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Extension of Planck's law to steady heat flux across nanoscale gaps

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Abstract Recent experiments report that the radiative heat conductance through a narrow vacuum gap between two flat surfaces increases as the inverse square of the width of the gap. Such a significant increase of thermal conductivity has attracted much interest because of numerous promising applications in nanoscale heat transfer and because of the lack of its theoretical explanation. It is shown here that the radiative heat transport across narrow layers can be described in terms of conventional theory adjusted to non-equilibrium structures with a steady heat flux.

1 Introduction

It is well known that moving electric charges radiate electromagnetic waves, that the electromagnetic fields affect the motion of electric charges, and that electrically charged electrons and protons perform perpetual thermal motion. Therefore, any material body radiates thermally agitated electromagnetic waves and at the same time it absorbs thermally agitated electromagnetic radiation arriving from other bodies.

Since electromagnetic waves propagate in a vacuum as well as in various materials, the last observation implies that bodies separated by a vacuum gap cannot be considered as thermally isolated because they can exchange energy and come to thermal equilibrium. Thus, if two material half-spaces are separated by a vacuum layer then they eventually come to identical temperatures after which the flows of energy from one half-space to the other and vice versa become equal, although they never vanish. The amount of heat that flows between half-spaces in equilibrium depends on the temperature and on the material properties of the half spaces. However, if both half-spaces can be considered as black bodies in the sense described in [1, Page 10], then the radiated and absorbed heat flows are determined solely by the temperature and are described by Planck's law of thermal radiation which determines the flows of the electromagnetic energy between bodies in thermal equilibrium.

Let two half-spaces A and B be separated by a vacuum gap and be maintained at temperatures T_A and T_B, respectively. Assume also that Q_A and Q_B are the fluxes of the thermally agitated electromagnetic fields radiated by the corresponding half-spaces. Then, in the equilibrium case we not only know that $T_A = T_B$ and $Q_A = Q_B$, but we can compute each of the mutually compensating fluxes Q_A and Q_B by Planck's theory of thermal radiation. On the contrary, if $T_A \neq T_B$ then Planck's theory cannot be used directly because it is valid only for equilibrium systems, and we do not know Q_A and Q_B nor do we have any beforehand information about the net flux $Q = Q_A - Q_B$.

If $\Delta T = T_A - T_B > 0$, then the heat flux Q_A from A to B exceeds the flux Q_B in the opposite direction and the thermal resistance of the gap between A and B can be defined as the ratio $R_T = \Delta T/(Q_A - Q_B)$. This resistance characterizes the ease of energy exchange between A and B, and it is clear that the stronger electromagnetic coupling between A and B is, the smaller will be the thermal resistance. On the other hand, the electromagnetic coupling between half-spaces is determined by the reflection coefficient r_e of electromagnetic waves propagating through the gap. Therefore, there must be a direct connection between the thermal resistance R_T of the gap and the electromagnetic

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reflection coefficient r_e of the gap. For example, if $r_e = 0$ then the electromagnetic fields in A and B consist of the same waves which implies that these fields have identical energy characteristics and, therefore, have equal temperatures. So, if there is no reflection of electromagnetic waves then there is no thermal resistance. In particular, an imaginary boundary between two portions of the same identical material may not have any thermal resistance, and common sense suggests that the thermal resistance of a gap between identical materials linearly depends on the reflection coefficient of electromagnetic radiation from that gap. Therefore, taking into account the well-known fact [2] that if the width H of the gap is sufficiently small then $r_e = O(H^2)$, and it is natural to expect that as $H \to 0$, the thermal resistance of the gap is of order $O(H^2)$ and, correspondingly, the thermal conductance increases as $O(1/H^2)$.

Despite the simplicity of the structure consisting of two half-spaces separated by a vacuum gap, it is not easy to get experimental confirmation of the above mentioned asymptotes because they may be observed only when the width of the gap is comparable with the dominant wavelength of thermally agitated electromagnetic radiation, which, at room temperature, is about 700 nm. However, in a recent series of measurements [3–5] experimentalists were able to observe a very high thermal conductance across a narrow vacuum gap between two identical glass plates which clearly increases as $O(1/H^2)$ when the width of the gap H decreases.

The theoretical analysis of radiative thermal transport across a narrow vacuum gap has also been a difficult problem, because it deals with a non-equilibrium state where the well-established laws of equilibrium statistical mechanics cannot be applied directly. However, we recently developed a method to deal with similar problems arising in the theory of heat transport by acoustic waves, usually referred to as phonons. In particular, in [6, 7] we describe the spectra of the energy density of an ensemble of acoustic waves with a steady heat flux, and using this information we obtained reasonable estimates of the so-called Kapitsa interface thermal resistance which has remained an open theoretical problem for about seven decades. Here we further develop our approach to problems of heat transport by electromagnetic waves, and we thereby get estimates of the thermal resistance of a narrow vacuum gap which agrees with recently published experimental data.

2 Electromagnetic fields in the presence of a steady heat flux

Let G be a completely isolated domain occupied by one or several material bodies in thermal equilibrium at temperature T. Then, every form of energy in this domain is balanced and, therefore, its thermal state can be studied by the methods of equilibrium statistical mechanics. In particular, the average electromagnetic energy E_G stored in G can be computed by the standard procedure [8] based on the representation of an arbitrary electromagnetic field in G as a superposition of eigen-fields each of which can be treated as a single quantum harmonic oscillator.

It is well known that the average thermal energy of any single quantum harmonic oscillator of an equilibrium ensemble has the value

$$P(\omega, T) = \frac{\hbar\omega}{\mathrm{e}^{\hbar\omega/\kappa T} - 1}, \tag{2.1}$$

which depends only on the frequency ω of the oscillator and on the temperature T of the ensemble [8, 9]. Therefore, the total thermal energy of the electromagnetic field in G has the value

$$E_G = \int_0^\infty P(\omega, T)\,\mathrm{d}N_G(\omega), \tag{2.2}$$

where $N_G(\omega)$ is the number of eigen-fields in G corresponding to the eigen-frequencies below ω, which means that $D(\omega)\,\mathrm{d}\omega = \mathrm{d}N_G(\omega)$ is the number of oscillators with frequencies from the band $(\omega, \omega + \mathrm{d}\omega)$. In general, there is no easy way to compute $N_G(\omega)$ for an arbitrary domain G and arbitrary frequency ω. However, if G is sufficiently smooth and its volume V_G is sufficiently large then $N_G(\omega)$ can be estimated by the asymptote

$$N_G(\omega) = 2\frac{V_G\,\omega^3}{6\pi^2 c^3} + o\left(\frac{V_G}{\lambda^3}\right), \quad \lambda = \frac{2\pi\omega}{c}, \tag{2.3}$$

which differs from the Weyl formula [10, Chap. VI, §4] by the additional factor “2” added to take into account both polarizations of the electromagnetic waves propagating through the medium with speed c. If the asymptote (2.3) is valid then $\mathrm{d}N_G(\omega)$ can be approximated as

$$\mathrm{d}N_G(\omega) = V_G\,D(\omega)\,\mathrm{d}\omega, \quad D(\omega) = \frac{\omega^2}{\pi^2 c^3}, \tag{2.4}$$

and formulas (2.2)–(2.4) can be re-arranged as

$$E_G = V_G\,E, \quad E = \int_0^\infty P(\omega, T)D(\omega)\,\mathrm{d}\omega, \tag{2.5}$$

where E is the energy density (energy per unit volume) of an equilibrium ensemble of electromagnetic fields.

If the ensemble of electromagnetic waves in the domain G is not in thermal equilibrium but instead has a small energy flux Q along the x-axis then it cannot be described by the formula (2.5) derived from the assumption of equilibrium. However, observing that this ensemble appears to be in equilibrium in a frame which moves along the x-axis with the speed $v \approx Q/E$ we can verify [6, 7] that its energy density can be represented as the sum

$$E = E^A + E^B, \tag{2.6}$$

where

$$E^A = \frac{1}{2}\{P(\omega + \omega\tau Q, T)\},\qquad E^B = \frac{1}{2}\{P(\omega - \omega\tau Q, T)\}, \tag{2.7}$$

are the energy densities radiated to the directions $x \to \infty$ and $x \to -\infty$, respectively, where

$$\tau \equiv \tau_\theta = \frac{\sin\theta}{cE}, \tag{2.8}$$

and the brackets $\{\cdot\}$ are used as an abbreviated notation for the integral

$$\{F\} = \int_0^{\pi/2}\left(\int_0^\infty F(\omega,\theta)D(\omega)\,\mathrm{d}\omega\right)\sin\theta\,\mathrm{d}\theta. \tag{2.9}$$

Obviously, if $Q = 0$ then $E^A = E^B = \frac{1}{2}E$ and (2.6)–(2.9) reduce to (2.5). However, if $Q \neq 0$ then applying Taylor expansions we get the order $o(Q)$ estimates

$$E^A = \frac{1}{2}\{P(\omega,T)\} + \frac{Q}{2}\{P'_\omega(\omega,T)\tau\omega\},\qquad E^B = \frac{1}{2}\{P(\omega,T)\} - \frac{Q}{2}\{P'_\omega(\omega,T)\tau\omega\}, \tag{2.10}$$

which show that the amounts of energy propagating in opposite directions are not equal, although the energy densities computed by (2.5) and (2.6) coincide with the accuracy of order $o(Q)$.

3 Transmission of a plane wave through a layer

Assume that the space (x, y, z) is filled by a material with the light speed c_0 and contains the layer $0 < x < H$ filled by another material with the light speed $c_1 = \gamma c_0$. Any electromagnetic field $\Xi(x, y, z)$ outside the gap $0 < x < H$ can be decomposed into the elementary waves

$$\Xi(x,y,z) = \begin{cases} A_-\mathrm{e}^{\mathrm{i}(e_x x + e_y y + e_z z)\omega/c_0} \\ \quad + B_-\mathrm{e}^{\mathrm{i}(-e_x x + e_y y + e_z z)\omega/c_0}, & \text{if } x < 0, \\ A_+\mathrm{e}^{\mathrm{i}(e_x x + e_y y + e_z z)\omega/c_0} \\ \quad + B_+\mathrm{e}^{\mathrm{i}(-e_x x + e_y y + e_z z)\omega/c_0}, & \text{if } x > H, \end{cases} \tag{3.1}$$

where $A_\pm$ and $B_\pm$ are as yet indefinite amplitudes, and

$$e_x = \sin\theta, \qquad e_y = \cos\theta\cos\phi, \qquad e_z = \cos\theta\sin\phi \tag{3.2}$$

are the components of the unit vector $\vec{e}$ which determines the direction of propagation of the waves (3.1) in terms of the spherical angles from the intervals $0 \le \theta \le \pi/2$ and $0 \le \phi \le 2\pi$. It is easy to see that two of the four coefficients $A_\pm$ and $B_\pm$ can have arbitrary values, and the two others are then determined by the requirement that the field Ξ satisfy continuity conditions at the gap boundaries.

To understand the relationship between $A_\pm$ and $B_\pm$ in (3.1) we consider a field

$$\xi(x,y,z) = \begin{cases} \mathrm{e}^{\mathrm{i}(e_x x + e_y y + e_z z)\omega/c_0} + \Gamma\mathrm{e}^{\mathrm{i}(-e_x x + e_y y + e_z z)\omega/c_0}, \\ \quad \text{if } x < 0, \\ a\,\mathrm{e}^{\mathrm{i}(d_x x + d_y y + d_z z)\omega/c_1} + b\,\mathrm{e}^{\mathrm{i}(-e_x x + e_y y + e_z z)\omega/c_1}, \\ \quad \text{if } 0 < x < H, \\ \Lambda\mathrm{e}^{\mathrm{i}(e_x x + e_y y + e_z z)\omega/c_0}, \quad \text{if } x > H, \end{cases} \tag{3.3}$$

where (e_x, e_y, e_z) and (d_x, d_y, d_z) are related by Snell's law, which implies that

$$d_x = \sqrt{1-\gamma^2\cos^2\theta}, \qquad d_y = \gamma e_y, \qquad d_z = \gamma e_z, \qquad \gamma = \frac{c_1}{c_0}. \tag{3.4}$$

Since $\xi(x, y, z)$ and its derivative $\xi'_x(x, y, z)$ are continuous at $x = 0$ and $x = H$ we find that

$$1 + \Gamma = a + b, \qquad a\mathrm{e}^{\mathrm{i}h_1} + b\mathrm{e}^{-\mathrm{i}h_1} = \Lambda\mathrm{e}^{\mathrm{i}h_0},\qquad 1 - \Gamma = \mu(a-b), \qquad \mu\left(a\mathrm{e}^{\mathrm{i}h_1} - b\mathrm{e}^{-\mathrm{i}h_1}\right) = \Lambda\mathrm{e}^{\mathrm{i}h_0}, \tag{3.5}$$

where

$$\mu \equiv \mu(\theta) = \frac{d_x c_0}{e_x c_1} = \frac{\sqrt{1-\gamma^2\cos^2\theta}}{\gamma\sin\theta} \tag{3.6}$$

and

$$h_0 = \frac{\omega}{c_0}H \equiv 2\pi\frac{H}{\lambda_0}, \qquad h_1 = \frac{\omega}{c_1}H \equiv 2\pi\frac{H}{\lambda_1}, \tag{3.7}$$

may be considered as the measures of H in terms of the wavelengths λ_0 and λ_1 in the main medium and in the layer, respectively. Next we convert the right-hand pair of equations from (3.5) to the form

$$2a\mu\mathrm{e}^{\mathrm{i}h_1} = (\mu+1)\Lambda\mathrm{e}^{\mathrm{i}h_0},\qquad 2b\mu\mathrm{e}^{-\mathrm{i}h_1} = (\mu-1)\Lambda\mathrm{e}^{\mathrm{i}h_0}, \tag{3.8}$$

and compute $(a+b)/(a-b)$ using (3.5), and then using (3.8). This leads to the equation

$$\mu\frac{1+\Gamma}{1-\Gamma} = \frac{(\mu+1)\mathrm{e}^{-\mathrm{i}h_1} + (\mu-1)\mathrm{e}^{\mathrm{i}h_1}}{(\mu+1)\mathrm{e}^{-\mathrm{i}h_1} - (\mu-1)\mathrm{e}^{\mathrm{i}h_1}} \equiv \frac{\mu\cos(h_1) - \mathrm{i}\sin(h_1)}{\cos(h_1) - \mathrm{i}\mu\sin(h_1)}, \tag{3.9}$$

with the solution

$$\Gamma \equiv \Gamma_{\omega,\theta} = \frac{\mathrm{i}(\mu^2-1)\sin(h_1)}{2\mu\cos(h_1) - \mathrm{i}(\mu^2+1)\sin(h_1)}. \tag{3.10}$$

It is obvious that Γ can always be represented in the form

$$\Gamma = r e^{i\sigma}, \tag{3.11}$$

where Γ approaches the asymptote

$$\Gamma = r e^{i\pi/2}, \qquad r \approx \frac{1}{2}\left(\mu - \frac{1}{\mu}\right) h_1 \equiv \pi \left(\mu - \frac{1}{\mu}\right) \frac{H}{\lambda_1}, \tag{3.12}$$

as the width H of the layer becomes small compared to the wavelength λ_1 in this layer.

From the properties of electromagnetic waves it follows that in the domain $x > H$ the field (3.3) contains only waves which propagate to the right and that the energy density of these waves has the value $E_+^A = |\Lambda|^2$. At the same time in the domain $x < 0$ this field contains waves which carry the energy density $E_1^A = 1$ to the right and the energy density $E_-^B = |\Gamma|^2$ to the left. These observations confirm the identity $|\Lambda|^2 = 1 - |\Gamma|^2$, and they also demonstrate that an arbitrary electromagnetic field of the type (3.1) cannot the composed from the fields (3.3) which do not include any waves propagating to the left in the domain $x > H$. However, it is easy to see that any field (3.1) admits the representation

$$\Xi(x, y, z) = u\xi(x, y, z) + v\widetilde{\xi}(x, y, z), \tag{3.13}$$

where u and v are constants and

$$\widetilde{\xi}(x, y, z) = \begin{cases} \widetilde{\Lambda} e^{i(-e_x x + e_y y + e_z z)\omega/c_0}, \\ \quad \text{if } x < 0, \\ \widetilde{b} e^{i(-e_x x + e_y y + e_z z)\omega/c_1} + \widetilde{a} e^{i(d_x x + d_y y + d_z z)\omega/c_1}, \\ \quad \text{if } 0 < x < H, \\ e^{i(-e_x x + e_y y + e_z z)\omega/c_0} + \widetilde{\Gamma} e^{i(e_x x + e_y y + e_z z)\omega/c_0}, \\ \quad \text{if } x > H, \end{cases} \tag{3.14}$$

appears as a mirror image of the field (3.3) with the reflection and transmission coefficients $\widetilde{\Gamma}$ and $\widetilde{\Lambda}$ related to Γ and Λ from (3.3) by the identities $|\widetilde{\Gamma}| = |\Gamma| = r$ and $|\widetilde{\Lambda}| = |\Lambda| = \sqrt{1 - r^2}$.

4 Computation of the heat flux though the gap

Let there be a steady heat flux Q in the structure considered above. Assume also that the parts $x < 0$ and $x > H$ of the medium are maintained at the temperatures T_- and T_+, respectively. Then, the energy densities of the electromagnetic fields in these domains can be represented by the thermodynamical formulas (2.6) the formulas (3.13) describing the wave properties of these fields.

Let E_-^A and E_-^B be the average energy densities of electromagnetic waves propagating in the domain $x < 0$ to the right and the left, respectively. Similarly, let E_+^A and E_+^B be the average energy densities of waves propagating to the right and the left in $x > H$. Then, the thermodynamical formulas (2.6) imply that

$$\begin{aligned} E_\pm^A &= \{P(\omega + \omega\tau Q, T_\pm)\}, \\ E_\pm^B &= \{P(\omega - \omega\tau Q, T_\pm)\}. \end{aligned} \tag{4.1}$$

On the other hand, to estimate the average energy densities of the electromagnetic fields in the domains $x < 0$ and $x > H$ we can consider them as statistical ensembles of the fields (3.13). In this case, the average energy densities E_-^A and E_-^B of the waves propagating in the domain $x < 0$ to the right and to the left have the values

$$E_-^A = \{u^2\}, \qquad E_-^B = \{r^2 u^2 + (1 - r^2) v^2\}, \tag{4.2}$$

where for simplicity we use a reduced notation $u^2 \equiv |u^2|$ and $v^2 \equiv |v|^2$. Similarly, the average energy densities E_+^A and E_+^B of the waves propagating to the right and to the left in the domain $x > H$ have the values

$$E_+^A = \{(1 - r^2) u^2\}, \qquad E_-^B = \{r^2 u^2 + (1 - r^2) v^2\}. \tag{4.3}$$

Then, assuming that $r = |R|$ is small, which is the case when the gap is narrow, we conclude that with the accuracy of the order $o(H)$ the formulas (4.2) and (4.3) can be reduced to the form

$$E_-^A = \{u^2\}, \qquad E_-^B = pH^2\{u^2\} + (1 - pH^2)\{v^2\}, \tag{4.4}$$

$$E_+^B = \{v^2\}, \qquad E_+^A = pH^2\{v^2\} + (1 - pH^2)\{u^2\}, \tag{4.5}$$

where p is some constant.

Next, combining (4.4) with (4.1) we get the equation

$$\{P(\omega - \omega\tau Q, T_-)\} = pH^2\{P(\omega + \omega\tau Q, T_-)\} + (1 - pH^2)\{P(\omega - \omega\tau Q, T_+)\}, \tag{4.6}$$

which is equivalent to

$$\{P(\omega - \omega\tau Q, T_+) - P(\omega - \omega\tau Q, T_-)\} = pH^2\{P(\omega - \omega\tau Q, T_+) - P(\omega + \omega\tau Q, T_-)\}. \tag{4.7}$$

Finally, applying Taylor expansions in (4.7) and ignoring higher-order terms we find that

$$\{P_T'(\omega, T_-)\}(1 - pH^2)\Delta T = -2pH^2 Q\{P_\omega'(\omega, T_-)\}, \tag{4.8}$$

which shows that as $H \to 0$, the thermal resistance

$$R_T \equiv \frac{\Delta T}{Q} = -\frac{2pH^2}{(1 - pH^2)} \frac{\{P_\omega'(\omega, T_-)\}}{\{P_T'(\omega, T_-)\}} \approx -2pH^2 \frac{\{P_\omega'(\omega, T_-)\}}{\{P_T'(\omega, T_-)\}} \tag{4.9}$$

decreases as $O(H^2)$, and hence the conductance increases as $O(1/H^2)$, which agrees with the observation from [3, 5].

5 Conclusion

It is shown that the significant increase of the radiative heat conductance between two flat surfaces separated by a narrow vacuum gap has a simple explanation in terms of the classical Planck's theory modified by a Doppler-like transform taking into account the presence of the steady heat flux. The analysis developed here is versatile and admits easy adaptations to other similar problems.

For example, if we set $\theta = \pi/2$ and remove all integrations with respect to θ from all previous formulas, then the resulting expressions will described the radiative heat transport through a H-long break in a one-dimensional medium. Although this simple configuration is not very practical, its analysis is important for understanding the underlying mechanisms of the radiative heat transport. Indeed, the decrease of the thermal resistance of a narrow gap in the three dimensional setting is often credited to the tunneling of evanescent or surface waves localized near the surfaces of the gap. But, since such waves do not exist in a one-dimensional structure, they cannot contribute to the $O(1/H^2)$ increase of thermal transport through a narrow break in a one-dimensional line which undermines the assertion that evanescent and surface waves are responsible for the intensive tunneling of heat across a gap.

Another important extension of the developed approach deals with the radiative heat transport across a narrow gap separating half-spaces made from different materials. This problem will be considered in detail in the next paper, but it is worth mentioning that in this case the thermal resistance of the gap of width H has the asymptote $R_K \approx \alpha + \beta H^2 + o(H^2)$, where α and β depend on material parameters of the half-spaces. If these materials coincide then $\alpha = 0$ and the heat conductance has the asymptote $C_T \approx 1/\beta H^2$, which becomes unbounded as $H \to 0$. However, if the materials are different then $\alpha \neq 0$ as the gap vanishes, and the heat conductance approaches a finite limit. It would be interesting for experimentalist to observe this prediction.

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