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The largest Condorcet domain on 8 alternatives

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Abstract

In this note, we report on a Condorcet domain of record-breaking size for $n = 8$ alternatives. We show that there exists a Condorcet domain of size 224 and that this is the largest possible size for 8 alternatives. Our search also shows that this domain is unique up to isomorphism. In this note we investigate properties of the new domain and relate them to various open problems and conjectures.

1 Introduction

Condorcet domains (CD), which are sets of linear orders giving rise to voting profiles with an acyclic pairwise majority relation, have been studied by mathematicians, economists, and mathematical social scientists since the 1950 s (Black 1948; Arrow 1951). Condorcet domains find use in Arrovian aggregation and social choice theory (Puppe and Slinko 2019; Lackner and Lackner 2017). In social choice theory, a Condorcet winner is a candidate who would win over every other candidate in a pairwise comparison by securing the majority of votes (Monjardet 2005). However, the existence of such a candidate is not always guaranteed, leading to the relevance of Condorcet Domains. A central question in this field has revolved around identifying large Condorcet domains, see Fishburn (1997); Galambos and Reiner (2008); Monjardet (2009); Danilov et al. (2012); Puppe and Slinko (2022); Karpov and Slinko (2022a); Karpov (2022).

A significant category of Condorcet domains is rooted in Fishburn's alternating scheme, which alternates between two restriction rules on a subset of candidates and has been employed to construct numerous maximum size Condorcet domains. We refer to such domains based on the alternating scheme as Fishburn domains.

Fishburn introduced a function $f(n)$ in Fishburn (1997), defined to be the maximum size of a Condorcet domain on a set of n alternatives, and posed the problem of determining the growth rate for $f(n)$. Fishburn also proved that for $n = 16$ the

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Fishburn domain is not the largest CD. This was followed by further research and bounds on $f(n)$ by Galambos and Reiner (2008); Danilov et al. (2012); Monjardet (2009). Karpov and Slinko extended and refined this work in Karpov and Slinko (2022b), as did Zhou and Riis (2023).

Although extensive research has been conducted, all known maximum-sized Condorcet domains have been built using components based on either Fishburn's alternating schemes or his replacement scheme. For instance, Karpov and Slinko (2022a) introduced a novel construction that enabled the creation of new Condorcet domains with unprecedented sizes. This allowed the authors to construct a Condorcet domain, superseding the size of Fishburn's domain for 13 alternatives. Recently, Zhou and Riis (2023) constructed Condorcet domains on 10 and 11 alternatives, superseding the size of the corresponding Fishburn domains.

This paper shows that $n = 8$ is the smallest number of alternatives for which the Fishburn domain (size 222) is not the largest and that there is a Condorcet domain of size 224. Furthermore, relying on extensive computer calculation on the super-computer Abisko at Umeå, we also established 224 as an upper bound and that there, up to isomorphism, is only one such Condorcet domain. The need for a supercomputer, and a carefully devised algorithm, reflects the fact that a naive search would lead to search-tree with more than 6^{112} vertices. We also analyse some of the properties of this new domain (Table 1).

2 Preliminaries

There are many equivalent definitions of Condorcet domains. In this paper, we adopt the definition proposed by Ward in Ward (1965). According to this definition, a Condorcet domain of degree $n \geq 3$ is a set of orderings of $X_n = \{1, 2, \ldots, n\}$ that satisfies certain local conditions.

Specifically, a Condorcet domain of degree $n = 3$ is defined as a set of orderings of X_3 that satisfies one of nine laws, denoted by xNi, where x is an element of X_3, and i is an integer between 1 and 3. The law xNi requires that x does not come in the i-th

Table 1 The Condorcet domains for 3 alternatives which contain the identity order

Triple	Rule assigned	Condorcet domains		Core
(i, j, k)	1N3	*ijk jik ikj kij*	Isomorphic	$\{ijk, ikj\}$
	2N3	*ijk jik jki kji*		$\{(ijk), (kji)\}$
	3N1	*ijk ikj jik jki*	Isomorphic	$\{ijk, jik\}$
	2N1	*ijk ikj kij kji*		$\{(ijk), (kji)\}$
	1N2	*ijk ikj jki kji*	Isomorphic	$\{ijk, ikj\}$
	3N2	*ijk jik kij kji*		$\{(ijk), (jik)\}$

Each rule assigned to the triplet (i, j, k) with i<j<k is associated with a CD (which is given on the same line). The CDs displayed fall into 3 isomorphism classes, and each CD has a core of size 2

position in any order in the Condorcet domain. For example, xN1 means that x may never come first, while xN3 means that x may never come last.

A Condorcet domain of degree $n > 3$ is a set A of orderings of X_n that satisfies the following property: the restriction of A to every subset of X_n of size 3 is a Condorcet domain. In other words, for every triple a, b, c of elements of X_n, one of the nine laws xNi must be satisfied, where $x \in a, b, c$. For example, cN2 would mean that c may not come between a and b in any orderings in A.

A *maximal Condorcet domain* of degree n is a Condorcet domain of degree n that is maximal under inclusion among the set of all Condorcet domains of degree n. A Maximum Condorcet domain is a Condorcet domain of the largest possible size for a given value of n.

To avoid repetition, we will use the acronyms CD and MCD, to refer to Condorcet domain and Maximal Condorcet domain respectively.

For the case of degree 3, there are nine MCDs, each corresponding to one of the nine different laws xNi. It is easy to verify that these nine MCDs contain exactly four elements: two transpositions and two even permutations (either the identity or a 3-cycle). Among the 9 MCDs of order 3, precisely six contain the identity order $1 > 2 > 3$ since the laws 1N1, 2N2, and 3N3 each rule out one CD of degree 3.

2.1 Transformations and isomorphism of condorcet domains

First, recall that each linear order L in a CD B may also be viewed as a finite sequence of integers, obtained by ordering the elements of X_n so that they are increasing according to L, or as the permutation which permutes the identity order on X_n to this sequence. We let S_n denote the set of all permutations on X_n.

Let $g \in S_n$ and $i \in X_n$. We define ig as $g(i)$; and if A is a sequence of elements of X_n we define Ag to be the sequence obtained by applying g to the elements of A in turn. If B is a CD, regarded as a set of sequences, we define Bg to be the set of sequences obtained by applying g to the sequences in B, and then Bg is also a CD. Specifically, if B satisfies the law xNi on a triple (a, b, c) for some $x \in \{a, b, c\}$, then Bg satisfies the law xgNi on the triple (ag, bg, cg). We call CDs B and Bg *isomorphic*. Therefore, two isomorphic CDs differ only by a relabelling of the elements of X_n.

The *core* of a CD B is the set of permutations $g \in B$ such that $Bg = B$. The core of a CD which contains the identity order B is a group. We provide a more detailed discussion of the core in Akello-Egwell et al. (2023).

It can be readily shown that for any Condorcet domain, the total number of 1N3 and 2N3 rules remains invariant under isomorphism. Likewise, this holds for the total number of 2N1 and 3N1 rules and the total number of 1N2 and 3N2 rules.

3 Search methodology

We developed an algorithm to generate all MCDs of a given degree n and size at least equal to a user-specified cutoff value (e.g. size ≥ 222 for $n = 8$). We implemented this algorithm in C in a serial version which is sufficient for $n \leq 6$, and a parallelized version that we used for $n = 7$ and 8. It is important to stress that this algorithm, unlike the one used by Zhou and Riis (2023), aims to construct all MCDs above some user-specified size.

Our algorithm works by starting with the unrestricted domain of all linear orders on n alternatives and then stepwise applying never laws $i\mathrm{N}p$ to those triples which do not already satisfy some such law. The algorithm works with *unitary* CDs, meaning CDs which contain the identity permutation. Since every CD is isomorphic to some unitary CD this is without loss of generality. However, by using unitary CDs we reduce the set of possible never laws from 9 to 6, thereby speeding up our search. We will next sketch some of the details required in order to see that the algorithm is complete, though at first inefficient, and then how to also make it efficient.

We define the *Condorcet tree* of rank n, which is a homogeneous rooted tree of valency 6 and depth $\binom{n}{3}$, as follows. The $\binom{n}{3}$ triples of elements of X_n are arranged in some order, so that the vertices of the tree at a given depth t are associated with the corresponding triple T_t. The six laws that a unitary CD may obey on a given triple are also ordered, and each child w of a non-leaf v of the tree is associated with one such law L_w. Every vertex v is associated with a set c_v of linear orders on X_n. If v is the root then c_v is the set of all orderings. If w is a child of v, where v has depth t, then c_w is obtained from c_v by removing those orderings that do not satisfy the law L_w when applied to T_t.

It is possible, in theory, to process the entire tree, depth first, constructing the sets c_v for every vertex v. Then the unitary MCDs of degree n, as well as many non-maximal CDs, are found among the sets c_v for the leaves v. In practice this is impracticable for $n > 5$ as the tree is too big.

For any leaf v the set c_v is a unitary CD, but these are not always maximal, and there will be very many duplicates. This arises from the fact that, as we move down the tree, the sets c_v will often not only obey the laws that have been explicitly applied on triples but may also obey laws on triples which are implied by the applied laws. Using this observation allows a massive reduction in the number of vertices that need to be processed, giving us a tree with 0, 1 or 6 descendants from v depending on whether c_v cannot be maximal or must be a duplicate, has an implied law, or is unrestricted by earlier laws. This is determined as follows.

When a vertex v of height t is processed the law that was enforced on each triple T_s for $s \leq t$ to define v - in other words the path from the root to v - is recorded, and c_v is constructed by taking c_u, where u is the parent of v, and deleting all elements that do not satisfy the corresponding never law N_v. For each $s \leq t + 1$ the set L_s of laws that all the elements of c_v obey when applied to the triple T_s is determined. If, for some $s \leq t$, the set L_s contains a law that precedes the law N_u, where u is the ancestor of v of depth s, then the vertex v is not processed any further, on the grounds of duplication, and its descendants are not visited. Otherwise, for each $s \leq t$, a law from L_s is selected, and the set of sequences that obey all these laws is computed. This set clearly contains c_v, and if, for some such selection of laws, this set strictly contains c_v then again c_v is not processed further. In this case, any unitary CD arising from a leaf descendant of v must either fail to be maximal, or will be a duplicate of a unitary MCD constructed from a descendant of another vertex of depth t. If v passes these tests, and L_{t+1} is non-empty, the only descendant of v that will be processed is the child w defined by the least element of L_{t+1}, and then $c_w = c_v$. Otherwise all children of v are processed.

The validity of these restrictions of the full Condorcet tree follows from a recursive argument which is given in full in Akello-Egwell et al. (2023).

4 Condorcet domains on 8 alternatives with size 224

Relying on extensive computer calculation on the super-computer Abisko at Umeå, we have established that:

Theorem 4.1 *The maximum size of a CD on* 8 *alternatives is* 224. *Up to isomorphism, there is only one such CD. This CD has a core of size* 4. *There are no MCDs of size* 223.

The largest Condorcet domain containing the identity permutation and its reverse for $n = 8$ *alternatives is the Fishburn domain, which has a size of* 222.

We aim to extend this with more precise counts and analysis of other large Condorcet domains on 8 alternatives in an upcoming paper.

Now let us investigate the properties of the MCD of size 224.

1. The Fishburn domain has size 222 and hence is not the maximum CD for $n = 8$ alternatives
2. There are 56 isomorphic Condorcet domains of size 224 which contain the identity order. Among these there is one special MCD we will refer to as D224, where each never-rule - except for the two triplets (123) and (678) - is 1N3 or 3N1. We display the rules for D224 in Table 2 and its linear orders in Table 3
3. The domain does not have maximal width, i.e. it does not contain a pair of reversed orders.
4. The domain is self-dual. That is, the domain is isomorphic to the domain obtained by reversing each of its linear orders.
5. The restriction of the domain to each triple of alternatives has size 4. This means that this domain is copious in the terminology of Slinko (2019) and is equivalent to the fact that the domain satisfies exactly one never-rule on each triple.
6. The domain is a peak-pit domain in the sense of Danilov et al. (2012), i.e. every triple satisfies a condition of either the form xN1 or xN3, for some x in the triple.
7. The authors of Karpov and Slinko (2022a) asked for examples of maximum CDs which are not peak-pit domains of maximal width. Our domain is the first known such example and shows that $n = 8$ is the smallest n for which this occurs.
8. The domain is connected (see Monjardet (2009) for the lengthy definition of this well used property.) This is in line with the conjecture from Puppe and Slinko (2022) that all maximal peak-pit CDs are connected.
9. The domain has a core of size 4, which is given in captions of Tables 2 and 3.

Table 2 Table of triplets and rules that produces the Condorcet domain D224 of size 224 for 8 alternatives

Triplets	Rules	Triplets	Rules	Triplets	Rules	Triplets	Rules
(1, 2, 3)	2N3	(1, 4, 8)	1N3	(2, 4, 7)	3N1	(3, 5, 8)	3N1
(1, 2, 4)	1N3	(1, 5, 6)	1N3	(2, 4, 8)	3N1	(3, 6, 7)	3N1
(1, 2, 5)	3N1	(1, 5, 7)	1N3	(2, 5, 6)	1N3	(3, 6, 8)	1N3
(1, 2, 6)	3N1	(1, 5, 8)	3N1	(2, 5, 7)	1N3	(3, 7, 8)	1N3
(1, 2, 7)	3N1	(1, 6, 7)	3N1	(2, 5, 8)	3N1	(4, 5, 6)	1N3
(1, 2, 8)	3N1	(1, 6, 8)	1N3	(2, 6, 7)	3N1	(4, 5, 7)	1N3
(1, 3, 4)	1N3	(1, 7, 8)	1N3	(2, 6, 8)	1N3	(4, 5, 8)	3N1
(1, 3, 5)	3N1	(2, 3, 4)	1N3	(2, 7, 8)	1N3	(4, 6, 7)	3N1
(1, 3, 6)	3N1	(2, 3, 5)	1N3	(3, 4, 5)	3N1	(4, 6, 8)	1N3
(1, 3, 7)	3N1	(2, 3, 6)	1N3	(3, 4, 6)	3N1	(4, 7, 8)	1N3
(1, 3, 8)	3N1	(2, 3, 7)	1N3	(3, 4, 7)	3N1	(5, 6, 7)	3N1
(1, 4, 5)	1N3	(2, 3, 8)	1N3	(3, 4, 8)	3N1	(5, 6, 8)	3N1
(1, 4, 6)	1N3	(2, 4, 5)	3N1	(3, 5, 6)	1N3	(5, 7, 8)	3N1
(1, 4, 7)	1N3	(2, 4, 6)	3N1	(3, 5, 7)	1N3	(6, 7, 8)	2N1

This specific CD is invariant under the action by the permutations group $G = \{\text{id}, (12)(34), (56)(78), (12)(34)(56)(78)\}$

5 Conclusion

In conclusion, our work has demonstrated a record-breaking maximum Condorcet domain for $n = 8$ alternatives, which is essentially unique (up to isomorphism and reversal). We have also investigated how our domain relates to various well-studied properties of MCDs. Our findings contribute to understanding the structure of Condorcet domains and have potential applications in voting theory and social choice.

Overall, our work highlights the importance of understanding the properties and structures of CDs in order to construct larger examples and might pave the way for future research in this area.

We also observe that some record-breaking CDs for $n = 8$ alternatives exhibit almost all rules of the form 1N3 and 3N1. These rules can be interpreted as a form of seeded voting. In such a system, for each set of three alternatives, a seeding is implemented to restrict the lowest-seeded alternative from being the highest-ranked preference or the highest-seeded alternative from being the lowest-ranked preference. A better understanding of the global effects of this type of local seeding could

Table 3 Permutation in Condorcet domain corresponding to the rules in table 2

Condorcet domain with 224 Permutations for 8 Alternatives

12345678	12345687	12345867	12345876	12346578	12346587	12346758	12346785	12354678	12354687
12354867	12354876	12358467	12358476	12364578	12364587	12364758	12364785	12367458	12367485
12435678	12435687	12435867	12435876	12436578	12436587	12436758	12436785	12453678	12453687
12453867	12453876	12458367	12458376	12463578	12463587	12463758	12463785	12467358	12467385
14235678	14235687	14235867	14235876	14236578	14236587	14236758	14236785	14253678	14253687
14253867	14253876	14258367	14258376	14263578	14263587	14263758	14263785	14267358	14267385
14523678	14523687	14523867	14523876	14528367	14528376	14582367	14582376	14623578	14623587
14623758	14623785	14627358	14627385	14672358	14672385	21345678	21345687	21345867	21345876
21346578	21346587	21346758	21346785	21354678	21354687	21354867	21354876	21358467	21358476
21364578	21364587	21364758	21364785	21367458	21367485	21435678	21435687	21435867	21435876
21436578	21436587	21436758	21436785	21453678	21453687	21453867	21453876	21458367	21458376
21463578	21463587	21463758	21463785	21467358	21467385	23145678	23145687	23145867	23145876
23146578	23146587	23146758	23146785	23154678	23154687	23154867	23154876	23158467	23158476
23164578	23164587	23164758	23164785	23167458	23167485	23514678	23514687	23514867	23514876
23518467	23518476	23581467	23581476	23614578	23614587	23614758	23614785	23617458	23617485
23671458	23671485	32145678	32145687	32145867	32145876	32146578	32146587	32146758	32146785
32154678	32154687	32154867	32154876	32158467	32158476	32164578	32164587	32164758	32164785
32167458	32167485	32514678	32514687	32514867	32514876	32518467	32518476	32581467	32581476
32614578	32614587	32614758	32614785	32617458	32617485	32671458	32671485	41235678	41235687
41235867	41235876	41236578	41236587	41236758	41236785	41253678	41253687	41253867	41253876
41258367	41258376	41263578	41263587	41263758	41263785	41267358	41267385	41523678	41523687
41523867	41523876	41528367	41528376	41582367	41582376	41623578	41623587	41623758	41623785
41627358	41627385	41672358	41672385						

The CD’s core consists of the underlined permutations 12345678 12346587, 21435678 and 21436587 serve as a foundation for future research, potentially offering insights into algorithmic fairness and impartiality in computer-supported decision-making.

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Declarations

Conflict of interest. The authors are listed alphabetically and declare no conflict of interest.

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