Social Choice and Welfare
https://doi.org/10.1007/s00355-023-01497-9

ORIGINAL PAPER

Christian Klamler's "A distance measure for choice functions" [Social Choice and Welfare 30 (2008) 419–425]: a correction

Davide Carpentiere[1] · Alfio Giarlotta[2] · Stephen Watson[3]

Received: 9 November 2022 / Accepted: 4 November 2023

Abstract
An extensive choice over X is a function assigning to any subset S of X a possibly empty subset of S. Klamler (Soc Choice Welf 30:419–425, 2008) shows that the operation of symmetric difference induces a metric on the family of extensive choices over X, and this metric is characterized by five axioms A1–A5. We provide counterexamples to Klamler's result, suggest a slight modification of axioms A4 and A5 to obtain a correct characterization, and finally observe that axiom A4 is redundant.

1 Introduction

An *extensive choice* over a set X is a map C that assigns to each menu $S \subseteq X$ a possibly empty submenu $C(S) \subseteq S$. Klamler (2008) defines a metric d_Δ on extensive choices over a finite set X, which is based on the symmetric difference Δ of sets (Kemeny 1959). Klamler's Theorem 1 states that d_Δ is the unique metric satisfying five axioms A1–A5. Here we provide counterexamples to this result, suggest a slight change for two of the five axioms to obtain a correct characterization, and show the redundancy of A4. To keep this note short and focused, we omit all main proofs (but they are available upon request).

✉ Alfio Giarlotta
alfio.giarlotta@unict.it

1 Department of Mathematics and Computer Science, University of Catania, Catania, Italy

2 Department of Economics and Business, University of Catania, Catania, Italy

3 Department of Mathematics and Statistics, York University, Toronto, Canada

Published online: 14 March 2024

2 Klamler's setting

Let X be a finite set of $n \geqslant 2$ alternatives, and $\mathscr{X}$ the family of all subsets of X (called *menus*). An *extensive choice* over X is a function $C\colon \mathscr{X} \to \mathscr{X}$ such that $C(S) \subseteq S$ for all $S \in \mathscr{X}$. We denote by $\mathsf{Choice}(X)$ the family of all extensive choices over X.

A *metric* on $\mathsf{Choice}(X)$ is a map $d\colon \mathsf{Choice}(X) \times \mathsf{Choice}(X) \to \mathbb{R}$ such that for all $C, C'C'' \in \mathsf{Choice}(X)$, the following usual properties hold:

[A0.1] $d(C, C') \geqslant 0$, and equality holds if and only if $C = C'$;
[A0.2] $d(C, C') = d(C', C)$;
[A0.3] $d(C, C') + d(C', C'') \geqslant d(C, C'')$.

Klamler (2008) studies metrics satisfying additional requirements. To state one of these properties, the notion of 'betweenness' (Alabayrak and Aleskerov 2000) is needed: for all $C, C', C'' \in \mathsf{Choice}(X)$, C' is *between* C and C'' if for any $S \in \mathscr{X}$,

$$C(S) \cap C''(S) \ \subseteq\ C'(S) \ \subseteq\ C(S) \cup C''(S).$$

Klamler (2008) proposes the following additional axioms for metrics on $\mathsf{Choice}(X)$:

[A1] $d(C, C') + d(C', C'') = d(C, C'')$ if and only if C' is between C and C'';
[A2] if $\widetilde{C}$ and $\widetilde{C}'$ result from, respectively, C and C' by the same permutation of alternatives, then $d(C, C') = d(\widetilde{C}, \widetilde{C}')$;
[A3] if C and C' agree on all (nonempty) menus in $\mathscr{X}$ except for a subfamily $\mathscr{X}' \subseteq \mathscr{X}$, then the distance $d(C, C')$ is determined exclusively from the choice sets over $\mathscr{X}'$;
[A4] if $C, C', \widetilde{C}, \widetilde{C}'$ only disagree on a menu $T \in \mathscr{X}$ such that $C(T) = \widetilde{C}(T) \cup S$ and $C'(T) = \widetilde{C}'(T) \cup S$ for some $S \subseteq T$, then $d(C, C') = d(\widetilde{C}, \widetilde{C}')$;
[A5] $\min\{d(C, C') : C \neq C'\} = 1$.

Axiom A1 is a strengthening of the triangular inequality A0.3. Axiom A2 is a condition of invariance under permutations. Axiom A3 is a separability property. Axiom A4 is a condition of translation-invariance. Axiom A5 selects a unit of measurement.

The analysis in Klamler (2008) aims to show that there is a unique metric on $\mathsf{Choice}(X)$ satisfying Axioms A1–A5. This metric is the map $d_\Delta\colon \mathsf{Choice}(X) \times \mathsf{Choice}(X) \to \mathbb{R}$, defined as follows for all extensive choices C, C':

$$d_\Delta(C, C') := \sum_{S \in \mathscr{X}} \left| C(S) \,\Delta\, C'(S) \right|.$$

The main result in Klamler (2008) is his Theorem 1:

Theorem 2.1 *The unique metric on* $\mathsf{Choice}(X)$ *satisfying Axioms A1–A5 is* d_Δ.

3 A correct characterization

Theorem 2.1 does not hold:

Example 3.1 Consider the following family $\mathfrak{C}$ of extensive choices over $X = \{x_1, x_2, x_3, x_4\}$:

$$\mathfrak{C} := \{C \in \mathsf{Choice}(X) : C(A) = \varnothing \text{ for all } A \neq X\}.$$

Define four extensive choices $C, C', \widetilde{C}, \widetilde{C}' \in \mathfrak{C}$ as follows:

$$C(X) := \{x_1, x_2, x_3\}, \quad C'(X) := \{x_2, x_3, x_4\}, \quad \widetilde{C}(X) := \{x_1, x_2\}, \quad \widetilde{C}'(X) := \{x_3, x_4\}.$$

Set $S := \{x_2, x_3\} \subseteq X$. Thus, the equalities $C(X) = \widetilde{C}(X) \cup S$ and $C'(X) = \widetilde{C}'(X) \cup S$ hold. However, $d_\Delta(C, C') = 2 \neq 4 = d_\Delta(\widetilde{C}, \widetilde{C}')$, and so Axiom A4 fails for d_Δ.

The gap in Klamler's proof of Theorem 2.1 is at the very end, when he claims that "*[...] and obviously d_Δ satisfies axioms A1–A5*". A minor modification of Axiom A4 —adding a requirement of disjointness— suffices to address the issue exhibited in Example 3.1:[1]

[A4′] if $C, C', \widetilde{C}, \widetilde{C}'$ only disagree on a menu $T \in \mathscr{X}$ such that $C(T) = \widetilde{C}(T) \cup S$ and $C'(T) = \widetilde{C}'(T) \cup S$ for some $S \subseteq T$ with $\widetilde{C}(T) \cap S = \varnothing$ and $\widetilde{C}'(T) \cap S = \varnothing$, then $d(C, C') = d(\widetilde{C}, \widetilde{C}')$.

Then, we have:

Lemma 3.2 *d_Δ is a metric on Choice(X) that satisfies Axioms A1, A2, A3, A4′, A5.*

Regrettably, Theorem 2.1 does not hold even with Axiom A4′ in place of Axiom A4, because uniqueness fails. In fact, there are infinitely many instances of metrics satisfying the modified set of axioms, as the next example shows.

Example 3.3 For any vector $\mathbf{k} = (k_1, \ldots, k_n)$ of positive integers such that $k_j = 1$ for some j, the function $d_\mathbf{k} \colon \mathsf{Choice}(X) \times \mathsf{Choice}(X) \to \mathbb{R}$ defined by

$$d_\mathbf{k}(C, C') := \sum_{S \in \mathscr{X}} k_{|S|} \cdot \left| C(S) \Delta C'(S) \right|$$

is a metric satisfying Axioms A1, A2, A3, A4′, A5.

The distance $d_\mathbf{k}$ described in Example 3.3 weights the cardinality of the symmetric difference of choice sets by $|X| = n$ parameters, which depend on the size of the menu. The distance d_Δ is the special case obtained by taking the constant vector $\mathbf{k} = (1, \ldots, 1)$. The gap in Klamler's proof lies at the beginning of his Lemma 2, where his statement "*Let us slightly abuse the notation [...]*" hides the dependency of the unit u on S.

[1] We thank a referee for suggesting this modification of Axiom A4. Note that in the original draft of the paper, we proposed the following alternative form of Axiom A4, which maintains the 'translation-invariance' spirit of Axiom A4 by taking symmetric difference in place of union:

[A4″] if $C, C', \widetilde{C}, \widetilde{C}'$ only disagree on a menu $T \in \mathscr{X}$ such that $C(T) = \widetilde{C}(T) \Delta S$ and $C'(T) = \widetilde{C}'(T) \Delta S$ for some $S \subseteq T$ with $S \cap \widetilde{C}(T) = \varnothing$ and $S \cap \widetilde{C}'(T) = \varnothing$, then $d(C, C') = d(\widetilde{C}, \widetilde{C}')$.

Observe that Axioms A4′ and A4″ are equivalent for our purposes.

To obtain uniqueness, we substitute A5 by the following axiom:

[A5′] For each menu S, the minimum distance between two choices that disagree only on S is 1.

Then, as announced, we have:

Theorem 3.4 *The unique metric on* $\mathsf{Choice}(X)$ *satisfying Axioms A1, A2, A3, A4′, A5′ is* d_Δ.

As a matter of fact, one can show that a sharper result holds, because Axiom A4′ is redundant:

Theorem 3.5 *The unique metric on* $\mathsf{Choice}(X)$ *satisfying Axioms A1, A2, A3, and A5′ is* d_Δ.

Acknowledgements The authors have no relevant financial or non-financial interests to disclose. The authors are grateful to two referees, an associate editor, and the managing editor François Maniquet for their careful comments and valuable suggestions. Alfio Giarlotta acknowledges the support of "Ministero dell'Istruzione, dell'Università e della Ricerca (MIUR)—PRIN 2017", project *Multiple Criteria Decision Analysis and Multiple Criteria Decision Theory*, grant 2017CY2NCA.

Funding Open access funding provided by Università degli Studi di Catania within the CRUI-CARE Agreement.

Data Availability Not applicable

References

Albayrak SR, Aleskerov F (2000) Convexity of choice function sets. Bogazici University Research Paper ISS/EC-2000-01

Kemeny JG (1959) Mathematics without numbers. Daedalus 88:577–591

Klamler C (2008) A distance measure for choice function. Soc Choice Welf 30:419–425