



Predictive inference of dual generalized order statistics from the inverse Weibull distribution

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Received: 16 December 2021 / Revised: 21 February 2022 / Accepted: 4 April 2022 /
Published online: 30 April 2022
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Abstract

In this paper, some predictive results of dual generalized order statistics (DGOSs) from the inverse Weibull distribution are obtained. For this goal, different predictive and reconstructive pivotal quantities are proposed. Moreover, several predictive and reconstructive intervals concerning DGOSs based on the inverse Weibull distribution are constructed. Furthermore, the maximum likelihood predictor as well as the predictive maximum likelihood estimates based on DGOSs are studied. Finally, simulation studies are carried out to assess the efficiency of the obtained results.

Keywords Dual generalized order statistics · Maximum likelihood predictor · Predictive interval · Probability coverage · Monte Carlo simulation

Mathematics Subject Classification 60G70 · 62E20 · 62F10 · 62G30 · 62G32 · 62N05

1 Introduction

Kamps (1995) introduced the generalized order statistics (GOSs) as a unified model of ascending ordered random variables. The GOSs have gotten a lot of attention in recent years. This is because such a concept describes random variables (RVs) in ascending order of magnitude, which has important applications and includes well-known concepts that have been treated separately in the statistical literature. Ordinary order statistics (OOSs), sequential order statistics (SOSs), Progressive type II censored order statistics (POSs), record values, k th record values, and Pfeifer's records are examples of the GOSs model.

Clearly, descending order RVs, such as lower record values, are not included in the GOSs model. DGOSs were first introduced by Burkschat et al. (2003) as a unified

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model of descending ordered RVs, similar to reversed OOSs, lower k -records and lower Pfeiffer records, through a combined approach. By analogy with Kamps (1995), the DGOSs, $X_{r,n,\underline{\gamma}}^{(D)}$, $r = 1, 2, \dots, n$, based on a continuous cumulative distribution function (CDF) F , were defined in Burkschat et al. (2003), as

$$X_{r,n,\underline{\gamma}}^{(D)} \stackrel{d}{=} F^{-1} \left(\prod_{j=1}^r B_j \right) \stackrel{d}{=} F^{-1} \left(\prod_{j=1}^r U_j^{* \frac{1}{\gamma_j}} \right), \quad r = 1, 2, \dots, n, \quad (1.1)$$

where, $\underline{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathfrak{R}_+^n$ is the vector of the model parameters with $\gamma_j = k + n - j + \sum_{i=j}^{n-1} m_i > 0$, $m_1, \dots, m_{n-1} \in \mathfrak{R}$, $\gamma_n = k > 0$, U_j^* , $j = 1, 2, \dots, n$ are independent standard uniform RVs, and $X \stackrel{d}{=} Y$ means X and Y have the same CDF. Hence, the strict relations $X_{1,n,\underline{\gamma}}^{(D)} > X_{2,n,\underline{\gamma}}^{(D)} > \dots > X_{n,n,\underline{\gamma}}^{(D)}$ hold almost surely. Interested readers can be referred to Ahsanullah (2004); Barakat and El-Adll (2009); Burkschat et al. (2003); Shah Imtiyaz et al. (2020) for more details on DGOSs.

Predicting future events based on past or current events is an important problem in statistics. In life testing problems, some failure times cannot be observed for various reasons and it is necessary to predict or reconstruct such failure times using a point or an interval. Clearly, OOSs play a significant role in predicting future observations and reconstructing previously unseen ones. For both frequentist and Bayesian approaches, many authors have studied point and interval predictions in statistical literature. Among them are Ahsanullah (1980), Al-Hussaini (1999), Al-Hussaini and Ahmad (2003), Al-Mutairi and Raqab (2020), David and Nagaraja (2003), Geisser (1993), Kaminsky and Rhodin (1985), Kotb and Raqab (2021), Lawless (1977), Nagaraja (1986) and Raqab (2001).

The first prediction result based on pivotal quantity is due to Lawless (1971), who applied the results of Sukhatme (1937) to construct confidence intervals for future OOSs from the exponential distribution. Lingappaiah (1973), defined a different pivotal quantity for the same purpose. Recent works on prediction and reconstruction based on pivotal quantities have been published by Aly (2015, 2016, 2022), Barakat et al. (2011, 2018, 2021), El-Adll (2011, 2021), El-Adll and Aly (2016a,b), among others. A general finite-sample method for predicting future observations from any arbitrary continuous distribution was proposed by Barakat et al. (2014). Later, Aly et al. (2019) extended this result to fractional record-values.

The Weibull distribution is one of the most widely used distributions in engineering, hydrology, ecology, medicine, the environment, and energy research. The inverse Weibull distribution, like the Weibull distribution, enables us to model long-tailed right-skewed data. The Inverse Weibull distribution is a special case of the generalized extreme value distribution, which is considered as an alternative to the Weibull distribution for modeling wind speed data. For some wind speed data measured in various locations and seasons, the inverse Weibull distribution outperforms the Weibull distribution in modeling. Since the Weibull distribution does not perform well in modeling wind speed data from various geographical regions around the world (e.g. Akgül et al. (2016); Wang et al. (2015)), the heavier right tail of the inverse Weibull distribution provides an advantage for modeling the right tail's extreme or outlying observations.

The probability density function (PDF) and CDF of the inverse Weibull distribution are respectively, given by

$$f(x) = \frac{\beta}{\sigma} \left(\frac{x}{\sigma}\right)^{-(\beta+1)} \exp\left[-\left(\frac{x}{\sigma}\right)^{-\beta}\right], \quad x > 0, \quad \beta, \sigma > 0, \quad (1.2)$$

and

$$F(x) = \exp\left[-\left(\frac{x}{\sigma}\right)^{-\beta}\right], \quad x > 0, \quad \beta, \sigma > 0. \quad (1.3)$$

It can be noted that the transformations, $Z_{r,n,\underline{\gamma}}^{(D)} = \log F\left(X_{r,n,\underline{\gamma}}^{(D)}\right)$, $r = 1, 2, \dots, n$, represent DGOSs based on the negative exponential distribution (NEXP(1)) with PDF and CDF, $g(x) = G(x) = e^x$, $x \leq 0$. In what follows, U_1 , U_2 , and U_3 will denote predictive pivotal quantities, while V_1 , V_2 , and V_3 denote reconstructive pivotal quantities. Moreover, in the DGOSs model considered here, it is assumed that $\gamma_i \neq \gamma_j$ for $i \neq j$, $1 \leq i, j \leq n$, which includes most of the important descending ordered RVs except for the lower record values. Furthermore, $W_{r,l} = Z_{r,n,\underline{\gamma}}^{(D)} - Z_{l,n,\underline{\gamma}}^{(D)} < 0$, for $1 \leq l < r < n$, and $T_{l,r} = \sum_{i=l+1}^r \gamma_i (Z_{i,n,\underline{\gamma}}^{(D)} - Z_{i-1,n,\underline{\gamma}}^{(D)})$ follows the negative gamma distribution with parameters $r-l, 1$, i.e., $T_{l,r} \sim N\Gamma(r-l, 1)$ with PDF

$$f_{T_{l,r}}(t) = \frac{1}{\Gamma(r-l)} (-t)^{r-l-1} e^t, \quad t < 0, \quad l < r.$$

The rest of this paper is organized as follows. In Sect. 2, three predictive pivotal quantities are suggested and their distributions are established. Section 3 is devoted to the reconstruction problem. In Sect. 4, the MLP as well as the PMLEs are discussed. Simulation studies are carried out in Sect. 5.

2 Prediction intervals of DGOSs

In this section, based on the knowledge of $X_{l,n,\underline{\gamma}}^{(D)}, \dots, X_{r,n,\underline{\gamma}}^{(D)}$, three predictive pivotal quantities of the unobserved sth, DGOS $X_{s,n,\underline{\gamma}}^{(D)}$, for $1 \leq l < r < s \leq n$, are proposed and their distributions are derived. Consequently, three predictive intervals of $X_{s,n,\underline{\gamma}}^{(D)}$ are constructed. The predictive pivotal quantities are

$$U_1 = 1 - \left(\frac{X_{s,n,\underline{\gamma}}^{(D)}}{X_{r,n,\underline{\gamma}}^{(D)}}\right)^{\beta}, \quad (2.1)$$

$$U_2 = \frac{\left(X_{s,n,\underline{\gamma}}^{(D)}\right)^{-\beta} - \left(X_{r,n,\underline{\gamma}}^{(D)}\right)^{-\beta}}{\sum_{i=l+1}^r \gamma_i \left(\left(X_{i,n,\underline{\gamma}}^{(D)}\right)^{-\beta} - \left(X_{i-1,n,\underline{\gamma}}^{(D)}\right)^{-\beta}\right)}, \quad (2.2)$$

$$U_3 = \frac{\left(X_{s,n,\underline{\gamma}}^{(D)}\right)^{-\beta} - \left(X_{r,n,\underline{\gamma}}^{(D)}\right)^{-\beta}}{\left(X_{r,n,\underline{\gamma}}^{(D)}\right)^{-\beta} - \left(X_{l,n,\underline{\gamma}}^{(D)}\right)^{-\beta}}. \quad (2.3)$$

The following lemma will be needed in the sequel which gives the marginal distributions of a single DGOS and marginal joint between two DGOSs.

Lemma 2.1 *Under the condition, $\gamma_i \neq \gamma_j$ for $i \neq j$, $1 \leq i, j \leq n$, the marginal PDF of the r th DGOS as well as the joint PDF of the r th and s th DGOSs are respectively, given by*

$$f_{X_{r,n,\underline{\gamma}}^{(D)}}(x_r) = C_r \sum_{i=1}^r a_i(r) F^{\gamma_i-1}(x_r) f(x_r), \quad x_r \in \mathbb{R}, \quad (2.4)$$

and

$$\begin{aligned} f_{X_{r,n,\underline{\gamma}}^{(D)}, X_{s,n,\underline{\gamma}}^{(D)}}(x_r, x_s) &= C_s \left[\sum_{i=r+1}^s a_i^{(r)}(s) \left(\frac{F(x_s)}{F(x_r)} \right)^{\gamma_i} \right] \\ &\times \left[\sum_{i=1}^r a_i(r) F^{\gamma_i}(x_r) \right] \frac{f(x_r)}{F(x_r)} \frac{f(x_s)}{F(x_s)}, \quad x_r > x_s, \quad r < s, \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} C_r &= \prod_{j=1}^r \gamma_j, \quad a_i(r) = \prod_{j=1, j \neq i}^r \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r, \\ a_i^{(r)}(s) &= \prod_{j=r+1, j \neq i}^s \frac{1}{\gamma_j - \gamma_i}, \quad r+1 \leq i \leq s. \end{aligned}$$

The proof of Lemma 2.1 is similar to the proof of Lemma 2.1 of Kamps and Cramer (2001) with appropriate adjustments.

Theorem 2.1 *The CDF of U_1 is given by*

$$F_{U_1}(u_1) = C_s \sum_{j=1}^r \sum_{i=r+1}^s \frac{a_i^{(r)}(s) a_j(r) u_1}{\gamma_j (\gamma_j - (\gamma_j - \gamma_i) u_1)}, \quad 0 \leq u_1 \leq 1. \quad (2.6)$$

A $100(1 - \tau)\%$ predictive interval for $X_{s,n,\underline{\gamma}}^{(D)}$ based on U_1 is

$$\left((1 - u_1)^{1/\beta} X_{r,n,\underline{\gamma}}^{(D)}, X_{r,n,\underline{\gamma}}^{(D)} \right),$$

where $u_1 = u_1(\tau)$ is such that $F_{U_1}(u_1) = 1 - \tau$.

Proof First, note that the pivotal quantity U_1 can be expressed as

$$U_1 = 1 - \left(\frac{X_{s,n,\underline{\gamma}}^{(D)}}{X_{r,n,\underline{\gamma}}^{(D)}} \right)^{\beta} = \frac{\left(X_{s,n,\underline{\gamma}}^{(D)}/\sigma \right)^{-\beta} - \left(X_{r,n,\underline{\gamma}}^{(D)}/\sigma \right)^{-\beta}}{\left(X_{s,n,\underline{\gamma}}^{(D)}/\sigma \right)^{-\beta}} = \frac{Z_{s,n,\underline{\gamma}}^{(D)} - Z_{r,n,\underline{\gamma}}^{(D)}}{Z_{s,n,\underline{\gamma}}^{(D)}}.$$

Clearly, $0 < U_1 < 1$. Therefore, for $0 < u_1 < 1$, we have

$$\begin{aligned} F(u_1) &= P(U_1 \leq u_1) = P\left(0 < 1 - \frac{Z_{r,n,\underline{\gamma}}^{(D)}}{Z_{s,n,\underline{\gamma}}^{(D)}} \leq u_1\right) \\ &= P\left(Z_{s,n,\underline{\gamma}}^{(D)} < Z_{r,n,\underline{\gamma}}^{(D)} \leq (1 - u_1)Z_{s,n,\underline{\gamma}}^{(D)}\right) \\ &= \int_{-\infty}^0 \int_{z_s}^{(1-u_1)z_s} f_{Z_{r,n,\underline{\gamma}}^{(D)}, Z_{s,n,\underline{\gamma}}^{(D)}}(z_r, z_s) dz_r dz_s. \end{aligned} \quad (2.7)$$

By the relation (2.5), the joint PDF of the r th and s th DGOSs based on the NEXP(1) can be simplified and written as

$$f_{Z_{r,n,\underline{\gamma}}^{(D)}, Z_{s,n,\underline{\gamma}}^{(D)}}(z_r, z_s) = C_s \sum_{j=1}^r \sum_{i=r+1}^s a_j(r) a_i^{(r)}(s) e^{\gamma_i z_s} e^{(\gamma_j - \gamma_i) z_r}, \quad -\infty < z_s < z_r < 0. \quad (2.8)$$

By (2.7) and (2.8) we obtain

$$\begin{aligned} F_{U_1}(u_1) &= C_s \sum_{j=1}^r \sum_{i=r+1}^s a_i^{(r)}(s) a_j(r) \int_{-\infty}^0 \int_{z_s}^{(1-u_1)z_s} e^{\gamma_i z_s} e^{(\gamma_j - \gamma_i) z_r} dz_r dz_s \\ &= C_s \sum_{j=1}^r \sum_{i=r+1}^s a_i^{(r)}(s) a_j(r) \left(\frac{1}{\gamma_j - \gamma_i} \right) \left(\frac{1}{\gamma_j - (\gamma_j - \gamma_i)u_1} - \frac{1}{\gamma_j} \right). \end{aligned}$$

After some algebraic calculations, we get the relation (2.6). The predictive intervals can be accomplished directly from the definition of the pivotal quantity U_1 . Hence, the theorem is proved.

Lemma 2.2 *The normalized spacings, $Y_i = \gamma_i \left(Z_{i,n,\underline{\gamma}}^{(D)} - Z_{i-1,n,\underline{\gamma}}^{(D)} \right)$, $i = 1, 2, \dots, n$, are independent and identically distributed (iid) RVs each of which has the NEXP(1) with $Z_{0,n,\underline{\gamma}}^{(D)} \equiv 0$. Moreover,*

$$Z_{r,n,\underline{\gamma}}^{(D)} \stackrel{d}{=} \sum_{i=1}^r \frac{Y_i}{\gamma_i}, \quad r = 1, 2, \dots, n.$$

Lemma 2.2, which is due to Burkschat et al. (2003), represents a fundamental tool for proving the next theorems.

Theorem 2.2 *The CDF of the predictive pivotal quantity U_2 is*

$$F_{U_2}(u_2) = 1 - \frac{C_s}{C_r} \sum_{i=r+1}^s \frac{a_i^{(r)}(s)}{\gamma_i} (1 + \gamma_i u_2)^{-(r-l)}, \quad u_2 \geq 0, \quad r > l \geq 0. \quad (2.9)$$

A $100(1 - \tau)\%$ predictive interval for $X_{s,n,\underline{\gamma}}^{(D)}$ is

$$\left(\left(\left(X_{r,n,\underline{\gamma}}^{(D)} \right)^{-\beta} + u_2 \sum_{i=l+1}^r \gamma_i \left(\left(X_{i,n,\underline{\gamma}}^{(D)} \right)^{-\beta} - \left(X_{i-1,n,\underline{\gamma}}^{(D)} \right)^{-\beta} \right) \right)^{-1/\beta}, X_{r,n,\underline{\gamma}}^{(D)} \right),$$

where $u_2 = u_2(\tau)$ satisfies the nonlinear equation, $F_{U_2}(u_2) = 1 - \tau$.

Proof The pivotal quantity U_2 can be written as

$$\begin{aligned} U_2 &= \frac{\left(X_{s,n,\underline{\gamma}}^{(D)} \right)^{-\beta} - \left(X_{r,n,\underline{\gamma}}^{(D)} \right)^{-\beta}}{\sum_{i=l+1}^r \gamma_i \left(\left(X_{i,n,\underline{\gamma}}^{(D)} \right)^{-\beta} - \left(X_{i-1,n,\underline{\gamma}}^{(D)} \right)^{-\beta} \right)}, \\ &= \frac{Z_{s,n,\underline{\gamma}}^{(D)} - Z_{r,n,\underline{\gamma}}^{(D)}}{T_{l,r}} = \frac{W_{r,s}}{T_{l,r}}, \quad 0 \leq l < r < s. \end{aligned}$$

By Lemma 2.2, it can be noted that $W_{r,s} = \sum_{i=r+1}^s Y_i / \gamma_i$ and $T_{l,r} = \sum_{i=l+1}^r Y_i$. Since Y_1, \dots, Y_n are independent, $W_{r,s}$ and $T_{l,r}$ are independent. The CDF of $W_{r,s}$ can be obtained as follows

$$\begin{aligned} F_{W_{r,s}}(w) &= P(W_{r,s} \leq w) = P\left(Z_{s,n,\underline{\gamma}}^{(D)} \leq Z_{r,n,\underline{\gamma}}^{(D)} + w\right) \\ &= \int_{-\infty}^0 \int_{-\infty}^{z_r+w} f_{Z_{r,n,\underline{\gamma}}^{(D)}, Z_{s,n,\underline{\gamma}}^{(D)}}(z_r, z_s) dz_s dz_r \\ &= C_s \sum_{j=1}^r \sum_{i=r+1}^s \frac{a_j(r) a_i^{(r)}(s)}{\gamma_i} \frac{e^{\gamma_i w}}{\gamma_j}, \quad w < 0. \end{aligned} \quad (2.10)$$

Consequently, the PDF of $W_{r,s}$ is given by

$$f_{W_{r,s}}(w) = C_s \left(\sum_{j=1}^r \frac{a_j(r)}{\gamma_j} \right) \left(\sum_{i=r+1}^s a_i^{(r)}(s) e^{\gamma_i w} \right) = \frac{C_s}{C_r} \sum_{i=r+1}^s a_i^{(r)}(s) e^{\gamma_i w}. \quad (2.11)$$

Therefore, by the independence of $W_{r,s}$ and $T_{l,r}$, coupled with the continuous version of the total law of probability, we get

$$\begin{aligned} F_{U_2}(u_2) &= P(0 < U_2 \leq u_2) = P(u_2 T_{l,r} \leq W_{r,s} < 0) \\ &= \int_{-\infty}^0 (1 - F_{W_{r,s}}(u_2 t)) f_{T_{l,r}}(t) dt \\ &= 1 - \frac{C_s}{C_r} \sum_{i=r+1}^s \frac{a_i^{(r)}(s)}{\gamma_i} (1 + \gamma_i u_2)^{-(r-l)}, \quad u_2 \geq 0, \end{aligned}$$

which is (2.9). The predictive interval is a direct consequence of the form of the pivotal quantity. This completes the proof of the theorem. \square

Theorem 2.3 *The CDF of the predictive pivotal quantity U_3 is given by*

$$F_{U_3}(u_3) = \frac{C_s}{C_l} \sum_{i=r+1}^s \sum_{j=l+1}^r \frac{a_i^{(r)}(s) a_j^{(l)}(r) u_3}{\gamma_j (\gamma_j + \gamma_i u_3)}, \quad u_3 \geq 0.$$

A $100(1 - \tau)\%$ predictive interval for $X_{s,n,\underline{\gamma}}^{(D)}$ is

$$\left(X_{r,n,\underline{\gamma}}^{(D)} \left(1 + u_3 \left(1 - \left(\frac{X_{l,n,\underline{\gamma}}^{(D)}}{X_{r,n,\underline{\gamma}}^{(D)}} \right)^{-\beta} \right) \right)^{-1/\beta}, X_{r,n,\underline{\gamma}}^{(D)} \right),$$

where $u_3 = u_3(\tau)$ is obtained by solving the nonlinear equation, $F_{U_3}(u_3) = 1 - \tau$.

Proof As we proceed in the previous theorems, the pivotal quantity U_3 can be formulated as

$$U_3 = \frac{Z_{s,n,\underline{\gamma}}^{(D)} - Z_{r,n,\underline{\gamma}}^{(D)}}{Z_{r,n,\underline{\gamma}}^{(D)} - Z_{l,n,\underline{\gamma}}^{(D)}} = \frac{W_{r,s}}{W_{l,r}}.$$

By Lemma (2.2), the RVs $W_{l,r} = \sum_{i=l+1}^r Y_i/\gamma_i$ and $W_{r,s} = \sum_{i=r+1}^s Y_i/\gamma_i$ are independent. Accordingly, the relation (2.11) yields

$$\begin{aligned} f_{W_{l,r}, W_{r,s}}(w_{l,r}, w_{r,s}) &= f_{W_{l,r}}(w_{l,r}) f_{W_{r,s}}(w_{r,s}) \\ &= \frac{C_s}{C_l} \sum_{i=r+1}^s \sum_{j=l+1}^r a_i^{(r)}(s) a_j^{(l)}(r) e^{\gamma_j w_{l,r}} e^{\gamma_i w_{r,s}}, \quad w_{l,r}, w_{r,s} < 0. \end{aligned} \quad (2.12)$$

Hence,

$$\begin{aligned} F_{U_3}(u_3) &= P(0 < U_3 \leq u_3) = P(u_3 W_{l,r} \leq W_{r,s} < 0) \\ &= \int_{-\infty}^0 \int_{u_3 w_{l,r}}^0 f_{W_{l,r}, W_{r,s}}(w_{l,r}, w_{r,s}) dw_{r,s} dw_{l,r} \end{aligned}$$

$$\begin{aligned}
&= \frac{C_s}{C_l} \sum_{i=r+1}^s \sum_{j=l+1}^r a_i^{(r)}(s) a_j^{(l)}(r) \int_{-\infty}^0 \int_{u_3 w_{l,r}}^0 e^{\gamma_j w_{l,r}} e^{\gamma_i w_{r,s}} dw_{l,r} dw_{r,s} \\
&= \frac{C_s}{C_l} \sum_{i=r+1}^s \sum_{j=l+1}^r \frac{a_i^{(r)}(s) a_j^{(l)}(r) v_3}{\gamma_j (\gamma_j + \gamma_i v_3)},
\end{aligned}$$

which was to be proved. The rest of the theorem is easy to prove. \square

3 Reconstructive intervals of DGOSs

This section is devoted to the reconstruction problem of DGOSs relying on the pivotal quantities approach. In this section, it is assumed that $X_{s,n,\underline{\gamma}}^{(D)}, \dots, X_{n,n,\underline{\gamma}}^{(D)}$ are observed and $X_{r,n,\underline{\gamma}}^{(D)}$, $r = s-1, s-2, \dots, 1$ are to be reconstructed. For this goal, four reconstructive pivotal quantities are proposed and their distributions are established. In what follows, a corollary to Theorem 2.1 and three theorems are presented without proof. Their proofs can be accomplished in the same manner as in Sect. 2.

Corollary 3.1 A $100(1-\tau)\%$ reconstructive interval of $X_{r,n,\underline{\gamma}}^{(D)}$ based on U_1 is

$$\left(X_{s,n,\underline{\gamma}}^{(D)}, (1-u_1)^{-1/\beta} X_{s,n,\underline{\gamma}}^{(D)} \right),$$

where $u_1 = u_1(\tau)$ satisfies the nonlinear equation $F_{U_1}(u_1) = 1 - \tau$, $0 < u_1 < 1$.

Theorem 3.1 The CDF of the pivotal quantity $V_1 = \frac{Z_{s,n,\underline{\gamma}}^{(D)} - Z_{r,n,\underline{\gamma}}^{(D)}}{Z_{r,n,\underline{\gamma}}^{(D)}}$ takes the form

$$F_{V_1}(v_1) = 1 - C_s \sum_{j=1}^r \sum_{i=r+1}^s \frac{a_j(r) a_i^{(r)}(s)}{\gamma_i (\gamma_j + \gamma_i v_1)}, \quad v_1 \geq 0.$$

Moreover, a $100(1-\tau)\%$ reconstructive interval for $X_{r,n,\underline{\gamma}}^{(D)}$ is

$$\left(X_{s,n,\underline{\gamma}}^{(D)}, (1+v_1)^{(1/\beta)} X_{s,n,\underline{\gamma}}^{(D)} \right),$$

where $v_1 = v_1(\tau)$ is the solution to the nonlinear equation, $F_{V_1}(v_1) = 1 - \tau$.

Theorem 3.2 The CDF of the reconstructive pivotal quantity $V_2 = \frac{Z_{s,n,\underline{\gamma}}^{(D)} - Z_{r,n,\underline{\gamma}}^{(D)}}{T_{s,n}}$ is given by

$$F_{V_2}(v_2) = 1 - \frac{C_s}{C_r} \sum_{i=r+1}^s \frac{a_i^{(r)}(s)}{\gamma_i} (1 + \gamma_i v_2)^{-(n-s)}, \quad v_2 \geq 0,$$

where $T_{s,n} = \sum_{i=s+1}^n \gamma_i (Z_{i,n,\underline{\gamma}}^{(D)} - Z_{i-1,n,\underline{\gamma}}^{(D)})$. Furthermore, a $100(1 - \tau)\%$ reconstructive interval of $X_{r,n,\underline{\gamma}}^{(D)}$ is

$$\left(\left(X_{s,n,\underline{\gamma}}^{(D)} \right)^{-\beta}, \left(\left(X_{s,n,\underline{\gamma}}^{(D)} \right)^{-\beta} - v_2 \sum_{i=s+1}^n \gamma_i \left(\left(X_{i,n,\underline{\gamma}}^{(D)} \right)^{-\beta} - \left(X_{i-1,n,\underline{\gamma}}^{(D)} \right)^{-\beta} \right) \right)^{-1/\beta} \right),$$

where $v_2 = v_2(\tau)$ can be obtained by solving the nonlinear equation, $F_{V_2}(v_2) = 1 - \tau$.

Theorem 3.3 The CDF of the reconstructive pivotal quantity, $V_3 = \frac{Z_{s,n,\underline{\gamma}}^{(D)} - Z_{r,n,\underline{\gamma}}^{(D)}}{Z_{n,n,\underline{\gamma}}^{(D)} - Z_{s,n,\underline{\gamma}}^{(D)}}$, is

$$F_{V_3}(v_3) = \frac{C_n}{C_r} \sum_{i=r+1}^s \sum_{j=s+1}^n \frac{a_i^{(r)}(s) a_j^{(s)}(n) v_3}{\gamma_j (\gamma_j + \gamma_i v_3)}, \quad v_3 \geq 0.$$

A $100(1 - \tau)\%$ confidence interval for $X_{r,n,\underline{\gamma}}^{(D)}$ is

$$\left(X_{s,n,\underline{\gamma}}^{(D)}, X_{s,n,\underline{\gamma}}^{(D)} \left(1 - v_3 \left(\left(\frac{X_{s,n,\underline{\gamma}}^{(D)}}{X_{n,n,\underline{\gamma}}^{(D)}} \right)^{\beta} - 1 \right) \right)^{-1/\beta} \right),$$

where $v_3 = v_3(\tau)$ is computed by solving, $F_{V_3}(v_3) = 1 - \tau$.

Remark 3.1

1. Clearly, all the predictive and reconstructive results of the inverse exponential distribution are obtained as special cases from the obtained results in Sects. 2 and 3 if $\beta = 1$.
2. The predictive and reconstructive intervals are free of the scale parameter σ , while this is not the case for the shape parameter β .
3. If the shape parameter β is known, the transformation, $Y^* = \left(\frac{X}{\sigma} \right)^{\beta}$ reduces the problem to the inverse exponential distribution.

The next section addresses the issue of the unknown parameters.

4 The MLP based on DGOSs

In this section, the MLEs and MLP, as well as the PMLEs based on the first r DGOSs, are studied. The following proposition is formulated in a general framework.

Proposition 4.1 The likelihood function based on the DGOSs, $X_{1,n,\underline{\gamma}}^{(D)}, \dots, X_{r,n,\underline{\gamma}}^{(D)}$, from any continuous DF, F is

$$L^*(\underline{\Theta}|\underline{\mathbf{x}}_r) = C_r \left(\prod_{i=1}^{r-1} F^{m_i}(x_i; \underline{\Theta}) f(x_i; \underline{\Theta}) \right) F^{\gamma_r-1}(x_r; \underline{\Theta}) f(x_r; \underline{\Theta}),$$

$$-\infty < x_r < \cdots < x_1 < \infty, \quad (4.1)$$

where $\underline{\Theta} = (\theta_1, \theta_2, \dots, \theta_d)$ is the vector of unknown parameters and $\underline{\mathbf{x}}_r = (x_1, x_2, \dots, x_r)$ denotes the first r observed DGOSs. Moreover, the predictive likelihood (PL^*) function of $X_{s,n,\underline{\gamma}}^{(D)}$ relying on $X_{1,n,\underline{\gamma}}^{(D)}, \dots, X_{r,n,\underline{\gamma}}^{(D)}$ is given by

$$PL^*(\underline{\Theta}, x_s|\underline{\mathbf{x}}_r) = C_s \left(\prod_{i=1}^{r-1} F^{m_i}(x_i; \underline{\Theta}) f(x_i; \underline{\Theta}) \right) F^{\gamma_r}(x_r; \underline{\Theta}) \left(\frac{f(x_r; \underline{\Theta}) f(x_s; \underline{\Theta})}{F(x_r; \underline{\Theta}) F(x_s; \underline{\Theta})} \right)$$

$$\times \sum_{j=r+1}^s a_j^{(r)}(s) \left(\frac{F(x_s; \underline{\Theta})}{F(x_r; \underline{\Theta})} \right)^{\gamma_j} \quad -\infty < x_s < x_r < \cdots < x_1 < \infty.$$

$$(4.2)$$

Proof According to Burkschat et al. (2003), after integrating the remaining variables, x_{r+1}, \dots, x_n , on the region $x_r > x_{r+1} > \cdots > x_n > -\infty$, the joint PDF of the first r DGOSs can be expressed as (4.1).

In view of Theorem 2.1 in Burkschat et al. (2003), the DGOSs form a Markov chain. Consequently, the conditional PDF of $X_{s,n,\underline{\gamma}}^{(D)}$ given that $X_{1,n,\underline{\gamma}}^{(D)} = x_1, \dots, X_{r,n,\underline{\gamma}}^{(D)} = x_r$ is equal to the conditional PDF of $X_{s,n,\underline{\gamma}}^{(D)}$ given that $X_{r,n,\underline{\gamma}}^{(D)} = x_r$. Following Kaminsky and Rhodin (1985); Barakat et al. (2018), Lemma 2.1 implies

$$f_{1,2,\dots,r,s}^{(D)}(x_1, \dots, x_r, x_s) = f_{1,2,\dots,r}^{(D)}(x_1, \dots, x_r) f_{X_{s,n,\underline{\gamma}}^{(D)}|X_{r,n,\underline{\gamma}}^{(D)}}(x_s|x_r)$$

$$= f_{1,2,\dots,r}^{(D)}(x_1, \dots, x_r) \frac{f_{X_{r,n,\underline{\gamma}}^{(D)}, X_{s,n,\underline{\gamma}}^{(D)}}(x_r, x_s)}{f_{X_{r,n,\underline{\gamma}}^{(D)}}(x_r)}$$

$$= \frac{C_s}{C_r} f_{1,2,\dots,r}^{(D)}(x_1, \dots, x_r) \sum_{j=r+1}^s a_j^{(r)}(s) \left(\frac{F(x_s)}{F(x_r)} \right)^{\gamma_j} \frac{f(x_s)}{F(x_s)}.$$

$$(4.3)$$

Hence, (4.2) follows directly from (4.1). This completes the proof. \square

For the inverse Weibull distribution, the log-likelihood function based on (4.1) can be simplified as

$$L(\sigma, \beta) \propto r \log \beta + r\beta \log \sigma - \beta \sum_{j=1}^r \log x_j - \sum_{j=1}^{r-1} (\gamma_j - \gamma_{j+1}) \left(\frac{x_j}{\sigma} \right)^{-\beta} - \gamma_r \left(\frac{x_r}{\sigma} \right)^{-\beta}.$$

$$(4.4)$$

The MLEs of σ and β can be obtained numerically using an iterative method like the Newton-Rophson method by solving the nonlinear equations

$$\frac{\partial L(\sigma, \beta)}{\partial \sigma} = 0 \quad \text{and} \quad \frac{\partial L(\sigma, \beta)}{\partial \beta} = 0. \quad (4.5)$$

If the scale parameter σ is known we have

$$\frac{\partial^2 L(\sigma, \beta)}{\partial \beta^2} = - \left[\frac{r}{\beta^2} + \sum_{j=1}^{r-1} (m_j + 1) x_j^* \left(\log \left(\frac{\sigma}{x_j} \right) \right)^2 + \gamma_r x_r^* \left(\log \left(\frac{\sigma}{x_r} \right) \right)^2 \right] < 0,$$

where $x_i^* = \left(\frac{x_i}{\sigma} \right)^{-\beta}$. This ensures that there exists a unique MLE of β (e.g. Mäkeläinen et al. (1981)). Similarly, the logarithm of the PL^* function can be written as

$$\begin{aligned} PL(x_s, \sigma, \beta) \propto & \log \left(\sum_{t=r+1}^s a_t^{(r)}(s) \exp(-\gamma_t (x_s^* - x_r^*)) \right) - \sum_{t=1}^{r-1} (\gamma_t - \gamma_{t+1}) x_t^* - \gamma_r x_r^* \\ & - (\beta + 1) \left(\sum_{t=1}^r \log x_t + \log x_s \right) + \beta(r + 1) \log \sigma + (r + 1) \log \beta, \end{aligned} \quad (4.6)$$

consequently, the MLP of x_s , as well as the PMLEs of σ and β , can be obtained numerically by solving the simultaneous equations

$$\frac{\partial PL(x_s, \sigma, \beta)}{\partial x_s} = 0, \quad \frac{\partial PL(x_s, \sigma, \beta)}{\partial \sigma} = 0, \quad \text{and} \quad \frac{\partial PL(x_s, \sigma, \beta)}{\partial \beta} = 0. \quad (4.7)$$

4.1 On the existence and uniqueness of the MLEs, MLP, and PMLEs

The main aim of this subsection is to discuss the existence and uniqueness of the MLEs, MLP, and PMLEs. Except in very limited circumstances, the analytical demonstration is a tough problem. Simulation can be used to provide an alternative solution for such problems. Clearly, the support of the inverse Weibull distribution does not depend on the distribution parameters, and the PDF is absolutely continuous in σ and β . Consequently, the function $L(\sigma, \beta)$ is the logarithm of a twice differentiable likelihood function with respect to σ and β in which (σ, β) varying in a connected open subset $\Theta \subset \mathbb{R}_+^2$. According to Mäkeläinen et al. (1981), there exists a unique MLEs if Hessian matrix $\mathbf{H}_L(\hat{\sigma}, \hat{\beta})$ of $L(\hat{\sigma}, \hat{\beta})$ is negative definite, where $\hat{\sigma}$ and $\hat{\beta}$ are the solutions of (4.5). The analytical derivation of the negative definite of the Hessian matrix is a difficult problem in most cases. Alternatively, in this work, a comprehensive simulation study based on 100,000 replicates is carried out to endorse the negative definite of the Hessian matrix for different values of the parameters of the selected models. Similar conclusions concerning the MLP and the PMLEs can be achieved via simulation. The

Table 1 The percentages of samples from which the Hessian matrices, $\mathbf{H}_L(\hat{\sigma}, \hat{\beta})$ and $\mathbf{H}_{PL}(\hat{x}_s, \hat{\sigma}, \hat{\beta})$, are negative definite for OOSs with selected values of σ and β

n	r	s	$\sigma = 10, \beta = 2$		$\sigma = 0.1, \beta = 0.5$		$\sigma = 50, \beta = 5$	
			$\mathbf{H}_L(\%)$	$\mathbf{H}_{PL}(\%)$	$\mathbf{H}_L(\%)$	$\mathbf{H}_{PL}(\%)$	$\mathbf{H}_L(\%)$	$\mathbf{H}_{PL}(\%)$
30	18	19		93.720		97.989		15.359
		22	99.928	95.854	98.670	97.486	99.664	97.544
		25		99.580		96.894		97.800
		28		99.508		95.886		98.003

Table 2 The percentages of samples from which the Hessian matrices, $\mathbf{H}_L(\hat{\sigma}, \hat{\beta})$ and $\mathbf{H}_{PL}(\hat{x}_s, \hat{\sigma}, \hat{\beta})$, are negative definite for SOSs with selected values of σ and β

n	r	s	$\sigma = 10, \beta = 2$		$\sigma = 0.1, \beta = 0.5$		$\sigma = 50, \beta = 5$	
			$\mathbf{H}_L(\%)$	$\mathbf{H}_{PL}(\%)$	$\mathbf{H}_L(\%)$	$\mathbf{H}_{PL}(\%)$	$\mathbf{H}_L(\%)$	$\mathbf{H}_{PL}(\%)$
30	18	19		98.550		98.484		29.671
		22	99.998	99.978	99.962	99.548	99.880	97.329
		25		99.991		99.486		98.103
		28		99.974		99.479		98.157

numerical solutions of (4.5) and (4.7) are obtained for each sample, after which the corresponding Hessian matrices of the obtained solutions are computed and they are checked to see if they are negative definite or not. The percentages of the samples from which Hessian matrices, $\mathbf{H}_L(\hat{\sigma}, \hat{\beta})$ and $\mathbf{H}_{PL}(\hat{x}_s, \hat{\sigma}, \hat{\beta})$, are negative definite, are shown in Tables 1 and 2 for OOSs and SOSs, respectively.

Remark 4.1 The simulation study, which is carried out for various values of r , s , and n (for brevity, we report selected values in Tables 1 and 2), reveals that:

1. In about 99% of the cases, the matrix \mathbf{H}_L is negative definite, which supports the existence of a unique MLEs of σ and β .
2. In at least 95% of the cases, the matrix \mathbf{H}_L is negative definite provided that $s > r + 1$, which backs up the existence of the MLP of $X_{s,n,\underline{\gamma}}^{(D)}$ and PMLEs of σ and β uniquely.
3. The OOSs and SOSs have no discernible differences.

4.2 The maximum likelihood reconstructor for the reversed OOSs

The maximum likelihood reconstructor (MLR) as well as the reconstructive maximum likelihood estimates (RMLEs) for the OOS are discussed in Asgharzadeh et al. (2012). After routine calculations, it can be shown that the reconstructive likelihood (RL^*) function of $X_{r:n}$, $r < s$ based on the reversed OOSs, $x_{s:n}, \dots, x_{n:n}$, takes the form

$$RL^*(x_r, \sigma, \beta | x_s, \dots, x_n) \propto \left(\prod_{j=s}^n f(x_j; \sigma, \beta) \right) (F(x_r; \sigma, \beta) - F(x_s; \sigma, \beta))^{s-r-1} (1 - F(x_r; \sigma, \beta))^{r-1} f(x_r; \sigma, \beta),$$

$x_r > x_s > \dots > x_n$. The log-likelihood function based on the inverse Weibull distribution can be written as

$$\begin{aligned} RL(x_r, \sigma, \beta) &\propto (n - s + 2)(\beta \log \sigma + \log \beta) - \sum_{j=s}^n x_j^* \\ &\quad - (\beta + 1) \sum_{j=s}^n \log x_j - x_r^* - (\beta + 1) \log x_r \\ &\quad + (s - r - 1) \log (e^{-x_r^*} - e^{-x_s^*}) + (r - 1) \log (1 - e^{-x_r^*}). \end{aligned}$$

The MLR of $X_{r:n}$, RMLEs of σ and β can be obtained numerically by solving the nonlinear system

$$\frac{\partial RL(x_r, \sigma, \beta)}{\partial x_r} = 0, \quad \frac{\partial RL(x_r, \sigma, \beta)}{\partial \sigma} = 0, \quad \text{and} \quad \frac{\partial RL(x_r, \sigma, \beta)}{\partial \beta} = 0. \quad (4.8)$$

Remark 4.2 In many practical situations, the parameters are unknown, and we have to replace them with their estimates. Consequently, some of the accuracy will be lost. In the next section, it is shown that when the unknown parameters are replaced with their estimates, the accuracy of the results is satisfactory compared with the ideal case of known parameters, provided that $s - r$ is not large. The comparison is based on the interval width and the coverage probability.

5 Numerical results

5.1 Simulation studies

In this section, simulation experiments are conducted to assess the efficiency of the obtained results in the preceding sections. For this aim, two special models from the DGOSs model are considered. The first one is the reversed OOSs with model parameters $\gamma_i = n - i + 1$, while the second one corresponding to the choice $\gamma_i = 2(n - i) + 1$ which may be interpreted as reversed SOSs. Here, two different cases are considered. In the first case, it is assumed that the inverse Weibull distribution parameters are known, with $\sigma = 10.0$ and $\beta = 2.0$ (Tables 3, 4, and 5). In the second case, the MLP is obtained and the parameters σ and β are replaced with their PMLEs (Tables 6, 7). In Table 8, the parameters σ and β are replaced with their RMLEs, which are obtained by (4.8). For comparison purposes, in the second case,

Table 3 Three 95% predictive intervals and their corresponding coverage probability of the reversed OOSs with parameters $\sigma = 10$ and $\beta = 2$

n	r	s	$X_{s+1:n}$	L_1	L_2	L_3	$X_{s:n}$	$X_{r:n}$	CP_1	CP_2	CP_3
30	18	19	9.880	9.469	9.466	9.465	10.339	10.822	94.955	94.909	94.916
		20	9.442	8.783	8.780	8.777	9.880	10.822	94.940	94.971	94.995
		21	9.016	8.221	8.217	8.214	9.442	10.822	94.949	94.931	94.921
		22	8.602	7.722	7.718	7.712	9.016	10.822	94.897	94.856	94.907
		23	8.194	7.263	7.260	7.251	8.602	10.822	95.059	95.054	95.085
		24	7.786	6.829	6.826	6.816	8.194	10.822	95.040	95.060	95.049
		25	7.371	6.409	6.406	6.395	7.786	10.822	94.980	95.015	94.975
		26	6.936	5.991	5.988	5.977	7.371	10.822	95.028	95.034	95.014
		27	6.463	5.562	5.560	5.549	6.936	10.822	95.046	95.025	95.029
		22	8.602	8.252	8.251	8.250	9.016	9.442	94.952	94.936	94.953
		23	8.194	7.631	7.631	7.627	8.602	9.442	94.962	94.994	94.978
		24	7.786	7.106	7.108	7.092	8.194	9.442	95.004	95.031	95.131
		25	7.371	6.624	6.627	6.618	7.786	9.442	95.056	95.026	95.029
		26	6.936	6.161	6.164	6.153	7.371	9.442	95.012	94.986	94.993
30	21	27	6.463	5.695	5.700	5.687	6.936	9.442	94.926	95.017	94.944
		25	7.371	7.057	7.057	7.055	7.786	8.194	94.900	94.900	94.920
		26	6.936	6.440	6.443	6.437	7.371	8.194	95.036	95.068	95.026
		27	6.463	5.887	5.893	5.883	6.936	8.194	95.052	95.078	95.032

we generate DGOSs from the inverse Weibull distribution with $\sigma = 10.0$ and $\beta = 2.0$ as in the first case.

5.2 Algorithms

In view of the results of Burkschat et al. (2003), the r th DGOS can be generated by the following algorithm:

Algorithm 1 (Generating dual generalized order statistics)

- Step 1. Choose the values of n , k , and the DGOSs model parameters, γ_i , $i = 1, 2, \dots, n$,
- Step 2. generate a random sample of size n say B_1, B_2, \dots, B_n , from beta distribution with CDF, $G(t) = t^{\gamma_j}$, $0 \leq t \leq 1$,
- Step 3. compute the r th DGOS from any continuous distribution by the relation

$$X_{r,n,\underline{\gamma}}^{(D)} = F^{-1} \left(\prod_{j=1}^r B_j \right), \quad r = 1, 2, \dots, n,$$

Table 4 Three 95% predictive intervals and their corresponding coverage probability of $X_{s,n,\gamma}^{(D)}$, the reversed SOSs ($\gamma_i = 2(n-i) + 1$), with parameters $\sigma = 10$, and $\beta = 2$

n	r	s	$X_{s+1,n,\gamma}^{(D)}$	L_1	L_2	L_3	$X_{s,n,\gamma}^{(D)}$	$X_{r,n,\gamma}^{(D)}$	CP_1	CP_2	CP_3
30	18	19	13.782	13.201	13.197	13.195	14.433	15.115	95.011	95.021	94.993
		20	13.154	12.230	12.225	12.221	13.782	15.115	95.050	95.042	95.069
		21	12.551	11.432	11.427	11.407	13.154	15.115	94.915	94.936	95.038
		22	11.956	10.722	10.718	10.708	12.551	15.115	94.900	94.901	94.952
		23	11.369	10.067	10.063	10.051	11.956	15.115	94.978	94.953	94.955
		24	10.777	9.445	9.441	9.427	11.369	15.115	94.933	94.986	94.917
		25	10.169	8.838	8.835	8.820	10.777	15.115	94.930	94.876	94.913
30	21	26	9.525	8.229	8.225	8.210	10.169	15.115	94.961	94.936	94.976
		27	8.808	7.592	7.590	7.574	9.525	15.115	95.027	94.907	94.987
		22	11.956	11.461	11.459	11.458	12.551	13.154	95.019	95.051	95.041
		23	11.369	10.576	10.577	10.570	11.956	13.154	94.977	94.968	94.958
		24	10.777	9.824	9.827	9.817	11.369	13.154	94.983	95.019	94.987
		25	10.169	9.129	9.134	9.120	10.777	13.154	95.009	95.071	95.022
		26	9.525	8.454	8.461	8.444	10.169	13.154	95.131	95.074	95.120
30	24	27	8.808	7.765	7.773	7.754	9.525	13.154	95.080	95.030	95.090
		25	10.169	9.729	9.729	9.727	10.777	11.369	95.073	95.113	95.086
		26	9.525	8.833	8.838	8.829	10.169	11.369	94.987	94.985	94.994
		27	8.808	8.018	8.028	8.013	9.525	11.369	94.893	94.912	94.909

Table 5 Two 95% reconstructive intervals and their corresponding coverage probability of the reversed OOSs based on the reconstructive pivotal quantities U_1 and V_1 with parameters $\sigma = 10$, and $\beta = 2$

n	s	r	$X_{s+1:n}$	U_{U_1}	U_{V_1}	$CP_{U_{U_1}}$	$CP_{U_{V_1}}$	$X_{r:n}$	$X_{r-1:n}$
30	12	11	14.612	17.32590	17.32590	94.953	94.953	15.495	16.500
		10	14.612	19.28550	19.28561	94.908	94.908	16.500	17.661
		9	14.612	21.40594	21.40584	95.005	95.005	17.661	19.040
		8	14.612	23.87873	23.87863	94.951	94.951	19.040	20.718
		7	14.612	26.91532	26.91237	95.027	95.024	20.718	22.837
		6	14.612	30.82946	30.82785	94.931	94.928	22.837	25.645
		5	14.612	36.21276	36.21321	94.868	94.869	25.645	29.616
		4	14.612	44.30626	44.30697	94.914	94.914	29.616	35.859
		3	14.612	58.32321	58.31664	95.010	95.010	35.859	48.310
		2	14.612	90.05960	90.05936	94.972	94.972	48.310	98.456
		8	17.661	21.94263	21.94263	94.982	94.982	19.040	20.718
		7	17.661	25.36668	25.36668	95.069	95.069	20.718	22.837
		6	17.661	29.49199	29.49198	95.011	95.011	22.837	25.645
		5	17.661	35.01182	35.01182	94.843	94.843	25.645	29.616
		4	17.661	43.19703	43.19700	94.955	94.955	29.616	35.859
		3	17.661	57.26681	57.26681	94.932	94.932	35.859	48.310
30	9	2	17.661	89.03582	89.03582	94.930	94.930	48.310	98.456
		5	22.837	31.62140	31.62140	94.882	94.882	25.645	29.616
		4	22.837	40.48920	40.48920	94.798	94.798	29.616	35.859
		3	22.837	54.90926	54.90926	94.955	94.955	35.859	48.310
		2	22.837	86.88093	86.88093	94.957	94.957	48.310	98.456

Step 4. for the inverse Weibull distribution, compute the r th DGOS from the formula

$$X_{r,n,\underline{\gamma}}^{(D)} = \sigma \left(- \sum_{i=1}^r \log B_j \right)^{-\frac{1}{\beta}}, \quad r = 1, 2, \dots, n.$$

Algorithm 2 (Constructing predictive (reconstructive) intervals and computing their coverage probability)

- Step 1. Determine the distribution parameters, σ and β ,
- Step 2. determine k , γ_i , and n , the number of DGOSs to be generated,
- Step 3. use Algorithm 1 to generate and store $M \times n$ arrays, each of which contains n DGOSs based on the inverse Weibull distribution, where M is the number of repetitions,
- Step 4. specify the number of observed DGOSs and the number of unknown DGOSs that required to be predicted or reconstructed,
- Step 5. apply Theorems 2.1, 2.2, and 2.3 to find the required quantiles q_i by solving the nonlinear equations $F_{U_i}(q_i) = 1 - \tau$, $i = 1, 2, 3$, for the prediction problem,

Table 6 The MLP, PMLEs, and three 95% predictive intervals with their corresponding coverage probability of the reversed OOSs, $X_{s:n}$, when the unknown parameters are replaced with their PMLEs

n	r	s	$X_{s+1:n}$	$\hat{X}_{s:n}$	$X_{s:n}$	L_1	L_2	L_3	U	CP_1	CP_2	CP_3	$\hat{\sigma}$	$\hat{\beta}$
30	18	19	9.87	10.65	10.32	9.41	9.41	9.42	10.80	94.49	94.52	94.53	10.19	1.99
		20	9.43	9.97	9.87	8.84	8.84	8.82	10.80	95.10	95.07	95.11	10.24	2.15
		21	9.00	9.54	9.43	8.32	8.31	8.30	10.80	95.20	95.11	95.12	10.25	2.18
		22	8.59	9.50	9.00	7.98	7.98	7.94	10.80	86.74	86.75	87.43	10.57	2.31
		23	8.19	8.97	8.59	7.51	7.51	7.47	10.80	87.52	87.64	88.32	10.46	2.27
		24	7.78	8.58	8.19	7.11	7.11	7.06	10.80	85.22	85.25	86.02	10.46	2.28
		25	7.36	8.18	7.78	6.71	6.71	6.66	10.80	83.80	83.86	84.61	10.44	2.27
30	21	26	6.93	7.78	7.36	6.32	6.32	6.27	10.80	82.40	82.40	83.31	10.43	2.28
		27	6.47	7.35	6.93	5.91	5.90	5.86	10.80	81.34	81.40	82.23	10.41	2.27
		22	8.59	9.40	9.00	8.27	8.27	8.27	9.43	93.15	93.02	93.30	10.26	2.08
		23	8.19	8.69	8.59	7.68	7.69	7.67	9.43	94.76	94.80	94.88	10.19	2.13
		24	7.78	8.29	8.19	7.18	7.19	7.16	9.43	95.38	95.60	95.54	10.20	2.15
		25	7.36	8.17	7.78	6.82	6.84	6.79	9.43	88.08	87.69	88.54	10.40	2.25
		26	6.93	7.67	7.36	6.36	6.37	6.33	9.43	87.77	87.28	88.24	10.32	2.23
30	24	27	6.47	7.26	6.93	5.92	5.94	5.89	9.43	85.91	85.72	86.65	10.32	2.23
		25	7.36	8.18	7.78	7.09	7.10	7.09	8.19	92.88	92.46	93.04	10.20	2.09
		26	6.93	7.51	7.36	6.50	6.51	6.49	8.19	94.42	94.39	94.38	10.18	2.13
		27	6.47	7.08	6.93	5.97	5.99	5.96	8.19	95.16	95.42	95.20	10.18	2.14

Table 7 The MLP, PMLEs, and three 95% predictive intervals with their corresponding coverage probability of the reversed SOSs ($\gamma_i = 2(n - i) + 1$), $X_{s,n,\underline{\gamma}}^{(D)}$, when the unknown parameters are replaced with their PMLEs

n	r	s	$X_{s+1,n,\underline{\gamma}}^{(D)}$	$\widehat{X}_{s,n,\underline{\gamma}}^{(D)}$	$X_{s,n,\underline{\gamma}}^{(D)}$	L_1	L_2	L_3	U	CP_1	CP_2	CP_3	$\widehat{\sigma}$	$\widehat{\beta}$
30	18	19	13.80	15.09	14.45	13.13	13.12	13.14	15.13	94.22	94.16	94.29	10.20	1.98
		20	13.17	14.01	13.80	12.37	12.37	12.34	15.13	94.73	94.72	94.79	10.52	2.18
		21	12.56	13.54	13.17	11.67	11.67	11.62	15.13	93.36	93.36	93.68	10.68	2.23
		22	11.97	13.11	12.56	11.07	11.07	11.01	15.13	89.48	89.59	90.06	10.84	2.27
		23	11.39	12.61	11.97	10.49	10.49	10.43	15.13	85.95	85.94	86.75	10.89	2.30
		24	10.80	12.00	11.39	9.89	9.89	9.83	15.13	84.68	84.65	85.33	10.84	2.29
		25	10.19	11.39	10.80	9.31	9.31	9.24	15.13	83.58	83.46	84.40	10.80	2.28
30	21	26	9.54	10.77	10.19	8.72	8.72	8.65	15.13	82.55	82.46	83.35	10.76	2.28
		27	8.82	10.12	9.54	8.10	8.10	8.04	15.13	80.89	81.15	81.91	10.72	2.27
		22	11.97	13.17	12.56	11.49	11.50	11.49	13.17	93.84	93.44	93.94	10.36	2.06
		23	11.39	12.11	11.97	10.68	10.69	10.66	13.17	95.12	95.12	95.10	10.33	2.13
		24	10.80	11.69	11.39	10.01	10.03	9.98	13.17	93.73	93.73	93.95	10.50	2.19
		25	10.19	11.25	10.80	9.40	9.43	9.37	13.17	90.29	90.11	90.67	10.63	2.23
		26	9.54	10.69	10.19	8.79	8.82	8.75	13.17	86.98	86.45	87.52	10.66	2.24
30	24	27	8.82	10.03	9.54	8.12	8.16	8.08	13.17	85.93	85.29	86.46	10.61	2.24
		25	10.19	11.39	10.80	9.81	9.82	9.80	11.39	93.06	92.73	93.24	10.35	2.09
		26	9.54	10.46	10.19	8.97	9.00	8.95	11.39	93.56	93.55	93.71	10.38	2.15
		27	8.82	9.84	9.54	8.19	8.23	8.17	11.39	93.44	93.42	93.59	10.39	2.16

Table 8 Two 95 % reconstructive intervals with their coverage probability, the MLR, and RMLEs of the reversed OOSs, $X_{r:n}$, based on the reconstructive pivotal quantities U_1 and V_1 when the parameters are unknown

n	s	r	L	U_{u_1}	U_{v_1}	$CP_{U_{u_1}}$	$CP_{U_{v_1}}$	$\hat{X}_{r:n}$	$X_{r:n}$	$\hat{\sigma}$	$\hat{\beta}$
30	12	11	14.62	17.50536	17.50536	95.839	95.839	15.50	16.50	10.18	1.90
		10	14.62	19.19934	19.19945	95.876	95.877	16.51	16.28	10.04	2.11
		9	14.62	21.27563	21.27554	95.988	95.988	17.67	17.43	10.04	2.10
		8	14.62	23.71133	23.71123	96.184	96.184	19.05	18.83	10.04	2.09
		7	14.62	26.72493	26.72203	96.306	96.300	20.73	20.54	10.04	2.09
		6	14.62	30.65209	30.65049	96.533	96.532	22.84	22.76	10.05	2.08
		5	14.62	36.14774	36.14819	96.788	96.788	25.62	25.78	10.06	2.07
		4	14.62	44.63857	44.63930	96.991	96.991	29.54	30.36	10.07	2.06
		3	14.62	60.27348	60.26640	97.403	97.401	35.76	38.90	10.09	2.04
		2	14.62	101.71202	101.71172	97.787	97.787	48.08	69.83	10.16	2.01
30	9	8	17.67	22.18812	22.18812	95.659	95.659	19.05	20.72	10.15	1.92
		7	17.67	25.27399	25.27399	95.865	95.865	20.73	20.48	10.05	2.09
		6	17.67	29.35946	29.35945	95.966	95.966	22.84	22.60	10.05	2.09
		5	17.67	34.89761	34.89761	96.242	96.242	25.62	25.57	10.05	2.08
		4	17.67	43.25189	43.25186	96.501	96.501	29.54	30.03	10.06	2.07
		3	17.67	58.25547	58.25548	96.827	96.827	35.76	38.40	10.08	2.05
		2	17.67	96.29435	96.29435	97.223	97.223	48.08	69.02	10.13	2.02
		5	22.84	32.11538	32.11538	95.546	95.546	25.62	29.54	10.13	1.93
		4	22.84	40.52436	40.52436	95.926	95.926	29.54	30.11	10.06	2.07
		3	22.84	55.39952	55.39952	96.267	96.267	35.76	38.18	10.07	2.06
30	6	2	22.84	91.21197	91.21197	96.669	96.669	48.08	68.57	10.11	2.03

- Step 6. for the reconstruction problem, apply Theorems 2.1, 3.1 (for small values of n) and Theorems 3.2, and 3.3 (for large values of n) to compute the required quantiles,
- Step 7. find the MLP and the PMLEs of the parameters based on the first r DGOSs, from (4.7),
- Step 8. from the obtained results of Sects. 2 and 3, compute the upper and lower limits of the predictive (reconstructive) intervals, when:
- (i) the true values of parameters are known and
 - (ii) the parameters are replaced with their PMLEs or RMLEs,
- Step 9. check whether the observed value of $X_{s,n,\underline{\gamma}}^{(D)}$ ($X_{r,n,\underline{\gamma}}^{(D)}$) did belong to the predictive (reconstructive) interval or not?
- Step 10. repeat Steps 7, 8, and 9 M times,
- Step 11. finally, compute the percentage of coverage probability, that is, the percent that the true value of the unobserved DGOS lies inside the predictive (reconstructive) interval, the average of the lower and upper limits.

Remark 5.1 1. The simulation studies are based on $M = 100,000$ replicates.
 2. All the computations in this paper are performed by Mathematica 12.3.

6 Conclusion

In this paper, some predictive results concerning DGOSs based on the inverse Weibull distribution were considered. More specifically, different predictive and reconstructive pivotal quantities were proposed and their exact distributions were derived. Accordingly, some predictive and reconstructive intervals were constructed. Moreover, the MLP and the PMLEs of DGOSs based on the inverse Weibull distribution were discussed. A comprehensive simulation study backs up the existence and uniqueness of the MLP and PMLEs (Tables 1 and 2). The simulation studies revealed that:

1. The probability coverage is closer to the theoretical level (95%) whenever the distribution parameters are known.
2. If the distribution parameters are unknown, the probability coverage is lower than the theoretical level. However, the predictive intervals become shorter.
3. In most cases, the lower limits of the three predictive intervals are close to each other up to at least one decimal place.
4. As $s - r$ increases, the interval width increases for all the predictive and reconstructive intervals.
5. The MLP, RML, PMLEs, and RMLEs perform well when they are compared with the true values, whenever $s - r$ is small.
6. For small samples, the reconstructive pivotal quantities U_1 and V_1 are recommended (see Tables 5 and 8).
7. If the sample size is greater than 30, the predictive pivotal quantity U_2 and the reconstructive pivotal quantities V_2 and V_3 are recommended.

Acknowledgements The author is grateful to Professor Werner G. Müller and the referees for their many valuable comments which improve the presentation of the paper substantially.

Funding Open access funding provided by The Science, Technology & Innovation Funding Authority (STDF) in cooperation with The Egyptian Knowledge Bank (EKB).

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References

- Ahsanullah M (1980) Linear prediction of record values for the two parameter exponential distribution. *Ann Inst Stat Math* 32(3):363–368. <https://doi.org/10.1007/BF02480340>
- Ahsanullah M (2004) Characterization of the uniform distribution by dual generalized order statistics. *Commun Stat* 33(12):2921–2928
- Akgül FG, Senoğlu B, Arslan T (2016) An alternative distribution to Weibull for modeling the wind speed data: inverse Weibull distribution. *Energy Convers Manage* 114:234–240
- Al-Hussaini EK (1999) Predicting observables from a general class of distributions. *J Stat Plan Inference* 79:79–91
- Al-Hussaini EK, Ahmad AB (2003) On Bayesian predictive distributions of generalized order statistics. *Metrika* 57(2):165–176
- Al-Mutairi JS, Raqab MZ (2020) Confidence intervals for quantiles based on samples of random sizes. *Stat Pap* 61(1):261–277. <https://doi.org/10.1007/s00362-017-0935-3>
- Aly AE (2015) Prediction intervals of future generalized order statistics based on generalized extreme value distribution. *Prob Stat Forum* 8:148–156
- Aly AE (2016) Prediction and reconstruction of future and missing unobservable modified Weibull lifetime based on generalized order statistics. *J Egyptian Math Soc* 24(2):309–318. <https://doi.org/10.1016/j.joems.2015.04.002>
- Aly AE (2022) Prediction of the exponential fractional upper record values. *Math Slovaca* 72(2):491–506. <https://doi.org/10.1515/ms-2022-0032>
- Aly AE, Barakat HM, El-Adll ME (2019) Prediction intervals of the record-values process. *Revstat Stat J* 17(3):401–427
- Asgharzadeh A, Ahmadi J, Mirzazadeh ZG, Valiollahi R (2012) Reconstruction of the past failure times for the proportional reversed hazard rate mode. *J Stat Comput Simul* 82(3):475–489. <https://doi.org/10.1080/00949655.2010.542550>
- Barakat HM, El-Adll ME (2009) Asymptotic theory of extreme dual generalized order statistics. *Statist Probab Lett* 79:1252–1259. <https://doi.org/10.1016/j.spl.2009.01.015>
- Barakat HM, El-Adll ME, Aly AE (2011) Exact prediction intervals for future exponential lifetime based on random generalized order statistics. *Comput Math Appl* 61(5):1366–1378. <https://doi.org/10.1016/j.camwa.2011.01.002>
- Barakat HM, El-Adll ME, Aly AE (2014) Prediction intervals of future observations for a sample of random size from any continuous distribution. *Math Comput Simul* 97:1–13. <https://doi.org/10.1016/j.matcom.2013.06.007>
- Barakat HM, Khaled OM, Ghonem HA (2020) Predicting future lifetime for mixture exponential distribution. *Commun Stat*. <https://doi.org/10.1080/03610918.2020.1715434>
- Barakat HM, Khaled OM, Ghonem HA (2021) New method for prediction of future order statistics. *Qual Technol Quant Manag* 18(10):101–116. <https://doi.org/10.1080/16843703.2020.1782087>

- Barakat HM, Nigm EM, El-Adll ME, Yusuf M (2018) Prediction of future exponential lifetime based on random number of generalized order statistics under a general set-up. *Stat Pap* 59(2):605–631. <https://doi.org/10.1007/s00362-016-0779-2>
- Burkschat M, Cramer E, Kamps U (2003) Dual generalized order statistics. *Metron* 61:13–26
- David HA, Nagaraja HN (2003) Order statistics, 3rd edn. Wiley, Hoboken, NJ
- El-Adll ME (2011) Predicting future lifetime based on random number of three parameters Weibull distribution. *Math Comput Simul* 81(9):1842–1854. <https://doi.org/10.1016/j.matcom.2011.02.003>
- El-Adll ME (2021) Inference for the two-parameter exponential distribution with generalized order statistics. *Math Popul Stud* 28(10):1–23. <https://doi.org/10.1080/08898480.2019.1681187>
- El-Adll ME, Aly AE (2016) Prediction intervals of future generalized order statistics from pareto distribution. *J Appl Stat Sci* 22(1–2):111–125
- El-Adll ME, Aly AE (2016) Reconstructing past fractional record values. *J Egyptian Math Soc* 24(4):622–628. <https://doi.org/10.1016/j.joems.2016.04.002>
- Geisser S (1993) Predictive inference: an introduction. Chapman and Hall, London
- Kaminsky KS, Rhodin LS (1985) Maximum likelihood prediction. *Ann Inst Stat Math* 37(1):507–517. <https://doi.org/10.1007/BF02481119>
- Kamps U (1995) A concept of generalized order statistics. Teubner, Stuttgart
- Kamps U, Cramer E (2001) On distributions of generalized order statistics. *Statistics* 35:269–280. <https://doi.org/10.1080/02331880108802736>
- Kotb MS, Raqab MZ (2021) Estimation of reliability for multi-component stress-strength model based on modified Weibull distribution. *Stat Pap* 62:2763–2797. <https://doi.org/10.1007/s00362-020-01213-0>
- Lawless JF (1971) A prediction problem concerning samples from the exponential distribution with applications in life testing. *Technometrics* 13:725–730. <https://doi.org/10.2307/1266949>
- Lawless JF (1977) Prediction intervals for the two parameter exponential distribution. *Technometrics* 19(4):469–472. <https://doi.org/10.2307/1267887>
- Lingappaiah GS (1973) Prediction in exponential life testing. *Can J Stat* 1:113–117. <https://doi.org/10.2307/3314650>
- Mäkeläinen T, Schmidt K, Styan GPH (1981) On the existence and uniqueness of the maximum likelihood estimate of a vector-valued parameter in fixed-size samples. *Ann Stat* 9(4):758–767. <https://doi.org/10.1214/aos/1176345516>
- Nagaraja HN (1986) Comparison of estimators and predictors from two-parameter exponential distribution. *Sankhy Ser B* 48:10–18
- Shah Imtiyaz A, Barakat HM, Khan AH (2020) Characterizations through generalized and dual generalized order statistics, with an application to statistical prediction problem. *Stat Probab Lett* 163:108782. <https://doi.org/10.1016/j.spl.2020.108782>
- Raqab ZM (2001) Optimal prediction-intervals for the exponential distribution based on generalized order statistics. *IEEE Trans Reliab* 50(1):112–115. <https://doi.org/10.1109/24.935025>
- Sukhatme BV (1937) Tests of significance for samples of the χ^2 population with two degrees of freedom. *Ann Eugen* 8:52–60
- Sultan KS, Abd Ellah AH (2006) Exact prediction interval for exponential lifetime based on random sample size. *Int J Comput Math* 83(12):867–878
- Wang J, Qin S, Jin S, Wu J (2015) Estimation methods review and analysis of offshore extreme wind speeds and wind energy resources. *Renew Sustain Energy Rev* 42:26–42