16,126,314,632,1364)	(21,133,327,649,1392)	(22,134,327,650,1396)
0.05	286.10 (307.49)	320.79 (327.37)	362.10 (363.84)	396.90 (394.71)	412.70 (410.44)
	(1,65,193,402,897)	(5,88,222,448,980)	(14,104,251,501,1087)	(20.115,278,550,1187)	(21,120,289,574,1221)
0.10	173.84 (165.38)	194.04 (182.92)	230.52 (222.70)	275.32 (271.34)	301.67 (297.68)
	(1,53,133,245,495)	(4,66,144,266,555)	(13,74,163,317,678)	(17,83,192,378,819)	(18,90,211,418,894)
0.25	66.93 (49.52)	67.62 (48.40)	72.75 (57.53)	86.95 (76.55)	101.85 (94.31)
	(1,30,60,95,158)	(4,33,59,92,158)	(9,33,58,97,185)	(11,33,65,117,240)	(10,35,74,138,292)
0.50	32.03 (20.20)	30.14 (17.45)	27.60 (16.34)	27.98 (18.90)	30.03 (22.34)
	(1,17,31,45,67)	(3,18,29,41,61)	(6,16,25,36,58)	(6,15,24,36,65)	(6,14,24,39,74)
0.75	21.08 (12.45)	19.31 (10.20)	16.63 (8.45)	15.39 (8.46)	15.29 (9.24)
	(1,12,21,30,42)	(3,12,19,26,37)	(4,11,16,22,32)	(4,9,14,20,31)	(4,9,13,20,33)
1.00	15.78 (8.97)	14.26 (7.14)	11.90 (5.52)	10.50 (5.04)	10.00 (5.15)
	(1,9,16,22,31)	(2,9,14,19,26)	(4,8,11,15,22)	(4,7,10,13,20)	(3,6,9,13,20)
1.25	12.67 (7.01)	11.36 (5.50)	9.29 (4.08)	8.00 (3.52)	7.42 (3.42)
	(1,7,13,18,24)	(2,7,11,15,21)	(3,6,9,12,16)	(3,6,8,10,14)	(3,5,7,9,14)
1.50	10.61 (5.76)	9.45 (4.46)	7.65 (3.23)	6.49 (2.69)	5.93 (2.52)
	(1,6,11,15,20)	(2,6,9,12,17)	(3,5,7,10,13)	(2,5,6,8,11)	(2,4,6,7,10)
2.00	8.06 (4.22)	7.13 (3.23)	5.71 (2.28)	4.75 (1.82)	4.27 (1.64)
	(1,5,8,11,15)	(2,5,7,9,13)	(2,4,6,7,10)	(2,3,5,6,8)	(2,3,4,5,7)
3.00	5.55 (2.76)	4.87 (2.09)	3.86 (1.44)	3.17 (1.11)	2.82 (0.97)
	(1,4,6,8,10)	(1,3,5,6,8)	(2,3,4,5,6)	(1,2,3,4,5)	(1,2,3,3,4)

5 Phase II Exp-EWMA chart

Up to this point, we have assumed that the quality characteristic of interest follows a normal distribution with known values of μ and σ. However, in many real-life applications, the IC values of μ and σ are unknown and they are estimated from an IC Phase I sample before on-line monitoring starts in Phase II. We note that the Phase I sample is chosen when the process is thought to be IC. The estimates of μ_0 and σ_0 are denoted by $\hat{\mu}_0$ and $\hat{\sigma}_0$, respectively and are used to estimate the starting value Z_0 and the control limits. Let X_{tj}, $t = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, be a sequence of IC Phase I observations which follow the $N(\mu_0, \sigma_0^2)$ distribution. The unbiased estimators for μ_0 and σ_0 are given as

$$\hat{\mu}_0 = \frac{\sum_{t=1}^{m} \sum_{j=1}^{n} X_{tj}}{m \cdot n}$$

and

$$\hat{\sigma}_0 = \frac{\sqrt{\frac{\sum_{t=1}^{m} \sum_{j=1}^{n} \left(X_{tj} - \overline{X}_t\right)^2}{m \cdot (n-1)}}}{c_{4,m}},$$

where $c_{4,m} = \frac{\sqrt{2}\Gamma\left(\frac{m(n-1)+1}{2}\right)}{\sqrt{m(n-1)} \cdot \Gamma\left(\frac{m(n-1)}{2}\right)}$; see, for example Malela-Majika et al. (2022). The control limits of the Exp-EWMA chart are calculated using (7) where the estimates of μ_0 and σ_0 are substituted for the unknown parameters.

The above methodology can used to implement the proposed chart in situations where the process parameters are unknown. Extensive simulations are required in order to investigate the effect of parameters estimation on the performance of the Exp-EWMA chart. This topic is out of the scope of this article and should be interest for future research.

6 Comparison study

In order to compare the OOC performance of different control charts, it is recommended to set a similar value of ARL_0, commonly 370 or 500, for all the charts. The chart with the smallest ARL_1 value in a specific shift is considered to be the most effective. Many authors, such as Chakraborti (2007), Teoh et al. (2017) and Qiao et al. (2022), recommended to compare different control charts using the OOC MRL (denoted as MRL_1) values by designing them under a similar value of MRL_0 because it provides a more meaningful interpretation than the ARL. Chan and Zhang (2000) suggested to take into account in the design of a chart not only the ARL_0, but also the SDRL_0 to avoid the large variation in the run-length distribution. In this article, the proposed Exp-EWMA chart is compared with the EWMA, CUSUM and EEWMA charts for both the zero-state and steady-state cases under similar IC run-length properties. For this reason, we chose the EWMA charts with $\lambda = 0.05, 0.10, 0.25, 0.50$ and the CUSUM charts with $k = 0.125, 0.5, 1$ (CUSUM charts optimally designed to detect shifts of 0.25, 1.00 and 2.00, respectively) and subsequently, we found the appropriate EEWMA and Exp-EWMA charts with similar values of ARL_0, $SDRL_0$ and 5th, 25th, 50th, 75th, 95th percentile points (regarded as P_5, P_{25}, P_{50}, P_{75} and P_{95}) of the IC run-length distribution. Moreover, the Exp-EWMA chart is compared with the EWMA (λ_∞) chart in order to contrast their performances.

Table 9 presents the IC run-length characteristics of the competing charts for both the zero-state and steady-state cases. We note that first, we compare the EWMA chart and its modifications, ie the EEWMA and Exp-EWMA charts, and then, we compare the Exp-EWMA and CUSUM charts. From Table 9, we observe that choosing the appropriate control charts with similar zero-state IC run-length properties, the corresponding steady-state properties are similar.

Table 10 shows the ARL and MRL (in the parenthesis) values of the competing EWMA, Exp-EWMA and EEWMA charts for the zero-state case. There are four quadruple of competing charts where the first EWMA, the EEWMA and the Exp-EWMA charts have similar IC run-length characteristics, while the second EWMA chart is designed with $\lambda_\infty = \lambda e^{-a^c}$. For example, in the first quadruple of competing charts, the EWMA ($\lambda = 0.05$, $L = 2.613$), the EEWMA ($\psi_1 = 0.07$, $\psi_2 = 0.03$, $L = 2.701$) and the Exp-EWMA ($\lambda = 0.06, a = 0.5, c = 0.5, H = 0.2986$) charts have similar IC run-length characteristics (see Table 9), while the other EWMA chart is designed with smoothing parameter equal to λ_∞ and has different IC run-length characteristics from the other two charts.

Comparing the Exp-EWMA and EWMA charts with similar IC run-length characteristics, it is seen that the Exp-EWMA ($a = 0.5$, $c = 0.5$, $\lambda = 0.06$) and Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.14$) charts are more effective than the EWMA ($\lambda = 0.05$) and EWMA ($\lambda = 0.10$) charts, respectively, for small shifts ($\delta \leq 0.50$), while for the rest of the shifts, the competing charts perform similarly. Moreover, the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.27$) chart has significantly smaller ARL_1 values than the EWMA ($\lambda = 0.25$) chart for $\delta \leq 1.00$ and smaller MRL_1 values for $\delta \leq 0.75$. Finally, the Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.40$) chart is superior to the EWMA ($\lambda = 0.50$) chart for $\delta \leq 1.50$. To sum up, we conclude that the proposed Exp-EWMA chart is more effective than the EWMA chart for small shifts when both of them are designed to detect a small shift quickly, while its superiority increases as the value of shift, that we want the charts to detect optimally, increases.

Similar conclusions, as previously, are observed by comparing the Exp-EWMA and EEWMA charts. We conclude that the Exp-EWMA ($a = 0.5$, $c = 0.5$, $\lambda = 0.06$) and Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.14$) charts are superior to the EEWMA ($\psi_1 = 0.07$, $\psi_2 = 0.03$) and EEWMA ($\psi_1 = 0.12$, $\psi_2 = 0.03$), respectively, for $\delta \leq 0.50$, while for the rest range of shifts, the competing charts have similar OOC behaviour. Furthermore, the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.27$) chart outperforms the EEWMA ($\psi_1 = 0.30$, $\psi_2 = 0.10$) chart for $\delta \leq 1.25$, while the two charts perform similarly for the rest range of shifts. Finally, the ARL_1 and MRL_1 values of the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.40$) chart are significantly smaller than those of the EEWMA ($\psi_1 = 0.60$, $\psi_2 = 0.20$) chart.

With regards to the comparison study between the Exp-EWMA and EWMA (λ_∞) charts, we observe that the Exp-EWMA chart has smaller ARL_1 and MRL_1 values,

Table 9 IC run-length characteristics of the competing charts for the zero-state and steady-state cases

Chart	Zero-state							Steady-state						
	ARL_0	$SDRL_0$	P_5	P_{25}	P_{50}	P_{75}	P_{95}	ARL_0	$SDRL_0$	P_5	P_{25}	P_{50}	P_{75}	P_{95}
EWMA ($\lambda = 0.05$, $L = 2.613$)	500.32	485.16	39	155	352	691	1472	481.76	485.60	19	136	335	669	1446
Exp-EWMA ($a = 0.5$, $c = 0.5$, $\lambda = 0.06$, $H = 0.2986$)	500.58	484.72	40	154	352	689	1466	475.35	485.45	15	131	329	662	1434
EEWMA ($\psi_1 = 0.07$, $\psi_2 = 0.03$, $L = 2.701$)	500.15	482.10	41	156	352	690	1457	480.08	484.63	19	135	335	668	1438
EWMA ($\lambda = 0.10$, $L = 2.813$)	500.16	489.83	34	152	351	692	1473	491.77	493.91	23	140	342	681	1471
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.14$, $H = 0.4273$)	500.45	491.12	36	151	350	692	1479	488.56	492.86	20	138	339	679	1464
EEWMA ($\psi_1 = 0.12$, $\psi_2 = 0.03$, $L = 2.860$)	500.40	489.79	34	152	350	694	1472	490.36	490.86	23	141	341	680	1467
EWMA ($\lambda = 0.25$, $L = 2.999$)	499.87	497.27	30	147	347	691	1491	496.71	494.79	25	143	345	689	1479
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.27$, $H = 0.6440$)	500.37	496.65	30	148	349	693	1485	498.80	500.68	23	142	347	691	1493
EEWMA ($\psi_1 = 0.30$, $\psi_2 = 0.10$, $L = 3.0384$)	500.36	496.58	30	147	348	692	1491	497.25	495.98	25	143	345	692	1485
EWMA ($\lambda = 0.50$, $L = 3.071$)	499.19	499.38	28	145	346	688	1499	498.62	499.43	26	144	346	691	1491
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.40$, $H = 0.8195$)	500.34	501.22	27	144	346	693	1502	502.41	502.01	24	143	348	699	1502
EEWMA ($\psi_1 = 0.60$, $\psi_2 = 0.20$, $L = 3.085$)	500.17	497.97	27	146	346	695	1503	501.28	497.31	26	147	348	695	1494
CUSUM($k = 0.125$, $H = 13.135$)	500.66	459.59	61	174	360	680	1423	415.54	455.72	1	87	273	589	1316
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.02$, $H = 0.1112$)	500.50	460.11	63	174	360	679	1410	409.44	453.41	1	81	267	582	1316
CUSUM($k = 0.5$, $H = 5.067$)	500.30	494.23	32	151	349	688	1488	492.53	495.90	22	138	340	684	1478
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.21$, $H = 0.5512$)	500.59	495.27	32	149	348	693	1485	495.15	495.61	22	141	345	687	1483
CUSUM($k = 1$, $H = 2.665$)	500.18	494.94	28	147	348	696	1498	496.64	496.47	26	143	345	688	1490
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.25$, $H = 0.6141$)	500.38	495.97	30	148	349	694	1481	497.68	499.11	23	141	346	690	1488

Table 10 Zero-state ARL and MRL (in the parenthesis) comparison between EWMA, EEWMA and Exp-EWMA charts

		EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA
	λ	0.05	0.06	0.030	–	0.10	0.14	0.052	–	0.25	0.27	0.10	–	0.50	0.40	0.147	–
	ψ_1	–	–	–	0.07	–	–	–	0.12	–	–	–	0.30	–	–	–	0.60
	ψ_2	–	–	–	0.03	–	–	–	0.03	–	–	–	0.10	–	–	–	0.20
	L	2.613	–	2.428	2.701	2.813	–	2.622	2.860	2.999	–	2.813	3.0384	3.071	–	2.901	3.085
	a	–	0.5	–	–	–	0.75	–	–	–	0.75	–	–	–	0.75	–	–
	c	–	0.5	–	–	–	0	–	–	–	0	–	–	–	0	–	–
δ	H	–	0.2986	–	–	–	0.4273	–	–	–	0.6440	–	–	–	0.8195	–	–
0.00		500.32	500.58	499.68	500.15	500.16	500.45	499.98	500.40	499.87	500.37	500.16	500.36	499.19	500.34	499.64	500.17
		(352)	(352)	(354)	(352)	(351)	(350)	(352)	(350)	(347)	(349)	(351)	(348)	(346)	(346)	(348)	(346)
0.05		412.58	392.06	393.23	406.99	438.58	413.69	413.42	435.27	468.51	437.92	438.58	466.14	483.64	450.95	449.44	485.05
		(293)	(277)	(281)	(289)	(309)	(292)	(294)	(308)	(327)	(307)	(309)	(325)	(337)	(313)	(315)	(337)
0.10		273.99	243.18	245.20	263.62	320.15	274.84	275.61	315.83	390.74	318.29	320.15	386.25	435.98	347.29	348.33	444.16
		(195)	(175)	(178)	(189)	(227)	(195)	(197)	(223)	(273)	(223)	(227)	(272)	(304)	(241)	(245)	(309)
0.25		84.09	74.35	76.59	81.43	106.47	83.14	84.59	103.52	169.96	104.29	106.47	166.63	254.18	125.38	127.28	275.14
		(64)	(59)	(61)	(63)	(77)	(63)	(64)	(76)	(120)	(75)	(77)	(118)	(177)	(88)	(91)	(191)
0.50		28.76	27.61	29.32	29.22	31.37	27.63	28.76	31.09	48.26	30.05	31.37	47.42	89.24	34.17	35.69	101.68
		(25)	(24)	(26)	(26)	(25)	(24)	(25)	(25)	(35)	(24)	(25)	(35)	(62)	(26)	(27)	(71)
0.75		16.34	16.14	17.63	16.97	15.82	15.35	16.27	15.94	20.12	14.83	15.82	20.22	35.81	15.47	16.54	41.98
		(15)	(15)	(16)	(16)	(14)	(14)	(15)	(14)	(15)	(13)	(14)	(16)	(25)	(13)	(14)	(30)
1.00		11.37	11.30	12.62	11.88	10.33	10.50	11.31	10.50	11.14	9.51	10.33	11.48	17.48	9.37	10.22	20.46
		(11)	(11)	(12)	(11)	(9)	(10)	(11)	(10)	(9)	(9)	(9)	(10)	(13)	(8)	(9)	(15)
1.25		8.73	8.64	9.84	9.11	7.66	7.95	8.67	7.79	7.41	6.94	7.66	7.71	10.01	6.59	7.32	11.61
		(8)	(8)	(9)	(9)	(7)	(7)	(8)	(7)	(6)	(6)	(7)	(7)	(8)	(6)	(7)	(9)

Table 10 continued

δ		EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA
	λ	0.05	0.06	0.030	–	0.10	0.14	0.052	–	0.25	0.27	0.10	–	0.50	0.40	0.147	–
	ψ_1	–	–	–	0.07	–	–	–	0.12	–	–	–	0.30	–	–	–	0.60
	ψ_2	–	–	–	0.03	–	–	–	0.03	–	–	–	0.10	–	–	–	0.20
	L	2.613	–	2.428	2.701	2.813	–	2.622	2.860	2.999	–	2.813	3.0384	3.071	–	2.901	3.085
	a	–	0.5	–	–	–	0.75	–	–	–	0.75	–	–	–	0.75	–	–
	c	–	0.5	–	–	–	0	–	–	–	0	–	–	–	0	–	–
	H	–	0.2986	–	–	–	0.4273	–	–	–	0.6440	–	–	–	0.8195	–	–
1.50		7.12	6.99	8.09	7.37	6.09	6.39	7.07	6.19	5.48	5.45	6.09	5.73	6.55	5.06	5.70	7.48
		(7)	(7)	(8)	(7)	(6)	(6)	(7)	(6)	(5)	(5)	(6)	(5)	(5)	(5)	(5)	(6)
2.00		5.23	5.06	6.01	5.29	4.37	4.60	5.19	4.39	3.64	3.83	4.37	3.74	3.65	3.46	4.00	4.00
		(5)	(5)	(6)	(5)	(4)	(4)	(5)	(4)	(3)	(4)	(4)	(4)	(3)	(3)	(4)	(3)
3.00		3.50	3.28	4.05	3.33	2.87	2.99	3.47	2.78	2.26	2.45	2.87	2.14	1.92	2.18	2.57	1.89
		(3)	(3)	(4)	(3)	(3)	(3)	(3)	(3)	(2)	(2)	(3)	(2)	(2)	(2)	(3)	(2)

especially for moderate to large shifts ($\delta \geq 0.50$), while it performs a slightly better for small shifts.

Table 11 presents the ARL and MRL values of the competing EWMA, Exp-EWMA and EEWMA charts for the steady-state case. The results of the comparison study between the Exp-EWMA and EWMA charts under similar IC run-length characteristics indicates that the Exp-EWMA ($a = 0.5, c = 0.5, \lambda = 0.06$) and Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.14$) charts are more effective than the EWMA ($\lambda = 0.05$) and EWMA ($\lambda = 0.10$) charts for $\delta \leq 0.25$ and $\delta \leq 0.50$ respectively, and vice versa for the rest of the shifts. Furthermore, the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.27$) and Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.40$) charts have better detection ability than the EWMA ($\lambda = 0.25$) and EWMA ($\lambda = 0.50$) charts for $\delta \leq 1.00$ and $\delta \leq 1.50$ respectively, while the EWMA charts perform better for larger shifts.

With regards to the results about the comparison between Exp-EWMA and EWMA (λ_∞) charts, it is seen that the two charts perform similarly for small to large shifts ($\delta \geq 0.25$). There are very small differences between the ARL_1 (and MRL_1) values due to the smaller ARL_0 (and MRL_0) of the EWMA charts.

Comparing the Exp-EWMA and EEWMA charts for the steady-state case, we observe that the Exp-EWMA ($a = 0.5, c = 0.5, \lambda = 0.06$) and Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.14$) charts are more sensitive than the EEWMA ($\psi_1 = 0.07, \psi_2 = 0.03$) and EEWMA ($\psi_1 = 0.12, \psi_2 = 0.03$) charts for $\delta \leq 0.25$ and $\delta \leq 0.50$ respectively, and vice versa for the rest of the shifts. Moreover, the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.27$) and Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.40$) charts are more effective than the EEWMA ($\psi_1 = 0.30, \psi_2 = 0.10$) and EEWMA ($\psi_1 = 0.60, \psi_2 = 0.20$) charts for $\delta \leq 1.00$ and $\delta \leq 1.50$ respectively, while the EEWMA charts perform better for larger shifts.

As a general conclusion from the above comparison study, the proposed Exp-EWMA chart is more effective than the EWMA and EEWMA charts for small to moderate shifts for both zero-state and steady-state cases when the charts are designed under similar IC run-length characteristics. Furthermore, comparing the proposed chart with the EWMA (λ_∞) chart, the first one is more sensitive for moderate to large shifts under the zero-state case, while the two charts have similar OOC performance under the steady-state case.

Table 12 shows the zero-state ARL and MRL values of the CUSUM and Exp-EWMA charts. It is seen that the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.02$) chart is more sensitive than the CUSUM($k = 0.125$) chart for $\delta \leq 0.25$, while both of them have the same detection ability for moderate and large shifts. Similar to the previous comparison study, the superiority of the Exp-EWMA chart versus the CUSUM chart increases as the value of δ^* increases. For example, the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.21$) chart is more effective than the CUSUM($k = 0.5$) chart for $\delta \leq 1.00$, while for the rest of the shifts, the competing charts perform similarly. Moreover, the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.25$) chart performs better than the CUSUM($k = 1$) chart for $\delta \leq 1.25$ and vice versa for $\delta \geq 2.00$.

The steady-state ARL and MRL values of the competing CUSUM and Exp-EWMA charts are presented in Table 13. From this table, we observe that the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.02$) chart performs better than the CUSUM($k = 0.125$) chart only for very small shifts ($\delta \leq 0.10$), while for the rest of the shifts, the CUSUM chart

Table 11 Steady-state ARL and MRL (in the parenthesis) comparison between EWMA and Exp-EWMA charts

		EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA
	λ	0.05	0.06	0.030	–	0.10	0.14	0.052	–	0.25	0.27	0.10	–	0.50	0.40	0.147	–
	ψ_1	–	–	–	0.07	–	–	–	0.12	–	–	–	0.30	–	–	–	0.60
	ψ_2	–	–	–	0.03	–	–	–	0.03	–	–	–	0.10	–	–	–	0.20
	L	2.613	–	2.428	2.701	2.813	–	2.622	2.860	2.999	–	2.813	3.0384	3.071	–	2.901	3.085
	a	–	0.5	–	–	–	0.75	–	–	–	0.75	–	–	–	0.75	–	–
	c	–	0.5	–	–	–	0	–	–	–	0	–	–	–	0	–	–
δ	H	–	0.2986	–	–	–	0.4273	–	–	–	0.6440	–	–	–	0.8195	–	–
0.00		481.76	475.35	466.17	480.08	491.77	488.56	481.93	490.36	496.71	498.80	491.77	497.25	498.62	502.41	494.24	501.28
		(335)	(329)	(322)	(335)	(342)	(339)	(335)	(341)	(345)	(347)	(342)	(345)	(346)	(348)	(343)	(348)
0.05		395.33	374.65	368.40	387.72	432.02	401.78	396.35	426.33	465.17	437.32	432.02	461.77	483.29	453.06	445.56	487.12
		(276)	(259)	(255)	(269)	(301)	(280)	(277)	(298)	(323)	(304)	(301)	(322)	(335)	(315)	(310)	(338)
0.10		261.66	232.82	229.86	250.12	313.49	266.40	263.29	307.27	387.08	316.60	313.49	383.77	434.77	349.08	343.22	445.08
		(183)	(166)	(164)	(176)	(219)	(187)	(185)	(215)	(269)	(221)	(219)	(268)	(302)	(244)	(241)	(308)
0.25		80.62	72.92	72.37	77.98	104.09	81.73	81.18	101.00	169.44	104.62	104.09	165.08	253.45	126.08	124.52	273.60
		(62)	(59)	(59)	(64)	(75)	(62)	(62)	(73)	(118)	(75)	(75)	(116)	(177)	(90)	(89)	(191)
0.50		27.68	28.23	28.09	28.16	30.41	27.82	27.71	30.20	47.93	30.50	30.41	47.07	88.56	35.12	34.86	101.32
		(24)	(26)	(26)	(25)	(24)	(24)	(24)	(24)	(35)	(24)	(24)	(34)	(62)	(27)	(27)	(71)
0.75		15.93	17.16	17.10	16.58	15.44	15.91	15.86	15.60	19.85	15.48	15.44	19.90	37.77	16.23	16.14	41.95
		(15)	(16)	(16)	(15)	(13)	(15)	(15)	(14)	(15)	(14)	(13)	(16)	(26)	(14)	(13)	(30)
1.00		11.11	12.34	12.30	11.66	10.09	11.08	11.05	10.28	10.97	10.13	10.09	11.32	17.35	10.05	10.02	20.44
		(11)	(12)	(12)	(11)	(9)	(11)	(10)	(9)	(9)	(9)	(9)	(9)	(13)	(9)	(9)	(15)
1.25		8.57	9.66	9.64	8.96	7.50	8.53	8.51	7.64	7.28	7.52	7.50	7.61	9.95	7.18	7.16	11.59
		(8)	(9)	(9)	(9)	(7)	(8)	(8)	(7)	(6)	(7)	(7)	(7)	(8)	(7)	(7)	(9)

Table 11 continued

		EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA	EWMA	Exp-EWMA	EWMA	EEWMA
	λ	0.05	0.06	0.030	–	0.10	0.14	0.052	–	0.25	0.27	0.10	–	0.50	0.40	0.147	–
	ψ_1	–	–	–	0.07	–	–	–	0.12	–	–	–	0.30	–	–	–	0.60
	ψ_2	–	–	–	0.03	–	–	–	0.03	–	–	–	0.10	–	–	–	0.20
	L	2.613	–	2.428	2.701	2.813	–	2.622	2.860	2.999	–	2.813	3.0384	3.071	–	2.901	3.085
	a	–	0.5	–	–	–	0.75	–	–	–	0.75	–	–	–	0.75	–	–
	c	–	0.5	–	–	–	0	–	–	–	0	–	–	–	0	–	–
δ	H	–	0.2986	–	–	–	0.4273	–	–	–	0.6440	–	–	–	0.8195	–	–
1.50		7.00	7.97	7.95	7.26	5.99	6.97	6.95	6.09	5.39	6.01	5.99	5.65	6.49	5.61	5.60	7.47
		(7)	(8)	(8)	(7)	(6)	(7)	(7)	(6)	(5)	(6)	(6)	(5)	(5)	(5)	(5)	(6)
2.00		5.17	5.95	5.93	5.25	4.31	5.14	5.13	4.33	3.57	4.32	4.31	3.69	3.59	3.94	3.93	3.96
		(5)	(6)	(6)	(5)	(4)	(5)	(5)	(4)	(3)	(4)	(4)	(3)	(3)	(4)	(4)	(3)
3.00		3.47	4.03	4.02	3.32	2.84	3.45	3.44	2.77	2.22	2.86	2.84	2.15	1.91	2.56	2.56	1.89
		(3)	(4)	(4)	(3)	(3)	(3)	(3)	(3)	(2)	(3)	(3)	(2)	(2)	(3)	(3)	(2)

Table 12 Zero-state ARL and MRL (in the parenthesis) comparison between CUSUM and Exp-EWMA charts

		CUSUM	Exp-EWMA	CUSUM	Exp-EWMA	CUSUM	Exp-EWMA
	a	–	0.75	–	0.75	–	0.75
	c	–	0	–	0	–	0
	λ	–	0.02	–	0.21	–	0.25
	k	0.125	–	0.5	–	1	–
δ	H	13.135	0.1112	5.067	0.5512	2.665	0.6141
0.00		500.66	500.50	500.30	500.59	500.18	500.38
		(360)	(360)	(349)	(348)	(348)	(349)
0.05		401.56	362.27	459.17	428.72	482.66	435.08
		(292)	(269)	(322)	(301)	(338)	(305)
0.10		253.27	212.78	369.64	299.61	434.31	311.66
		(189)	(166)	(261)	(211)	(304)	(219)
0.25		83.34	75.38	145.29	94.09	249.73	101.01
		(70)	(66)	(103)	(211)	(175)	(73)
0.50		34.63	34.07	38.92	28.61	81.63	29.51
		(32)	(32)	(29)	(23)	(57)	(24)
0.75		21.66	21.72	17.29	14.83	30.74	14.81
		(21)	(21)	(14)	(13)	(22)	(13)
1.00		15.77	15.89	10.52	9.79	14.65	9.58
		(15)	(15)	(9)	(9)	(11)	(9)
1.25		12.41	12.52	7.50	7.27	8.55	7.04
		(12)	(12)	(7)	(7)	(7)	(6)
1.50		10.25	10.32	5.83	5.77	5.76	5.55
		(10)	(10)	(5)	(5)	(5)	(5)
2.00		7.64	7.64	4.07	4.10	3.43	3.91
		(7)	(7)	(4)	(4)	(3)	(4)
3.00		5.14	5.06	2.60	2.64	1.94	2.50
		(5)	(5)	(3)	(3)	(2)	(2)

is more effective. Moreover, the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.21$) chart is more sensitive than the CUSUM($k = 0.5$) chart for $\delta \leq 0.75$, while the latter chart has a slightly better OOC performance for the rest of the shifts. Finally, the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.25$) chart outperforms the CUSUM($k = 1$) chart for $\delta \leq 1.25$ and vice versa for larger shifts.

7 IC robustness

According to Human et al. (2011), the IC robustness of a control chart is a very important issue for its proper design implementation. A control chart is considered to be IC robust if its run-length distribution remains unchanged or nearly unchanged

Table 13 Steady-state ARL and MRL (in the parenthesis) comparison between CUSUM and Exp-EWMA charts

		CUSUM	Exp-EWMA	CUSUM	Exp-EWMA	CUSUM	Exp-EWMA
	a	–	0.75	–	0.75	–	0.75
	c	–	0	–	0	–	0
	λ	–	0.02	–	0.21	–	0.25
	k	0.125	–	0.5	–	1	–
δ	H	13.135	0.1112	5.067	0.5512	2.665	0.6141
0.00		415.54	409.44	492.53	495.15	496.64	497.68
		(273)	(267)	(340)	(345)	(345)	(346)
0.05		330.30	300.48	451.55	423.75	479.23	433.19
		(220)	(204)	(313)	(296)	(331)	(302)
0.10		207.47	180.15	363.34	296.31	432.11	310.03
		(143)	(137)	(251)	(207)	(299)	(217)
0.25		65.20	68.05	141.37	93.86	246.56	101.00
		(53)	(61)	(99)	(69)	(173)	(73)
0.50		26.15	32.29	36.84	28.91	80.83	29.91
		(24)	(31)	(28)	(24)	(57)	(24)
0.75		16.19	21.16	16.15	15.45	30.51	15.45
		(16)	(21)	(13)	(14)	(22)	(14)
1.00		11.78	15.81	9.72	10.40	14.36	10.20
		(12)	(16)	(9)	(10)	(11)	(9)
1.25		9.28	12.68	6.89	7.85	8.38	7.61
		(9)	(13)	(6)	(7)	(7)	(7)
1.50		7.69	10.60	5.34	6.33	5.61	6.10
		(8)	(11)	(5)	(6)	(5)	(6)
2.00		5.77	8.04	3.72	4.61	3.31	4.41
		(6)	(8)	(4)	(4)	(3)	(4)
3.00		3.94	5.53	2.40	3.07	1.88	2.92
		(4)	(6)	(2)	(3)	(2)	(3)

when the assumption of the underlying distribution, which is usually normality, is violated (Rocke 1989).

As we considered in the previous sections, the proposed chart has been designed assuming that the underlying distribution is the normal one. In order to study the IC robustness of the Exp-EWMA chart to non-normality and compare it with those of EWMA, EEWMA and CUSUM charts, we consider the following distributions: (i) the Student t_ν distribution with degrees of freedom $\nu = 4$ and 8, (ii) the logistic $LG\left(0, \sqrt{3}/\pi\right)$ distribution, (iii) the double exponential of Laplace $L\left(0, 1/\sqrt{2}\right)$ distribution and (iv) the gamma $G(\alpha, \beta)$ distribution with $\alpha = 1, 2, 4$ and $\beta = 1$. We note that the Student and logistic distributions are symmetric with heavier tails than the normal distribution, while the Laplace distribution is symmetric with fatter tails than the normal. Moreover, the gamma distribution is positively skewed and was chosen to investigate the effect of skewness.

Table 14 presents the ARL_0 values of the control charts mentioned in the previous section for the above distributions. From this table, we conclude the following:

1. The proposed chart is more IC robust than the EWMA and EEWMA charts, while the EWMA chart has better IC robustness ability than the EEWMA chart.
2. The Exp-EWMA chart is more IC robust as the value of smoothing parameter λ decreases. For example, the ARL_0 values of the Exp-EWMA ($a = 0.5, c = 0.5, \lambda = 0.06$), Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.14$), Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.27$) and Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.40$) charts under the $LG\left(0, \sqrt{3}/\pi\right)$ distribution are 490.07, 472.41, 422.84 and 374.48, respectively. Similar findings can be seen for the EWMA and EEWMA charts.
3. In the case of $G(\alpha, \beta)$ distribution, the Exp-EWMA chart is more IC robust as the value of parameter α increases. For example, the ARL_0 values of the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.14$) chart under the $G(1, 1)$, $G(2, 1)$ and $G(4, 1)$ distributions are 460.31, 481.99 and 490.83, respectively. This happens because the gamma distribution approaches the normal one for large values of α.
4. Comparing the IC robustness of the Exp-EWMA and CUSUM charts, we observe that the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.02$) and CUSUM($k = 0.125$) charts have similar IC robustness abilities, while the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.21$) and Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.25$) charts are more IC robust than the CUSUM($k = 0.5$) and CUSUM($k = 1$) charts, respectively.

8 Illustrative Examples

In this section, we present two examples, one with simulated dataset and one with real dataset, in order to demonstrate the application of the proposed chart.

8.1 Simulated dataset

In this example, we consider 10 individual measurements from a $N(0, 1)$ distribution and 40 from a $N(0.25, 1)$ distribution. The IC values of the process mean and standard deviation are $\mu_0 = 0$ and $\sigma_0 = 1$. Thus, the process is IC for the first 10 observations and a shift of $\delta = 0.25$ in the process mean is added to the last 40 observations (steady-state case). Table 15 presents the dataset as X_t. Setting an $ARL_0 \approx 500$, we construct the EWMA ($\lambda = 0.05, L = 2.613$), the CUSUM($k = 0.125, H = 13.135$), the Exp-EWMA ($a = 0.5, c = 0.5, \lambda = 0.06, H = 0.2986$) and EEWMA ($\psi_1 = 0.07, \psi_2 = 0.03, L = 2.701$) charts. The EWMA, Exp-EWMA and EEWMA charts have similar IC run-length characteristics (see Table 9), while the specific CUSUM chart is optimally designed to detect a shift of $\delta = 0.25$. The charting statistics E_t, C_t^+ (only the upper statistic), Z_t and EE_t of the EWMA, CUSUM, Exp-EWMA and EEWMA charts are also provided in Table 15, where those that lie above the UCL are in bold print. The control charts are displayed in Fig. 2. The Exp-EWMA chart

Table 14 IC robustness of different control charts

	$N(0,1)$	t_4	t_8	$LG(0,\sqrt{3}/\pi)$	$L(0,1/\sqrt{2})$	$G(1,1)$	$G(2,1)$	$G(4,1)$
EWMA ($\lambda = 0.05$, $L = 2.613$)	500.32	432.86	469.54	474.06	444.78	465.75	484.26	493.33
Exp-EWMA ($a = 0.5$, $c = 0.5$, $\lambda = 0.06$, $H = 0.2986$)	500.58	482.35	487.57	490.07	482.47	512.76	503.90	502.32
EEWMA ($\psi_1 = 0.07$, $\psi_2 = 0.03$, $L = 2.701$)	500.15	299.04	399.21	407.05	327.82	326.20	384.56	431.41
EWMA ($\lambda = 0.10$, $L = 2.813$)	500.16	333.01	418.26	423.15	352.83	323.07	386.05	434.39
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.14$, $H = 0.4273$)	500.45	428.31	467.86	472.41	440.52	460.31	481.99	490.83
EEWMA ($\psi_1 = 0.12$, $\psi_2 = 0.03$, $L = 2.860$)	500.40	264.13	369.80	376.32	285.00	253.61	319.03	383.75
EWMA ($\lambda = 0.25$, $L = 2.999$)	499.87	187.29	293.29	296.13	197.52	153.13	203.38	269.38
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.27$, $H = 0.6440$)	500.37	331.99	417.70	422.84	351.69	321.85	385.23	433.70
EEWMA ($\psi_1 = 0.30$, $\psi_2 = 0.10$, $L = 3.0384$)	500.36	136.44	224.62	227.50	140.82	110.83	150.10	205.12
EWMA ($\lambda = 0.50$, $L = 3.071$)	499.19	113.65	185.80	188.51	112.76	85.58	115.73	159.90
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.40$, $H = 0.8195$)	500.34	265.90	368.85	374.78	282.40	235.63	300.90	368.14
EEWMA ($\psi_1 = 0.60$, $\psi_2 = 0.20$, $L = 3.085$)	500.17	93.76	154.05	154.90	90.32	70.02	93.42	130.69
CUSUM ($k = 0.125$, $H = 13.135$)	500.66	521.63	503.67	503.32	509.78	503.01	500.73	499.33
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.02$, $H = 0.1112$)	500.50	521.19	499.69	505.84	511.38	515.24	501.04	500.76
CUSUM ($k = 0.5$, $H = 5.067$)	500.30	263.32	372.18	381.55	291.90	193.48	254.80	326.85
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.21$, $H = 0.5512$)	500.59	371.51	439.73	444.78	390.03	379.49	430.73	464.61
CUSUM ($k = 1$, $H = 2.665$)	500.18	119.54	197.69	201.43	119.35	81.50	108.85	152.76
Exp-EWMA ($a = 0.75$, $c = 0$, $\lambda = 0.25$, $H = 0.6141$)	500.38	343.99	425.26	430.85	364.00	339.80	399.56	443.40

Table 15 Simulated dataset and charting statistics

t	X_t	E_t	C_t^+	Z_t	EE_t
1	0.502	0.025	0.377	0.021	0.035
2	−0.132	0.017	0.121	0.016	0.009
3	0.079	0.020	0.075	0.018	0.019
4	−0.887	−0.025	0.000	−0.012	−0.047
5	−0.117	−0.030	0.000	−0.016	−0.026
6	−0.319	−0.044	0.000	−0.025	−0.044
7	0.582	−0.013	0.457	−0.006	0.008
8	−0.715	−0.048	0.000	−0.028	−0.060
9	0.825	−0.004	0.700	−0.002	0.022
10	0.360	0.014	0.935	0.010	0.021
11	0.340	0.030	1.150	0.020	0.033
12	0.346	0.046	1.371	0.030	0.046
13	0.048	0.046	1.295	0.030	0.037
14	0.990	0.093	2.159	0.060	0.104
15	0.373	0.107	2.408	0.069	0.096
16	0.221	0.113	2.503	0.074	0.096
17	−0.139	0.100	2.240	0.067	0.076
18	0.761	0.133	2.876	0.089	0.131
19	−0.664	0.094	2.087	0.066	0.056
20	2.560	0.217	4.522	0.141	0.253
21	−0.188	0.197	4.209	0.131	0.153
22	1.014	0.238	5.098	0.158	0.223
23	0.512	0.251	5.485	0.169	0.220
24	1.023	0.290	6.383	0.195	0.267
25	−0.564	0.247	5.694	0.172	0.186
26	−0.188	0.225	5.381	0.161	0.183
27	−0.470	0.191	4.785	0.142	0.148
28	0.481	0.205	5.141	0.152	0.190
29	−0.908	0.149	4.109	0.120	0.104
30	0.497	0.167	4.481	0.131	0.162
31	0.159	0.166	4.514	0.132	0.152
32	2.007	0.258	6.397	0.189	0.282
33	0.112	0.251	6.384	0.186	0.218
34	0.139	0.246	6.398	0.185	0.216
35	−0.440	0.211	5.833	0.166	0.172
36	0.028	0.202	5.736	0.162	0.180
37	0.433	0.214	6.044	0.170	0.203
38	0.667	0.236	6.586	0.185	0.228
39	1.315	0.290	7.777	0.219	0.291

Table 15 continued

t	X_t	E_t	C_t^+	Z_t	EE_t
40	1.220	0.337	8.872	0.249	0.325
41	0.148	0.327	8.895	0.246	0.286
42	1.653	0.394	10.423	0.288	0.386
43	−1.527	0.298	8.772	0.234	0.214
44	0.873	0.326	9.519	0.253	0.312
45	−0.272	0.296	9.122	0.237	0.255
46	1.572	0.360	10.569	0.277	0.363
47	−0.113	0.337	10.331	0.265	0.293
48	1.569	0.398	11.775	**0.304**	0.395
49	0.294	0.393	11.944	**0.304**	0.352
50	−1.629	0.292	10.190	0.246	0.215

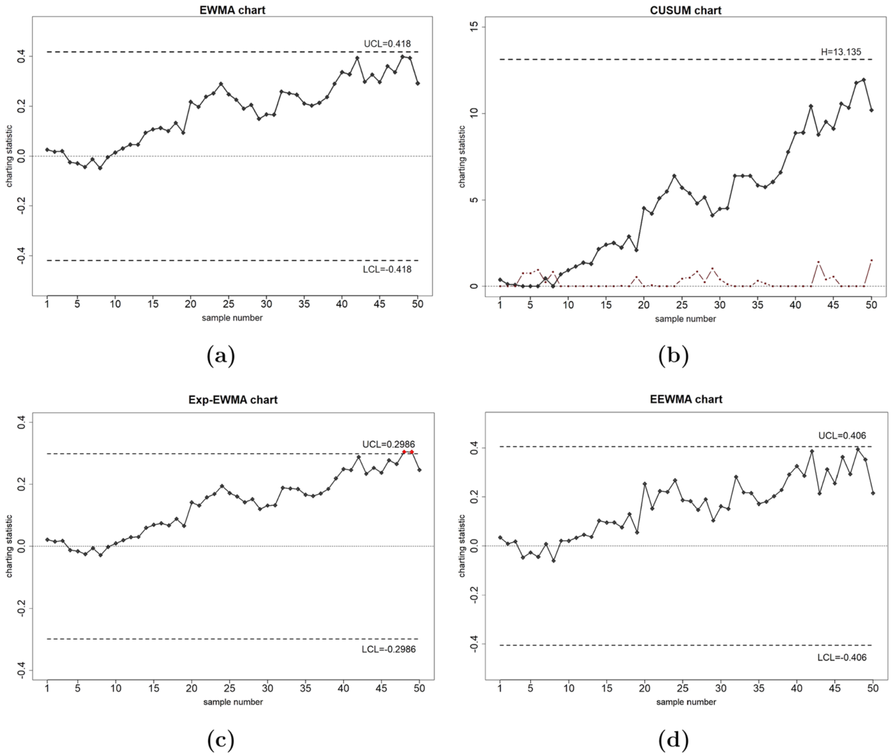

Fig. 2 Control charts for simulated dataset **a** EWMA **b** CUSUM **c** Exp-EWMA **d** EEWMA

gives OOC signals at the 48th and 49th charting statistics while the other charts cannot detect the shift.

8.2 Real dataset

In this example, we use a dataset, given by Montgomery (2013), about the inside diameter measurements (in mm) from automobile engine piston rings. The dataset is provided in Table 16 and consists of 40 samples each of size $n = 5$. The first 25 samples represent the Phase I process and are used to compute the IC process mean and standard deviation; those are $\mu_0 = 74.001$mm and $\sigma_0 = 0.009424$mm. The second 15 samples represent the Phase II process. Setting an $\text{ARL}_0 \approx 500$, we construct the EWMA ($\lambda = 0.50, L = 3.071$), the CUSUM($k = 1, h = 2.665$), the Exp-EWMA ($a = 0.75, c = 0, \lambda = 0.40, H = 0.8195$) and the EEWMA ($\psi_1 = 0.60$, $\psi_2 = 0.20$, $L = 3.085$) charts. The EWMA, Exp-EWMA and EEWMA charts have similar IC run-length characteristics, while the CUSUM chart is optimally designed to detect a shift of $\delta = 2.00$. The charting statistics of the control charts are also presented in Table 16, where the bold prints indicate those that lie above the UCL. The control charts are displayed in Fig. 3. We observe that all charts detect the shift after 35 samples. Thus, in this example, the proposed chart performs similarly to its competitors.

Conclusion

In this article, we extended the EWMA chart and proposed the exponentiated EWMA (Exp-EWMA) chart where its smoothing parameter is not constant, but changes with the sample number. Although the construction of the Exp-EWMA chart is more complicated than that of the traditional EWMA chart, the proposed chart is found more effective especially for more shifts and more IC robust for non-normal distributions.

The existing EWMA and PM charts are special cases of the new chart. Performing the Monte Carlo simulation approach, we computed the zero-state and steady-state run-length characteristics of the Exp-EWMA chart using the asymptotic control limits. A comparison study with the EWMA, EEWMA and CUSUM charts under similar IC run-length characteristics indicated that the proposed Exp-EWMA chart has better detection ability for small shifts when the competing charts are designed to detect a small shift quickly. The superiority of the Exp-EWMA chart versus its competitors increases as the amount of shift, that we want the charts to detect quickly, increases. Moreover, the Exp-EWMA chart is superior to the EWMA (λ_∞) chart for the zero-state case, while the two charts perform similarly for the steady-state case. Finally, we investigated the IC robustness of the proposed chart for several non-normal distributions and we showed that the Exp-EWMA chart has better IC robustness ability than its competitors.

The current paper may be extended to design the Exp-EWMA chart for monitoring changes in the process variance, as well as changes in both location and scale parameters. Moreover, the effect of parameters estimation on the performance of

Table 16 Piston ring diameter data and values of charting statistics

t	X_1	X_2	X_3	X_4	X_5	$\overline{X}_t$	E_t	C_t^+	Z_t	EE_t
1	74.030	74.002	74.019	73.992	74.008	74.010	74.006	0.005	74.003	74.006
2	73.995	73.992	74.001	74.011	74.004	74.001	74.003	0.001	74.002	74.002
3	73.988	74.024	74.021	74.005	74.002	74.008	74.006	0.003	74.003	74.006
4	74.002	73.996	73.993	74.015	74.009	74.003	74.004	0.001	74.003	74.004
5	73.992	74.007	74.015	73.989	74.014	74.003	74.000	0.000	74.003	74.004
6	74.009	73.994	73.997	73.985	73.993	73.996	74.000	0.000	74.002	73.999
7	73.995	74.006	73.994	74.000	74.005	74.000	73.998	0.000	74.002	74.000
8	73.985	74.003	73.993	74.015	73.988	73.997	74.001	0.000	74.001	73.998
9	74.008	73.995	74.009	74.005	74.004	74.004	74.000	0.000	74.002	74.002
10	73.998	74.000	73.990	74.007	73.995	73.998	73.997	0.000	74.001	73.999
11	73.994	73.998	73.994	73.995	73.990	73.994	73.999	0.000	74.000	73.996
12	74.004	74.000	74.007	74.000	73.996	74.001	73.998	0.000	74.000	74.000
13	73.983	74.002	73.998	73.997	74.012	73.998	73.994	0.000	74.000	73.998
14	74.006	73.967	73.994	74.000	73.984	73.990	74.000	0.000	73.998	73.993
15	74.012	74.014	73.998	73.999	74.007	74.006	73.999	0.001	73.999	74.002
16	74.000	73.984	74.005	73.998	73.996	73.997	74.000	0.000	73.999	73.998
17	73.994	74.012	73.986	74.005	74.007	74.001	74.000	0.000	73.999	74.000
18	74.006	74.010	74.018	74.003	74.000	74.007	74.003	0.002	74.001	74.004
19	73.984	74.002	74.003	74.005	73.997	73.998	74.001	0.000	74.000	74.000
20	74.000	74.010	74.013	74.020	74.003	74.009	74.005	0.004	74.001	74.006

Table 16 continued

t	X_1	X_2	X_3	X_4	X_5	$\overline{X}_t$	E_t	C_t^+	Z_t	EE_t
21	73.982	74.001	74.015	74.005	73.996	74.000	74.002	0.000	74.001	74.002
22	74.004	73.999	73.990	74.006	74.009	74.002	74.002	0.000	74.001	74.002
23	74.010	73.989	73.990	74.009	74.014	74.002	74.002	0.000	74.001	74.002
24	74.015	74.008	73.993	74.000	74.010	74.005	74.004	0.000	74.002	74.004
25	73.982	73.984	73.995	74.017	74.013	73.998	74.001	0.000	74.001	74.000
26	74.012	74.015	74.030	73.986	74.000	74.009	74.005	0.004	74.003	74.006
27	73.995	74.010	73.990	74.015	74.001	74.002	74.003	0.001	74.002	74.003
28	73.987	73.999	73.985	74.000	73.990	73.992	73.998	0.000	74.001	73.997
29	74.008	74.010	74.003	73.991	74.006	74.004	74.001	0.000	74.001	74.002
30	74.003	74.000	74.001	73.986	73.997	73.997	73.999	0.000	74.001	73.999
31	73.994	74.003	74.015	74.020	74.004	74.007	74.003	0.002	74.002	74.004
32	74.008	74.002	74.018	73.995	74.005	74.006	74.004	0.003	74.002	74.005
33	74.001	74.004	73.990	73.996	73.998	73.998	74.001	0.000	74.002	74.000
34	74.015	74.000	74.016	74.025	74.000	74.011	74.006	0.006	74.003	74.007
35	74.030	74.005	74.000	74.016	74.012	74.013	**74.010**	**0.014**	**74.005**	74.010
36	74.001	73.990	73.995	74.010	74.024	74.004	74.007	**0.012**	74.004	74.006
37	74.015	74.020	74.024	74.005	74.019	74.017	**74.012**	**0.024**	**74.006**	74.013
38	74.035	74.010	74.012	74.015	74.026	74.020	**74.016**	**0.039**	**74.008**	74.016
39	74.017	74.013	74.036	74.025	74.026	74.023	**74.019**	**0.057**	**74.010**	74.020
40	74.010	74.005	74.029	74.000	74.020	74.013	**74.016**	**0.064**	**74.011**	74.015

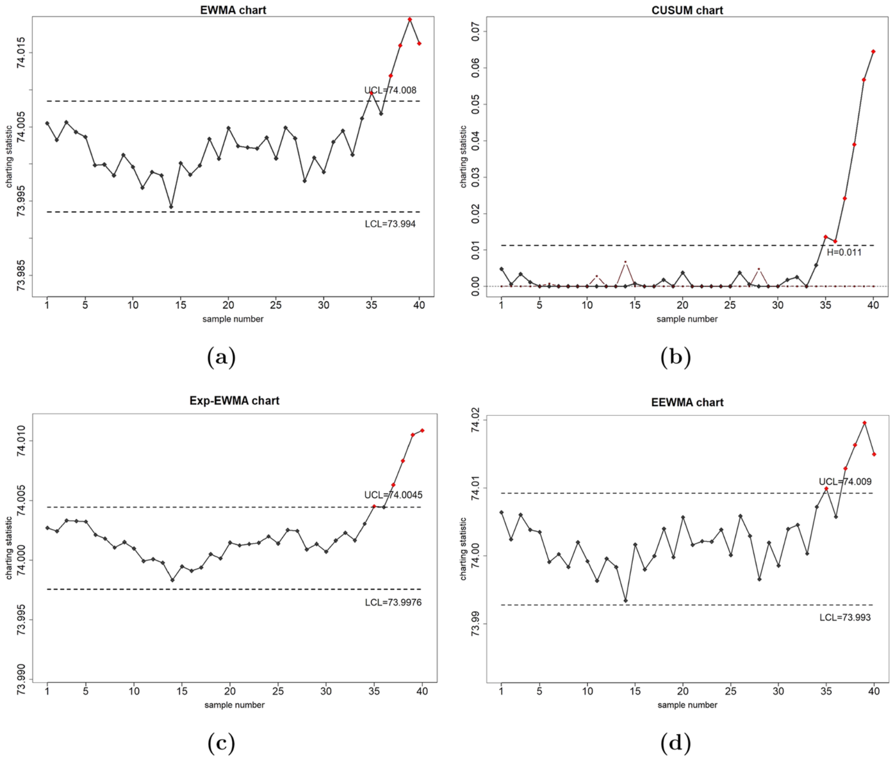

Fig. 3 Control charts for for the piston rings dataset **a** EWMA **b** CUSUM **c** Exp-EWMA **d** EEWMA

the Exp-EWMA chart is proposed as a future topic. Another interest topic of future research should be the investigation of optimally designed Exp-EWMA charts for specific shifts as there are three design parameters (λ, a, c).

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Appendix A

Proof of $E(Z_t|IC) = \mu_0$.

The charting statistic of the proposed Exp-EWMA chart is given by

$$Z_t = \lambda e^{-a^{\left(c+\frac{1}{t}\right)}}\overline{X}_t + \sum_{i=1}^{t-1} \lambda e^{-a^{\left(c+\frac{1}{i}\right)}} \left(\prod_{j=i}^{t-1}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j+1}\right)}}\right)\right)\overline{X}_i + \prod_{j=1}^{t}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j}\right)}}\right)\mu_0$$

Moreover, it is to noted that $E(\overline{X}_t|IC) = \mu_0$. Therefore,

$$\begin{aligned}
E(Z_t|IC) &= \lambda e^{-a^{\left(c+\frac{1}{t}\right)}} E(\overline{X}_t) + \sum_{i=1}^{t-1} \lambda e^{-a^{\left(c+\frac{1}{i}\right)}} \left(\prod_{j=i}^{t-1}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j+1}\right)}}\right)\right) E(\overline{X}_i) \\
&\quad + \prod_{j=1}^{t}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j}\right)}}\right)\mu_0 \\
&= \left[\lambda e^{-a^{\left(c+\frac{1}{t}\right)}} + \sum_{i=1}^{t-1} \lambda e^{-a^{\left(c+\frac{1}{i}\right)}} \left(\prod_{j=i}^{t-1}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j+1}\right)}}\right)\right)\right. \\
&\quad \left. + \prod_{j=1}^{t}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j}\right)}}\right)\right]\mu_0 \\
&= \left[\lambda e^{-a^{\left(c+\frac{1}{t}\right)}} + \sum_{i=1}^{t-1}\left[\left(\prod_{j=i+1}^{t}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j}\right)}}\right)\right) - \left(\prod_{j=i}^{t}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j}\right)}}\right)\right)\right]\right. \\
&\quad \left. + \prod_{j=1}^{t}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j}\right)}}\right)\right]\mu_0 = \mu_0.
\end{aligned}$$

Hence proved.

Appendix B

Convergence of $\lim_{t\to+\infty} Var(Z_t|IC)$.
The IC variance of the statistic Z_t is given by

$$Var(Z_t|IC) = \left[\left(\lambda e^{-a^{\left(c+\frac{1}{t}\right)}}\right)^2 + \sum_{i=1}^{t-1}\left\{\lambda e^{-a^{\left(c+\frac{1}{i}\right)}} \left(\prod_{j=i}^{t-1}\left(1-\lambda e^{-a^{\left(c+\frac{1}{j+1}\right)}}\right)\right)\right\}^2\right]\frac{\sigma_0^2}{n},$$

where $0 < \lambda \leq 1$ and $a, c \geq 0$ We note that

$$a^c \leq a^{(c+\frac{1}{j})} \leq a^{c+1}, \quad \text{if } a > 1,$$
$$a^{c+1} \leq a^{(c+\frac{1}{j})} \leq a^c, \text{ if } 0 < a < 1.$$

This implies

$$m_1 \leq e^{-a^{(c+\frac{1}{j})}} \leq M_1, \quad \text{if } a > 1,$$
$$M_1 \leq e^{-a^{(c+\frac{1}{j})}} \leq m_1, \text{ if } 0 < a < 1,$$

where $m_1 = e^{-a^c}$ and $M_1 = e^{-a^{c+1}}$.
For $a > 1$, we have

$$\frac{n}{\sigma_0^2} Var(Z_t|IC) \leq \lambda^2 M_1^2 + \sum_{i=1}^{t-1} \lambda^2 M_1^2 (1 - \lambda m_1)^{2(t-i)} = \lambda^2 M_1^2 \sum_{i=0}^{t-1} (1 - \lambda m_1)^{2i}.$$

Therefore,

$$\frac{n}{\sigma_0^2} Var(Z_\infty|IC) \leq \lambda^2 M_1^2 \sum_{i=0}^{\infty} (1 - \lambda m_1)^{2i} = \frac{\lambda^2 M_1^2}{1 - (1 - \lambda m_1)^2}.$$

Moreover,

$$\frac{n}{\sigma_0^2} Var(Z_t|IC) \geq \lambda^2 m_1^2 + \sum_{i=1}^{t-1} \lambda^2 m_1^2 (1 - \lambda M_1)^{2(t-i)} = \lambda^2 m_1^2 \sum_{i=0}^{t-1} (1 - \lambda M_1)^{2i}.$$

Therefore,

$$\frac{n}{\sigma_0^2} Var(Z_\infty|IC) \geq \lambda^2 m_1^2 \sum_{i=0}^{\infty} (1 - \lambda M_1)^{2i} = \frac{\lambda^2 m_1^2}{1 - (1 - \lambda M_1)^2}.$$

Combining the above, we get

$$\frac{\lambda^2 m_1^2}{1 - (1 - \lambda M_1)^2} \leq \frac{n}{\sigma_0^2} Var(Z_\infty|IC) \leq \frac{\lambda^2 M_1^2}{1 - (1 - \lambda m_1)^2}.$$

Similarly, for $0 < a < 1$, we get

$$\frac{\lambda^2 M_1^2}{1 - (1 - \lambda m_1)^2} \leq \frac{n}{\sigma_0^2} Var(Z_\infty|IC) \leq \frac{\lambda^2 m_1^2}{1 - (1 - \lambda M_1)^2}.$$

Appendix C

Progressive mean as a special case of the Exp-EWMA chart

The charting statistic of the PM chart is denoted as

$$PM_t = \frac{\sum_{k=1}^{t} \overline{X}_k}{t},$$

and the typical L-sigma control limits are denoted as

$$UCL_t/LCL_t = \mu_0 \pm L\frac{\sigma_0}{\sqrt{t}}, CL = \mu_0,$$

Abbas et al. (2013) suggested the use of an arbitrary function $g(t) = t^{0.2}$ in the control limits, so that they are narrower for large values of t.

It is to be noted that for $f(t) = \ln(t)$, we have

$$e^{-f(t)} = \frac{1}{t}$$

and

$$e^{-f(i)} \prod_{j=i}^{t-1} \left(1 - e^{-f(j+1)}\right) = \frac{1}{i} \prod_{j=i}^{t-1} \left(1 - \frac{1}{j+1}\right) = \frac{1}{t}.$$

Therefore, we can write

$$Z_t = \frac{\sum_{k=1}^{t} \overline{X}_k}{t}$$

Thus, for $f(t) = \ln(t)$, the Exp-EWMA chart reduces to the PM chart.

References

Abbas N, Riaz M, Does RJMM (2013a) Mixed exponentially weighted moving average-cumulative sum charts for process monitoring. Qual Reliab Eng Int 29(3):345–56

Abbas N, Zafar RF, Riaz M, Hussain Z (2013b) Progressive mean control chart for monitoring process location parameter. Qual Reliab Eng Int 29(3):357–367

Abbas N, Riaz M, Does RJMM (2014) An EWMA-type control chart for monitoring the process mean using auxiliary information. Commun Stat 43(16):3485–3498

Capizzi G, Masarotto G (2003) An adaptive exponentially weighted moving average control chart. Technometrics 45(3):199–207

Chakraborti S (2007) Run length distribution and percentiles: the Shewhart chart with unknown parameters. Qual Eng 19(2):119–127

Chan LK, Zhang J (2000) Some issues in the design of EWMA charts. Commun Stat 29(1):201–217

Chandrasekaran S, English JR, Disney RL (1995) Modeling and analysis of EWMA control schemes with variance-adjusted control limits. IIE Trans 27(3):282–290

Crowder SV (1987) A simple method for studying run - length distributions of exponentially weighted moving average charts. Technometrics 29(4):401–407
Crowder SV (1989) Design of exponentially weighted moving average schemes. J Qual Technol 21(3):155–162
Human SW, Kritzinger P, Chakraborti S (2011) Robustness of the EWMA control chart for individual observations. J Appl Stat 38(10):2071–2087
Khan N, Aslam M, Jun C-H (2017) Design of a control chart using a modified EWMA statistic. Qual Reliab Eng Int 33(5):1095–1104
Knoth S (2005) Fast initial response features for EWMA control charts. Stat Pap 46:47–64
Knoth S (2015) Run length quantiles of EWMA control charts monitoring normal mean or/and variance. Int J Prod Res 53(15):4629–4647
Lucas JM, Saccucci MS (1990) Exponentially weighted moving average control schemes: properties and enhancements. Technometrics 32(1):1–12
Malela-Majika J-C, Shongwe SC, Castagliola P, Mutambayi RM (2022) A novel single composite Shewhart-EWMA control chart for monitoring the process mean. Qual Reliab Eng Int 38(4):1760–1789
Montgomery DC (2013) Introduction to statistical quality control, 7th edn. Wiley, New York
Naveed M, Azam M, Khan N, Aslam M (2018) Design of a control chart using extended EWMA statistic. Technologies 6(4):108
Naveed M, Azam M, Khan N, Aslam M (2020) Designing a control chart of extended EWMA statistic based on multiple dependent state sampling. J Appl Stat 47(8):1482–1492
Naveed M, Azam M, Khan N, Aslam M, Albassam M (2021) Designing of control chart of extended EWMA statistic using repetitive sampling scheme. Ain Shams Eng J 12(1):1049–1058
Nawaz MS, Azam M, Aslam M (2021) EWMA and DEWMA repetitive control charts under non-normal processes. J Appl Stat 48(1):4–40
Page ES (1954) Continuous inspection schemes. Biometrika 41(1/2):100–115
Patel AK, Divecha J (2011) Modified exponentially weighted moving average (EWMA) control chart for an analytical process data. J Chem Eng 2:12–20
Qiao YL, Sun JS, Castagliola P, Hu XL (2022) Optimal design of one-sided exponential EWMA charts based on median run length and expected median run length. Commun Stat 51(9):2887–2907
Roberts SW (1959) Control chart tests based on geometric moving averages. Technometrics 1(3):239–250
Rocke DM (1989) Robust control charts. Technometrics 31(2):173–184
Saccucci MS, Lucas JM (1990) Average run lengths for exponentially weighted moving average control schemes using the Markov chain approach. J Qual Technol 22(2):154–162
Shamma SE, Shamma AK (1992) Development and evaluation of control charts using double exponentially weighted moving averages. Int J Qual Reliab Manag 9(6):18–25
Sheu SH, Lin TC (2003) The generally weighted moving average control chart for detecting small shifts in the process mean. Qual Eng 16(2):209–231
Shewhart WA (1926) Quality control charts. Bell Syst Tech J 5(4):593–603
Steiner SH (1999) EWMA control charts with time-varying control limits and fast initial response. J Qual Technol 31(1):75–86
Teoh WL, Chong JK, Khoo MBC, Castagliola P, Yeong WC (2017) Optimal designs of the variable sample size $\overline{X}$ chart based on median run length and expected median run length. Qual Reliab Eng Int 33(1):121–34