

# Convergence theorems of willingness-to-pay and willingness-to-accept for nonmarket goods

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**Abstract** This paper shows that the disparity between an agent's willingness-to-pay and willingness-to-accept for a nonmarket good becomes small and the two values converge to the same limit only if the agent's initial consumption level of the numéraire is large enough. A necessary and sufficient condition for convergences is provided, and a formula is provided to compute the limit value directly from a utility function. These convergence results are derived when the nonmarket goods are indivisible and qualitatively differentiated, and then extended for the divisible case.

## 1 Introduction

For an agent and a public good in a community, the agent's *willingness-to-pay* (WTP) is the maximum amount of money the agent will pay in exchange for the public good, and the *willingness-to-accept* (WTA) is the minimum amount of money the agent will accept to forgo the public good. Since the two concepts are generally applicable for a large class of nonmarket goods, including not only conventional public goods but also services of environmental resources or amenities, the two concepts are applied in

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a standard cost–benefit analysis for evaluating alternative public projects of supplying nonmarket goods in a community.<sup>1</sup>

However, smallness of the disparity between WTP and WTA is crucial for the consistent evaluation of nonmarket goods on the list of public projects, even when the public projects do not influence a market price vector. Namely, if the disparity is very small, since the individual monetary value of a nonmarket good is well-defined almost uniquely, we can derive a total benefit of a nonmarket good by aggregating all the individual values and the two aggregated measures result in the same rankings for the nonmarket goods;<sup>2</sup> otherwise, it is possible that the two aggregated measures result in the opposite rankings for some nonmarket goods, even if the individual welfare rankings are well defined, independent of the choice of measure.

More specifically, let us consider a community with agent 1 and agent 2, where no public facility is constructed initially. We assume that both agents hold three units of money and their utility functions are defined on the public facilities and the personal consumption levels of money. Agent  $i$ 's utility function is denoted by  $U^i$  for  $i = 1, 2$ , and the initial state of public facility in the community is denoted by  $q_0$ , i.e.,  $q_0$  means no public facility. Now, the community has two alternative projects for constructing new public facilities; one is constructing a swimming pool ( $P$ ) and the other is constructing an archaeology museum ( $M$ ). For simplicity, we assume that the cost for constructing each facility is 0 units of money. Hence, the best facility is the one which maximizes the total (aggregated) benefits over the two agents. For agent  $i$  ( $i = 1, 2$ ), WTP for the swimming pool ( $P$ ) is defined by the amount of money  $c^i$  such that  $U^i(P, 3 - c^i) = U^i(q_0, 3)$ , and WTA for  $P$  is defined by the amount of money  $e^i$  such that  $U^i(P, 3) = U^i(q_0, 3 + e^i)$ . The willingness-to-pay and WTA for  $P$  are denoted by  $WTP^i(P)$  and  $WTA^i(P)$ , respectively. We can similarly define  $WTP^i(M)$  and  $WTA^i(M)$  for the museum.

As a benchmark case, let us consider quasi-linear utility functions  $U^1$  and  $U^2$  such that

$$\begin{aligned} U^1(q_0, x) &= x, & U^1(P, x) &= x + 1.3 & \text{and} & U^1(M, x) &= x + 3; \\ U^2(q_0, x) &= x, & U^2(P, x) &= x + 2 & \text{and} & U^2(M, x) &= x + 1, \end{aligned}$$

where  $x$  is an amount of money. In the quasi-linear case, there is no disparity between WTP and WTA, and thus the individual (monetary) values of nonmarket goods are well-defined exactly.<sup>3</sup> Hence, we can derive the total benefits of a nonmarket good by aggregating all the individual values, and we have that

<sup>1</sup> For comprehensive surveys, see [Bockstael and Freeman \(2005\)](#) and [Carson and Hanemann \(2005\)](#).

<sup>2</sup> In the neo-classical setting, it is well known that the aggregated measures are consistent with the Hicks–Kaldor compensation tests only when all utility functions are Gorman-type. See [Blackorby and Donaldson \(1985\)](#).

<sup>3</sup> Concretely, we have that  $WTP^1(P) = WTA^1(P) = 1.3 < WTP^1(M) = WTA^1(M) = 3.0$ , and that  $WTP^2(P) = WTA^2(P) = 2.0 > WTP^2(M) = WTA^2(M) = 1.0$ . These values are independent of agents' initial holdings of money.

$$\begin{aligned} \text{WTP}^1(P) + \text{WTP}^2(P) &= 3.3 < \text{WTP}^1(M) + \text{WTP}^2(M) = 4.0; \\ \text{WTA}^1(P) + \text{WTA}^2(P) &= 3.3 < \text{WTA}^1(M) + \text{WTA}^2(M) = 4.0. \end{aligned}$$

Namely, both of the welfare measures conclude that the archaeology museum is better than the swimming pool, i.e., the archaeology museum is the best facility in this case.

Let us consider the next case: supposing that the agent 2 will go to swimming pool every day and agent 2’s health condition will be improved if the pool is constructed. Consequently, agent 2’s utility function  $U^2(P, x) = x + 2$  is replaced with  $U^2(P, x) = x + 2 + (1 - e^{-x}) = x - e^{-x} + 3$  in which  $(1 - e^{-x})$  represents the health effect. This effect can be recognized as a (positive) income effect in Mäler (1974) definition of normal good,<sup>4</sup> which ensures generally that  $\text{WTP} < \text{WTA}$ . In this case, it holds that  $\text{WTP}^2(P) = 2.43 < \text{WTA}^2(P) = 2.95$  as shown in Fig. 1, and that the pool is better than the museum in agent 2’s welfare ranking independent of the choice of measure, i.e.,  $\text{WTP}^2(P) > \text{WTP}^2(M)$  and  $\text{WTA}^2(P) > \text{WTA}^2(M)$ . Although all the individual rankings are well-defined, we have that

$$\begin{aligned} \text{WTP}^1(P) + \text{WTP}^2(P) &= 3.73 < \text{WTP}^1(M) + \text{WTP}^2(M) = 4.0; \\ \text{WTA}^1(P) + \text{WTA}^2(P) &= 4.25 > \text{WTA}^1(M) + \text{WTA}^2(M) = 4.0, \end{aligned}$$

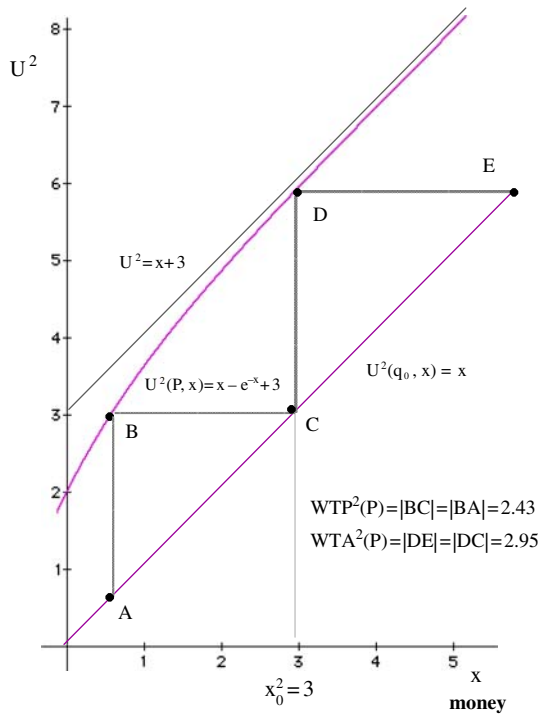
which implies that the two aggregated welfare measures result in opposite rankings: the values of aggregate WTP indicate that the archaeology museum is better than the swimming pool, but the values of aggregate WTA indicate that the swimming pool is better than the archaeology museum. Thus, the smallness of the disparity between WTP and WTA is necessary for the consistent evaluation of nonmarket goods in this example.

Next, let us suppose that agent  $i$ ’s initial holdings of money is a variable and denote it by  $x_0^i$  for  $i = 1, 2$ . Since  $\text{WTP}^i(P)$  and  $\text{WTA}^i(P)$  depend on agent  $i$ ’s initial state  $(q_0, x_0^i)$ , we denote them by  $\text{WTP}^i(P; q_0, x_0^i)$  and  $\text{WTA}^i(P; q_0, x_0^i)$ . Similarly, we use the notation  $\text{WTP}^i(M; q_0, x_0^i)$  and  $\text{WTA}^i(M; q_0, x_0^i)$  in case of  $M$ . When  $x_0^2$  is 5, it holds approximately that  $\text{WTP}^2(P; q_0, 5) = 2.88 < \text{WTA}^2(P; q_0, 5) = 3.00$ , and that

$$\begin{aligned} \text{WTP}^1(P; q_0, 3) + \text{WTP}^2(P; q_0, 5) &= 4.18 > \text{WTP}^1(M; q_0, 3) + \text{WTP}^2(M; q_0, 5) = 4.0; \\ \text{WTA}^1(P; q_0, 3) + \text{WTA}^2(P; q_0, 5) &= 4.30 > \text{WTA}^1(M; q_0, 3) + \text{WTA}^2(M; q_0, 5) = 4.0. \end{aligned}$$

This implies that the two welfare measures result in the same rankings. Mathematically, we can prove that both of the two values  $\text{WTP}^2(P; q_0, x_0^2)$  and  $\text{WTA}^2(P; q_0, x_0^2)$  converge to 3, when  $x_0^2$  is sufficiently large, since  $U^2(P, x) = x + 3$  is an asymptote of  $U^2(P, x) = x - e^{-x} + 3$  as shown in Fig 1. This example tells us that the disparity is negligibly small and consistent evaluations of nonmarket goods are derived when the initial holdings of money (or the initial consumption level of money) is sufficiently large, even though the utility function is not quasi-linear. As the main objective of this paper, we derive the utility condition under which WTP and WTA converge to the same limit when the initial holdings of money is sufficiently large in a single-agent

<sup>4</sup> Mäler (1974, Ch. 4, Section 10, (Eq. 26)) introduces this concept, but he names it convexity.



**Fig. 1** Limit values of  $WTP^2(P)$  and  $WTA^2(P)$  :  $\lim_{x_0^2 \rightarrow +\infty} WTP^2(P; q_0, x_0^2) = \lim_{x_0^2 \rightarrow +\infty} WTA^2(P; q_0, x_0^2) = 3$

setting, assuming that the nonmarket goods are normal as assumed for the swimming pool in the above example.

In the next section, we show that  $WTP$  and  $WTA$  for *indivisible* nonmarket goods are well defined under two standard conditions for a non-linear utility function.<sup>5</sup> Then we introduce the formal definition of a normal good due to [Mäler \(1974\)](#) and show that the normal good condition implies the inequality  $WTP < WTA$ , and that the neutral good condition, or equivalently the quasi-linearity condition, implies  $WTP = WTA$  as shown in above example.

Assuming the normal good condition, [Sect. 3](#) shows that the magnitude of disparity between  $WTP$  and  $WTA$  is larger than a positive number on any compact subset of the

<sup>5</sup> [Mäler \(1974, Ch. 4, Section 10\)](#) defines  $WTP$  and  $WTA$  for indivisible nonmarket goods as well as divisible nonmarket goods, since the indivisibility is an intrinsic property of some nonmarket goods as in case of a park or national defense. More on the indivisibility, see [Samuelson and Nordhaus \(1998, Ch.18, Section C\)](#). Even for (potential) public goods on a planning process of local government, the indivisibility is also an intrinsic property, since the size of a public good is pre-determined by the population of community. For the population aspect of nonmarket goods including the income distribution, see [Ebert \(2003, Sect. 2\)](#). The formal analysis for the disparity of  $WTP$  and  $WTA$  has not been done under the indivisibility with general non-linear utility functions. Exceptionally, [Hanemann \(1999b\)](#); [Erlander \(2005\)](#) and [Dagsvik et al. \(2006\)](#) analyze the welfare properties of the discrete choice model, but their approaches are essentially different from ours.

consumption set (Proposition 2), which implies that a limiting operation is *necessary* for deriving the small disparity of WTP and WTA or the convergence of WTP and WTA. Then, we examine the relation between WTP and WTA, letting the initial holdings of money be large. A necessary and sufficient condition for the convergence is given in terms of utilities under the normality condition (Theorem 1). The condition for the convergence is that the nonmarket good can be replaced with some amount of money, independent of the initial holdings of money. More concretely, when an agent initially consumes a nonmarket good and an amount of money, the condition requires that the agent should prefer the state where the nonmarket good is replaced with an additional amount of money to the initial state and that the additional amount of money should be determined independent of the initial consumption level of money. In the case that  $U^2(P, x) = x - e^{-x} + 3$  and  $U^2(q_0, x) = x$ ,  $P$  is replaceable with three units of money in this sense, since  $U^2(P, x_0^2) < U^2(q_0, x_0^2 + 3) = x_0^2 + 3$  for all  $x_0^2 \geq 0$  as shown in Fig. 1.<sup>6</sup>

Consequently, for the convergence of WTP and WTA, or the smallness of income effects, it is not sufficient to let the initial holdings of money be large: we have to additionally assume the replaceability condition on the utility function, although replaceability is weaker than neutrality. Once the replaceability condition is included, WTP and WTA converge to the same limit value, and hence the well-definedness of the economic value of nonmarket good is obtained, when the initial holdings of money is sufficiently large.

Assuming that the utility function is smooth, Sect. 4 provides a formula for computing the limit values of WTP and WTA directly from the utility function (Theorem 2). Using this formula, some numerical examples are presented. These examples reveal that the convergence theorem (Theorem 1) is not vacuous, i.e., there exist utility functions satisfying all the conditions of the theorem. In Sect. 5, the convergence results are extended for Mäler's (1974, Ch. 4, Section 10) original case where the quality of indivisible nonmarket goods is measured by one variable (Theorem 3). This case can be recognized as the standard neo-classical case where there is just one type of a nonmarket good and the nonmarket good is homogeneous and perfectly divisible.

## 2 Willingness-to-pay and willingness-to-accept for indivisible nonmarket goods

This section introduces a single-agent setting with indivisible nonmarket goods and perfectly divisible money (a numéraire or composite commodity), and the two measures for evaluating the individual benefits for nonmarket goods in terms of the money. We assume that the set of (potential) nonmarket goods is a finite set containing a specific member  $q_0$  which corresponds to “*status quo*”. In this paper, the member  $q_0$  is called the *null* nonmarket good. Denote the set of all nonmarket goods by  $Q = \{q_0, q_1, \dots, q_n\}$ ,<sup>7</sup>

<sup>6</sup> The monotonicity condition and Archimedean condition imply that  $U^2(P, x_0^2) < U^2(q_0, x_0^2 + \delta)$  for some  $\delta > 0$ , which is weaker than the replaceability condition. See Footnote 10 in Sect. 3. A condition like the replaceability condition has been introduced by Cook and Graham (1977), but their definition of replaceability is different from ours, see Hanemann (1999a, footnote 24, p. 65).

<sup>7</sup> The nonmarket goods in  $Q$  can be anything—levels of output of a specific public good, vectors of output levels of many different public goods, or alternative public goods as in the previous section.

and denote the set of *non-null* nonmarket goods by  $Q_+ = \{q_1, \dots, q_n\}$ . We consider an agent who initially holds an amount of money  $x_0 \geq 0$ . The pair  $(q_0, x_0)$  is called *initial situation* and  $(q_0, x_0)$  is the reference point of the two welfare measures which will be defined below. The agent's utility function  $U$  is defined on all the pairs  $(q, x)$  where  $q \in Q$  is a nonmarket good and  $x \in [0, +\infty)$  is an amount of money. The consumption set is denoted by  $Q \times X$  where  $X \equiv [0, +\infty)$ , and the utility function  $U(q, x)$  is assumed to be a real-valued function on  $Q \times X$  satisfying the following standard conditions:

**A<sub>1</sub>**(Continuity and monotonicity): For each  $q \in Q$ ,  $U(q, x)$  is *continuous* and *increasing* in  $x$ .

**A<sub>2</sub>**(Archimedean with desirability): For each  $(q, x) \in Q_+ \times X$ , there exists some  $\delta > 0$  such that  $U(q, x) = U(q_0, x + \delta)$ .

**A<sub>2</sub>** is an Archimedean condition, and along with **A<sub>1</sub>** it implies that  $q_0$  is the least desirable member of  $Q$ , which follows from  $U(q, x) = U(q_0, x + \delta) > U(q_0, x)$ . Although **A<sub>1</sub>** and **A<sub>2</sub>** are always assumed for a utility function in the main part of this paper, the desirability condition in **A<sub>2</sub>** is not crucial for our main results. In fact, the results are applied for some undesirable nonmarket goods such as pollutants in Sect. 5.

For simplicity, a member in the set of all non-null nonmarket goods  $Q_+$  is called a non-null good in the remainder of this paper. At the initial situation  $(q_0, x_0)$ , the *willingness-to-pay* (WTP) for a non-null good  $q \in Q_+$  is defined by the amount of money  $c \in X$  such that

$$U(q_0, x_0) = U(q, x_0 - c), \quad (1)$$

and the *willingness-to-accept* (WTA) *compensation required* for a non-null good  $q \in Q_+$  is defined by the amount of money  $e \in X$  such that

$$U(q_0, x_0 + e) = U(q, x_0). \quad (2)$$

The willingness-to-pay and willingness-to-accept for  $q \in Q_+$  at  $(q_0, x_0)$  are denoted by  $WTP(q; q_0, x_0)$  and  $WTA(q; q_0, x_0)$ , respectively. Under the conditions of **A<sub>1</sub>** and **A<sub>2</sub>**, which are assumed throughout in this paper except for Sect. 5, we have the following lemma:

**Lemma 1** *For each non-null good  $q \in Q_+$ , the following five assertions hold:*

- (i) *There exists a unique amount of money  $\lambda_q > 0$  such that  $U(q_0, \lambda_q) = U(q, 0)$ .*
- (ii)  *$WTP(q; q_0, x_0)$  exists uniquely and  $WTP(q; q_0, x_0) > 0$  is continuous for  $x_0 \in [\lambda_q, +\infty)$ .*
- (iii)  *$WTA(q; q_0, x_0)$  exists uniquely and  $WTA(q; q_0, x_0) > 0$  is continuous for  $x_0 \in X$ .*
- (iv)  *$WTA(q; q_0, x_0) = WTP(q; q_0, x_0 + WTA(q; q_0, x_0))$  for all  $x_0 \in [\lambda_q, +\infty)$ .*
- (v) *Let  $V$  be an equivalent utility function of  $U$ , i.e.,  $V$  is a utility function satisfying the condition:  $V(q, x) \geq V(q^*, x^*) \Leftrightarrow \{U(q, x)\} \geq \{U(q^*, x^*)\}$  for all  $(q, x), (q^*, x^*) \in Q \times X$ . Then the two values,  $WTP(q; q_0, x_0)$  and  $WTA(q; q_0, x_0)$  are invariant if they are re-computed by  $V$ .*

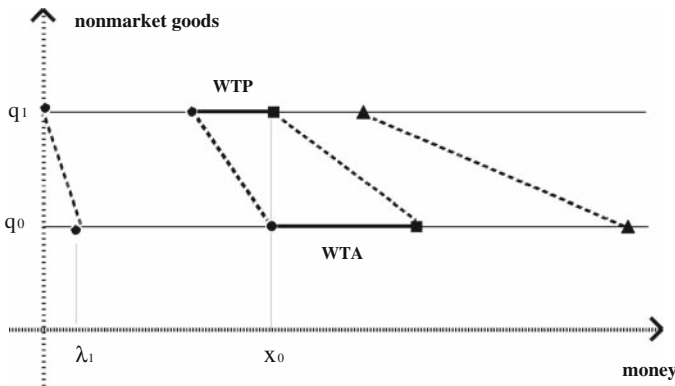


Fig. 2 Indifference map of normal good

Lemma 1 is proved in the Appendix. Assertion (ii) states that the well-definedness of  $WTP(q; q_0, x_0)$  is ensured on domain  $x_0 \in [\lambda_q, +\infty)$ , and it has positiveness and continuity properties on the domain, and Assertion (iii) states that the well-definedness of  $WTA(q; q_0, x_0)$  is ensured for all  $x_0 \in X$  and it also has positiveness and continuity properties. Hence, we can apply the limit operation  $x_0 \rightarrow +\infty$  for  $WTP(q; q_0, x_0)$  and  $WTA(q; q_0, x_0)$ , since both values are well defined for all  $x_0 \in [\lambda_q, +\infty)$ . Assertion (iv) is a restatement of Cook and Graham (1977, (2)) in our setting,<sup>8</sup> and Assertion (v) means that WTP and WTA are determined by the ordinal properties of utility function. Let us consider some numerical examples.

*Example 1* Suppose that  $Q = \{q_0, q_1\}$  and  $x_0 = 10,000$ , and define a utility function  $U$  by  $U(q_0, x) = \sqrt{x}$  and  $U(q_1, x) = \sqrt{x} + 40$  for all  $x \in X$ . Then it holds that  $\lambda_1 = 1,600$  and that  $WTP(q_1; q_0, x_0) = 6,400 < WTA(q_1; q_0, x_0) = 9,600$ . The indifference map is given in Fig. 2.

*Example 2* Suppose that  $Q = \{q_0, q_1\}$  and  $x_0 = 10,000$ , and define a utility function  $U$  by  $U(q_0, x) = x$  and  $U(q_1, x) = x + 3,000$  for all  $x \in X$ . Then it holds that  $\lambda_1 = 3,000$  and that  $WTP(q_1; q_0, x_0) = WTA(q_1; q_0, x_0) = 3,000$ . See Fig. 3 for the indifference map.

*Example 3* Suppose that  $Q = \{q_0, q_1\}$  and  $x_0 = 10,000$ , and define a utility function  $U$  by  $U(q_0, x) = x\sqrt{x}$  and  $U(q_1, x) = x\sqrt{x} + 10,000$  for all  $x \in X$ . Then it holds that  $\lambda_1 \approx 464$  and that  $WTP(q_1; q_0, x_0) \approx 66.78 > WTA(q_1; q_0, x_0) \approx 66.56$ . See Fig. 4.

Depending on the utility function, WTP can be larger than, smaller than, or equal to WTA. The next proposition provides a sufficient condition for each case. In order to state the proposition, we need some definitions:

**Normality of non-null good** A non-null good  $q \in Q_+$  is defined to be a *normal* good if and only if the following condition holds: If  $U(q, x^*) = U(q_0, x)$ , then  $U(q, x^* + \delta) > U(q_0, x + \delta)$  for all  $\delta > 0$ .

<sup>8</sup> See also Hanemann (1999a, Proposition 3.6).

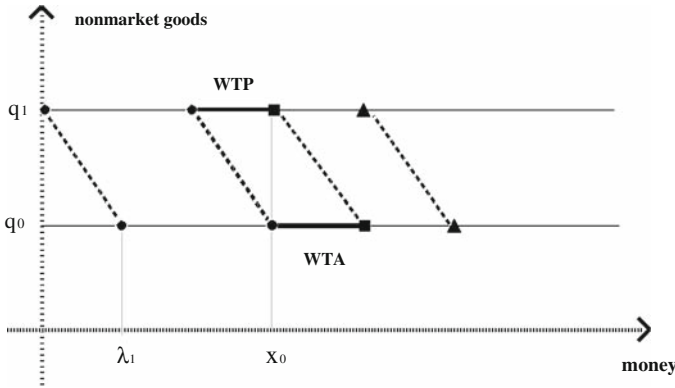


Fig. 3 Indifference map of neutral good

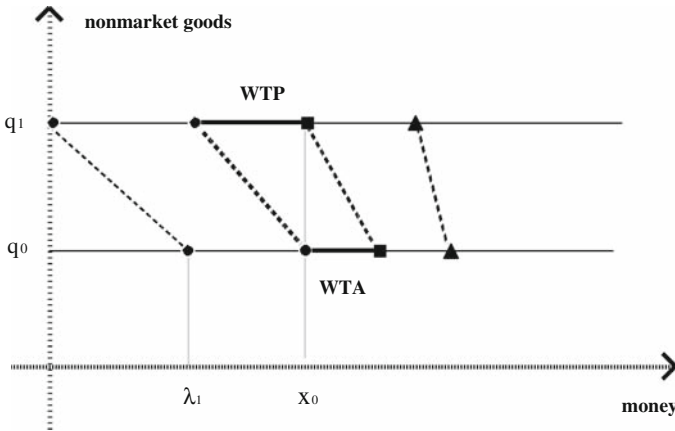


Fig. 4 Indifference map of inferior good

This definition is introduced by Mäler (1974, Ch. 4, Section 10, (26)). If the normal nonmarket goods are sold in a hypothetical competitive market, and if the agent attempts to buy at most one good, then the agent’s demand shifts from  $q_0$  to  $q$ , or remains the same whenever the agent’s initial holdings of money (income) is increased, and hence our definition of normality is the discrete counterparts of Hicksian definition of normal good in a neo-classical setting with continuum variables.

**Neutrality of non-null good** A non-null good  $q \in Q_+$  is defined to be a *neutral* good if and only if the following condition holds: If  $U(q, x^*) = U(q_0, x)$ , then  $U(q, x^* + \delta) = U(q_0, x + \delta)$  for all  $\delta > 0$ .

This condition means that the indifference curves are parallel as shown in Fig. 3, and we can easily show that there exists a quasi-linear utility function which is equivalent to  $U$ . (For a proof, see Mas-Colell et al. (1995, Definition 3.B.7 and Exercise 3.C.5.) Hence, our definition of neutrality is the discrete counterpart of Hicksian definition of neutrality, which is a direct consequence of the quasi-linearity.



**Inferiority of non-null good** A non-null good  $q \in Q_+$  is defined to be an *inferior* good if and only if the following condition holds: If  $U(q, x^*) = U(q_0, x)$ , then  $U(q, x^* + \delta) < U(q_0, x + \delta)$  for all  $\delta > 0$ .

Then we can state the proposition as follows:<sup>9</sup>

**Proposition 1** *Let  $q$  be a non-null good in  $Q_+$ , and suppose that  $x_0$  is greater than  $\lambda_q$ .*

- (A) *If  $q$  is normal, then the following assertions hold:*
  - (i)  $WTP(q; q_0, x_0 + \delta) > WTP(q; q_0, x_0)$  for all  $\delta > 0$ ,
  - (ii)  $WTA(q; q_0, x_0 + \delta) > WTA(q; q_0, x_0)$  for all  $\delta > 0$ ,
  - (iii)  $WTA(q; q_0, x_0) > WTP(q; q_0, x_0)$ .
- (B) *If  $q$  is neutral, then the two measures, WTP and WTA in Assertions (i–iii) have the same value, i.e., all the inequalities in Assertions (i–iii) are replaced with equalities.*
- (C) *If  $q$  is inferior, then the opposite inequalities in Assertions (i–iii) hold.*

Proposition 1 is proved in Sect. 6. Assertions (i) and (ii) mean that  $WTP(q; q_0, x_0)$  and  $WTA(q; q_0, x_0)$  are increasing functions of the initial holdings of money  $x_0$ , respectively. Assertion (iii) means that  $WTA(q; q_0, x_0)$  is always larger than  $WTP(q; q_0, x_0)$  as shown in Fig. 1, and then the economic value for  $q$  in terms of money is not determined exactly for  $U$  under the normality. Hence, Proposition 1 implies that the primal source of disparity  $WTA > WTP$  is the normality of nonmarket good, and the disparity is equivalent to the monotonicity of WTP and WTA with respect to the initial holdings of money.

### 3 Convergence of WTP and WTA under normality

In this section, we assume additionally that nonmarket goods in  $Q_+$  are normal as defined above, which is a sufficient condition for the inequality  $WTA > WTP$ , and then we investigate the magnitude of that disparity.

Since a continuous real-valued function on a compact set has its minimizer within the compact set, we have the following proposition as a direct consequence of Proposition 1(Aiii) and Lemma 1(ii, iii):

**Proposition 2** *Suppose that a non-null good  $q \in Q_+$  is normal. For any compact interval  $[z_1, z_2]$  in  $X$  with  $z_1 \geq \lambda_q$ , there exists  $\varepsilon > 0$  such that*

$$WTA(q; q_0, x_0) - WTP(q; q_0, x_0) > \varepsilon \quad \text{for all } x_0 \in [z_1, z_2].$$

Proposition 2 states that, for any normal good  $q \in Q_+$ , the difference between  $WTP(q; q_0, x_0)$  and  $WTA(q; q_0, x_0)$  is uniformly bounded from below whenever

<sup>9</sup> These assertions are the discrete counterparts of well-known assertions in the perfect divisible case as shown by Cook and Graham (1977), Loehman (1991, Theorem 2), Ebert (1993), Hanemann (1999a, Proposition 3.6) and Latham (1999, Proposition 3). See Kaneko et al. (2006, Assumption E and Lemma 2.3(2)) for further discussion of the normal good condition.

$x_0$  belongs to a compact interval  $[z_1, z_2]$  in  $X$ . This proposition implies that we must introduce a limit operation on  $x_0$  to eliminate the disparity between WTP and WTA.

Fixing a normal good  $q \in Q_+$ , let us investigate the asymptotic behaviors of  $WTP(q; q_0, x_0)$  and  $WTA(q; q_0, x_0)$  when  $x_0 \rightarrow +\infty$ . Since  $WTP(q; q_0, x_0)$  is increasing with respect to  $x_0$  by Proposition 1(Ai), there are just two possible cases:  $0 < \lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) < +\infty$ , or  $\lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) = +\infty$ . Since  $WTA(q; q_0, x_0)$  also has the monotonicity property, there are the same two cases for WTA, i.e., convergence or divergence. Hence, there are four possible cases:

- (I)  $0 < \lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) < +\infty$  and  $0 < \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) < +\infty$ ;
- (II)  $0 < \lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) < +\infty$  and  $\lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) = +\infty$ ;
- (III)  $\lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) = +\infty$  and  $0 < \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) < +\infty$ ;
- (IV)  $\lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) = +\infty$  and  $\lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) = +\infty$ .

However, we have the following proposition which tells us that there are just two possible cases, (I) or (IV):

**Proposition 3** *If a non-null good  $q \in Q_+$  is normal, then it holds that*

$$\lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0).$$

The equality above includes the case that  $\lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) = +\infty$ .

The proof of Proposition 3 is given in Sect. 6. For example, let us reconsider the utility function  $U$  in Example 1. This  $U$  satisfies the normality condition on  $q_1$ , but it holds that

$$\lim_{x_0 \rightarrow +\infty} WTA(q_1; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} (80(\sqrt{x_0} + 1), 600) = +\infty,$$

which implies that  $U$  satisfies all the conditions for Case (IV). Consider another example:

*Example 4* As in the example of Sect. 1, setting  $Q = \{q_0, q_1\}$ , the utility function  $U^*$  is defined by

$$U^*(q_0, x) = x \quad \text{and} \quad U^*(q_1, x) = x + 3 - e^{-x} \quad \text{for all } x \geq 0.$$

One can easily show that  $U^*$  satisfies the normality condition on  $q_1$ , and the graph of  $U^*(q_1, x)$  can be drawn as  $U^2(P, x)$  on Fig. 1. It holds by the definition of WTA that

$$WTA^*(q_1; q_0, x_0) = U^*(q_1, x_0) - U^*(q_0, x_0) = 3 - e^{-x_0},$$

i.e.,  $WTA^*(q_1; q_0, x_0)$  coincides with the difference of utility values,  $U^*(q_1, x_0)$  and  $U^*(q_0, x_0)$ . Hence, it holds that

$$\lim_{x_0 \rightarrow +\infty} WTA^*(q_1; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} (3 - e^{-x_0}) = 3, \tag{3}$$

and that  $U^*$  satisfies all the conditions for Case (I).

For example, given  $q \in Q_+$ , one may infer from Example 4 that  $WTA(q; q_0, x_0)$  converges to a limit when  $x_0 \rightarrow +\infty$ , if  $WTA(q; q_0, x_0)$  is bounded from above. Because  $WTA(q; q_0, x_0) > 0$  is an increasing function of  $x_0$  by Proposition 1 (Aii), we can prove the convergence by applying the following well-known theorem:

**Theorem of the monotone limit** (See Howie 2001, Theorem 3.1, p. 74, or Simon and Blume 1994, Theorem 29.2, p. 805): *If a function  $f: [0, +\infty) \rightarrow [0, +\infty)$  is increasing and bounded from above, then  $f$  has a limit as  $x \rightarrow +\infty$ , i.e.,  $\lim_{x \rightarrow +\infty} f(x) < +\infty$ . If additionally  $f(x) > 0$  for some  $x \geq 0$ , then  $0 < \lim_{x \rightarrow +\infty} f(x) < +\infty$ .*

Practically, in order to ensure the boundedness of  $WTA(q; q_0, x_0)$ , one can assume the following condition on the utility function  $U$ :

**Replaceability of non-null good  $q \in Q_+$**  There exists  $K_q \geq 0$  such that  $U(q_0, x + K_q) \geq U(q, x)$  for all  $x \geq 0$ .

Replaceability means that  $q$  is replaceable for some amount of money,  $K_q$ , independent of the initial holdings of money.<sup>10</sup> Since  $U(q_0, x_0 + WTA(q; q_0, x_0)) = U(q, x_0)$  by the definition of  $WTA$ , it holds under the replaceability condition that

$$U(q_0, x_0 + WTA(q; q_0, x_0)) = U(q, x_0) \leq U(q_0, x_0 + K_q) \quad \text{for all } x_0.$$

By this and the contrapositive of monotonicity condition in  $A_1$  we have that  $WTA(q; q_0, x_0) \leq K_q$  for all  $x_0$  and that  $WTA(q; q_0, x_0)$  is bounded. Hence, it follows from the theorem of the monotone limit that  $0 < \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) < +\infty$ . Thus, we observe that the replaceability condition is *sufficient* for the convergence of  $WTA$  under the normality condition.

Conversely, suppose that  $0 < \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) < +\infty$ , and set  $K_q = 1 + \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0)$ . Since  $\lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) < K_q$  implies  $WTA(q; q_0, x) < K_q$  for sufficiently large  $x$ , it holds by Proposition 1(Aiii) that  $K_q > WTA(q; q_0, x)$  for all  $x$ . Hence it holds by the definition of  $WTA$  and the monotonicity in  $A_1$  that  $U(q_0, x) = U(q_0, x + WTA(q; q_0, x)) < U(q_0, x + K_q)$ , which implies that the replaceability condition is *necessary* for the convergence of  $WTA$ . Thus, the replaceability condition is necessary and sufficient for the convergence of  $WTA$  under the normality condition.

<sup>10</sup> Since the Archimedean condition in  $A_2$  implies that  $q$  is replaceable for some amount of money *depending* on the initial holdings of money, the replaceability is not implied by  $A_2$ . In fact, the utility function  $U$  in Example 1 satisfies  $A_2$ , but it does not satisfy the replaceability.

The arguments above and Proposition 3 together imply the following theorem:

**Theorem 1** [Convergence of WTP and WTA] *If a non-null good  $q \in Q_+$  is normal, then the following four statements are mutually equivalent:*

- (i)  $q$  is a replaceable good;
- (ii)  $0 < \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) < +\infty$ ;
- (iii)  $0 < \lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) < +\infty$ ; <sup>11</sup>
- (iv)  $0 < \lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) < +\infty$ .

Theorem 1 implies that the necessary and sufficient condition for the convergence of WTP and WTA is that the nonmarket good is replaceable with money under the normality condition. Thus, for the convergence of WTP and WTA, it is not sufficient to let the initial holdings of money be large, and we have to additionally assume the replaceability condition on the utility function; otherwise, it holds that  $\lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) = +\infty$ . More formally, as a direct consequence of Theorem 1 and Proposition 3, we have the following corollary:

**Corollary 1** [Divergences of WTP and WTA] *If a non-null good  $q \in Q_+$  is normal, then the following three statements are mutually equivalent:*

- (i)  $q$  is an irreplaceable good, i.e., for each  $x \geq 0$  there exists some  $L_x \geq 0$  such that  $U(q, L_x) > U(q_0, x + L_x)$ ; <sup>12</sup>
- (ii)  $\lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) = +\infty$ ;
- (iii)  $\lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) = +\infty$ .

#### 4 Direct computation of the limit values

This section provides formulas to compute the limit values of WTP and WTA directly from the utility function  $U$ , without employing the formulas that defining  $WTP(q; q_0, x_0)$  and  $WTA(q; q_0, x_0)$  for all  $x_0$ , respectively. In order to state the formulas we need a definition and a lemma. For a given utility function  $U(q, x)$  on  $Q \times X$ , denoting  $f(x) \equiv U(q_0, x)$  for all  $x$ , the *normalized* utility function  $V$  of  $U$  is defined by

<sup>11</sup> Furthermore, under the normality condition, the statement (iii) implies that the marginal values of  $WTP(q; q_0, x_0)$  with respect to  $x_0$  converges to 0 as  $x_0 \rightarrow +\infty$ , i.e.,

$$\lim_{x_0 \rightarrow +\infty} |WTP(q; q_0, x_0 + 1) - WTP(q; q_0, x_0)| = 0.$$

Hence, the implausible phenomenon in Milgrom (1993, p. 430) does not occur as long as (iii) holds.

<sup>12</sup> By the definition of irreplaceable good, a non-null good  $q$  is irreplaceable if and only if  $q$  is not replaceable. Suppose that  $q$  is irreplaceable for an agent who owns  $q$  in a hypothetical competitive market of  $q$ . Then, no matter how large price  $x > 0$  is given, the agent does not sell  $q$  if the agent's initial holdings of money is  $L_x$ . In Example 1,  $q_1$  is irreplaceable and  $L_x$  can be set by  $L_x = [(x - 1, 600)/80]^2 + 1$  for all  $x > 0$ .

$$V(q, x) = f^{-1}(U(q, x)) \quad \text{for all } (q, x) \in Q \times X, \tag{4}$$

where the well-definedness of  $f^{-1}$  is ensured by  $\mathbf{A}_1$  and  $\mathbf{A}_2$ . Then we have a lemma:

**Lemma 2** *Let  $U^a$  and  $U^b$  be two utility functions on  $Q \times X$ , and let  $V^a$  and  $V^b$  be the normalized utility functions of  $U^a$  and  $U^b$ , respectively. If  $U^a$  is equivalent to  $U^b$ , then  $V^a$  coincides with  $V^b$ .*

Lemma 2 is proved in the Appendix. By Lemma 2, we can consistently assume a cardinal property on the normalized utility function, since it is a representative of an equivalent class of utility functions. Specifically, we consider the following properties:

**Concavity** A smooth utility function  $V(q, x)$  on  $Q \times X$  is *concave* if and only if  $d^2V(q, x)/dx^2 < 0$  for all  $q \in Q_+$  and all  $x > 0$ ; <sup>13</sup>

**Uniformly boundedness of the differences**  $V(q, x)$  satisfies the *uniformly boundedness of the differences* at  $q \in Q_+$  if and only if there exists  $M_q > 0$  such that  $V(q, x) - V(q_0, x) < M_q$  for all  $x > M_q$ .

The main result of this section is the following theorem:

**Theorem 2** *Suppose that  $U(q, x)$  is a smooth utility function on  $Q \times X$ , and that the normalized utility function  $V(q, x)$  of  $U(q, x)$  is concave and satisfies the uniformly boundedness of the differences at all  $q \in Q_+$ . Then  $U$  and  $V$  satisfy all the conditions in Theorem 1 for all  $q \in Q_+$ , and the following two assertions hold:*

- (i) 
$$0 < \lim_{x \rightarrow +\infty} WTP(q; q_0, x) = \lim_{x \rightarrow +\infty} WTA(q; q_0, x) = \lim_{x \rightarrow +\infty} [V(q, x) - x] < +\infty \quad \text{for all } q \in Q_+,$$
- (ii) *for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that*

$$\max_{(q,x) \in Q_+ \times [\delta, +\infty)} |v(q) + x - V(q, x)| < \varepsilon,$$

where  $v(q) = \lim_{x \rightarrow +\infty} WTP(q; q_0, x) = \lim_{x \rightarrow +\infty} WTA(q; q_0, x)$ . Namely, setting  $v(q_0) = 0$ , the utility function  $V^*(q, x)$  defined by  $V^*(q, x) = v(q) + x$  is the limit function of the utility function  $V(q, x)$  when  $x \rightarrow +\infty$ .

Theorem 2 is proved in Sect. 6. For a given utility function in an elementary function form, we can easily check whether the two conditions hold or not, and then we can construct an example which satisfies all the conditions of Theorem 2. The utility function  $U^*$  in Example 4 satisfies all the conditions in Theorem 2, which implies that the convergence theorem (Theorem 1) is non-void, i.e., there exist utility functions satisfying all the conditions in the theorem. In Example 4, a linear utility function  $V^*$  such that

$$V^*(q_i, x) = v(q_i) + x,$$

<sup>13</sup> A real-valued function  $V(q, x)$  on  $Q \times X$  is *smooth* if  $V(q, x)$  is  $C^2$  with respect to  $x$  for all  $q$ , i.e.,  $V(q, x)$  is continuous and continuously second differentiable for  $x$ .

where  $v(q_0) = 0$  and  $v(q_1) = 3$ , is derived as a limit function of the non-linear utility function  $U^*$  when  $x \rightarrow +\infty$ . Hence, a linear utility function (as introduced in Example 2) used in the discrete choice model or random utility model<sup>14</sup> can be interpreted as the limit utility function of some non-linear utility function  $V$  in Theorem 2.<sup>15</sup>

However, there is a non-linear utility function which does not satisfy the replaceability condition in Theorem 1. See the next example:

*Example 5* (A Box–Cox type utility function)<sup>16</sup> Define a utility function  $U$  on  $\{q_0, q_1\} \times X$  by

$$U(q_0, x) = 2(\sqrt{x} - 1) \quad \text{and} \quad U(q_1, x) = 1 + 2(\sqrt{x} - 1) \quad \text{for all } x \in X.$$

This  $U$  is equivalent to  $V(q, x) = f^{-1}(U(q, x))$  where  $f(x) = U(q_0, x) = 2(\sqrt{x} - 1)$ . By a simple computation, we have that

$$V(q_0, x) = x \quad \text{and} \quad V(q_1, x) = x + \sqrt{x} + 1/4 \quad \text{for all } x \in X.$$

Hence,  $U$  and  $V$  satisfy the normality condition on  $q_1$ . Since we can prove by almost the same arguments in deriving (3) in Example 4 that

$$\lim_{x_0 \rightarrow +\infty} \text{WTA}(q_1; q_0, x_0) = \lim_{x \rightarrow +\infty} [V(q_1, x) - x] = \lim_{x \rightarrow +\infty} (\sqrt{x} + 1/4) = +\infty,$$

Assertion (ii) of Theorem 1 does not hold, and hence we have by Theorem 1 ( $i \Leftrightarrow ii$ ) that  $U$  and  $V$  do not satisfy the replaceability condition.

### 5 Extensions

We extend the convergence results (Theorems 1 and 2) to the case where the quality of an indivisible nonmarket good is measured by one real variable  $q \in [0, +\infty)$  such as the size of a public facility. This case corresponds to Mäler’s (1974, Ch. 4, Section 10) original case. Specifically, the consumption set,  $Q \times X$  is  $[0, +\infty) \times [0, +\infty)$ , and we assume that the initial level of the quality is fixed and denoted by  $q_0 \geq 0$ . In order to state the next theorem, we need some definitions for a *smooth* utility function  $U$  on  $Q \times X$ :<sup>17</sup>

**Monotonicity:**  $U_q(q, x) > 0$  and  $U_x(q, x) > 0$  for all  $(q, x) \in Q \times X$ ,

<sup>14</sup> Moreover, a piece-wise linear approximation for  $U$  also satisfies all the conditions in the theorem, and the piece-wise linear utility function is applicable for the logit models, see Morey et al. (2003).

<sup>15</sup> In Example 4, the economic value  $v(q_1)$  consists of not only the generic value  $U^*(q_1, 0)$ , but also the accumulated marginal values  $\int_0^{+\infty} U^{*'}(q_1, x)dx$ , i.e.,  $v(q_1) = U^*(q_1, 0) + \int_0^{+\infty} U^{*'}(q_1, x)dx$ . In the case of park or public library, the first and second terms are recognized as the existence value and use value, respectively, and Theorem 2 simply provides the intergrability condition for the use value.

<sup>16</sup> For the Box–Cox type utility function, see Carson and Hanemann (2005).

<sup>17</sup> A real-valued function  $f$  on  $[0, +\infty) \times [0, +\infty)$  is *smooth* if  $f$  is  $C^2$ , i.e.,  $f$  is continuous on  $[0, +\infty) \times [0, +\infty)$  and all second partial derivatives:  $f_{xy}(x, y)$ ,  $f_{yx}(x, y)$ ,  $f_{xx}(x, y)$ , and  $f_{yy}(x, y)$  are continuous on  $(0, +\infty) \times (0, +\infty)$

**Hicksian normality of  $q$  :**  $U_x(q, x) \cdot U_{qx}(q, x) - U_q(q, x) \cdot U_{xx}(q, x) > 0$  for all  $(q, x) \gg (0, 0)$ ,

**Existence of positive limit marginal utility:**  $0 < \lim_{x \rightarrow +\infty} U_x(q_0, x) < +\infty$ .

As a main result of this section, we have the following theorem:

**Theorem 3** *Suppose that a smooth utility function  $U(q, x)$  on  $Q \times X$  satisfies the following conditions:*

- (i) *Monotonicity;*
- (ii) *Hicksian normality of  $q$ ;*
- (iii) *Existence of positive limit marginal utility;*
- (iv) *Uniformly boundedness of the differences at any  $q > q_0$  as defined in Sect. 4.*

*Then  $U$  satisfies all the conditions in Theorem 1 for any  $q > q_0$ , and it holds that*

$$\begin{aligned}
 0 < \lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) &= \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) \\
 &= \frac{\lim_{x \rightarrow +\infty} [U(q, x) - U(q_0, x)]}{\lim_{x \rightarrow +\infty} U_x(q_0, x)} < +\infty \text{ for any } q > q_0.
 \end{aligned}
 \tag{5}$$

Theorem 3 is proved in Sect. 6. The leading two conditions are ordinal ones, but latter two conditions state *cardinal* properties of a utility function as the conditions in Theorem 2. Now, suppose that a smooth utility function  $U$  satisfies the monotonicity and Hicksian normality conditions. Then we can prove easily that  $U$  satisfies the replaceability condition if and only if there exists *some* equivalent utility function  $U^*$  of  $U$  satisfying the two cardinal conditions such as the normalized utility function in Theorem 2,<sup>18</sup> although it does not hold that *all* equivalent utility functions satisfy the two cardinal conditions. Moreover, we can prove, using Theorem 3 and Lemma 1(vi), that the limit values of WTP and WTA are well defined, independent of the selection of the utility function  $U^*$ , i.e., it holds that

$$\begin{aligned}
 \frac{\lim_{x \rightarrow +\infty} [U(q, x) - U(q_0, x)]}{\lim_{x \rightarrow +\infty} U_x(q_0, x)} &= \lim_{x_0 \rightarrow +\infty} WTP(q; q_0, x_0) \\
 &= \lim_{x_0 \rightarrow +\infty} WTA(q; q_0, x_0) = \frac{\lim_{x \rightarrow +\infty} [U^*(q, x) - U^*(q_0, x)]}{\lim_{x \rightarrow +\infty} U_x^*(q_0, x)}.
 \end{aligned}$$

Hence we can consistently derive the limit values, although we utilize some cardinal properties of a utility function. Some examples are given as follows.<sup>19</sup>

<sup>18</sup> We can prove easily that a smooth equivalent utility function  $U^*$  of  $U$  satisfies all the conditions in Theorem 3 if and only if there is a smooth and increasing real-valued function  $g$  on  $U(Q \times X)$  such that  $U^*(q, x) = g(U(q, x))$  for all  $(q, x) \in Q \times X$  with  $0 < \lim_{y \rightarrow +\infty} g'(y) < +\infty$ , where  $U(Q \times X) = \{u : u = U(q, x) \text{ for some } (q, x) \in Q \times X\}$ .

<sup>19</sup> A Cobb–Douglas type function satisfies  $A_1$  and  $A_2$  on the interior of  $Q \times X$ , but it does not when the boundary is included. Even in the interior, there is no Cobb–Douglas utility function which satisfies all the conditions in Theorem 3.

*Example 6* (additively separable utility function) A utility function

$$U(q, x) = \sqrt{q} + \sqrt{x} + \log(x + 1) + x \quad \text{for all } (q, x) \in Q \times X$$

satisfies all conditions in Theorem 3, and then it holds that

$$\lim_{x_0 \rightarrow +\infty} \text{WTP}(q; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) = \sqrt{q} - \sqrt{q_0} \quad \text{for all } q > q_0.$$

*Example 7* (non-separable utility function) A utility function

$$U(q, x) = \sqrt{q}(2 - e^{-x}) + \sqrt{x} + x \quad \text{for all } (q, x) \in Q \times X$$

satisfies all conditions in Theorem 3, and then it holds that

$$\lim_{x_0 \rightarrow +\infty} \text{WTP}(q; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) = 2(\sqrt{q} - \sqrt{q_0}) \quad \text{for all } q > q_0.$$

*Example 8* (CES-like utility function) Let us consider a utility function

$$U(q, x) = [\alpha(q + a)^\rho + \beta(x + b)^\rho]^{1/\rho} + \delta q + \varepsilon x \quad \text{for all } (q, x) \in Q \times X$$

where  $\alpha > 0, \beta > 0, a > 0, b > 0, \rho < 0, \delta \geq 0$  and  $\varepsilon > 0$ . This  $U$  satisfies all conditions in Theorem 3, and then it holds that

$$\lim_{x_0 \rightarrow +\infty} \text{WTP}(q; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) = \frac{\alpha^{1/\rho} + \delta}{\varepsilon} (q - q_0) \quad \text{for all } q > q_0.$$

Interpreting  $q \in Q$  as a quantity variable, this case can be recognized as a standard neo-classical case where there is just one type of nonmarket good and the nonmarket good is homogeneous and perfectly divisible.

Finally, let us consider the case of undesirable nonmarket goods such as a pollutant in the discrete setting.<sup>20</sup> Let  $q$  be an undesirable nonmarket good. The condition  $\mathbf{A}_2$  should be replaced with the following condition:

$\mathbf{A}_2^*$  (Archimedean and undesirability): There exists some  $\delta > 0$  such that  $U(q, x + \delta) = U(q_0, x)$ .

Although this condition implies that WTP and WTA are always negative, we can directly apply Theorems 1 and 2 for these goods under  $\mathbf{A}_1$  and  $\mathbf{A}_2^*$ , as long as these goods are inferior as defined in Sect. 2. For example, define a utility function  $U$  on  $\{q_0, q_1\} \times X$  by  $U(q_0, x) = x$  and  $U(q_1, x) = x - 3 + e^{-x}$ . Then it holds that

$$\lim_{x_0 \rightarrow +\infty} \text{WTP}(q_1; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} \text{WTA}(q_1; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} [U(q_1, x_0) - x_0] = -3.$$

<sup>20</sup> In a continuum setting with detrimental goods, Ebert (1993, Sect. 5) has derived characterizing results of WTP and WTA for the detrimental goods.



### 6 Proofs

*Proof of Proposition 1 (A)* (i) Fix any  $\delta > 0$ . It holds by (1) that  $U(q, x_0 - \text{WTP}(q; q_0, x_0)) = U(q_0, x_0)$ . Then it holds by the normality condition that  $U(q, x_0 - \text{WTP}(q; q_0, x_0) + \delta) > U(q_0, x_0 + \delta)$ . Moreover, it holds by (1) that  $U(q_0, x_0 + \delta) = U(q, x_0 + \delta - \text{WTP}(q; q_0, x_0 + \delta))$ . Hence, we have that  $U(q, x_0 - \text{WTP}(q; q_0, x_0) + \delta) > U(q, x_0 + \delta - \text{WTP}(q; q_0, x_0 + \delta))$ . It follows from this and  $A_1$  that  $x_0 - \text{WTP}(q; q_0, x_0) + \delta > x_0 + \delta - \text{WTP}(q; q_0, x_0 + \delta)$  and  $\text{WTP}(q; q_0, x_0 + \delta) > \text{WTP}(q; q_0, x_0)$ .

(ii) We can prove Assertion (ii) by almost the same manner as in the proof of Assertion (i).

(iii) It holds by Lemma 1(iv) that  $\text{WTA}(q; q_0, x_0) = \text{WTP}(q; q_0, x_0 + \text{WTA}(q; q_0, x_0))$ . Hence, we have by Lemma 1(ii, iii) that  $\text{WTA}(q; q_0, x_0) = \text{WTP}(q; q_0, x_0 + \text{WTA}(q; q_0, x_0)) > \text{WTP}(q; q_0, x_0)$ . (B, C) We can prove (B, C) by almost the same manner as in the proof of (A). □

*Proof of Proposition 3* Suppose that Case (II) holds, i.e.,

$$0 < \lim_{x_0 \rightarrow +\infty} \text{WTP}(q; q_0, x_0) < +\infty, \tag{6}$$

$$\text{and } \lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) = +\infty. \tag{7}$$

Since  $\lim_{x_0 \rightarrow +\infty} [x_0 + \text{WTA}(q; q_0, x_0)] = +\infty$ , it holds by (6) that

$$\lim_{x_0 \rightarrow +\infty} \text{WTP}(q; q_0, x_0 + \text{WTA}(q; q_0, x_0)) < +\infty.$$

Hence it holds by Lemma 1(iv) that

$$\lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} \text{WTP}(q; q_0, x_0 + \text{WTA}(q; q_0, x_0)) < +\infty,$$

which contradicts (7). Hence, Case (II) does not hold.

If  $\lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) < +\infty$ , then it holds by Proposition 1(Aiii) that  $\lim_{x_0 \rightarrow +\infty} \text{WTP}(q; q_0, x_0) < +\infty$ . Hence Case (III) does not hold. □

*Proof of Theorem 2* (i) We need the following lemma proved in the Appendix:

**Lemma 3** *U and V satisfy the normality condition on all  $q \in Q_+$ .*

Fix any  $q \in Q_+$ . It holds by Lemma 1(iii), Lemma 3 and Proposition 1(Aii) that  $\text{WTA}(q; q_0, x_0) > 0$  is increasing in  $x_0$ . Moreover, we have by the uniformly boundedness condition that  $\text{WTA}(q; q_0, x_0)$  is uniformly bounded from above when  $x_0 > M_q$ . Thus, it holds by the theorem of the monotone limit that  $0 < \lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) < +\infty$ . It holds by this, (2) and (4) that

$$\begin{aligned} 0 < \lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) &= \lim_{x_0 \rightarrow +\infty} [V(q, x) - V(q_0, x)] \\ &= \lim_{x_0 \rightarrow +\infty} [V(q, x_0) - x_0] < +\infty. \end{aligned}$$

Moreover, it holds by Lemma 3 and Theorem 1 that

$$0 < \lim_{x_0 \rightarrow +\infty} \text{WTP}(q; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) = \lim_{x_0 \rightarrow +\infty} [V(q, x_0) - x_0] < +\infty.$$

(ii) Assertion (ii) is a direct consequence of Assertion (i). □

*Proof of Theorem 3* Fix any  $q > q_0$ . On  $\{q, q_0\} \times X$  we can easily prove that  $U$  satisfies  $\mathbf{A}_1$  and the desirability in  $\mathbf{A}_2$ . It holds by the existence of positive limit marginal utility that  $U$  satisfies Archimedean in  $\mathbf{A}_2$ . We need the following lemma proved in the Appendix:

**Lemma 4**  $U$  satisfies the normality condition on  $q$ .

Define a utility function  $V$  by  $V(q, x) = f^{-1}(U(q, x))$  for all  $(q, x) \in \{q, q_0\} \times X$  where  $f(x) = U(q_0, x)$ . By the existence of positive limit marginal utility, we have that

$$0 < \lim_{x \rightarrow +\infty} f'(x) = \lim_{x \rightarrow +\infty} U_x(q_0, x) < +\infty. \tag{8}$$

Then it holds by Mean Value Theorem that

$$\lim_{x \rightarrow +\infty} f''(x) = 0. \tag{9}$$

By (2) and the definition of  $V$ , it holds that  $\text{WTA}(q; q_0, x_0) = V(q, x_0) - x_0$  for all  $x_0$ . Then it follows from Taylor's Formula and Inverse Function Theorem that

$$\begin{aligned} \text{WTA}(q; q_0, x_0) &= V(q, x_0) - x_0 = f^{-1}(U(q, x_0)) - x_0 \\ &= f^{-1}([U(q, x_0) - U(q_0, x_0)] + U(q_0, x_0)) - x_0 \\ &= f^{-1}(U(q_0, x_0)) + [U(q, x_0) - U(q_0, x_0)]/f'(x_0) \\ &\quad + [U(q, x_0) - U(q_0, x_0)]^2 f''(x_0 + \xi)/2[f'(x_0)]^3 - x_0 \\ &= [U(q, x_0) - U(q_0, x_0)]/f'(x_0) \\ &\quad + [U(q, x_0) - U(q_0, x_0)]^2 f''(x_0 + \xi)/2[f'(x_0)]^3 \end{aligned} \tag{10}$$

where  $\xi \in [0, U(q, x_0) - U(q_0, x_0)]$ . It holds by (8), (9) and the boundedness condition that

$$\lim_{x_0 \rightarrow +\infty} [U(q, x_0) - U(q_0, x_0)]^2 f''(x_0 + \xi)/2[f'(x_0)]^3 = 0, \tag{11}$$

and hence there exists  $M^* > 0$  such that  $\text{WTA}(q; q_0, x_0)$  is uniformly bounded from above when  $x_0 > M^*$ . Moreover, it holds by Lemma 1(iii), Lemma 4 and Proposition 1(Aii) that  $\text{WTA}(q; q_0, x_0) > 0$  is increasing in  $x_0$ . Thus, it holds by the theorem of the monotone limit, (8), (10) and (11) that

$$\begin{aligned} 0 < \lim_{x_0 \rightarrow +\infty} \text{WTA}(q; q_0, x_0) &= \lim_{x_0 \rightarrow +\infty} [U(q, x_0) - U(q_0, x_0)]/f'(x_0) \\ &= \lim_{x \rightarrow +\infty} [U(q, x) - U(q_0, x)]/\lim_{x \rightarrow +\infty} U_x(q_0, x) < +\infty. \end{aligned}$$

Furthermore, it holds by Lemma 4 and Theorem 1 that (5) in Theorem 3 holds. □

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### Appendix

*Proof of Lemma 1* (i) For any  $q \in Q_+$ , it follows from **A**<sub>1</sub> and **A**<sub>2</sub> that there uniquely exists  $\lambda_q \geq 0$  such that  $U(q_0, \lambda_q) = U(q, 0)$ , in which the uniqueness is ensured by **A**<sub>1</sub>.

(ii) Since  $x_0 \geq \lambda_q$ , it holds by **A**<sub>1</sub> and **A**<sub>2</sub> that  $U(q, x_0) > U(q_0, x_0) \geq U(q_0, \lambda_q) = U(q, 0)$ . It follows from the Intermediate Value Theorem and **A**<sub>1</sub> that there uniquely exists  $z \geq 0$  such that  $U(q_0, x_0) = U(q, z)$  and  $x_0 > z$ , which implies that  $WTP(q; q_0, x_0) > 0$  exists uniquely. Set  $f(x) = U(q_0, x)$  and  $g(x) = U(q, x)$ . Since  $f$  and  $g$  are continuous and increasing by **A**<sub>1</sub>, it holds by (1) that  $f(x_0) = g(x_0 - WTP(q; q_0, x_0))$ , which implies that  $WTP(q; q_0, x_0) = x_0 - g^{-1}(f(x_0))$  and that  $WTP(q; q_0, x_0)$  is continuous in  $x_0$ ,

(iii) It holds by **A**<sub>1</sub> that  $U(q, x_0) > U(q_0, x_0)$ . It follows from **A**<sub>1</sub> and **A**<sub>2</sub> that there exists uniquely  $e > 0$  such that  $U(q, x_0) = U(q_0, x_0 + e)$ , which implies that  $WTA(q; q_0, x_0) > 0$  exists uniquely. The continuity can be shown by almost the same manner as in Case (ii).

(iv) Since  $WTA(q; q_0, x_0) > 0$  by lemma 1(iii), it holds by the desirability in **A**<sub>2</sub>, the monotonicity in **A**<sub>1</sub> and (1) that  $U(q, x_0 + WTA(q; q_0, x_0)) > U(q_0, x_0 + WTA(q; q_0, x_0)) > U(q_0, x_0) = U(q, x_0 - WTP(q; q_0, x_0))$ . Then it holds by **A**<sub>1</sub> that there uniquely exists  $\alpha \in (0, WTA(q; q_0, x_0) + WTP(q; q_0, x_0))$  such that

$$U(q_0, x_0 + WTA(q; q_0, x_0)) = U(q, x_0 + WTA(q; q_0, x_0) - \alpha). \tag{12}$$

We have by (12) and (1) that

$$WTP(q; q_0, x_0 + WTA(q; q_0, x_0)) = \alpha. \tag{13}$$

Hence, it holds by (2) and (12) that  $U(q, x_0) = U(q_0, x_0 + WTA(q; q_0, x_0)) = U(q, x_0 + WTA(q; q_0, x_0) - \alpha)$ , which implies that  $x_0 = x_0 + WTA(q; q_0, x_0) - \alpha$  and  $WTA(q; q_0, x_0) = \alpha$ . Thus, we have by (13) that  $WTA(q; q_0, x_0) = WTP(q; q_0, x_0 + WTA(q; q_0, x_0))$ .

(v) We can easily prove Assertion (v). □

*Proof of Lemma 2* Set  $f(x) \equiv U^a(q_0, x)$  and  $g(x) \equiv U^b(q_0, x)$  for all  $x \geq 0$ . Since  $f^{-1}$  and  $g^{-1}$  are increasing by **A**<sub>1</sub>,  $U^a$  and  $U^b$  are equivalent to  $V^a = f^{-1} \circ U^a$  and  $V^b = g^{-1} \circ U^b$ , respectively, Since  $U^a$  is equivalent to  $U^b$ , we have that

$$V^a \text{ is equivalent to } V^b. \tag{14}$$

Moreover, it holds by (4) that

$$\begin{aligned} V^a(q_0, x) &= f^{-1}(U^a(q_0, x)) = f^{-1}(f(x)) = x = g^{-1}(g(x)) \\ &= g^{-1}(U^b(q_0, x)) = V^b(q_0, x) \quad \text{for all } x \end{aligned} \tag{15}$$

Now, suppose that  $V^a \neq V^b$ . By (15), we can assume without loss of generality that

$$V^a(q_1, x^*) > V^b(q_1, x^*) \quad \text{for some } x^* > 0.$$

Set  $z^* = [V^a(q_1, x^*) - V^b(q_1, x^*)]/2$ . Then we have by (15) that  $z^* = V^a(q_0, z^*) = V^b(q_1, z^*)$ . Thus,  $V^a(q_1, x^*) > V^a(q_0, z^*)$  and  $V^b(q_1, z^*) < V^b(q_0, x^*)$ , which contradicts with (14). □

*Proof of Lemma 3* At first, we will prove that

$$V'(q, x) \equiv dV(q, x)/dx \geq 1 \text{ for all } (q, x) \in Q_+ \times X. \tag{16}$$

Suppose that  $V'(q^*, x^*) < 1$  for some  $(q^*, x^*) \in Q_+ \times X$ . Define  $y^*$  by

$$y^* = x^* + [(V(q^*, x^*) - x^*)/(1 - V'(q^*, x^*))]. \tag{17}$$

By **A<sub>2</sub>**, we have that  $V(q^*, x^*) > V(q_0, x^*) = x^*$  and  $y^* > x^*$ . Since  $V(q^*, x)$  is concave with respect to  $x$ , it holds by  $V'(q^*, x^*) < 1$  and (17) that

$$\begin{aligned} V(q^*, y^*) &= V(q^*, x^*) + \int_{x^*}^{y^*} V'(q^*, x)dx < V(q^*, x^*) + V'(q^*, x^*) \int_{x^*}^{y^*} 1 \, dx \\ &= V(q^*, x^*) + V'(q^*, x^*)[(V(q^*, x^*) - x^*)/(1 - V'(q^*, x^*))] \\ &= (V(q^*, x^*) - x^*V'(q^*, x^*))/(1 - V'(q^*, x^*)) \\ &= (V(q^*, x^*) - x^*V'(q^*, x^*))/(1 - V'(q^*, x^*)) + (x^* - x^*)/(1 - V'(q^*, x^*)) \\ &= x^* + [(V(q^*, x^*) - x^*)/(1 - V'(q^*, x^*))] = y^* = V(q_0, y^*). \end{aligned}$$

This contradicts **A<sub>2</sub>**. Hence (16) holds. Now, fix any  $q \in Q_+$ , and suppose that

$$V(q, z^*) = V(q_0, z). \tag{18}$$

Fix any  $\delta > 0$ . It holds by (16) and the concavity of  $V(q, x)$  that

$$\begin{aligned} V(q, z^* + \delta) &= V(q, z^*) + \int_{z^*}^{z^*+\delta} V'(q, x)dx > V(q, z^*) + \int_{z^*}^{z^*+\delta} 1 \, dx \\ &= V(q, z^*) + \delta. \end{aligned}$$

Hence, it holds by (18) and (4) that

$$V(q, z^* + \delta) > V(q, z^*) + \delta = V(q_0, z) + \delta = V(q_0, x + \delta),$$

which means that  $V$  satisfies the normality condition on  $q$ . Since  $V$  is equivalent to  $U$ ,  $U$  also satisfies the normality condition on  $q$ .  $\square$

*Proof of Lemma 4* Suppose that  $U(q, z) = U(q_0, x)$  for  $z, x \in X$ , and fix any  $\delta > 0$ . Since  $U$  satisfies  $A_2$ ,  $U(q, z + \delta) = U(q_0, x + \varepsilon)$  for some  $\varepsilon > 0$ . There remains to show  $\varepsilon > \delta$ . Since  $U$  is smooth, there is a smooth indifference path  $I : [q_0, q] \rightarrow X$  connecting  $(q_0, X)$  and  $(q, z)$ , i.e.,  $I(q_0) = x$  and  $I(q) = z$ . Similarly, there is a smooth indifference path  $J : [q_0, q] \rightarrow X$  connecting  $(q_0, x + \varepsilon)$  and  $(q, z + \delta)$ . It holds by the monotonicity that  $x > z$  and  $x + \varepsilon > (z + \delta)$ . Then we have that

$$x - z = \int_{q_0}^q |I(t)/dt| dt = \int_{q_0}^q \text{MRS}(t, I(t)) dt; \tag{19}$$

$$x + \varepsilon - (z + \delta) = \int_{q_0}^q |J(t)/dt| dt = \int_{q_0}^q \text{MRS}(t, J(t)) dt, \tag{20}$$

where  $\text{MRS}(r, s) \equiv U_q(r, s)/U_x(r, s)$ . Since it holds by the Hicksian normality that

$$\begin{aligned} \partial \text{MRS}(r, s)/\partial x &= [U_x(r, s)]^{-2} \cdot [U_x(r, s) \cdot U_{qx}(r, s) - U_q(r, s) \cdot U_{xx}(r, s)] \\ &> 0 \quad \text{for all } (r, s) \gg (0, 0), \end{aligned}$$

and since  $I(t) < J(t)$  by the monotonicity and  $\delta > 0$ , we have that

$$\text{MRS}(t, I(t)) < \text{MRS}(t, J(t)) \quad \text{for all } t \in (q_0, q).$$

This implies that  $\int_{q_0}^q \text{MRS}(t, I(t)) dt < \int_{q_0}^q \text{MRS}(t, J(t)) dt$ . Hence, we have by (19) and (20) that  $x - z < x + \varepsilon - (z + \delta)$  and  $\varepsilon > \delta$ . Thus,  $U$  satisfies the normality condition on  $q$ .  $\square$

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