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REGULAR ARTICLE

Some practical and theoretical issues related to the quantile estimators

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Abstract
The paper contains the comparative analysis of the efficiency of different qunatile estimators for various distributions. Additionally, we show strong consistency of different quantile estimators and we study the Bahadur representation for each of the quantile estimators, when the sample is taken from NA, φ, ρ^*, ρ-mixing population.

Keywords Quantile · Estimator · Bahadur representation · Dependent mixing random variables

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a distribution function F. The p-th quantile of F is defined as

$$Q_p = \inf\{x : F(x) \geq p\},$$

where $0 < p < 1$.

Quantiles play an important role in finance, modeling and statistics. In practical applications, quantile estimators are fundamental, which was noticed quite early, e.g. at the paper Galton (1889).

In the literature, there are numerous quantiles estimators. Some quantile estimators were developed for specific distributions, whereas others were designed to be "distribution-free" (in other words—nonparametric estimators), with no assumption about the population density function.

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Let $(X_{(1)}, X_{(2)}, ..., X_{(n)})$ be the ordered sample of $(X_1, ..., X_n)$ and $\lfloor x \rfloor$ denotes an integer part of x. Dielman et al. (1994) presented eight different nonparametric estimators of a quantile:

- Weighted average at $X_{(\lfloor np+0.5 \rfloor)}$

$$E_1 = (0.5 + \lfloor np + 0.5 \rfloor - np) X_{(\lfloor np+0.5 \rfloor)} + (0.5 - \lfloor np + 0.5 \rfloor + np) X_{(\lfloor np+0.5 \rfloor + 1)},$$

where $0.5 \leq np \leq (n - 0.5)$.

Remark 1 It follows from the condition $0.5 \leq np \leq (n - 0.5)$ that $n \geq \max \left\{ \frac{0.5}{1-p}, \frac{1}{2p} \right\}$.

- Weighted average at $X_{(\lfloor np \rfloor)}$

$$E_2 = (1 - (np - \lfloor np \rfloor)) X_{(\lfloor np \rfloor)} + (np - \lfloor np \rfloor) X_{(\lfloor np \rfloor + 1)}.$$

- Lower empirical cumulative distribution function (CDF) value

$$E_3 = X_{(\lfloor np \rfloor)}.$$

- Upper empirical cumulative distribution function (CDF) value

$$E_4 = X_{(\lfloor np \rfloor + 1)}.$$

- Observation numbered closest to np

$$E_5 = \begin{cases} X_{(\lfloor np \rfloor)}, & \text{if } np - \lfloor np \rfloor < 0.5 \\ X_{(\lfloor np \rfloor + 1)}, & \text{if } np - \lfloor np \rfloor \geq 0.5 \end{cases}.$$

- Empirical cumulative distribution function (CDF)

$$E_6 = \begin{cases} X_{(\lfloor np \rfloor)}, & \text{if } np - \lfloor np \rfloor = 0 \\ X_{(\lfloor np \rfloor + 1)}, & \text{if } np - \lfloor np \rfloor > 0 \end{cases}.$$

- Weighted average at $X_{(\lfloor (n+1)p \rfloor)}$

$$E_7 = (1 - ((n+1)p - \lfloor (n+1)p \rfloor)) X_{(\lfloor (n+1)p \rfloor)} + ((n+1)p - \lfloor (n+1)p \rfloor) X_{(\lfloor (n+1)p \rfloor + 1)}.$$

- Empirical cumulative distribution function (CDF) with averaging

$$E_8 = \begin{cases} \frac{X_{(\lfloor np \rfloor)} + X_{(\lfloor np \rfloor + 1)}}{2}, & \text{if } np - \lfloor np \rfloor = 0 \\ X_{(\lfloor np \rfloor + 1)}, & \text{if } np - \lfloor np \rfloor > 0 \end{cases}.$$

The most common estimator studied in many papers is E_4. Bahadur (1966) first established an elegant representation for a sample quantile in terms of the empirical distribution function based on independent and identically distributed samples.

Let $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I[X_i \leq x], x \in \mathbb{R}, n \geq 1$ be the empirical distribution function for the sample $(X_1, X_2, \ldots, X_n)$.

Theorem 1 (Bahadur (1966)) *Let* $0 < p < 1$ *and* $\{X_n, n \geq 1\}$ *be a sequence of independent identically distributed random variables with the distribution function* F. *Assume that* F *has at least two derivatives at some neighborhood of* Q_p *and* $F'(Q_p) = f(Q_p) > 0$. *Then*

$$E_4 = Q_p - \frac{F_n(Q_p) - p}{f(Q_p)} + O\left(n^{-\frac{3}{4}} \log n\right) \quad a.s. \tag{1}$$

In next years, many researchers have studied the Bahadur representation for sample quantiles for dependent sequences. This is very important problem for practical applications, because in practice we often deal with samples with different dependency structures. Sen (1972); Babu and Singh (1978); Yoshihara (1995); Yang et al. (2019) and Wu et al. (2021) obtained the Bahadur represenation for φ-mixing sequences, Sun (2006); Wang et al. (2011) and Zhang et al. (2014) got the Bahadur representation for α-mixing sequences and Xing and Yang (2019) for ψ-mixing sequences. For negatively dependent structures Ling (2008); Xing and Yang (2011) and Xu et al. (2013) considered this problem for negatively associated (NA) sequences and Li et al. (2011) studied it for negatively orthant dependent (NOD) random variables.

Below, we present the definitions of four types of dependence of random variables that will be considered in this work.

Definition 1 (Joag-Dev and Proschan (1983)) A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, ..., n\}$, we have

$$\text{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0,$$

wherever f_1 and f_2 are coordinatewise nondecreasing, provided the covariance exists. An infinite family of random variables is said to be NA if every finite subfamily is NA.

Many authors have investigated NA's statistical properties. For example, Joag-Dev and Proschan (1983) studied NA's fundamental properties, Yang (2003) investigated uniformly asymptotic normality of regression weighted estimator for NA samples, Liang and Jing (2005) presented asymptotic properties of the estimation of nonparametric regression model based on NA sequences, Liang et al. (2006) studied asymptotic properties of the estimation of semiparametric regression model based on a linear process with NA innovations.

Remark 2 Increasing functions defined on disjoint subsets of a set of NA random variables are NA random variables.

Example 1 (Joag-Dev and Proschan (1983)) Let $\mathbf{Z} = (Z_1, ..., Z_k)$ be a vector having a multinomial distribution, obtained by taking only one observation. Thus only one Z_i is 1 while the rest are zero. The NA property for $\mathbf{Z}$ trivially follows from Definition 1. Since the general multinomial is a convolution of independent copies of $\mathbf{Z}$, the closure property (Remark 2) establishes NA in this case.

Definition 2 (Rozanov and Volkonski (1959)) A sequence of random variables $\{X_n, n \geq 1\}$ is said to be φ-mixing if

$$\varphi(n) = \sup_{m\geq 1} \sup_{A\in\mathcal{F}_1^m, B\in\mathcal{F}_{m+n}^\infty, P(A)>0} \big|P(B|A) - P(B)\big| \to 0,$$

as $n \to \infty$, where $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$.

φ-mixing property was studied by many researchers. Utev (1990) studied the central limit theorem, Chen et al. (2009) investigated total convergence of the sequences, Yang et al. (2012) obtained the Berry-Esseen bound. The more information on φ-mixing properties one can find in Billingsley (1968) in chapter 4.

Example 2 (Wu et al. (2021)) Let $\{\epsilon_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with zero mean and a finite variance. Define

$$X_n = \sum_{k=0}^{m} a_k \epsilon_{n-k}$$

for some positive integer m and constants a_k, $k = 0, 1, ..., m$. Then $\{X_n, n \geq 1\}$ is known as a moving average process with older m. It can be verified that $\{X_n, n \geq 1\}$ is a φ-mixing process.

Definition 3 (Bradley (1992)) A sequence of random variables $\{X_n, n \geq 1\}$ is said to be ρ^*-mixing, if

$$\rho^*(n) = \sup_{S,T\subset\mathbb{N}, \mathrm{dist}(S,T)\geq n} \{\rho(S, T)\} \to 0,$$

as $n \to \infty$, where

$$\rho(S, T) = \sup_{X\in L_2(\sigma(S)), Y\in L_2(\sigma(T))} \left\{ \frac{|Cov(X, Y)|}{\sqrt{Var(X)Var(Y)}} \right\},$$

$\mathrm{dist}(S, T) = \min_{i\in S, j\in T} |j - i|$ and $\sigma(S)$ and $\sigma(T)$ are the σ-fields generated by $\{X_i, i \in S\}$ and $\{X_j, j \in T\}$, respectively.

ρ^*-mixing random variables are a well-described and repeatedly studied structure. Bradley (1992) obtained the central limit theorem, Bryc and Smolenski (1993); Peligrad and Gut (1999); Utev and Peligrad (2003) presented the moment inequalities and Sung (2010) analysed the complete convergance of weighted sums for ρ^*-mixing sequences of random variables.

Remark 3 Note that increasing functions defined on a disjoint subset of a ρ^*-mixing field $\{X_k, k \in N^d\}$ with mixing coefficients $\rho^*(s)$ are also ρ^*-mixing with coefficients not greater that $\rho^*(s)$.

Example 3 (Wang et al. (2019)) Let $\{X_n, n \geq 1\}$ be a strictly stationary, finite-state, irreducible and aperiodic Markov chain. Then it is a ρ^*- mixing process with $\rho^*(k) = o(e^{-Ck})$ for some $C > 0$.

Definition 4 (Kolmogorov and Rozanov (1960)) A sequence of random variables $\{X_n, n \geq 1\}$ is said to be ρ-mixing if

$$\rho(n) = \sup_{k\geq 1, X\in L^2(\mathcal{F}_1^k), Y\in L^2(\mathcal{F}_{k+n}^\infty)} \frac{|\mathrm{Cov}(X,Y)|}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \to 0,$$

as $n \to \infty$, where $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$ and $L^2(\cdot)$ is a set of real-valued square-integrable functions.

The ρ-mixing condition was introduced by Kolmogorov and Rozanov (1960). Shao (1995) in his paper explored the central limit theorem, the law of large numbers and the complete convergence of ρ-mixing sequences.

Example 4 (Peligrad (1987)) Suppose $\{Y_k, k \geq 1\}$ and $\{Z_k, k \geq 1\}$ are independent random variables with the identical standard normal distribution function F and consider the sequence

$$X_k^{(\alpha)} = Y_k - Y_{k-1} + \sqrt{\alpha} Z_k.$$

Because of $\rho(2) = 0$, sequence $\{X_k^{(\alpha)}, k \geq 1\}$ is ρ-mixing with $\sigma_n^2 = \mathrm{Var}\left(\sum_{k=1}^n X_k^{(\alpha)}\right) = 2 + n\alpha$, $\inf_n \frac{\sigma_n^2}{n} = \alpha$ and $E(X_k^{(\alpha)})^2 = 2 + \alpha$.

The following Rosenthal-type inequality will be important in further considerations:

$$E \max_{1\leq m\leq n} \left| \sum_{k=1}^m X_k \right|^q \leq C_q \left\{ \sum_{k=1}^n E|X_k|^q + \left(\sum_{k=1}^n EX_k^2 \right)^{\frac{q}{2}} \right\}, \tag{2}$$

where $C_q > 0$, $q \geq 2$, $EX_n = 0$ and $E|X_n|^q < \infty$ for every $n \geq 1$.

Remark 4 Inequality (2) is true for NA random variables (Shao and Su (1999)), for ρ^*-mixing random variables (Peligrad and Gut (1999)) and for φ-mixing random variables with some additional condition on mixing coefficients $\varphi(n)$, $n \geq 1$ (Wu et al. (2021)).

For ρ-mixing random variables, the following inequality plays the same role as (2).

Lemma 1 *(Shao (1995)) Let $q \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of ρ-mixing random variables. Assume that $EX_i = 0$, $E|X_n|^q < \infty$ and*

$$\sum_{n=1}^{\infty} \rho^{\frac{2}{q}}(2^n) < \infty.$$

Then there exists a positive constant $K = K(q, \rho(\cdot))$ depending only on q and $\rho(\cdot)$ such that for any $k \geq 0$, $n \geq 1$,

$$E \max_{1 \leq n} \left| \sum_{i=k+1}^{k+n} X_i \right|^q \leq K \left\{ n \max_{k < i \leq k+n} E|X_i|^q + \left(n \max_{k < i \leq k+n} EX_i^2 \right)^{\frac{q}{2}} \right\}.$$

The first aim of this paper is a comparative analysis of the effectiveness of each of the estimators, depending on the distribution from which the sample is drawn. For this purpose, in section 2 we will compare the fit of the values of the estimators presented above obtained for samples taken from populations with different distributions to the theoretical value. The second aim is to show the strong consistency of each of the estimators and to study that the Bahadur representation holds for each of the estimators when the sample is taken from a NA, φ, ρ^* or ρ-mixing population.

2 Comparative analysis of estimators

In the comparative analysis of the estimators well-known probability distributions were used:

- Normal distribution with $\mu = 0, \sigma = 4$,
- Student's t distribution with 3 degrees of freedom,
- Weibull distribution with scale $\lambda = 1$ and shape $k = 5$,
- Uniform distribution on the interval [0,2],
- χ^2 distribution with 3 degrees of freedom,
- Exponential distribution with $\lambda = 1$.

The analysis was carried out for the sample sizes: $n \in \{50, 150, 555, 1130, 2165\}$ and for the different values of $p \in \{0.025, 0.5, 0.975\}$. Using the R software, in each case the teoretical value of the quantile was generated. Next, samples of sizes $n \in \{50, 150, 555, 1130, 2165\}$ were taken and the quantile estimator was calculated. The experiment was repeated 1000 times in each case and a mean square error (MSE) was generated.

In the normal distribution case, the best results were obtained for the estimators E_4, E_6 and E_8. The estimator E_1 was the only one that gave better results for small p.

Table 1 Estimator fit quality for $p = 0.025$

	Normal	Student's t	Weibull	Uniform	χ^2	Exponential
E_1	++	++	+++	++	++	+
E_2	+	+	+	+++	+++	++
E_3	−	−	−	+++	+	++
E_4	+++	+++	+++	+	−	−
E_5	−	−	−	+++	+	++
E_6	+++	+++	+++	+	−	+++
E_7	+	+	++	+++	+++	++
E_8	+++	+++	+++	+	−	−

For $p = 0.5$, the results for each estimator are similar. The results for the Student's t distribution are similar to the normal distribution. Again, the best results were obtained for the estimators E_4, E_6, E_8. The estimators E_2, E_3 and E_5 showed much worse results for small p than the others estimators and showed quite good results for high p. The results for $p = 0.5$ are, again, similar for each estimator. In the case of Weibull distribution, the errors are much smaller than in the normal and Student's t distribution. Only E_1 gives a better result for small p than for large p. As the sample grows, the differences in the errors for individual estimators are practically non-existent. For the uniform distribution the errors are also smaller than for the normal and Student's t distribution. In this case, the estimator E_7 gave the best results. Additionally, E_1, E_2, E_3 and E_5 gave a better result for small p than the others. For χ^2 distribution the best results gave E_2. Also good results were obtained for the estimators E_4, E_5, E_6 and E_8. Each of the estimators performs worse at high p than at low p. For the exponential distribution the greatest differences in the results are seen for high p. The best results are given by the estimator E_2, but the estimators E_4, E_5, E_6 and E_8 also give quite good results.

The Tables 1 and 2 contain an assessment of the fit of each estimator depending on the distribution for, respectively, $p = 0.025$ and $p = 0.975$. Within a given distribution, estimator with the best fit gets three pluses and estimator with the worst fit gets minus. The errors for a given distribution was analogous for all estimators for $p = 0.5$.

In conclusion, none of the estimators performed equally well for different distributions for low and high p. Different estimators turn out to be the best depending on a given distribution. Only for $p = 0.5$ the fit of each estimator is at a very similar level for a given distribution. Common conclusion for each estimator is that as the sample size increases, the error decreases and the results for each estimator are very similar. However, it should be noted that the financial and life situation does not always allow to obtain a sample of 1000 or 2000 elements. That is why it is so important to choose an appropriate estimator for a given distribution especially when the research is based on a small sample.

Table 2 Estimator fit quality for $p = 0.975$

	Normal	Student's t	Weibull	Uniform	χ^2	Exponential
E_1	–	+	–	+	–	–
E_2	++	+++	+++	++	+++	+++
E_3	+	++	+	–	+	+
E_4	+++	++	+++	++	++	++
E_5	+++	++	+++	++	++	++
E_6	+++	++	+++	++	++	++
E_7	+	–	++	+++	–	–
E_8	+++	++	+++	++	++	++

3 Strong consistency and Bahadur representation

In this section we will study a strong consistency of a quantile estimator and the Bahadur representation for sample quantiles. Further, detailed considerations will be carried out for the estimator E_1.

Theorem 2 *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with one of the following dependency structures:*

1. *NA dependence,*
2. *φ-mixing dependence with coefficients satifying $\sum_{n=1}^{\infty} \varphi(2^n) < \infty$,*
3. *ρ^*-mixing dependence,*
4. *ρ-mixing dependence with coefficients satifying $\sum_{n=1}^{\infty} \rho^{\frac{2}{q}}(2^n) < \infty$, for $q > \frac{1}{\delta}$ for $0 < \delta < \frac{1}{2}$.*

Let $\{X_n, n \geq 1\}$ be indentically distributed with a common distribution function F and a quantile Q_p. Assume that F possesses a positive continuous density f in some neighborhood $\mathfrak{D}_p$ of Q_p such that $0 < \sup\{f(x); x \in \mathfrak{D}_p\} < \infty$. Moreover, we assume that $f'(x)$ is defined in some neighborhood $\mathfrak{D}_p$ of Q_p,

$$|f'(x)| < M, \quad x \in \mathfrak{D}_p, \quad M \in \mathbb{R}. \tag{3}$$

Then for any $0 < \delta < \frac{1}{2}$

$$P\Big(E_1 - Q_p = o(n^{-\frac{1}{2}+\delta}), \;\; as \;\; n \to \infty\Big) = 1. \tag{4}$$

Proof It it easy to see that $\forall\, \varepsilon > 0$

$$\left\{|E_1 - Q_p| \geq \varepsilon n^{-\frac{1}{2}+\delta}\right\}$$
$$= \left\{E_1 \geq Q_p + \varepsilon n^{-\frac{1}{2}+\delta}\right\} \cup \left\{E_1 \leq Q_p - \varepsilon n^{-\frac{1}{2}+\delta}\right\} = A_n^1 \cup A_n^2.$$

One can obtain that

$$X_{\lfloor np+0.5\rfloor} \leq E_1 \leq X_{\lfloor np+0.5\rfloor+1}. \tag{5}$$

Put $\xi_{ni} = I(X_i \leq Q_p + \varepsilon n^{-\frac{1}{2}+\delta}) - F(Q_p + \varepsilon n^{-\frac{1}{2}+\delta})$ for $1 \leq i \leq n$. Therefore, we get

$$A_n^1 = \left\{E_1 \geq Q_p + \varepsilon n^{-\frac{1}{2}+\delta}\right\} \subset \left\{\sum_{i=1}^{n} I(X_i \leq Q_p + \varepsilon n^{-\frac{1}{2}+\delta}) < \lfloor np + 0.5\rfloor + 1\right\}$$

$$= \left\{\sum_{i=1}^{n} \xi_{ni} < \lfloor np + 0.5\rfloor + 1 - nF(Q_p + \varepsilon n^{-\frac{1}{2}+\delta})\right\}. \tag{6}$$

Note that, using Taylor's expansion:

$$F(Q_p + \varepsilon n^{-\frac{1}{2}+\delta}) = p + f(Q_p)\varepsilon n^{-\frac{1}{2}+\delta} + o(n^{-\frac{1}{2}+\delta})$$

we can obtain that there exists some constant $c(\varepsilon) > 0$, depending only on $\varepsilon > 0$, such that for a sufficiently large n

$$\left\{\sum_{i=1}^{n} \xi_{ni} < \lfloor np + 0.5\rfloor + 1 - np - \varepsilon f(Q_p) n^{\frac{1}{2}+\delta} + o(n^{\frac{1}{2}+\delta})\right\}$$
$$\subset \left\{\sum_{i=1}^{n} \xi_{ni} < -c(\varepsilon) n^{\frac{1}{2}+\delta}\right\} \tag{7}$$

Hence, on the basis of Eqs. (6),(7), Markov's inequality and Eq. (2) (for NA, φ and ρ^*-mixing random variables) or Lemma 1 (for ρ-mixing random variables), for $r > \dfrac{1}{\delta}$ where $0 < \delta < \dfrac{1}{2}$, we get the following estimation

$$\sum_{n=1}^{\infty} P\left(E_1 \geq Q_p + \varepsilon n^{-\frac{1}{2}+\delta}\right) \leq \sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} \xi_{ni} < -c(\varepsilon) n^{\frac{1}{2}+\delta}\right)$$
$$\leq \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} \xi_{ni}\right| > c(\varepsilon) n^{\frac{1}{2}+\delta}\right) \leq C \sum_{n=1}^{\infty} n^{-(\frac{1}{2}+\delta) r} E\left|\sum_{i=1}^{n} \xi_{ni}\right|^{r}$$
$$\leq C \sum_{n=1}^{\infty} n^{-(\frac{1}{2}+\delta) r}\left[(n E \xi_{n1}^{2})^{\frac{r}{2}} + n E|\xi_{n1}|^{r}\right] \leq C \sum_{n=1}^{\infty} n^{-\delta r} < \infty.$$

We can carry out analogous considerations for A_n^2 using the lower estimate of E_1 in the inequality Eq. (5). Hence, we get $\sum_{n=1}^{\infty} P\left[E_1 \leq Q_p - \varepsilon n^{-\frac{1}{2}+\delta}\right] < \infty$. By the Borel-Cantelli lemma we get thesis Eq. (4).

□

Remark 5 One can obtain analogous results for the estimator:

- E_2—by assumption that $X_{(\lfloor np \rfloor)} \leq E_2 \leq X_{(\lfloor np \rfloor + 1)}$,
- E_3—calculations are similar to the estimator E_4, which was considered in the paper (Dudek and Kuczmaszewska (2022)),
- E_5 and E_6—calculations are analogous to the estimators E_3 and E_4, because the proof does not depend on the value of $np - \lfloor np \rfloor$,
- E_7—by the assumption that $X_{(\lfloor (n+1)p \rfloor))} \leq E_7 \leq X_{(\lfloor (n+1)p \rfloor + 1))}$,
- E_8—it is a combination of the estimator E_4 and E_3.

Now, let us focus on the Bahadur representation of sample quantiles.

Theorem 3 *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with one of the following dependency structures:*

1. *NA dependence,*
2. *ρ^*-mixing dependence.*

Let $\{X_n, n \geq 1\}$ be indentically distributed with a common distribution function F and quantile Q_p. Assume that F possesses a positive continuous density f in some neighborhood $\mathfrak{D}_p$ of Q_p such that

$$0 < \sup\{f(x); x \in \mathfrak{D}_p\} < \infty. \tag{8}$$

Then for any $\delta > 0$

$$P\left(\sup_{x \in \mathfrak{I}_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| = O(n^{-\frac{1}{2}+\delta}), \quad n \to \infty\right) = 1,$$

where $\mathfrak{I}_n = \left[Q_p - c_0 n^{-\frac{1}{4}+\delta}, Q_p + c_0 n^{-\frac{1}{4}+\delta}\right]$ for some $c_0 > 0$.

Proof Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences defined as follows

$$a_n = c_0 n^{-\frac{1}{4}+\delta} \quad \text{for some } \; c_0 > 0, \quad b_n = \lfloor n^{\frac{1}{4}} \rfloor + 1$$

and

$$G_n(x) = F_n(x) - F_n(Q_p) - F(x) + p.$$

For each $n \in \mathbb{N}$ and any integer j we define

$$\eta_{j,n} = Q_p + j a_n b_n^{-1}, \quad \alpha_{j,n} = F(\eta_{j+1,n}) - F(\eta_{j,n}) \quad \text{and} \quad \mathfrak{I}_{j,n} = [\eta_{j,n}, \eta_{j+1,n}].$$

Since F_n and F are nondecreasing we get for $x \in \mathfrak{I}_{j,n}$

$$G_n(x) \leq F_n(\eta_{j+1,n}) - F_n(Q_p) - F(\eta_{j,n}) + p \leq G_n(\eta_{j+1,n}) + \alpha_{j,n}$$

and

$$G_n(x) \geq F_n(\eta_{j,n}) - F_n(Q_p) - F(\eta_{j+1,n}) + p \geq G_n(\eta_{j,n}) - \alpha_{j,n}.$$

Hence

$$\sup_{x \in \mathfrak{I}_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| \leq \max_{-b_n \leq j \leq b_n} \{|G_n(\eta_{j,n})|\} + \max_{-b_n \leq j \leq b_n - 1} \{\alpha_{j,n}\}.$$

It is easy to see that by The Mean Value Theorem and (8) we have

$$\alpha_{j,n} = F(\eta_{j+1,n}) - F(\eta_{j,n}) \leq C(\eta_{j+1,n} - \eta_{j,n}) = C a_n b_n^{-1} \leq C n^{-\frac{1}{2}+\delta}.$$

Therefore we obtain

$$\sum_{n=1}^{\infty} P\left(\sup_{x \in \mathfrak{I}_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| \geq c_0 n^{-\frac{1}{2}+\delta} \right)$$
$$\leq C \sum_{n=1}^{\infty} P\left(\max_{-b_n \leq j \leq b_n} |G_n(\eta_{j,n})| \geq \frac{c_0}{2} n^{-\frac{1}{2}+\delta} \right).$$

Moreover, we note that

$$G_n(\eta_{j,n}) = F_n(\eta_{j,n}) - F_n(Q_p) - F(\eta_{j,n}) + p = \frac{1}{n} \sum_{i=1}^{n} \left(Y_i^{Q_p} - Y_i^{(j,n)} \right)$$

where $Y_i^{Q_p} = E(I[X_i \leq Q_p]) - I[X_i \leq Q_p]$ and $Y_i^{(j,n)} = E(I[X_i \leq \eta_{j,n}]) - I[X_i \leq \eta_{j,n}]$, $-b_n \leq j \leq b_n$ are respectively NA or ρ^*-mixing random variables.

It follows from the Markov's inequality and Eq. (2) (for NA and ρ^*-mixing random variables) that for $r > \max\left\{2, \dfrac{5}{4\delta}\right\}$ where $\delta > 0$ we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\sup_{x\in\mathfrak{I}_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| \geq c_0 n^{-\frac{1}{2}+\delta}\right)\\
&\leq C \sum_{n=1}^{\infty} \sum_{j=-b_n}^{b_n} P\left(|G_n(\eta_{j,n})| \geq \frac{c_0}{2} n^{-\frac{1}{2}+\delta}\right)\\
&= C \sum_{n=1}^{\infty} \sum_{j=-b_n}^{b_n} P\left(|\sum_{i=1}^{n}\left(Y_i^{Q_p} - Y_i^{(j,n)}\right)| \geq \frac{c_0}{2} n^{\frac{1}{2}+\delta}\right)\\
&\leq C \sum_{n=1}^{\infty} \sum_{j=-b_n}^{b_n} P\left(|\sum_{i=1}^{n} Y_i^{Q_p}| + |\sum_{i=1}^{n} Y_i^{(j,n)}| \geq \frac{c_0}{2} n^{\frac{1}{2}+\delta}\right)\\
&\leq C \sum_{n=1}^{\infty} \sum_{j=-b_n}^{b_n} \left(P\left(|\sum_{i=1}^{n} Y_i^{Q_p}| \geq \frac{c_0}{4} n^{\frac{1}{2}+\delta}\right) + P\left(|\sum_{i=1}^{n} Y_i^{(j,n)}| \geq \frac{c_0}{4} n^{\frac{1}{2}+\delta}\right)\right)\\
&\leq C \sum_{n=1}^{\infty} \sum_{j=-b_n}^{b_n} \left[\frac{E\left(|\sum_{i=1}^{n} Y_i^{Q_p}|\right)^r}{(n^{\frac{1}{2}+\delta})^r} + \frac{E\left(|\sum_{i=1}^{n} Y_i^{(j,n)}|\right)^r}{(n^{\frac{1}{2}+\delta})^r}\right]\\
&\leq C \sum_{n=1}^{\infty} 2b_n n^{-\delta r} \leq C \sum_{n=1}^{\infty} n^{\frac{1}{4}-\delta r} < \infty.
\end{aligned}
$$

By the Borel-Cantelli lemma we get thesis. □

Theorem 4 *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with one of the following dependency structures:*

1. *φ-mixing dependence with coefficients satifying $\sum_{n=1}^{\infty} \varphi(2^n) < \infty$,*
2. *ρ-mixing dependence with coefficients satifying $\sum_{n=1}^{\infty} \rho^{\frac{2}{q}}(2^n) < \infty$, for $q > \max\left\{2, \frac{5}{2\delta}\right\}$ for $\delta > 0$.*

Let $\{X_n, n \geq 1\}$ be indentically distributed with a common distribution function F and quantile Q_p. Assume that F possesses a positive continuous density f in some neighborhood $\mathfrak{D}_p$ of Q_p such that

$$0 < \sup\{f(x); x \in \mathfrak{D}_p\} < \infty. \tag{9}$$

Then for any $\delta > 0$

$$P\left(\sup_{x\in\Im_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| = O(n^{-\frac{3}{4}+\delta}), \quad n \to \infty\right) = 1,$$

where $\Im_n = \left[Q_p - c_0 n^{-\frac{1}{2}+\delta}, Q_p + c_0 n^{-\frac{1}{2}+\delta}\right]$ *for some* $c_0 > 0$

Proof The proof of Theorem 4 is very similar to the proof of Theorem 3. Analogously, let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences defined as follows

$$a_n = c_0 n^{-\frac{1}{2}+\delta} \quad \text{for some } \ c_0 > 0, \quad b_n = \lfloor n^{\frac{1}{4}} \rfloor + 1$$

and

$$G_n(x) = F_n(x) - F_n(Q_p) - F(x) + p.$$

Let $\eta_{j,n}$, $\alpha_{j,n}$ and $\Im_{j,n}$ be defined as in Theorem 3.
As it was shown in Theorem 3, it is easy to see that

$$\alpha_{j,n} = F(\eta_{j+1,n}) - F(\eta_{j,n}) \leq C(\eta_{j+1,n} - \eta_{j,n}) = C a_n b_n^{-1} \leq C n^{-\frac{3}{4}+\delta}.$$

Therefore we get

$$\sum_{n=1}^{\infty} P\left(\sup_{x\in\Im_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| \geq c_0 n^{-\frac{3}{4}+\delta}\right)$$
$$\leq C \sum_{n=1}^{\infty} P\left(\max_{-b_n \leq j \leq b_n} |G_n(\eta_{j,n})| \geq \frac{c_0}{2} n^{-\frac{3}{4}+\delta}\right).$$

Next, we have

$$G_n(\eta_{j,n}) = F_n(\eta_{j,n}) - F_n(Q_p) - F(\eta_{j,n}) + p = \frac{1}{n} \sum_{i=1}^{n} \left(Y_i^{(j,n)} - EY_i^{(j,n)}\right),$$

where $Y_i^{(j,n)} = I[Q_p < X_i \leq \eta_{j,n}]$, $-b_n \leq j \leq b_n$ are respectively φ or ρ-mixing random variables.

Moreover, it is easy to obtain by The Mean Value Theorem that for $r \geq 2$

$$E|Y_i^{(j,n)}|^r \leq C j a_n b_n^{-1}.$$

It follows from the Markov's inequality and Eq. (2) (for φ-mixing random variables) or Lemma 1 (for ρ-mixing random variables) that for $r > \max\left\{2, \dfrac{5}{2\delta}\right\}$ where $\delta > 0$

we have

$$\sum_{n=1}^{\infty} P\left(\sup_{x\in\mathfrak{I}_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| \geq c_0 n^{-\frac{3}{4}+\delta}\right)$$

$$\leq C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} P\left(|G_n(\eta_{j,n})| \geq \frac{c_0}{2} n^{-\frac{3}{4}+\delta}\right)$$

$$= C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} P\left(|\sum_{i=1}^{n}\left(Y_i^{(j,n)} - EY_i^{(j,n)}\right)| \geq \frac{c_0}{2} n^{\frac{1}{4}+\delta}\right)$$

$$\leq C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} n^{-(\frac{1}{4}+\delta)r} E|\sum_{i=1}^{n}\left(Y_i^{(j,n)} - EY_i^{(j,n)}\right)|^r$$

$$\leq C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} n^{-(\frac{1}{4}+\delta)r}\left[\left(nE(Y_1^{(j,n)})^2\right)^{\frac{r}{2}} + nE|Y_1^{(j,n)}|^r\right]$$

$$\leq C\sum_{n=1}^{\infty}\sum_{j=-b_n}^{b_n} n^{-(\frac{1}{4}+\delta)r}\left(nja_nb_n^{-1}\right)^{\frac{r}{2}} \leq C\sum_{n=1}^{\infty} n^{-\frac{\delta r}{2}+\frac{1}{4}} < \infty,$$

By the Borel-Cantelli lemma we get thesis. □

Theorem 5 *Suppose that assumptions of Theorem 2 hold. Then for any* $0 < \delta < \dfrac{1}{2}$ *we have,*

$$P\left(E_1 = Q_p - \frac{F_n(Q_p) - p}{f(Q_p)} + O(n^{-\frac{1}{2}+\delta}),\ \ as\ \ n \to \infty\right) = 1 \qquad (10)$$

for NA and ρ^**-mixing random variables.*

Proof One can note that

$$F_n(E_1) \leq n^{-1}(\lfloor np + 0.5\rfloor + 1) = p + O(n^{-1}).$$

By Taylor's expansion we obtain for $0 < \theta < 1$

$$F(E_1) = p + f(Q_p)(E_1 - Q_p) + \frac{1}{2} f'(Q_p + \theta(E_1 - Q_p))(E_1 - Q_p)^2.$$

From Eq. (3) and Theorem 2 for $0 < \delta < \dfrac{1}{2}$ it follows that

$$|F_n(E_1) - F(E_1) + f(Q_p)(E_1 - Q_p)|$$

$$\leq \frac{1}{2}|f'(Q_p + \theta(E_1 - Q_p))|(E_1 - Q_p)^2 + O(n^{-1}) = o(n^{-1+2\delta}). \tag{11}$$

By Eq. (11) and Theorem 3 for $0 < \delta < \frac{1}{2}$ we get that with probability 1,

$$\begin{aligned}
&|f(Q_p)(E_1 - Q_p) + F_n(Q_p) - p| \\
&\quad \leq |F_n(E_1) - F(E_1) + f(Q_p)(E_1 - Q_p)| + |F_n(E_1) - F(E_1) - (F_n(Q_p) - p)| \\
&\quad \leq o(n^{-1+2\delta}) + \sup_{x \in \Im_n} |F_n(x) - F(x) - (F_n(Q_p) - p)| = O(n^{-\frac{1}{2}+\delta}),
\end{aligned}$$

which gives that $f(Q_p)(E_1 - Q_p) + F_n(Q_p) - p = O(n^{-\frac{1}{2}+\delta})$, when $n \to \infty$. Hence, we get Eq. (10). □

Theorem 6 *Suppose that assumptions of Theorem 2 hold. Then for any* $0 < \delta < \frac{1}{4}$ *we get*

$$P\left(E_1 = Q_p - \frac{F_n(Q_p) - p}{f(Q_p)} + O(n^{-\frac{3}{4}+\delta}), \ \ as \ \ n \to \infty\right) = 1 \tag{12}$$

for φ and ρ-mixing random variables.

Proof Analogously as in the proof of Theorem 5, using Eq. (11) and Theorem 4 for $0 < \delta < \frac{1}{4}$ we get that with probability 1,

$$|f(Q_p)(E_1 - Q_p) + F_n(Q_p) - p| = O(n^{-\frac{3}{4}+\delta}).$$

Hence, we get Eq. (12). □

Remark 6 The calculations for the estimators E_2, E_4, E_5, E_6, E_7 and E_8 proceed analogously through the condition that $F_n(E_i) = p + O(n^{-1})$, where $i = 2, 4, 5, 6, 7, 8$. Assuming that $F_n(E_3) \leq p$, one can obtain analogous results for the estimator E_3.

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