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A family of condorcet domains that are single-peaked on a circle

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Abstract
Fishburn's alternating scheme domains occupy a special place in the theory of Condorcet domains. Karpov (2023) generalised these domains and made an interesting observation proving that all of them are single-peaked on a circle. However, an important point that all generalised Fishburn domains are maximal Condorcet domain remained unproved. We fill this gap and suggest a new combinatorial interpretation of generalised Fishburn's domains which provide a constructive proof of single-peakedness of these domains on a circle. We show that classical single-peaked domains and single-dipped domains as well as Fishburn's alternating scheme domains belong to this family of domains while single-crossing domains do not.

1 Introduction

A Condorcet domain is a set of linear orders on a given set of alternatives such that, if all voters of a certain society are known to have preferences over those alternatives represented by linear orders from that set, the pairwise majority relation of this society is acyclic. Maximal Condorcet domains historically attracted a special attention since they represent a compromise which allows a society to always have transitive collective decisions and, under this constraint, provide voters with as much individual freedom as possible. Thus the question: "How large a Condorcet domain can be?" has attracted even more attention. Kim et al. (1992) identified this problem as a major unsolved problem in the mathematical social sciences.

Craven (1992) conjectured that the single-peaked domain containing 2^{n-1} orders was the largest. Fishburn (1996) was the first to venture beyond this barrier: based on example by Monjardet (2009) he constructed Condorcet domains which are asymp-

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totically n times larger. He called them alternating scheme domains but now they are known as Fishburn's domains.

With the introduction by Karpov (2023) of generalised Fishburn's domains we have gained a new insight into the structure of the universe of Condorcet domains. It appeared that Fishburn's domains and single-peaked domains are close relatives and are the two extremes of a certain spectrum of Condorcet domains. In this paper we investigate the domains from this spectrum and show that they are all single-peaked on a circle.

Let us briefly touch some of the basics of Condorcet domains. More information about them can be found in Monjardet (2009); Puppe and Slinko (2024).

One of the best known Condorcet domains is the domain of single-peaked linear orders on a line spectrum Black (1958). Recently Peters and Lackner (2020) generalised this domain to single-peaked domains on a circle. And, although so generalised domains are not necessarily Condorcet, as will be demonstrated in this paper, they have a role to play in the theory of Condorcet domains too.

Intuitively, a domain is single-peaked on a circle if all the alternatives can be placed on a circle so that, for every order of the domain, we can 'cut' the circle once so that the given order becomes single-peaked on the resulting line spectrum. The location of the cutting point may differ for different orders of the domain.

By $\mathcal{L}(A)$ we will denote all linear orders on the set of alternatives A which will always be assumed to be finite. For a linear order $v \in \mathcal{L}(A)$ and two alternatives $x, y \in A$ we write $x \succ_v y$ if v ranks x higher than y. The set of alternatives A is often taken as $[n] = \{1, 2, \ldots, n\}$. Up to isomorphisms, for $n = 3$ there are only three maximal Condorcet domains:

$$\mathcal{D}_1 = \{123, 312, 132, 321\},\ \mathcal{D}_2 = \{123, 231, 132, 321\},\ \mathcal{D}_3 = \{123, 213, 231, 321\}.$$

The domain $\mathcal{D}_1$ on the left contains all the linear orders on [3] in which 2 is never ranked first, the domain $\mathcal{D}_2$ in the middle contains all the linear orders on [3] in which 1 is never ranked second, and domain $\mathcal{D}_3$ on the right contains all the linear orders on [3] in which 2 is never ranked last. Following Monjardet (2009), we denote these conditions as $2N_{\{1,2,3\}}1$, and $1N_{\{1,2,3\}}2$, and $2N_{\{1,2,3\}}3$, respectively.

Definition 1 Any condition of type $xN_{\{a,b,c\}}i$ with $x \in \{a, b, c\}$ and $i \in \{1, 2, 3\}$ is called a *never condition* since it being applied to a domain $\mathcal{D}$ requires that in orders of the restriction $\mathcal{D}|_{\{a,b,c\}}$ of $\mathcal{D}$ to $\{a, b, c\}$ alternative x never takes ith position. We say that a subset $\mathcal{N}$ of

$$\{xN_{\{a,b,c\}}i \mid \{a, b, c\} \subseteq A,\ x \in \{a, b, c\} \text{ and } i \in \{1, 2, 3\}\}$$

is a *complete set of never-conditions* if $\mathcal{N}$ contains exactly one never condition for every triple a, b, c of elements of A.

The following criterion is a well-known characterisation of Condorcet domains that goes back to Sen (1966). See also Theorem 1(d) in Puppe and Slinko (2019) and references therein.

Criterion 1 *A domain of linear orders $\mathcal{D} \subseteq \mathcal{L}(A)$ is a Condorcet domain if and only if it satisfies a complete set of never conditions.*

The following property of Condorcet domains was shown to be very important.

Definition 2 (Slinko 2019) A Condorcet domain $\mathcal{D}$ is *copious* if for any triple of alternatives $a, b, c \in A$ the restriction $\mathcal{D}|_{\{a,b,c\}}$ of this domain to this triple has four distinct orders, that is, $|\mathcal{D}|_{\{a,b,c\}}| = 4$.

For $n \geq 5$ not all maximal Condorcet domains are copious Slinko (2019). We note that, if a Condorcet domain is copious, then it satisfies a unique complete set of never conditions. Copiousness is often an important step in proving maximality of the domain.

Proposition 1 *Let $\mathcal{N}$ be a complete set of never conditions and $\mathcal{D}(\mathcal{N})$ is the set of all linear orders from $\mathcal{L}(A)$ that satisfy $\mathcal{N}$. If $\mathcal{D}(\mathcal{N})$ is copious, then $\mathcal{D}(\mathcal{N})$ is a maximal Condorcet domain.*

Proof Suppose $\mathcal{D}(\mathcal{N})$ is copious but not maximal. Then there exists a linear order u such that $\mathcal{D}' = \mathcal{D}(\mathcal{N}) \cup \{u\}$ is a larger Condorcet domain. Since $u \notin \mathcal{D}(\mathcal{N})$ for a certain triple of alternatives a, b, c the domain $\mathcal{D}'|_{\{a,b,c\}}$ contains an order on a, b, c which is not in $\mathcal{D}|_{\{a,b,c\}}$. But then $\mathcal{D}'|_{\{a,b,c\}}$ contains five orders on a, b, c which is not possible as it would not be a Condorcet domain. $\square$

Many Condorcet domains are defined relative to some sort of *societal axis*, also called *spectrum*. In politics it is often referred to as left-right spectrum of political opinions.

Definition 3 Let $A = \{a_1, \ldots, a_n\}$. A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is said to be (classical) single-peaked if there exists a societal axis (spectrum)

$$a_1 \lhd a_2 \lhd \cdots \lhd a_n$$

such that for every linear order $v \in \mathcal{L}(A)$ and $a \in A$ the upper contour set $U(a, v) = \{b \in A \mid b \succ_v a\}$ is convex relative to the spectrum. By $SP_n(\lhd)$ we will denote the domain of all single-peaked orders on $\lhd$.

Up to isomorphisms A is often taken as $[n]$ and the societal axis as $1 < 2 < \cdots < n$.

Never conditions allow us to define useful classes of Condorcet domains as was pioneered by Peter Fishburn (1996) who introduced the so-called alternating scheme of never conditions and constructed Condorcet domains of large order. Karpov (2023) generalised his scheme as follows.

Definition 4 (Karpov 2023) Let $A = [n]$. A complete set of never conditions is said to be a *generalised alternating scheme*, if for some subset $K \subseteq [2, \ldots, n-1]$ and for all $1 \leq i < j < k \leq n$ we have

$$jN_{\{i,j,k\}}3, \text{ if } j \in K, \text{ and } jN_{\{i,j,k\}}1, \text{ if } j \notin K. \tag{1}$$

The domain which consists of all linear orders satisfying the generalised alternating scheme is called the *generalised Fishburn's domain* or *GF-domain*.

Fishburn's original alternating scheme has K equal to the set of even numbers in $[2, \ldots, n-1]$. The GF-domain constructed using a subset $K \subseteq [2, \ldots, n-1]$ will be denoted as F_K. Every GF-domain has orders $\bar{e} = 12 \ldots n$ and $n \ldots 21$ as they obviously satisfy conditions (1). Domains with this property are said to have *maximal width* (Puppe, 2018).

2 Generalised Fishburn's domains and their combinatorial representation

The idea of this representation comes from an example in Danilov et al. (2010).

A set of n vertices on a circle, some white and some black, are numbered by integers $1, 2, \ldots, n$. We will often identify the vertices with the numbers on them.

Definition 5 An arrangement of n black and white vertices on a circle numbered by integers $1, 2, ..., n$ (not becessarily in any particular order) will be called a *necklace* and the vertices themselves will be called *beads*.

Definition 6 A set of beads $X \subseteq [n]$ is said to be *white convex* (or simply *w-convex*) if

(a) X does not consist of a single black bead;
(b) There does not exist $i < j < k$ such that $i, k \in X$, $j \notin X$ and j is white;
(c) X is an arc of the circle.

Definition 7 A *flag* of w-convex sets is a sequence $X_1, \ldots, X_n$ of w-convex sets

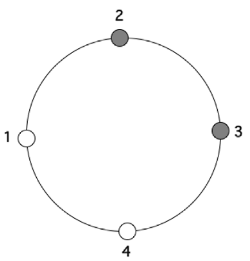
$$X_1 \subset X_2 \subset \cdots \subset X_n = [n], \tag{2}$$

where $|X_k| = k$.

Any flag (2) of w-convex sets defines a linear order $v = x_1 x_2 \ldots x_n$ on $[n]$, where $\{x_i\} = X_i \setminus X_{i-1}$ (for convenience we assume that $X_0 = \emptyset$).

Definition 8 Given a necklace S the domain $\mathcal{D}(S)$ is the set of all linear orders corresponding to flags of w-convex sets.

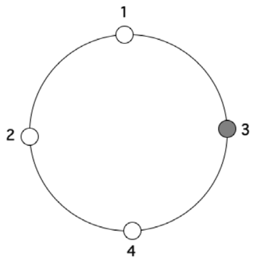
Example 1 Consider now the necklace S presented on the picture on the left:



1	1	1	1	4	4	4	4
2	2	4	4	1	1	3	3
3	4	2	3	2	3	1	2
4	3	3	2	3	2	2	1

This is a single-dipped domain relative to the spectrum $1 \lhd 2 \lhd 3 \lhd 4$ or $4 \lhd 3 \lhd 2 \lhd 1$.

Example 2 Consider now the necklace S presented on the picture on the left:



$$\begin{array}{ccccccccc}
1 & 1 & 2 & 2 & 2 & 2 & 4 & 4 & 4 \\
2 & 2 & 1 & 1 & 4 & 4 & 2 & 2 & 3 \\
3 & 4 & 3 & 4 & 1 & 3 & 1 & 3 & 2 \\
4 & 3 & 4 & 3 & 3 & 1 & 3 & 1 & 1
\end{array}$$

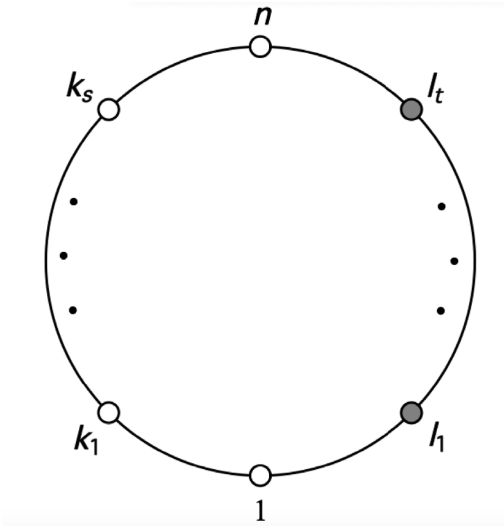
Then domain $\mathcal{D}(S)$ is given by the array on the right. This is the Fishburn domain relative to the spectrum $1 \lhd 2 \lhd 3 \lhd 4$.

Our example shows that the construction is promising and generating maximal GF-domains. Let us generalise these examples and offer a new combinatorial representation of GF-domains from which we will deduce their maximality.

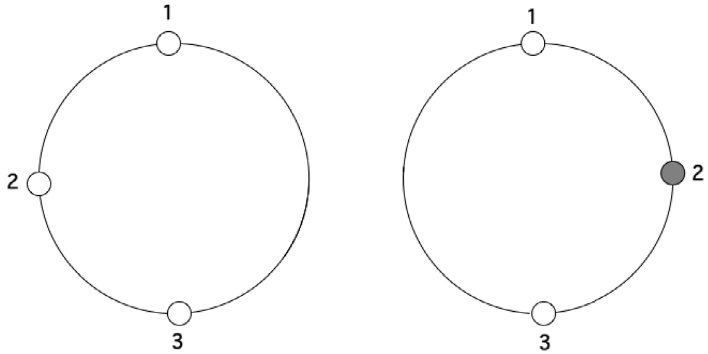
Let $K \subseteq [2, \dots, n-1]$ and $L = [2, \dots, n-1] \setminus K$ be two complementary subsets of $[2, \dots, n-1]$. Let $k_1 < \dots < k_s$ and $\ell_1 < \dots < \ell_t$ be ordered elements of K and L, respectively, where $s + t = n - 2$. Consider the following spectrum on the circle

$$1 \lhd k_1 \lhd \dots \lhd k_s \lhd n \lhd \ell_t \lhd \dots \lhd \ell_1 \lhd 1. \tag{3}$$

Mark beads $1, k_1, \dots, k_s, n$ white and $\ell_1, \dots, \ell_t$ black to obtain a necklace S_K.



Example 3 For $n = 3$ we have two options: one with $K_1 = \emptyset$ and another with $K_2 = \{2\}$. Respectively we have two necklaces S_{K_1} and S_{K_2}:



Then $\mathcal{D}(S_{K_1}) = \{123, 213, 231, 321\}$ and $\mathcal{D}(S_{K_2}) = \{123, 132, 312, 321\}$ which are F_{K_1} and F_{K_2}, respectively. These are single-peaked and single-dipped domains.

Proposition 2 *If $K = [2, \ldots, n-1]$, then $\mathcal{D}(S_K) = F_K$ is the classical single-peaked domain.*

Proof Since $L = \emptyset$ in S_K there are no black beads and beads 1 and n are neighbours on the circle. The only w-convex set containing both of them is the longer arc Z with endbeads 1 and n, that is, $\{1, 2, \ldots, n\}$. Any arc $X \subseteq Z$ is w-convex. So w-convex sets coincide with the upper contour sets of the classical single-peaked domain with the spectrum $1 \lhd 2 \lhd \ldots \lhd n-1 \lhd n$. □

Lemma 1 *In domain $\mathcal{D}(S_K)$:*

(i) *A black bead a satisfies the never-top condition in any triple $\{a, b, c\}$ such that $b < a < c$, that is $aN_{\{a,b,c\}}1$.*
(ii) *A white bead a satisfies the never-bottom condition in any triple $\{a, b, c\}$ such that $b < a < c$, that is $aN_{\{a,b,c\}}3$.*

Proof (i) Suppose $a \in L \subseteq [2, \ldots, n]$ is black and there is a linear order v in $\mathcal{D}(S_K)$ whose restriction to subset $\{a, b, c\}$ is abc with $b < a < c$. Let X be the first w-convex set from the flag corresponding to v that contains a. Then we need to consider two cases: 1) $a = \ell_i$ and $X = \{\ell_i, \ldots, \ell_1, 1, k_1, \ldots, k_j\}$ does not contain b and c (here it is possible that $j = 0$). We note that $b \notin L$ as $b < \ell_i$ and not in X, thus $b = k_s$ with $s > j$. But then $1 < k_s < \ell_i$ and X is not w-convex. The case 2) $a = \ell_i$ and $X = \{\ell_i, \ldots, \ell_t, n, k_s, \ldots, k_j\}$ does not contain b and c is similar.

(ii) Suppose $a \in K \subseteq [2, \ldots, n]$ white and a certain linear order v in $\mathcal{D}(S_K)$ has restriction bca to subset $\{a, b, c\}$ is with $b < a < c$. Let X be the largest w-convex set in the flag corresponding to v that still does not contain a. Then it contains b and c which contradicts to w-convexity of X. □

Theorem 1 $\mathcal{D}(S_K) = F_K$.

Proof By Lemma 1 we know that $\mathcal{D}(S_K) \subseteq F_K$. Let us prove the converse. Let $v = a_1 \ldots a_n \in F_K$. We define the k-th ideal of v as $\mathrm{Id}_k(v) = \{a_1, \ldots, a_k\}$. It is enough to show that for any $k \in [n]$ the set $\mathrm{Id}_k(v)$ is w-convex. We need to check conditions (a)–(c) of Definition 6.

We note that, due to (1) $a_1 \in K$ and must be white. Hence condition (a) of Definition 6 is satisfied.

Next, we have to prove (b). Suppose the contrary. Then there exist $a, b, c \in [n]$ with $a, c \in \mathrm{Id}_k(v)$ and $b \notin \mathrm{Id}_k(v)$ satisfying $a < b < c$ and $b \in K$ is white. Then the restriction of v onto $\{a, b, c\}$ is acb or cab in violation of $bN_{\{a,b,c\}}3$.

To prove (c) suppose, first, that both 1 and n are not in $\mathrm{Id}_k(v)$. Then no black bead can be in $\mathrm{Id}_k(v)$. Indeed, if $\ell \in \mathrm{Id}_k(v)$ is black, then either $\ell 1n$ or $\ell n1$ is in the restriction of v onto $\{1, \ell, n\}$ which contradicts $\ell N_{\{1,\ell,n\}}1$. Then, due to (b), $\mathrm{Id}_k(v)$ is an arc. Without loss of generality we assume now that $1 \in \mathrm{Id}_k(v)$. Then for some $k_j \in \mathrm{Id}_k(v)$ the white beads in $\mathrm{Id}_k(v)$ form an arc $\{1, \ldots, k_j\}$. Let k_j be maximal with this property. The white arc $\{k_{j+1}, \ldots, n\}$ has no intersection with $\mathrm{Id}_k(v)$ which implies that the black arc $\{\ell_p \mid k_j < \ell_p < n\}$ is also has empty intersection with $\mathrm{Id}_k(v)$.

If $\mathrm{Id}_k(v)$ is not arc, then we must have $\ell_k \in \mathrm{Id}_k(v)$ for some $\ell_i \notin \mathrm{Id}_k(v)$ with $1 < \ell_i < \ell_k$. But such a case would contradict to $\ell_k N_{\{\ell_i,\ell_k,n\}}1$.

This contradiction proves the theorem. □

Lemma 2 *For any $K \subseteq [2, \ldots, n-1]$ the domain F_K is copious.*

***Proof*.** We will use Theorem 1 and consider $\mathcal{D}(S_K)$ instead. Let $a, b, c \in [n]$ with $a < b < c$. We need to consider several cases.

1. a, b, c are all white. Then $a = k_p, b = k_s, c = k_r$ with $p < s < r$. The following sets are w-convex:

$$\{k_p\}, \{k_s\}, \{k_r\}, \{k_p, \ldots, k_s\}, \{k_s, \ldots, k_r\}, \{k_p, \ldots, k_s, \ldots, k_r\}.$$

   Thus, abc, cba, bac, bca all belong to the restriction of $\mathcal{D}(S_K)$ onto $\{a, b, c\}$.
2. a, b, c are all black. Note that every arc containing K is w-convex. Suppose $a = \ell_p$, $b = \ell_q$, $c = \ell_r$ with $p < q < r$. Let

$$K' = K \cup \{1\} \cup \{n\} \cup \{\ell_1, \ldots, \ell_{p-1}\} \cup \{\ell_t, \ldots, \ell_{r-1}\}.$$

   Then the sequence of $K' \cup \{\ell_r\} \subset K' \cup \{\ell_r, \ell_p\} \subset [n]$ gives us cab and the sequence $K' \cup \{\ell_p\} \subset K' \cup \{\ell_r, \ell_p\} \subset [n]$ gives us acb. Also, the sequence $K' \cup \{\ell_r\} \subset K' \cup \{\ell_r, \ldots, \ell_q\} \subset [n]$ gives cba and the sequence $K' \cup \{\ell_p\} \subset K' \cup \{\ell_p, \ldots, \ell_q\} \subset [n]$ gives abc. Hence we have four suborders in $\mathcal{D}(S_K)|_{\{a,b,c\}}$.
3. a is white; b, c are black. Then obviously, abc and acb belong to the restriction of $\mathcal{D}(S_K)$ onto $\{a, b, c\}$. But also in the restriction of $\mathcal{D}(S_K)$ onto $\{n, a, b, c\}$ we have $ncba$ and $ncab$, hence cba and cab belong to $\mathcal{D}(S_K)|_{\{a,b,c\}}$, so this restriction has four suborders.
4. b is white; a, c are black. Then bac and bca are in $\mathcal{D}(S_K)|_{\{a,b,c\}}$ as well as $ncba$ and $1abc$ (or $1cba$ and $nabc$ belong to the restrictions of $\mathcal{D}(S_K)$ onto $\{n, a, b, c\}$ and $\{1, a, b, c\}$, respectively. Hence cba and abc are in $\mathcal{D}(S_K)|_{\{a,b,c\}}$ and this restriction has four suborders as well.

5. a is black; b, c are white. Then bca, cba, bac and $1abc$ belong to respective restrictions, so four suborders.
6. a and b are black and c is white. Then cba and cab belong to $\mathcal{D}(S_K)|_{\{a,b,c\}}$ together with $1abc$ and $1acb$. These are all possible cases. □

Combining Proposition 1 with Lemma 2 we get

Theorem 2 *For any $K \subseteq [2, \ldots, n-1]$ the domain F_K is a maximal Condorcet domain.*

The universal domain $\mathcal{L}(A)$ has many representations. One of the most useful ones is by the *permutohedron* of order n (Monjardet, 2009), whose vertices are labeled by the permutations of $[n]$ from the symmetric group S_n. Two vertices are connected by an edge if their permutations differ in only two neighbouring places. Domains can be considered as a subgraphs of the permutohedron.

Definition 9 A domain $\mathcal{D}$ of maximal width is called *semi-connected* if the two completely reversed orders e and $\bar{e}$ from $\mathcal{D}$ can be connected by a shortest path (geodesic path) in the permutohedron so that all vertices of this path belong to $\mathcal{D}$. It is *directly connected*, if any two orders of a domain are connected by a shortest path in the permutohedron that stays within the domain.

Maximality of GF-domains has a number of profound consequences.

Theorem 3 *Every GF-domain $\mathcal{D}$ is a directly connected domain of maximal width.*

Proof We have already noticed that GF-domains have maximal width containing $12\ldots n$ and $n\ldots 21$. By their definition, they are also the so-called peak-pit domains which means that they satisfy a complete set of never-top and never-bottom conditions. By Theorem 2 of Danilov et al. (2012) maximality of $\mathcal{D}$ implies that this is a tiling domain and, in particular, it is semi-connected. It has been observed in Puppe (2016) (Proposition A.1) that maximal semi-connected domains are directly connected. □

Let us now give a formal definition of a domain single-peaked on a circle.

Definition 10 (Peters and Lackner 2020) A linear order $v \in \mathcal{L}(A)$ is said to be *single-peaked on a circle*, if alternatives from A can be placed on a circle

$$a_1 \lhd a_2 \lhd \cdots \lhd a_n \lhd a_1$$

in anticlockwise order so that for every alternative $a \in A$ the upper counter set $U(a, v) = \{b \in A \mid b \succ_v a\}$ is a contiguous arc of the circle.

A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is said to be *single-peaked on a circle* if there exists an arrangement of alternatives on that circle such that each order of $\mathcal{D}$ is single-peaked on a circle relative to their common arrangement of alternatives.

Our Theorem 1 as a corollary provides a constructive proof of the following theorem.

Corollary 1 (Karpov 2023) *Every GF-domain F_K is single-peaked on a circle.*

Proof By Theorem 1 F_K is isomorphic to $\mathcal{D}(S_K)$. The statement now follows from the fact that any upper contour set of this domain is a contiguous arc of the necklace. □

Karpov's original proof was based on the characterisation of single-peaked on a circle domains by means of forbidden configurations given in Peters and Lackner (2020).

As we have seen two prominent members of the class of peak-pit maximal Condorcet domains of maximal width, namely, single-peaked on a line and Fishburn's domains, are single-peaked on a circle. The question may be asked: Are all peak-pit maximal Condorcet domains of maximal width are single-peaked on a circle? The answer, however, is negative.

Theorem 4 *For any $n \geq 4$ single-crossing maximal Condorcet domains are not single-peaked on a circle.*

Proof Slinko et al. (2021) characterised all single-crossing maximal Condorcet domains in terms of the relay structure. They showed that there are, up to relabeling the alternatives, exactly two single-crossing maximal Condorcet domains, one represented by a top-down relay and the other by a bottom-up relay which are flip-isomorphic (one obtained from the other by reversing all orders). In the top-down relay linear orders are arranged in a sequence $v_1, v_2, v_3, \ldots$, so that moving from left to right 1 initially moves from top to bottom being swapped with $2, 3, \ldots, n$. Then n starts to move up being swapped sequentially with $n-1, n-2, \ldots, 2$. We will thus have $\mathrm{Id}_2(v_1) = \{1, 2\}$, $\mathrm{Id}_2(v_3) = \{2, 3\}$, $\mathrm{Id}_2(v_{2n-3}) = \{2, n\}$. But it is impossible to have such three arcs $\{1, 2\}, \{2, 3\}, \{2, n\}$ on a circle as 2 can have only two neighbours. □

Here is an example of a top-down relay for $n = 4$

$$\begin{matrix} 1 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 1 & 3 & 3 & 4 & 2 & 3 \\ 3 & 3 & 1 & 4 & 3 & 3 & 2 \\ 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{matrix}$$

In this proof effectively we spotted in any single-crossing maximal Condorcet domain one of the forbidden configurations described in Peters and Lackner (2020).

3 Conclusion and future work

Now we know that the single-peaked domain, Fishburn's domain and the single-dipped domain are members of the same family with single-peaked and single-dipped being the two extremes. It would be interesting to investigate how the size of domain F_K depends on K. It is well-known that when $K = [2, \ldots, n-1]$ or $K = \emptyset$ (the case of classical single-peaked and single-dipped domains) we have $|F_K| = 2^{n-1}$ and when K is the set of even numbers in $[2, \ldots, n-1]$ (the case of classical Fishburn's domain). Galambos and Reiner (2008) gave the exact formula for the cardinality of F_K:

$$|F_K| = (n+3)2^{n-3} - \begin{cases} (n-\frac{3}{2})\binom{n-2}{\frac{n}{2}-1} & \text{for even } n; \\ (\frac{n-1}{2})\binom{n-1}{\frac{n-1}{2}} & \text{for odd } n. \end{cases}$$

It is reasonable to conjecture that these are the most extreme cases and the cardinality of $|F_K|$ for various K must be somewhere in between 2^{n-1} and the cardinality of Fishburn's domain. However, since Fishburn's domain is not the largest peak-pit domain of maximal width for at least $n \geq 34$ (Karpov and Slinko, 2023), we do not even know if Fishburn's domains are the largest among all GF-domains.

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Declarations

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