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On the Trajectory of a Light Small Rigid Body in an Incompressible Viscous Fluid

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Abstract
In this paper, we study the dynamics of a small rigid body in a viscous incompressible fluid in dimension two and three. More precisely we investigate the trajectory of the rigid body in the limit when its mass and its size tend to zero. We show that the velocity of the center of mass of the rigid body coincides with the background fluid velocity in the limit. We are able to consider the limit when the volume of the rigid bodies converges to zero while their densities are a fixed constant.

Keywords PDEs · Fluid-structure interaction · Asymptotic limit · Navier-Stokes · Rigid body

Mathematics Subject Classification 35Q30 · 35Q70 · 70E15

1 Introduction

In this paper, we study the interaction of a "small light" rigid body with an incompressible viscous fluid in dimension two and three. The system fluid plus rigid body occupies the domain $\mathbb{R}^d$ for $d = 2, 3$. The unknowns of the problem are the position of the rigid body $\mathcal{S}(t)$, an open, bounded, connected, simply connected subset of $\mathbb{R}^d$ with smooth boundary, and the velocity of the fluid $u_{\mathcal{F}}$ which is defined on the fluid domain $\mathcal{F}(t) = \mathbb{R}^d \setminus \overline{\mathcal{S}(t)}$ with values in $\mathbb{R}^d$. Moreover, the equations that $u_{\mathcal{F}}$ satisfies are the incompressible Navier–Stokes equations

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$$
\begin{aligned}
\partial_t u_{\mathcal{F}} + \operatorname{div}(u_{\mathcal{F}} \otimes u_{\mathcal{F}}) - \nu \Delta u_{\mathcal{F}} - \nabla p_{\mathcal{F}} &= 0 && \text{for } x \in \mathcal{F}(t), \\
\operatorname{div}(u_{\mathcal{F}}) &= 0 && \text{for } x \in \mathcal{F}(t), \\
u_{\mathcal{F}} &= u_{\mathcal{S}} && \text{for } x \in \partial\mathcal{S}(t), \\
|u_{\mathcal{F}}| &\longrightarrow 0 && \text{as } |x| \longrightarrow +\infty,
\end{aligned} \tag{1}
$$

where $u_{\mathcal{S}}$ is the velocity of the rigid body and $\nu > 0$ is the viscosity coefficient. We assume that the rigid body has constant density $\rho_{\mathcal{S}} \in \mathbb{R}$ with $\rho_{\mathcal{S}} > 0$ and it occupies the volume $\mathcal{S}^{in}$ at initial time. $\mathcal{S}^{in}$ is an open, bounded, connected, simply connected subset of $\mathbb{R}^d$ with smooth boundary.

The motion of the rigid body is completely determined by the dynamics of the center of mass and of the angular rotation. Recall that the mass and the center of mass of the rigid body are defined, respectively, by

$$
m = \int_{\mathcal{S}^{in}} \rho_{\mathcal{S}} \, \mathrm{d}x \quad \text{and} \quad h^{in} = \frac{1}{m} \int_{\mathcal{S}^{in}} \rho_{\mathcal{S}} x \, \mathrm{d}x.
$$

We denote by $h(t)$ the position of the center of mass at time t and by $Q(t)$ the special orthogonal matrix that characterizes the rotation around the center of mass from the initial configuration. Then, the volume occupied by the rigid body at time t is

$$
\mathcal{S}(t) = \left\{ y \text{ such that } Q^T(t)(y - h(t)) \in \mathcal{S}^{in} \right\}.
$$

In dimension $d = 3$, its velocity is

$$
\begin{aligned}
u_{\mathcal{S}} &= \frac{\mathrm{d}}{\mathrm{d}t}(h(t) + Q(t)x)\Big|_{x = Q^T(t)(y - h(t))} = h'(t) + Q'(t)Q^T(t)(y - h(t)) \\
&= \ell(t) + \omega(t) \times (y - h(t))
\end{aligned} \tag{2}
$$

where we denote $\ell(t) = h'(t)$. Moreover, from the fact that $Q(t)$ is a rotation matrix, $Q'(t)Q^T(t)$ is skew symmetric and it can be identified with a vector $\omega(t) \in \mathbb{R}^3$ through the relation

$$
Q'(t)Q^T(t)x = \omega(t) \times x
$$

where $x \in \mathbb{R}^3$. The evolution of ℓ and ω follows Newton's laws that read

$$
\begin{aligned}
m\ell'(t) &= -\oint_{\partial\mathcal{S}(t)} \Sigma(u_{\mathcal{F}}, p_{\mathcal{F}}) n \, \mathrm{d}s \\
\mathcal{J}(t)\omega'(t) &= \mathcal{J}(t)\omega(t) \times \omega(t) - \oint_{\partial\mathcal{S}(t)} (x - h(t)) \times \Sigma(u_{\mathcal{F}}, p_{\mathcal{F}}) n \, \mathrm{d}s.
\end{aligned} \tag{3}
$$

In the above equations, Σ is the stress tensor

$$
\Sigma(u, p) = 2\nu D(u) - p\mathbb{I} \quad \text{where} \quad D(u) = \frac{\nabla u + (\nabla u)^T}{2}
$$

and $\mathcal{J}$ is the inertia momentum, which is given through the formula

$$\mathcal{J}(t) = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} \left[|x - h(t)|^2 \mathbb{I} - (x - h(t)) \otimes (x - h(t)) \right] \mathrm{d}x = Q(t)\mathcal{J}(0)Q^T(t).$$

Finally the initial conditions are

$$u_{\mathcal{F}}(0) = u_{\mathcal{F}}^{in}, \quad \ell(0) = \ell^{in} \quad \text{and } \omega(0) = \omega^{in}, \tag{4}$$

such that they satisfy the compatibility conditions

$$\operatorname{div}(u_{\mathcal{F}}^{in}) = 0 \text{ in } \mathcal{F}^{in} \quad \text{and} \quad u_{\mathcal{F}}^{in} = \ell^{in} + \omega^{in} \times (x - h^{in}) \quad \text{in } \partial\mathcal{S}^{in}. \tag{5}$$

Moreover, without loss of generality we have $h(0) = 0$ and $Q(0) = \mathbb{I}$.

In the case of dimension $d = 2$, Eqs. (2) and (3) simplify, in fact $Q'(t)Q^T(t)$ is skew symmetric and it can be identify with a scalar quantity $\omega(t)$ through the relation

$$Q'(t)Q^T(t)x = \omega(t)x^{\perp}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $x^{\perp} = (-x_2, x_1)^T$. In particular, (2) becomes

$$u_{\mathcal{S}} = \ell(t) + \omega(t)(x - h(t))^{\perp}.$$

Newton's laws read

$$\begin{aligned} m\ell'(t) &= -\oint_{\partial\mathcal{S}(t)} \Sigma(u_{\mathcal{F}}, p_{\mathcal{F}})n \,\mathrm{d}s \\ \mathcal{J}\omega'(t) &= -\oint_{\partial\mathcal{S}(t)} (x - h(t))^{\perp} \cdot \Sigma(u_{\mathcal{F}}, \rho_{\mathcal{F}})n \,\mathrm{d}s \end{aligned}$$

and $\mathcal{J}$ is time independent.

System (1)–(3)–(4) has been widely studied. The first works on the existence of Hopf–Leray-type weak solutions are Judakov (1974) and Serre (1987) where the fluid plus rigid body occupies $\mathbb{R}^3$; in other words, $\mathcal{F}(t) \cup \overline{\mathcal{S}(t)} = \mathbb{R}^3$. These results were then extended in Gunzburger et al. (2000), Conca et al. (1999), Desjardins and Esteban (1999), Feireisl (2002, 2003). Uniqueness was shown in Glass and Sueur (2015) in dimension two and in Muha et al. (2021) in dimension three under Prodi–Serrin conditions. Regularity was studied in dimension three under Prodi–Serrin conditions in Muha et al. (2022). Well-posedness of strong solutions in Hilbert space setting was proved in Grandmont and Maday (2000), Takahashi (2003), Takahashi and Tucsnak (2004) and in the Banach space setting in Geissert et al. (2013), Maity and Tucsnak (2018). Notice that similar results hold in the case the Navier slip boundary conditions are prescribed on $\partial\mathcal{S}$, and see Planas and Sueur (2014), Gérard-Varet and Hillairet (2014), Bravin (2019), Al Baba et al. (2021), Chemetov et al. (2019).

Let us now introduce a small parameter $\varepsilon > 0$, and for $\bar{x} \in \mathbb{R}^d$, let $\mathcal{S}_{\varepsilon}^{in} \subset B_{\varepsilon}(\bar{x})$ be a sequence of initial configurations of the rigid bodies. In this paper, we study the dynamics of the rigid body when ε converges to zero. Let us recall that it has already been shown that the fluid is not influenced by the presence of a "small" rigid body. In fact, in Lacave and Takahashi (2017), He and Iftimie (2019), He and Iftimie (2021) and Feireisl et al. (2022), the authors showed that given a sequence of solutions $(u_{\mathcal{F},\varepsilon}, \ell_\varepsilon, \omega_\varepsilon)$ to system (1)–(3)–(4), then there exists a subsequence of fluid velocity $u_{\mathcal{F},\varepsilon}$ that converges, in some appropriate weak norms, to u that satisfies the Navier–Stokes system

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) - \nu \Delta u - \nabla p = 0 \quad & \text{for} x \in \mathbb{R}^d, \\ \operatorname{div}(u) = 0 \quad & \text{for } x \in \mathbb{R}^d, \\ |u| \longrightarrow 0 \quad & \text{as } |x| \longrightarrow +\infty. \end{aligned} \tag{6}$$

These results hold under some mild assumptions on m_ε, $\mathcal{S}_\varepsilon^{in}$ and on the convergence of the initial data.

Similar results are available also for compressible fluid see Bravin and Nečasová (2023), Feireisl et al. (2022) and Section 6 of Feireisl et al. (2022). Moreover, we proved in Bravin and Nečasová (2022) that under some lower bounds on the masses

$$m_\varepsilon/\varepsilon^{1/2} \longrightarrow +\infty \text{ for d = 3} \quad \text{and} \quad m_\varepsilon \geq C > 0 \text{ for d = 2}$$

the "small" rigid body will move with constant initial velocity. In this paper, we show that under the assumptions

$$\begin{aligned} m_\varepsilon/|\mathcal{S}_\varepsilon^{in}|^{1/3} \longrightarrow 0 \quad \text{and} \quad |\mathcal{S}_\varepsilon^{in}|/\varepsilon^{9/2} \longrightarrow +\infty \quad & \text{if d = 3}, \\ m_\varepsilon/|\mathcal{S}_\varepsilon^{in}|^{\delta} \longrightarrow 0 \quad \text{and} \quad |\mathcal{S}_\varepsilon^{in}|^{\delta}/\varepsilon^{\tilde{\delta}} \longrightarrow +\infty \text{ for some } \delta > 0 \text{ and } 0 < \tilde{\delta} < 1 \quad & \text{if d = 2} \end{aligned} \tag{7}$$

and appropriate convergence of the initial data, the "small" rigid body follows the fluid flow. The assumptions (7) are used in a crucial way in (24) and (27).

Using the fact that $m_\varepsilon = \rho_{\mathcal{S},\varepsilon}|\mathcal{S}_\varepsilon^{in}|$, assumption (7) rewrites

$$\begin{aligned} \rho_{\mathcal{S},\varepsilon}|\mathcal{S}_\varepsilon^{in}|^{2/3} \longrightarrow 0 \quad \text{and} \quad |\mathcal{S}_\varepsilon^{in}|/\varepsilon^{9/2} \longrightarrow +\infty \quad & \text{if } d = 3, \\ \rho_{\mathcal{S},\varepsilon}|\mathcal{S}_\varepsilon^{in}|^{1-\delta} \longrightarrow 0 \quad \text{and} \quad |\mathcal{S}_\varepsilon^{in}|^{\delta}/\varepsilon^{\tilde{\delta}} \longrightarrow +\infty \text{for some } \delta > 0 \text{ and } 0 < \tilde{\delta} < 1 \quad & \text{if } d = 2. \end{aligned}$$

The main result of this work shows a different behavior of the "small light" rigid body respect to the case where the fluid is assumed to be inviscid, i.e., the viscosity coefficient $\nu = 0$. In fact, a "small" rigid body is not accelerated by an inviscid incompressible fluid, see Section 1.4 of Glass et al. (2014b). In a series of works, Glass et al. (2014a, b, 2016, 2018, 2019), Glass and Sueur (2019), the authors studied the interaction of a two-dimensional incompressible inviscid fluid modeled by the Euler equations with a rigid body that shrinks to point particle. In particular, they showed that the dynamics of the point particle is characterized either by a point vortex-type equation if the mass of the rigid body converges to zero or by a second-order equation involving the Kutta–Joukowski-type force if the mass of the rigid body converges to a positive real number.

In the above results, the authors prescribe a nonzero circulation $\gamma \neq 0$ around the rigid body at initial time. This assumption implies that the initial velocity field behaves like a point vortex

$$\gamma \frac{(x - h^{in})^{\perp}}{2\pi |x - h^{in}|^2} \tag{8}$$

closed to the point particle, where h^{in} is the initial position of the point particle. The sequence of initial data that have been considered in Glass et al. (2014a, b, 2016, 2018, 2019), Glass and Sueur (2019), does not converge in L^2, which is the classical energy space for the initial data of the Navier–Stokes system.

To compare, Glass et al. (2014a, b, 2016, 2018, 2019), Glass and Sueur (2019), with our result where the initial data converge in L^2, we have to consider only the case $\gamma = 0$. In this case, the point particle is not influence by the fluid in the sense that it move with constant velocity, see Section 1.4 of Glass et al. (2014b).

Let us recall that some works have been done to study well-posedness for the system (1)–(3)–(4) where it is possible to consider initial fluid velocity of the form (8), see Bravin (2020) and Ferriere and Hillairet (2023).

In contrast to the works (Glass et al. 2014a, b, 2016, 2018, 2019; Glass and Sueur 2019), we take advantage of the viscous term in an essential way to show our result. This partially explains why it is expected that the point particle behaves differently in the case the external fluid is viscous or inviscid. The key idea in our result is to notice that the L^2-norm of the velocity of the center of mass of a rigid body is bounded by

$$\int_0^T \int_{\mathbb{R}^d} |\nabla u_{\mathcal{F}}|^2 \, \mathrm{d}x, \tag{9}$$

in the case the rigid body has constant density. See for more details (20). Moreover, the quantity (9) appears naturally in the Leray-type energy estimate for the system (1)–(3)–(4). We then compare the solutions of system (1)–(3)–(4) with a regular solution of the Navier–Stokes system via a relative energy estimate. The result then follows if we assume "well-prepared" initial data.

Let us notice that this result can be extended to the case of finitely many "small" rigid bodies following the same strategy and using a restriction operator introduced in Feireisl et al. (2022). In the case of infinite many "small" rigid bodies, the problem is open and it is not clear how to extend the approach presented here. Finally, in the case where the fluid has density not constant, not even the well-posedness issue for the couple system fluid plus rigid body is studied in the literature.

To conclude, we present a short outlook of the remaining part of the paper. In Sect. 2, we introduce the definition of weak solution for system (1)–(3)–(4) and the main result. In Sect. 3, we introduce a restriction operator, we state a relative energy inequality and we use these tools to show the main result. In Sect. 4, we present the proof of the relative energy inequality. Finally we introduce appropriate Bogovskiĭ operators that follow the rigid bodies in "Appendix A" and we present an appropriate extension of the Sobolev embedding to our setting in "Appendix B."

2 Definition of Weak Solutions and Main Result

In this section, we recall the concept of finite energy weak solutions for the system (1)–(3)–(4). Then we state the main result of the paper.

We start with some notations. For an open subset $\mathcal{O} \subset \mathbb{R}^n$ with smooth boundary, where $n \in \mathbb{N}\backslash\{0\}$, for $p \in [1, +\infty]$ and for $k \in \mathbb{N}\backslash\{0\}$, we denote by $L^p(\mathcal{O})$, $W^{k,p}(\mathcal{O})$ the classical Lebesque and Sobolev spaces. In the special case of $p = 2$, we use the notation $H^k(\mathcal{O})$ to denote $W^{k,2}(\mathcal{O})$. To short the notation in the estimates, we use L^p_x, $W^{k,p}_x$, L^p_t, $W^{k,p}_t$ instead of $L^p(\mathbb{R}^d)$, $W^{k,p}(\mathbb{R}^d)$, $L^p(0, T)$, $W^{k,p}(0, T)$, respectively, where $T > 0$ is a fixed time. The symbols $L^q_t(L^p_x)$, $L^q_t(W^{k,p}_x)$ denote the Bochner spaces $L^q(0, T, L^p(\mathbb{R}^d))$, $L^q(0, T, W^{k,p}(\mathbb{R}^d))$ for $q \in [1, \infty]$ and $p \in [1, +\infty)$, while for $q = +\infty$ we use the notations $\|.\|_{L^q_t(L^\infty_x)}$, $\|.\|_{L^q_t(W^{k,\infty}_x)}$ to denote the corresponding norms. Following (He and Iftimie 2021), let us denote by

$$\rho = \chi_{\mathcal{F}(t)} + \rho_{\mathcal{S}}\chi_{\mathcal{S}(t)}$$

the extension by 1 of the density of the rigid body. Here, for a set $A \subset \mathbb{R}^d$, we denote by χ_A the indicator function of A, more precisely $\chi_A(x) = 1$ for $x \in A$ and 0 elsewhere. Similarly we define the global velocity

$$u = u_{\mathcal{F}}\chi_{\mathcal{F}(t)} + u_{\mathcal{S}}\chi_{\mathcal{S}(t)} = u\chi_{\mathcal{F}(t)} + (\ell(t) + \omega \times (x - h(t)))\,\chi_{\mathcal{S}(t)}.$$

Notice that if $u^{in} \in L^2(\mathcal{F}(0))$, then the compatibility conditions (5) on the initial data imply that div $(u^{in}) = 0$ in an appropriate weak sense.

After all this preliminary, we introduce the definitions of regular solution to the Navier–Stokes system (6) and of Hopf–Leray-type weak solution for system (1)–(3)–(4).

Theorem 1 *Let $k \geq d/2 + 1$ be an odd number and let $u^{in} \in H^k(\mathbb{R}^d)$ be such that* div $(u^{in}) = 0$*. Then, for some $T > 0$, there exists a unique regular solution (u, p) to the Navier–Stokes system* (6) *in the sense that*

$$u \in \bigcap_{l=0}^{\frac{k+1}{2}} H^l(0, T; H^{k+1-2l}(\mathbb{R}^d)) \cap L^\infty(0, T; H^k(\mathbb{R}^d)),$$

$$p \in L^2_{loc}(\mathbb{R}^d) \text{ such that } \nabla p \in \bigcap_{l=0}^{\frac{k-1}{2}} H^l(0, T; H^{k-1-2l}(\mathbb{R}^d)),$$

and (u, p) satisfies (6) *pointwise. Moreover, in dimension two T can be chosen arbitrarily big. In dimension three, T can be chosen arbitrarily big if the initial datum u^{in} is sufficiently small in the norm $H^k(\mathbb{R}^d)$.*

The proof of the above result is classical and it is a consequence of the parabolic structure of the Stokes system plus the fact that the nonlinearity is a small perturbation for short times. We move to the definition of weak solutions for system (1)–(3)–(4).

Definition 1 Let $\mathcal{S}^{in}$ and $\rho_{\mathcal{S}}^{in}$ be the initial position and density of the rigid body, let $(u_{\mathcal{F}}^{in}, \ell^{in}, \omega^{in})$ satisfying the hypothesis (5) and such that $u^{in} \in L^2(\mathbb{R}^d)$. Then a triple $(u_{\mathcal{F}}, \ell, \omega)$ is a Hopf–Leray weak solution for the system(1)–(3)–(4) associated with the initial data $\mathcal{S}^{in}$, $\rho_{\mathcal{S}}^{in}$, $u_{\mathcal{F}}^{in}$, ℓ^{in} and ω^{in}, if

- the functions $u_{\mathcal{F}}$, ℓ and ω satisfy

$$\ell \in L^\infty(\mathbb{R}^+;\mathbb{R}^d), \quad \omega \in L^\infty(\mathbb{R}^+;\mathbb{R}^{2d-3})$$
$$u_{\mathcal{F}} \in L^\infty(\mathbb{R}^+; L^2(\mathcal{F}(t))) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathcal{F}(t))), \quad \text{and} \quad u \in C_w(\mathbb{R}^+; L^2(\mathbb{R}^d));$$

- the vector field u is divergence free in $\mathbb{R}^d$ with $D(u) = 0$ in $\mathcal{S}(t)$;
- the vector field u satisfies the equation in the following sense:

$$-\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \rho u \cdot (\partial_t \varphi + (u\cdot\nabla)\varphi) - 2\nu D(u) : D(\varphi)\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^d} \rho^{in} u^{in} \cdot \varphi(0,.)\,\mathrm{d}x, \tag{10}$$

  for any test function $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ such that div $(\varphi) = 0$ and $D(\varphi) = 0$ in $\mathcal{S}(t)$.
- The following energy inequality holds

$$\int_{\mathbb{R}^d} \rho(t,.)|u(t,.)|^2\,\mathrm{d}x + 4\nu\int_0^t\int_{\mathbb{R}^d} |D(u)|^2\,\mathrm{d}x\mathrm{d}t \le \int_{\mathbb{R}^d} \rho|u^{in}|^2\,\mathrm{d}x, \tag{11}$$

  for almost any time $t \in \mathbb{R}^+$.

The existence of weak solutions for the system (1)–(3)–(4) is now classical and can be found, for example, in Feireisl (2002).

Theorem 2 *For initial data $\mathcal{S}^{in}$, $\rho_{\mathcal{S}}^{in}$, $u_{\mathcal{F}}^{in}$, ℓ^{in} and ω^{in} satisfying the hypothesis* (5) *and such that $u^{in} \in L^2(\mathbb{R}^d)$, there exist a Hopf–Leray weak solution $(u_{\mathcal{F}}, \ell, \omega)$ of the system* (1)–(3)–(4).

Now let us introduce a small parameter $\varepsilon > 0$ that "controls" the size of the rigid body in the sense that $\mathcal{S}_\varepsilon^{in} \subset B_\varepsilon(0)$. We study the dynamics of the rigid body as ε goes to zero for solutions of the system (1)–(3)–(4) under some assumptions on the initial data $\rho_{\mathcal{S},\varepsilon}$, $u_{\mathcal{F},\varepsilon}^{in}$, ℓ_ε^{in} and ω_ε^{in}. In particular, we show that the "small" rigid body follows the fluid flow in the limit. This result can be resumed as follows.

Theorem 3 *Let $(u_{\mathcal{F}\varepsilon}, \ell_\varepsilon, \omega_\varepsilon)$ be a sequence of Hopf–Leray weak solutions in the sense of Definition* 1*, corresponding to the initial data $\mathcal{S}_\varepsilon^{in}$, $\rho_{\mathcal{S},\varepsilon}^{in}$, $u_{\mathcal{F},\varepsilon}^{in}$, ℓ_ε^{in}, ω_ε^{in} satisfying the hypothesis* (5) *and such that $u_{\mathcal{F}_\varepsilon,\varepsilon}^{in} \in L^2(\mathcal{F}^{in})$. Let (u, p) be a regular solution, in the sense of Theorem* 1*, to the Navier–Stokes equations in $[0, T] \times \mathbb{R}^d$ with initial data $u^{in} \in H^k(\mathbb{R}^d)$ for an odd $k > d/2 + 1$. If we assume that*

- *the rigid body $\mathcal{S}_\varepsilon^{in} \subset B_\varepsilon(0)$;*
- *m_ε and $\mathcal{S}_\varepsilon^{in}$ satisfy* (7)*;*

- *for* $d=3$, $\|u_\varepsilon^{in}-u^{in}\|^2_{L^2(\mathcal{F}_\varepsilon)}/|\mathcal{S}_\varepsilon^{in}|^{1/3}\longrightarrow 0$, $m_\varepsilon|\ell_\varepsilon^{in}-u^{in}(h_\varepsilon(0))|^2/|\mathcal{S}_\varepsilon^{in}|^{1/3}\longrightarrow 0$ *and* $\omega_\varepsilon^{in}\cdot\mathcal{J}_\varepsilon^{in}\omega_\varepsilon^{in}/|\mathcal{S}_\varepsilon^{in}|^{1/3}\longrightarrow 0$;
- *for* $d=2$, $\|u_\varepsilon^{in}-u^{in}\|^2_{L^2(\mathcal{F}_\varepsilon)}/|\mathcal{S}_\varepsilon^{in}|^{\delta}\longrightarrow 0$, $m_\varepsilon|\ell_\varepsilon^{in}-u^{in}(h_\varepsilon(0))|^2/|\mathcal{S}_\varepsilon^{in}|^{\delta}\longrightarrow 0$ *and* $\mathcal{J}_\varepsilon^{in}|\omega_\varepsilon^{in}|^2/|\mathcal{S}_\varepsilon^{in}|^{\delta}\longrightarrow 0$ *where* $\delta>0$ *is the same of condition* (7),

then up to subsequence

$$h_\varepsilon \xrightarrow{w} h \quad \text{in } H^1(0,T) \quad \text{and} \quad \ell_\varepsilon \longrightarrow u(t,h(t)) \quad \text{in } L^2(0,T),$$

where h_ε *is the center of mass of the rigid body* $\mathcal{S}_\varepsilon$. *Moreover,*

$$h(t)=h(0)+\int_0^t u(\tau,h(\tau))\,\mathrm{d}\tau.$$

Let us notice that in dimension two the time of existence of regular solutions is arbitrarily big. In this case, the convergence of h_ε and ℓ_ε holds in any compact interval. In dimension three, the existence of global regular solutions is an open problem but there exist local in time solutions and they are global in time for small initial data.

3 Proof of the Main Result

The proof of Theorem 3 is based on a relative energy inequality that is stated in Lemma 2. We present the proof of Lemma 2 in a separate section because it is technical. The plan for this section is to recall the definition of the restriction operator R_ε, to state Lemma 2 and to prove Theorem 3. In the remaining part of the paper, we set $\nu=1$ to simplify the notation.

3.1 The Restriction Operator

Let us introduce a restriction operator introduced in Feireisl et al. (2022). Consider a cutoff $\eta\in C^\infty(\mathbb{R}^d)$ such that $0\le\eta\le 1$, $\eta=0$ in $B_1(0)$ and $\eta=1$ in $\mathbb{R}^d\setminus B_2(0)$, moreover we introduce $\eta_\varepsilon(x)=\eta(x/\varepsilon)$. The restriction operator is defined as

$$\begin{aligned}R_\varepsilon[\varphi](t,x)=&\,\eta_\varepsilon(x-h_\varepsilon(t))\varphi(t,x)+(1-\eta_\varepsilon(x-h_\varepsilon(t)))\varphi(t,h_\varepsilon(t))\\&+\mathcal{B}_\varepsilon[\operatorname{div}(\eta_\varepsilon(x-h_\varepsilon(t))\varphi(t,x)+(1-\eta_\varepsilon(x-h_\varepsilon(t)))\varphi(t,h_\varepsilon(t)))],\end{aligned}$$

where $\mathcal{B}_\varepsilon[f](x)=\varepsilon\mathcal{B}_1[f(\varepsilon y)](x/\varepsilon)$ and $\mathcal{B}_1$ is a Bogovskiĭ operator in $B_2(0)\backslash B_1(0)$. See "Appendix A" for more details.

To simplify the notation for any regular enough function φ, we denote by $\bar\varphi^\varepsilon(t)=\varphi(t,h_\varepsilon(t))$. This allows us to rewrite the restriction operator in the more compact form

$$R_\varepsilon[\varphi]=\eta_\varepsilon\varphi+(1-\eta_\varepsilon)\bar\varphi^\varepsilon+\mathcal{B}_\varepsilon[\operatorname{div}(\eta_\varepsilon\varphi+(1-\eta_\varepsilon)\bar\varphi^\varepsilon)].$$

In the following lemma, we resume some properties of R_ε.

Lemma 1 *The restriction operator R_ε satisfies the following properties. For any $\varphi \in C^0(\mathbb{R}^d)$ such that* $\operatorname{div}(\varphi) = 0$*,*

$$\operatorname{div}(R_\varepsilon[\varphi]) = 0 \quad \text{in } \mathbb{R}^d, \quad R_\varepsilon[\varphi] = \varphi(t, h_\varepsilon(t)) \quad \text{in } \mathcal{S}_\varepsilon(t).$$

Moreover, for $p \in [1, +\infty)$, it holds

$$\|R_\varepsilon[\varphi] - \varphi\|_{L^p(\mathbb{R}^d)} \le C\varepsilon^{d/p}\|\varphi\|_{L^\infty(\mathbb{R}^d)} \quad \text{and} \quad \|R_\varepsilon[\varphi] - \varphi\|_{L^\infty(\mathbb{R}^d)} \le C\|\varphi\|_{L^\infty(\mathbb{R}^d)}. \tag{12}$$

Finally, if $\varphi \in W^{1,\infty}_x(\mathbb{R}^d)$ such that $\operatorname{div}(\varphi) = 0$ *and if $p \in (1, +\infty)$,*

$$\|\nabla R_\varepsilon[\varphi] - \nabla\varphi\|_{L^p(\mathbb{R}^d)} \le C_p\varepsilon^{d/p}\|\varphi\|_{W^{1,\infty}(\mathbb{R}^d)}. \tag{13}$$

Proof Let us show estimates (12)–(13). We have

$$\|R_\varepsilon[\varphi] - \varphi\|_{L^p_x} \le \|R_\varepsilon[\varphi] - \eta_\varepsilon\varphi\|_{L^p_x} + \|\eta_\varepsilon\varphi - \varphi\|_{L^p_x} \le \|R_\varepsilon[\varphi] - \eta_\varepsilon\varphi\|_{L^p_x} + C\varepsilon^{d/p}\|\varphi\|_{L^\infty_x}.$$

It is then enough to show

$$\|R_\varepsilon[\varphi] - \eta_\varepsilon\varphi\|_{L^p_x} \le C\varepsilon^{d/p}\|\varphi\|_{L^\infty_x}. \tag{14}$$

Using the definition of R_ε and the fact that $1 - \eta_\varepsilon$ and $\mathcal{B}_\varepsilon$ are supported in a ball of radius 2ε, we have

$$\begin{aligned}\|R_\varepsilon[\varphi] - \eta_\varepsilon\varphi\|_{L^1_x} &= \|(1-\eta_\varepsilon)\bar{\varphi}^\varepsilon + \mathcal{B}_\varepsilon[\operatorname{div}(\eta_\varepsilon\varphi + (1-\eta_\varepsilon)\bar{\varphi}^\varepsilon)]\|_{L^1_x} \\ &\le C\varepsilon^d\|\varphi\|_{L^\infty_x} + C\varepsilon\|\mathcal{B}_\varepsilon[\operatorname{div}(\eta_\varepsilon\varphi + (1-\eta_\varepsilon)\bar{\varphi}^\varepsilon)]\|_{L^{d/(d-1)}_x} \\ &\le C\varepsilon^d\|\varphi\|_{L^\infty_x} + C\varepsilon\|\operatorname{div}(\eta_\varepsilon\varphi + (1-\eta_\varepsilon)\bar{\varphi}^\varepsilon)\|_{L^1_x}.\end{aligned}$$

Using the fact that φ is divergence free, we rewrite

$$\operatorname{div}(\eta_\varepsilon\varphi + (1-\eta_\varepsilon)\bar{\varphi}^\varepsilon) = \nabla\eta_\varepsilon \cdot (\varphi - \bar{\varphi}^\varepsilon).$$

This allows us to deduce

$$\begin{aligned}\|R_\varepsilon[\varphi] - \eta_\varepsilon\varphi\|_{L^1_x} &\le C\varepsilon^d\|\varphi\|_{L^\infty_x} + C\varepsilon\|\nabla\eta_\varepsilon(\varphi - \bar{\varphi}^\varepsilon)\|_{L^1_x} \\ &\le C\varepsilon^d\|\varphi\|_{L^\infty_x} + C\varepsilon\|\nabla\eta_\varepsilon\|_{L^1}\|\varphi\|_{L^\infty_x} \\ &\le C\varepsilon^d\|\varphi\|_{L^\infty_x}.\end{aligned} \tag{15}$$

Similarly

$$\begin{aligned}\|R_\varepsilon[\varphi] - \eta_\varepsilon\varphi\|_{L^\infty_x} &= \|(1-\eta_\varepsilon)\bar{\varphi}^\varepsilon + \mathcal{B}_\varepsilon[\operatorname{div}(\eta_\varepsilon\varphi + (1-\eta_\varepsilon)\bar{\varphi}^\varepsilon)]\|_{L^\infty_x} \\ &\le C\|\varphi\|_{L^\infty_x} + C\|\mathcal{B}_\varepsilon[\operatorname{div}(\eta_\varepsilon\varphi + (1-\eta_\varepsilon)\bar{\varphi}^\varepsilon)]\|_{L^\infty_x}\end{aligned}$$

$$\leq C\|\varphi\|_{L^\infty_x} + C\|\mathrm{div}\,(\eta_\varepsilon\varphi + (1-\eta_\varepsilon)\bar{\varphi}^\varepsilon)\|_{L^d_x}$$
$$\leq C\|\varphi\|_{L^\infty_x} + C\|\nabla\eta_\varepsilon\|_{L^d}\|\varphi\|_{L^\infty_x}$$
$$\leq C\|\varphi\|_{L^\infty_x}. \tag{16}$$

From (15)–(16) and interpolation inequality, we deduce (14).

To show estimate (13), it is enough to show

$$\|\nabla R_\varepsilon[\varphi] - \eta_\varepsilon\nabla\varphi\|_{L^p_x} \leq C_p\varepsilon^{d/p}\|\varphi\|_{L^\infty_x}. \tag{17}$$

Notice that

$$\nabla R_\varepsilon[\varphi] - \eta_\varepsilon\nabla\varphi = \nabla\eta_\varepsilon(\varphi - \bar{\varphi}^\varepsilon) + \nabla\mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(\varphi - \bar{\varphi}^\varepsilon)]$$

and

$$\|\nabla\eta_\varepsilon(\varphi - \bar{\varphi}^\varepsilon)\|_{L^p_x} \leq \left\| |. - h_\varepsilon(t)|\eta_\varepsilon(.)\frac{\varphi(.) - \bar{\varphi}^\varepsilon}{|. - h_\varepsilon(t)|}\right\|_{L^p_x}$$
$$\leq \||. - h_\varepsilon(t)|\eta_\varepsilon(.)\|_{L^p_x}\left\|\frac{\varphi(.) - \bar{\varphi}^\varepsilon}{|. - h_\varepsilon(t)|}\right\|_{L^\infty_x}$$
$$\leq C\varepsilon^{d/p}\|\varphi\|_{W^{1,\infty}_x}.$$

The above computations and estimates allow us to deduce

$$\|\nabla R_\varepsilon[\varphi] - \eta_\varepsilon\nabla\varphi\|_{L^p_x} \leq \|\nabla\eta_\varepsilon(\varphi - \bar{\varphi}^\varepsilon)\|_{L^p_x} + \|\nabla\mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(\varphi - \bar{\varphi}^\varepsilon)]\|_{L^p_x}$$
$$\leq C_p\|\nabla\eta_\varepsilon(\varphi - \bar{\varphi}^\varepsilon)\|_{L^p_x}$$
$$\leq C_p\varepsilon^{d/p}\|\varphi\|_{W^{1,\infty}_x}.$$

□

Remark 1 Let us notice that in the proof of the above lemma we show also inequality (14) that reads

$$\|R_\varepsilon[\varphi] - \eta_\varepsilon\varphi\|_{L^p(\mathbb{R}^d)} \leq C\varepsilon^{d/p}\|\varphi\|_{L^\infty_x} \tag{18}$$

for any $\varphi \in C^0(\mathbb{R}^d)$ such that $\mathrm{div}\,(\varphi) = 0$ and for $p \in [1, +\infty]$ and estimate (17) that reads

$$\|\nabla R_\varepsilon[\varphi] - \eta_\varepsilon\nabla\varphi\|_{L^p(\mathbb{R}^d)} \leq C\varepsilon^{d/p}\|\varphi\|_{W^{1,\infty}_x}. \tag{19}$$

for any $\varphi \in W^{1,\infty}_x(\mathbb{R}^d)$ such that $\mathrm{div}\,(\varphi) = 0$ and if $p \in (1, +\infty)$.

We will now present the relative energy inequality.

3.2 Relative Energy Inequality

We will use the restriction operator introduced in the previous section to deduce a relative energy inequality.

Lemma 2 *Under the hypothesis of Theorem 3, we have*

$$\int_{\mathbb{R}^d} \rho_\varepsilon |u_\varepsilon(t,.) - R_\varepsilon[u(t,.)]|^2 \,\mathrm{d}x + 4\int_0^T \int_{\mathbb{R}^d} |D(u_\varepsilon - R_\varepsilon[u])|^2 \,\mathrm{d}x\mathrm{d}t \\ \leq \int_{\mathbb{R}^d} \rho_\varepsilon^{in} |u_\varepsilon^{in} - R_\varepsilon[u^{in}]|^2 \,\mathrm{d}x + C\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon |u_\varepsilon - R_\varepsilon[u]|^2 \,\mathrm{d}x\mathrm{d}t + Rest_\varepsilon,$$

where

$$|Rest_\varepsilon| \leq C(m_\varepsilon + |\mathcal{S}_\varepsilon| + \varepsilon^{3/2} + \varepsilon^3 |\mathcal{S}_\varepsilon^{in}|^{-1/3}), \quad \text{for } d = 3,$$
$$|Rest_\varepsilon| \leq C(m_\varepsilon + |\mathcal{S}_\varepsilon| + \varepsilon^{\tilde{\delta}} + \varepsilon^{1+\tilde{\delta}} |\mathcal{S}_\varepsilon^{in}|^{-\delta}), \quad \text{for } d = 2,$$

with C independent of ε.

The above lemma is the key estimate to deduce Theorem 3. We will prove Lemma 2 in Sect. 4.

3.3 Proof of Theorem 3

Let us show Theorem 3 with the help of Lemma 2.

Proof of Theorem 3 Let us show Theorem 3 in dimension three and explain at the end how to adapt the proof in dimension two. First of all, let us notice that

$$\int_{\mathcal{S}_\varepsilon(t)} |u_\varepsilon(t,x) - R_\varepsilon[u](t,x)|^2 \,\mathrm{d}x = \int_{\mathcal{S}_\varepsilon(t)} |\ell_\varepsilon(t) + \omega_\varepsilon(t) \times (x - h_\varepsilon(t)) - u(t, h_\varepsilon(t))|^2 \,\mathrm{d}x \\ = |\mathcal{S}_\varepsilon||\ell_\varepsilon(t) - u(t,h_\varepsilon(t))|^2 + \frac{1}{\rho_{\mathcal{S},\varepsilon}} \omega_\varepsilon(t) \cdot \mathcal{J}_\varepsilon(t)\omega_\varepsilon(t) \geq |\mathcal{S}_\varepsilon||\ell_\varepsilon(t) - u(t,h_\varepsilon(t))|^2, \tag{20}$$

where we used in an essential manner the fact that $\rho_{\mathcal{S},\varepsilon}$ is constant in $\mathcal{S}_\varepsilon$ in the second equality and the fact that $\mathcal{J}_\varepsilon$ is semi-positive definite in the last inequality. This allows us to estimate

$$|\mathcal{S}_\varepsilon^{in}|^{1/2} \|\ell_\varepsilon - u(t,h_\varepsilon(t))\|_{L^2_t} \leq \|u_\varepsilon - R_\varepsilon[u]\|_{L^2_t(L^2(\mathcal{S}_\varepsilon(t)))} \\ \leq |\mathcal{S}_\varepsilon^{in}|^{1/3} \|u_\varepsilon - R_\varepsilon[u]\|_{L^2_t(L^6(\mathcal{S}_\varepsilon(t)))} \\ \leq C|\mathcal{S}_\varepsilon^{in}|^{1/3} \|D(u_\varepsilon - R_\varepsilon[u])\|_{L^2_t(L^2(\mathbb{R}^3))}.$$

We can rewrite the above inequality as

$$|\mathcal{S}_\varepsilon^{in}|^{1/3} \|\ell_\varepsilon - u(t,h_\varepsilon(t))\|_{L^2_t}^2 \leq C\|D(u_\varepsilon - R_\varepsilon[u])\|_{L^2_t(L^2(\mathbb{R}^3))}^2. \tag{21}$$

From Lemma 2,

$$
\begin{aligned}
&|\mathcal{S}_\varepsilon^{in}|^{-1/3}\int_{\mathbb{R}^3}\rho_\varepsilon|u_\varepsilon(t,.)-R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x+C^{-1}\|\ell_\varepsilon-u(t,h_\varepsilon(t))\|_{L_t^2}^2\\
&\quad\le|\mathcal{S}_\varepsilon^{in}|^{-1/3}\int_{\mathbb{R}^3}\rho_\varepsilon|u_\varepsilon(t,.)-R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x+4|\mathcal{S}_\varepsilon^{in}|^{-1/3}\\
&\qquad\int_0^T\int_{\mathbb{R}^3}|D(u_\varepsilon-R_\varepsilon[u])|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad\le|\mathcal{S}_\varepsilon^{in}|^{-1/3}\int_{\mathbb{R}^3}\rho_\varepsilon^{in}|u_\varepsilon^{in}-R_\varepsilon[u^{in}]|^2\,\mathrm{d}x+C|\mathcal{S}_\varepsilon^{in}|^{-1/3}\\
&\qquad\int_0^T\int_{\mathbb{R}^3}\rho_\varepsilon|u_\varepsilon-R_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t+|\mathcal{S}_\varepsilon^{in}|^{-1/3}Rest_\varepsilon.
\end{aligned}
$$

Grömwall's inequality implies that

$$
\|\ell_\varepsilon-u(t,h_\varepsilon(t))\|_{L_t^2}^2\le\left(|\mathcal{S}_\varepsilon^{in}|^{-1/3}\int_{\mathbb{R}^3}\rho_\varepsilon|u_\varepsilon^{in}-R_\varepsilon[u^{in}]|^2\,\mathrm{d}x+|\mathcal{S}_\varepsilon^{in}|^{-1/3}|Rest_\varepsilon|\right)e^{TC}. \tag{22}
$$

Notice that

$$
\begin{aligned}
\int_{\mathcal{F}_\varepsilon(t)}|u_\varepsilon^{in}-R_\varepsilon[u^{in}]|^2\,\mathrm{d}x&\le\int_{\mathcal{F}_\varepsilon(t)}|u_\varepsilon^{in}-u^{in}|^2\,\mathrm{d}x+2\int_{\mathcal{F}_\varepsilon(t)}u_\varepsilon^{in}\cdot(u^{in}-R_\varepsilon[u^{in}])\,\mathrm{d}x\\
&\quad-\int_{\mathcal{F}_\varepsilon(t)}(u^{in}-R_\varepsilon[u^{in}])\cdot R_\varepsilon[u^{in}]\,\mathrm{d}x\\
&\quad-\int_{\mathcal{F}_\varepsilon(t)}u^{in}\cdot(u^{in}-R_\varepsilon[u^{in}])\,\mathrm{d}x\\
&\le\int_{\mathcal{F}_\varepsilon(t)}|u_\varepsilon^{in}-u^{in}|^2\,\mathrm{d}x+C\varepsilon^{3/2},
\end{aligned}
$$

where we used some Hölder inequalities and (12) for $d=3$, $p=2$ and $\varphi=u^{in}$.

The above inequality implies

$$
\begin{aligned}
\frac{1}{|\mathcal{S}_\varepsilon^{in}|^{1/3}}\int_{\mathbb{R}^3}\rho_\varepsilon^{in}|u_\varepsilon^{in}-R_\varepsilon[u^{in}]|^2\,\mathrm{d}x&\le\frac{1}{|\mathcal{S}_\varepsilon^{in}|^{1/3}}\int_{\mathcal{F}_\varepsilon(t)}|u_\varepsilon^{in}-u^{in}|^2\,\mathrm{d}x\\
&\quad+\frac{m_\varepsilon}{|\mathcal{S}_\varepsilon^{in}|^{1/3}}|\ell_\varepsilon^{in}-u^{in}(h_\varepsilon(0))|^2\\
&\quad+\frac{1}{|\mathcal{S}_\varepsilon^{in}|^{1/3}}\omega_\varepsilon^{in}\cdot\mathcal{J}_\varepsilon^{in}\omega_\varepsilon^{in}+C\frac{\varepsilon^{3/2}}{|\mathcal{S}_\varepsilon^{in}|^{1/3}}\longrightarrow0,
\end{aligned}
\tag{23}
$$

as $\varepsilon\longrightarrow0$, where we used the assumptions

$$
\|u_\varepsilon^{in}-u^{in}\|_{L^2(\mathcal{F}_\varepsilon)}^2/|\mathcal{S}_\varepsilon^{in}|^{1/3}\longrightarrow0,\quad m_\varepsilon|\ell_\varepsilon^{in}-u^{in}(h_\varepsilon(0))|^2/|\mathcal{S}_\varepsilon^{in}|^{1/3}\longrightarrow0
$$

$$\text{and} \quad \omega_\varepsilon^{in} \cdot \mathcal{J}_\varepsilon^{in} \omega_\varepsilon^{in} / |\mathcal{S}_\varepsilon^{in}|^{1/3} \longrightarrow 0.$$

Lemma 2 ensures that

$$|\mathcal{S}_\varepsilon^{in}|^{-1/3} |Rest_\varepsilon| \leq |\mathcal{S}_\varepsilon^{in}|^{-1/3} |C(m_\varepsilon + |\mathcal{S}_\varepsilon| + \varepsilon^{3/2} + \varepsilon^3 |\mathcal{S}_\varepsilon^{in}|^{-1/3}) \longrightarrow 0, \quad (24)$$

as $\varepsilon \longrightarrow 0$, from hypothesis

$$\frac{m_\varepsilon}{|\mathcal{S}_\varepsilon^{in}|^{1/3}} \longrightarrow 0 \quad \text{and} \quad \frac{|\mathcal{S}_\varepsilon^{in}|}{\varepsilon^{9/2}} \longrightarrow +\infty.$$

The estimate (22), together with (23) and (24), ensures that

$$\|\ell_\varepsilon\|_{L_t^2} \leq C,$$

which implies that h_ε is uniformly bounded in $W_t^{1,2}$. Up to subsequence

$$h_\varepsilon \xrightarrow{w} h \text{ in } W_t^{1,2},$$

in particular, it converges strongly in C_t^0. To show the convergence of ℓ_ε, we rewrite

$$\ell_\varepsilon(t) - u(t, h(t)) = u_\varepsilon(t, h_\varepsilon(t)) - u(t, h_\varepsilon(t)) + u(t, h_\varepsilon(t)) - u(t, h(t)).$$

Inequality (22), together with (23), implies that

$$\ell_\varepsilon - u(t, h_\varepsilon(t)) \to 0 \quad \text{in } L_t^2. \quad (25)$$

Using that $u \in L^\infty(0, T; W_x^{1,\infty})$ and $h_\varepsilon \to h$ in C_t^0, we have

$$|u(t, h_\varepsilon(t)) - u(t, h(t))| \leq \|\nabla u\|_{L^\infty(0,T;L_x^\infty)} |h_\varepsilon(t) - h(t)| \longrightarrow 0, \quad (26)$$

for almost any $t \in [0, T]$. The convergence (25) and (26) imply

$$\ell_\varepsilon \longrightarrow u(t, h(t)) \quad \text{in } L_t^2.$$

Finally we pass to the limit in the equation

$$h_\varepsilon(t) = h_\varepsilon(0) + \int_0^t \ell_\varepsilon(\tau)\, \mathrm{d}\tau$$

to deduce

$$h(t) = h(0) + \int_0^t u(\tau, h(\tau))\, \mathrm{d}\tau.$$

Let us now move to the case of dimension two. First of all, let us notice that

$$\begin{aligned}|\mathcal{S}_\varepsilon^{in}|^{1/2}\|\ell_\varepsilon - u(t,h_\varepsilon(t))\|_{L^2_t} &\le \|u_\varepsilon - R_\varepsilon[u]\|_{L^2_t(L^2(\mathcal{S}_\varepsilon(t)))}\\ &\le |\mathcal{S}_\varepsilon^{in}|^{1/q}\|u_\varepsilon - R_\varepsilon[u]\|_{L^2_t(L^p(\mathcal{S}_\varepsilon(t)))}\\ &\le C_p|\mathcal{S}_\varepsilon^{in}|^{1/q}(\|u_\varepsilon - R_\varepsilon[u]\|_{L^2_t(L^2(\mathcal{F}_\varepsilon(t)))} + \|\nabla(u_\varepsilon - R_\varepsilon[u])\|_{L^2_t(L^2(\mathbb{R}^2))})\\ &\le C_p|\mathcal{S}_\varepsilon^{in}|^{1/q}(\|u_\varepsilon - R_\varepsilon[u]\|_{L^2_t(L^2(\mathcal{F}_\varepsilon(t)))} + \sqrt{2}\|D(u_\varepsilon - R_\varepsilon[u])\|_{L^2_t(L^2(\mathbb{R}^2))}),\end{aligned}$$

where $1/p + 1/q = 1/2$. We used Lemma 4 in the third inequality, and in the last one, we apply the Korn's identity

$$\int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x = 2\int_{\mathbb{R}^2} |Dv|^2 \,\mathrm{d}x,$$

which holds for any $v \in H^1(\mathbb{R}^2)$ such that $\mathrm{div}(v) = 0$. The Korn's equality can be verified using the density of $C_c^\infty(\mathbb{R}^2)$ functions in $H^1(\mathbb{R}^2)$. We deduce that

$$\begin{aligned}|\mathcal{S}_\varepsilon^{in}|^{1/p}\|\ell_\varepsilon - u(t,h_\varepsilon(t))\|_{L^2_t} \le C_p(&\|u_\varepsilon - R_\varepsilon[u]\|_{L^2_t(L^2(\mathcal{F}_\varepsilon(t)))}\\ &+ \sqrt{2}\|D(u_\varepsilon - R_\varepsilon[u])\|_{L^2_t(L^2(\mathbb{R}^2))}),\end{aligned}$$

for any $p < \infty$.

Choose now $1/p = \delta$. Using Lemma 2, we deduce that

$$\begin{aligned}&|\mathcal{S}_\varepsilon^{in}|^{-\delta}\int_{\mathbb{R}^2}\rho_\varepsilon|u_\varepsilon(t,.) - R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x + \frac{2}{C_P}\|\ell_\varepsilon - u(t,h_\varepsilon(t))\|_{L^2_t}\\ &\quad\le |\mathcal{S}_\varepsilon^{in}|^{-\delta}\int_{\mathbb{R}^2}\rho_\varepsilon|u_\varepsilon(t,.) - R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x + 4|\mathcal{S}_\varepsilon^{in}|^{-\delta}\int_0^T\int_{\mathbb{R}^2}|D(u_\varepsilon - R_\varepsilon[u])|^2\,\mathrm{d}x\mathrm{d}t\\ &\qquad + 2\int_0^T\int_{\mathcal{F}_\varepsilon(t)}|u_\varepsilon - R_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t\\ &\quad\le |\mathcal{S}_\varepsilon^{in}|^{-\delta}\int_{\mathbb{R}^2}\rho_\varepsilon^{in}|u_\varepsilon^{in} - R_\varepsilon[u^{in}]|^2\,\mathrm{d}x + (C+2)\,|\mathcal{S}_\varepsilon^{in}|^{-\delta}\int_0^T\int_{\mathbb{R}^2}\rho_\varepsilon|u_\varepsilon - R_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t\\ &\qquad + |\mathcal{S}_\varepsilon^{in}|^{-\delta}Rest_\varepsilon.\end{aligned}$$

Grömwall's inequality implies that

$$\|\ell_\varepsilon - u(t,h_\varepsilon(t))\|^2_{L^2_t} \le \tilde{C}\left(|\mathcal{S}_\varepsilon^{in}|^{-\delta}\int_{\mathbb{R}^2}\rho_\varepsilon^{in}|u_\varepsilon^{in} - R_\varepsilon[u^{in}]|^2\,\mathrm{d}x + |\mathcal{S}_\varepsilon^{in}|^{-\delta}|Rest_\varepsilon|\right)e^{T(C+2)}. \tag{27}$$

Using the assumptions (7) and following the same strategy as in the case of dimension three, we prove the theorem. □

4 Proof of Lemma 2

Let us now show Lemma 2.

Proof of Lemma 2 By Definition 1, any solution $(u_{\mathcal{F},\varepsilon}, \ell_\varepsilon, \omega_\varepsilon)$ satisfies the energy inequality (11) that reads

$$m_\varepsilon|\ell_\varepsilon(t)|^2 + \omega_\varepsilon(t)\cdot\mathcal{J}_\varepsilon(t)\omega_\varepsilon(t) + \int_{\mathcal{F}_\varepsilon(t)}|u_\varepsilon|^2\,\mathrm{d}x + 4\nu\int_0^t\int_{\mathbb{R}^d}|D(u_\varepsilon)|^2\,\mathrm{d}x\mathrm{d}t$$
$$\le m_\varepsilon|\ell_\varepsilon^{in}|^2 + \omega_\varepsilon^{in}\cdot\mathcal{J}_\varepsilon^{in}\omega_\varepsilon^{in} + \int_{\mathcal{F}_\varepsilon^{in}}|u_\varepsilon^{in}|^2\,\mathrm{d}x \le C. \tag{28}$$

Moreover, the right-hand side of the above inequality is bounded uniformly in ε due to the assumptions on the initial data stated on points two and three in Theorem 3. In the same spirit as estimate (21), we deduce in dimension three that

$$|\mathcal{S}_\varepsilon^{in}|^{1/6}\|\ell_\varepsilon\|_{L^2_t} \le C\|D(u_\varepsilon)\|_{L^2_t(L^2(\mathbb{R}^3))} \le C. \tag{29}$$

By the hypothesis of Theorem 3 that $u^{in} \in H^k$ for $k > d/2+1$, there exist a unique local solution in dimension three and a global solution in dimension two of the Navier–Stokes equations such that

$$u \in L^\infty(0,T;H^k(\mathbb{R}^d)) \cap L^2(0,T;H^{k+1}(\mathbb{R}^d)).$$

This solution satisfies the energy equality

$$\int_{\mathbb{R}^d}|u|^2\,\mathrm{d}x + \nu\int_0^t|\nabla u|^2\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^d}\left|u^{in}\right|^2\,\mathrm{d}x. \tag{30}$$

With this choice of k, we have also the bounds

$$\|\partial_t u\|_{L^2_t(L^\infty_x)} + \|u\|_{L^\infty_t(W^{1,\infty}_x)} + \|\nabla p\|_{L^2_t(L^\infty_x)} \le C. \tag{31}$$

The above bound will be implicitly used in many of the estimates to prove this lemma.

To deduce the relative energy inequality, let us start by computing

$$\int_{\mathbb{R}^d}\rho_\varepsilon|u_\varepsilon(t,.) - R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x = \int_{\mathbb{R}^d}\rho_\varepsilon|u_\varepsilon(t,.)|^2\,\mathrm{d}x - 2\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon(t,.)\cdot R_\varepsilon[(u(t,.)]\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^d}\rho_\varepsilon|R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x$$
$$\le \int_{\mathbb{R}^d}\rho_\varepsilon^{in}|u_\varepsilon^{in}|^2\,\mathrm{d}x - 4\int_0^T\int_{\mathbb{R}^d}|Du_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t - 2\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon(t,.)\cdot\partial_t R_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t$$
$$-2\int_0^T\int_{\mathcal{F}_\varepsilon(t)}u_\varepsilon\otimes u_\varepsilon : \nabla R_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t + 4\int_0^T\int_{R^d}Du_\varepsilon : DR_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&- 2\int_{\mathbb{R}^d} \rho_\varepsilon^{in} u_\varepsilon^{in} \cdot R_\varepsilon[u^{in}]\,\mathrm{d}x + \int_{\mathbb{R}^d} \rho_\varepsilon^{in} |R_\varepsilon[u^{in}]|^2\,\mathrm{d}x - 4\int_0^T \int_{\mathbb{R}^d} |DR_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t \\
&+ \int_{\mathbb{R}^d} \rho_\varepsilon |R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} |u(t,.)|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} |u^{in}|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho_\varepsilon^{in} |R_\varepsilon[u^{in}]|^2\,\mathrm{d}x \\
&+ 4\int_0^T \int_{\mathbb{R}^d} |DR_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t - 4\int_{\mathbb{R}^d} |Du|^2\,\mathrm{d}x,
\end{aligned}
$$

where in the inequality we use (28), (10) and (30) that are, respectively, the energy inequality for u_ε, the weak formulation satisfied by u_ε and the energy equality satisfied by u. After bringing on the left-hand side some terms involving Du_ε and $DR_\varepsilon[u]$, we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^d} \rho_\varepsilon |u_\varepsilon(t,.) - R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x + 4\int_0^T \int_{\mathbb{R}^d} |D(u_\varepsilon - R_\varepsilon[u])|^2\,\mathrm{d}x\mathrm{d}t& \\
\leq \int_{\mathbb{R}^d} \rho_\varepsilon^{in} |u_\varepsilon^{in} - R_\varepsilon[u^{in}]|^2\,\mathrm{d}x + \widetilde{Rest_\varepsilon}&
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{Rest_\varepsilon} =& -2\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon u_\varepsilon(t,.) \cdot \partial_t R_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t - 2\int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \otimes u_\varepsilon : \nabla R_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t \\
&- 4\int_0^T \int_{\mathbb{R}^d} Du_\varepsilon : DR_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} \rho_\varepsilon |R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} |u(t,.)|^2\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^d} |u^{in}|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho_\varepsilon^{in} |R_\varepsilon[u^{in}]|^2\,\mathrm{d}x + 4\int_0^T \int_{\mathbb{R}^d} |DR_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t \\
&- 4\int_{\mathbb{R}^d} |Du|^2\,\mathrm{d}x.
\end{aligned}
$$

It remains to estimate $|\widetilde{Rest_\varepsilon}|$. To do that, we decompose the remainder $\widetilde{Rest_\varepsilon} = Rest_\varepsilon^1 + Rest_\varepsilon^2$ where

$$
\begin{aligned}
Rest_\varepsilon^2 =& \int_{\mathbb{R}^d} \rho_\varepsilon |R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} |u(t,.)|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} |u^{in}|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho_\varepsilon^{in} |R_\varepsilon[u^{in}]|^2\,\mathrm{d}x \\
&+ 4\int_0^T \int_{\mathbb{R}^d} |DR_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t - 4\int_{\mathbb{R}^d} |Du|^2\,\mathrm{d}x.
\end{aligned}
$$

We start by estimating the terms

$$
\begin{aligned}
\left| \int_{\mathbb{R}^d} \rho_\varepsilon |R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} |u(t,.)|^2\,\mathrm{d}x \right| &\leq (m_\varepsilon + |\mathcal{S}_\varepsilon^{in}|)\|u\|_{L_x^\infty}^2 \\
&+ \left| \int_{\mathcal{F}_\varepsilon(t)} |R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x - \int_{\mathcal{F}_\varepsilon(t)} |u(t,.)|^2\,\mathrm{d}x \right|.
\end{aligned}
$$

To tackle the last term on the right-hand side, we notice that $R_\varepsilon[u(t,.)] - u(t,.)$ is supported in $B_{2\varepsilon}(h_\varepsilon(t))$ and that

$$\|R_\varepsilon[u]\|_{L^\infty_x} \le C\|u\|_{L^\infty_x}$$

from (12).

The two above observations allow us to estimate

$$\begin{aligned}\left|\int_{\mathcal{F}_\varepsilon(t)} |R_\varepsilon[u(t,.)]|^2\,\mathrm{d}x - \int_{\mathcal{F}_\varepsilon(t)} |u(t,.)|^2\,\mathrm{d}x\right| &\le \left|\int_{\mathcal{F}_\varepsilon(t)} R_\varepsilon[u(t,.)](R_\varepsilon[u(t,.)] - u(t,.))\,\mathrm{d}x\right| \\ &\quad + \left|\int_{\mathcal{F}_\varepsilon(t)} (R_\varepsilon[u(t,.)] - u(t,.))u(t,.)\,\mathrm{d}x\right| \\ &\le C\varepsilon^d\|u\|^2_{L^\infty_x}.\end{aligned}$$

We deduce that

$$\left|\int_{\mathbb{R}^d} \rho_\varepsilon|R_\varepsilon[u(t,.])|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} |u(t,.)|^2\,\mathrm{d}x\right| \le C(m_\varepsilon + |\mathcal{S}^{in}_\varepsilon| + \varepsilon^d). \tag{32}$$

We estimate the third and fourth terms of $Rest^2_\varepsilon$ analogously, and we deduce

$$\left|\int_{\mathbb{R}^d} |u^{in}|^2\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho^{in}_\varepsilon|R_\varepsilon[u^{in}]|^2\,\mathrm{d}x\right| \le C(m_\varepsilon + |\mathcal{S}^{in}_\varepsilon| + \varepsilon^d). \tag{33}$$

We are left with the estimate of

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^d} |DR_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbb{R}^d} |Du|^2\,\mathrm{d}x\mathrm{d}t\right| &\le \left|\int_0^T\int_{\mathbb{R}^d} DR_\varepsilon[u] : D(R_\varepsilon[u] - u)\,\mathrm{d}x\mathrm{d}t\right| \\ &\quad + \left|\int_0^T\int_{\mathbb{R}^d} D(R_\varepsilon[u] - u) : Du\,\mathrm{d}x\mathrm{d}t\right|\end{aligned}$$

From (13), we have

$$\|DR_\varepsilon[u] - D(u)\|_{L^2_t(L^2_x)} \le C(\|u\|_{L^2_t(W^{1,\infty}_x)})\varepsilon^{d/2},$$

which also implies

$$\|DR_\varepsilon[u]\|_{L^2_t(L^2_x)} \le C\left(\|u\|_{L^2_t(H^1_x)} + \|u\|_{L^2_t(W^{1,\infty}_x)}\right).$$

We deduce that

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^d} |DR_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbb{R}^d} |Du|^2\,\mathrm{d}x\mathrm{d}t\right| \\ &\quad\le (\|DR_\varepsilon[u]\|_{L^2_t(L^2_x)} + \|Du\|_{L^2_t(L^2_x)})(\|D(R_\varepsilon[u] - u)\|_{L^2_t(L^2_x)}) &(34)\\ &\quad\le C\varepsilon^{d/2}. &(35)\end{aligned}$$

Collecting (32)–(33)–(35), we have

$$|Rest_\varepsilon^2| \le C(m_\varepsilon + \varepsilon^{d/2} + |\mathcal{S}_\varepsilon^{in}|). \tag{36}$$

We now consider the more difficult term $Rest_\varepsilon^1$, which reads

$$\begin{aligned} Rest_\varepsilon^1 = &-2\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon u_\varepsilon(t,.) \cdot \partial_t R_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t - 2\int_0^T \int_{\mathbb{R}^d} u_\varepsilon \otimes u_\varepsilon : \nabla R_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t \\ &- 4\int_0^T \int_{\mathbb{R}^d} Du_\varepsilon : DR_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

To tackle this term, we compute the time derivative of $R_\varepsilon[u]$ as follows

$$\begin{aligned} \partial_t R_\varepsilon[u] = &-\ell_\varepsilon \cdot \nabla\eta_\varepsilon u + \eta_\varepsilon \partial_t u + \ell_\varepsilon \cdot \nabla\eta_\varepsilon \bar{u}^\varepsilon + (1-\eta_\varepsilon)\partial_t \bar{u}^\varepsilon + (1-\eta_\varepsilon)\ell_\varepsilon \cdot \nabla\bar{u}^\varepsilon \\ &- \ell_\varepsilon \cdot \nabla\mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(u-\bar{u}^\varepsilon)] + \mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(\partial_t u - \partial_t\bar{u}^\varepsilon + \ell_\varepsilon\cdot\nabla u - \ell_\varepsilon \cdot \nabla\bar{u}^\varepsilon))], \end{aligned}$$

where we used equations (6) satisfied by u. Let us notice that in the above expression there is no time derivative of η_ε inside $\mathcal{B}_\varepsilon$ because $\mathcal{B}_\varepsilon$ follows the rigid body as η_ε. Let us rewrite

$$\partial_t R_\varepsilon[u] = \sum_{i=1}^5 I_i,$$

where

$$\begin{aligned} &I_1 = \eta_\varepsilon \partial_t u, \quad I_2 = -\ell_\varepsilon \cdot \nabla\eta_\varepsilon u + \ell_\varepsilon \cdot \nabla\eta_\varepsilon \bar{u}^\varepsilon, \quad I_3 = (1-\eta_\varepsilon)\partial_t\bar{u}^\varepsilon + (1-\eta_\varepsilon)\ell_\varepsilon \cdot \nabla\bar{u}^\varepsilon, \\ &I_4 = -\ell_\varepsilon \cdot \nabla\mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(u-\bar{u}^\varepsilon)], \quad \text{and} \quad I_5 = \mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(\partial_t u - \partial_t\bar{u}^\varepsilon + \ell_\varepsilon\cdot\nabla u - \ell_\varepsilon\cdot\nabla\bar{u}^\varepsilon))]. \end{aligned}$$

To tackle the term I_1, we use the equation satisfied by u. So, let us start by estimating the other terms. From now on, the estimates depend on the dimension so let us focus on the case of dimension three. In the last part of the proof, we explain how to adapted the estimates in the case the dimension is two. To tackle the term involving I_2, we notice that the support of $\nabla\eta_\varepsilon$ is included in $B_{2\varepsilon}(h_\varepsilon(t))$. A direct application of the Hölder inequality implies that

$$\begin{aligned} \left|\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon u_\varepsilon \cdot I_2\,\mathrm{d}x\mathrm{d}t\right| &= \left|\int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot (-\ell_\varepsilon \cdot \nabla\eta_\varepsilon u + \ell_\varepsilon \cdot \nabla\eta_\varepsilon\bar{u}^\varepsilon)\,\mathrm{d}x\mathrm{d}t\right| \\ &= \left|\int_0^T \int_{B_{2\varepsilon}(h_\varepsilon(t))} u_\varepsilon \cdot \left(\ell_\varepsilon \cdot (|x-h_\varepsilon(t)|\nabla\eta_\varepsilon)\frac{u(\tau,x)-u(\tau,h_\varepsilon(\tau))}{|x-h_\varepsilon(\tau)|}\right)\mathrm{d}x\mathrm{d}t\right| \\ &\le \|u_\varepsilon\|_{L_t^2(L_x^6)}\|\ell_\varepsilon\|_{L_t^2}\||x-h_\varepsilon|\nabla\eta_\varepsilon\|_{L_x^\infty}\|u\|_{L_t^\infty(W_x^{1,\infty})}\|1\|_{L^{6/5}(B_{2\varepsilon}(h_\varepsilon(t)))} \\ &\le C\varepsilon^{5/2}|\mathcal{S}_\varepsilon^{in}|^{-1/6}. \end{aligned} \tag{37}$$

In the last inequality, we used that $\|u_\varepsilon\|_{L^2_t(L^6_x)}$ is uniformly bounded due to (28). We used (29) to estimate $\|\ell_\varepsilon\|_{L^2_t}$. The term $\||x-h_\varepsilon|\nabla\eta_\varepsilon\|_{L^\infty_x} \le C$ by scaling argument. Inequality (31) ensures that $\|u\|_{L^\infty_t(W^{1,\infty}_x)}$ is finite. Finally $\|1\|_{L^{6/5}(B_{2\varepsilon}(h_\varepsilon(t)))} \le C\varepsilon^{5/2}$.

The term I_3 is the only one which is not zero in $\mathcal{S}_\varepsilon$. We have

$$\left|\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon\cdot I_3\,\mathrm{d}x\mathrm{d}t\right| = \left|m_\varepsilon\int_0^T \ell_\varepsilon\cdot\partial_t\bar{u}^\varepsilon\,\mathrm{d}t + \ell_\varepsilon\cdot\nabla\bar{u}^\varepsilon + \int_0^T\int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon\cdot(1-\eta_\varepsilon)(\partial_t\bar{u}^\varepsilon+\ell_\varepsilon\cdot\nabla\bar{u}^\varepsilon)\,\mathrm{d}x\mathrm{d}t\right|. \tag{38}$$

To tackle the right-hand side, we notice that

$$\begin{aligned}\left|m_\varepsilon\int_0^T\ell_\varepsilon\cdot\partial_t\bar{u}^\varepsilon\,\mathrm{d}t\right| &\le \left|m_\varepsilon\int_0^T(\ell_\varepsilon-\bar{u}^\varepsilon)\cdot\partial_t\bar{u}^\varepsilon\,\mathrm{d}t + m_\varepsilon\int_0^T\bar{u}^\varepsilon\cdot\partial_t\bar{u}^\varepsilon\,\mathrm{d}t\right|\\ &\le \frac12\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon|u_\varepsilon-R_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t + m_\varepsilon\|\partial_t u\|_{L^2_t(L^\infty_x)}(\|\partial_t u\|_{L^2_t(L^\infty_x)}+\|u\|_{L^2_t(L^\infty_x)}),\end{aligned} \tag{39}$$

and similarly,

$$\left|m_\varepsilon\int_0^T\ell_\varepsilon\cdot\nabla\bar{u}^\varepsilon\,\mathrm{d}t\right| \le \frac12\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon|u_\varepsilon-R_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t + m_\varepsilon(\|\nabla u\|_{L^\infty_x}+\|u\|_{L^2_t(L^\infty_x)}). \tag{40}$$

Moreover,

$$\begin{aligned}&\left|\int_0^T\int_{\mathcal{F}_\varepsilon(t)}u_\varepsilon\cdot(1-\eta_\varepsilon)(\partial_t\bar{u}^\varepsilon+\ell_\varepsilon\cdot\nabla\bar{u}^\varepsilon)\,\mathrm{d}x\mathrm{d}t\right|\\ &\qquad\le \|u_\varepsilon\|_{L^2_t(L^6_x)}\left(\|\partial_t u\|_{L^2_t(L^\infty_x)}+\|\ell_\varepsilon\|_{L^2_t}\|\nabla u\|_{L^\infty_t(L^\infty_x)}\right)\varepsilon^{5/2}.\end{aligned} \tag{41}$$

Equality (38) and estimates (39)–(40)–(41) imply that

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon\cdot I_3\,\mathrm{d}x\mathrm{d}t\right| &\le \int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon|u_\varepsilon-R_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t\\ &\quad+C\left(m_\varepsilon+\varepsilon^{5/2}+\varepsilon^{5/2}|\mathcal{S}^{in}_\varepsilon|^{-1/6}\right).\end{aligned} \tag{42}$$

For the term I_4, we proceed as follows.

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon\cdot I_4\,\mathrm{d}x\mathrm{d}t\right| &= \left|\int_0^T\int_{\mathcal{F}_\varepsilon(t)}u_\varepsilon\cdot\ell_\varepsilon\cdot\nabla\mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(u-\bar{u}^\varepsilon)]\,\mathrm{d}x\mathrm{d}t\right|\\ &\le \|u_\varepsilon\|_{L^2_t(L^6_x)}\|\ell_\varepsilon\|_{L^2_t}\|\nabla\mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(u-\bar{u}^\varepsilon)]\|_{L^\infty_t(L^{6/5}_x)}\\ &\le C\|u_\varepsilon\|_{L^2_t(L^6_x)}\|\ell_\varepsilon\|_{L^2_t}\|\nabla\eta_\varepsilon(u-\bar{u}^\varepsilon)\|_{L^\infty_t(L^{6/5}_x)}\end{aligned}$$

$$\leq C\|u_\varepsilon\|_{L^2_t(L^6_x)}\|\ell_\varepsilon\|_{L^2_t}\|(x-h_\varepsilon)\nabla\eta_\varepsilon\|_{L^\infty_x}\|u\|_{L^\infty_t(W^{1,\infty}_x)}\varepsilon^{5/2}$$
$$\leq C\varepsilon^{5/2}|\mathcal{S}^{in}_\varepsilon|^{-1/6}. \tag{43}$$

Let us tackle I_5 as follows

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon\cdot I_5\,\mathrm{d}x\mathrm{d}t\right| &= \left|\int_0^T\int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon\cdot\mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(\partial_t u-\partial_t\bar{u}^\varepsilon+\ell_\varepsilon\cdot\nabla u-\ell_\varepsilon\cdot\nabla\bar{u}^\varepsilon))]\,\mathrm{d}x\mathrm{d}t\right|\\ &\leq C\|u_\varepsilon\|_{L^2_t(L^6_x)}\|\mathcal{B}_\varepsilon[\nabla\eta_\varepsilon(\partial_t u-\partial_t\bar{u}^\varepsilon+\ell_\varepsilon\cdot\nabla u-\ell_\varepsilon\cdot\nabla\bar{u}^\varepsilon))]\|_{L^2_t(L^2_x)}\varepsilon\\ &\leq C\|u_\varepsilon\|_{L^2_t(L^6_x)}\|\nabla\eta_\varepsilon(\partial_t u-\partial_t\bar{u}^\varepsilon+\ell_\varepsilon\cdot\nabla u-\ell_\varepsilon\cdot\nabla\bar{u}^\varepsilon))\|_{L^2_t(L^{6/5}_x)}\varepsilon\\ &\leq C\|u_\varepsilon\|_{L^2_t(L^6_x)}\|\nabla\eta_\varepsilon\|_{L^3_x}\Big(\|\partial_t u\|_{L^2_t(L^\infty_x)}+\|\ell_\varepsilon\|_{L^2_t}\|\nabla u\|_{L^\infty_t(L^\infty_x)}\Big)\varepsilon^{5/2}\\ &\leq C(\varepsilon^{5/2}+\varepsilon^{5/2}|\mathcal{S}^{in}_\varepsilon|^{-1/6}).\end{aligned} \tag{44}$$

We will now consider I_1. Recall that u is a regular solution, we rewrite

$$\eta_\varepsilon\partial_t u = -\eta_\varepsilon u\cdot\nabla u+\eta_\varepsilon\Delta u+\eta_\varepsilon\nabla p,$$

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon\cdot I_1\,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon\eta_\varepsilon u_\varepsilon\cdot\partial_t u\,\mathrm{d}x\mathrm{d}t\\ &= -\int_0^T\int_{\mathbb{R}^d}\eta_\varepsilon u_\varepsilon\cdot(u\cdot\nabla)u-\eta_\varepsilon u_\varepsilon\cdot\Delta u+\eta_\varepsilon u_\varepsilon\cdot\nabla p\,\mathrm{d}x\mathrm{d}t\\ &= -\int_0^T\int_{\mathbb{R}^d}\eta_\varepsilon u_\varepsilon\otimes u:\nabla u+2\eta_\varepsilon Du_\varepsilon:Du\,\mathrm{d}x\mathrm{d}t\\ &\quad -\int_0^T\int_{\mathbb{R}^d}(u_\varepsilon\otimes\nabla\eta_\varepsilon+\nabla\eta_\varepsilon\otimes u_\varepsilon):Du-u_\varepsilon\cdot\nabla\eta_\varepsilon(p-\bar{p}^\varepsilon)\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

where $\bar{p}^\varepsilon = p(t,h_\varepsilon(t))$. We can now rewrite the remainder using the above computations and deduce

$$\begin{aligned}Rest^1_\varepsilon &= -2\sum_{i=1}^5\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon\cdot I_i\,\mathrm{d}x\mathrm{d}t-2\int_0^T\int_{\mathcal{F}_\varepsilon(t)}u_\varepsilon\otimes u_\varepsilon:\nabla R_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t\\ &\quad -4\int_0^T\int_{\mathbb{R}^d}Du_\varepsilon:DR_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t\\ &= -2\sum_{i=2}^5\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon\cdot I_i\,\mathrm{d}x\mathrm{d}t-2\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon u_\varepsilon\cdot I_1\,\mathrm{d}x\mathrm{d}t\\ &\quad -2\int_0^T\int_{\mathbb{R}^d}u_\varepsilon\otimes u_\varepsilon:\nabla R_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t-4\int_0^T\int_{\mathbb{R}^d}Du_\varepsilon:DR_\varepsilon[u]\,\mathrm{d}x\mathrm{d}t\\ &= -2\sum_{j=1}^5 J_j\end{aligned} \tag{45}$$

where

$$J_1 = \sum_{i=2}^{5} \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon u_\varepsilon \cdot I_i \,\mathrm{d}x\mathrm{d}t,$$
$$J_2 = \int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \otimes u_\varepsilon : \nabla R_\varepsilon[u] - \eta_\varepsilon u_\varepsilon \otimes u : \nabla u \,\mathrm{d}x\mathrm{d}t,$$
$$J_3 = \int_0^T \int_{\mathbb{R}^d} (u_\varepsilon \otimes \nabla\eta_\varepsilon + \nabla\eta_\varepsilon \otimes u_\varepsilon) : Du \,\mathrm{d}x\mathrm{d}t,$$
$$J_4 = 2\int_0^T \int_{\mathbb{R}^d} Du_\varepsilon : DR_\varepsilon[u] - \eta_\varepsilon Du_\varepsilon : Du \,\mathrm{d}x\mathrm{d}t \quad \text{and} \quad J_5 = -\int_0^T \int_{\mathbb{R}^d} u_\varepsilon \cdot \nabla\eta_\varepsilon p \,\mathrm{d}x\mathrm{d}t.$$

Inequalities (37)–(42)–(43)–(44) imply that

$$|J_1| \le C\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon |u_\varepsilon - R_\varepsilon[u]|^2 \,\mathrm{d}x\mathrm{d}t + C(m_\varepsilon + \varepsilon^{5/2}|\mathcal{S}_\varepsilon^{in}|^{-1/6}). \tag{46}$$

To tackled J_2, we start by rewriting it as

$$\begin{aligned}
J_2 &= \int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \otimes u_\varepsilon : \nabla R_\varepsilon[u] - \eta_\varepsilon u_\varepsilon \otimes u : \nabla u \,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \otimes u_\varepsilon : (\nabla R_\varepsilon[u] - \eta_\varepsilon \nabla u) \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathcal{F}_\varepsilon(t)} \eta_\varepsilon u_\varepsilon \otimes (u_\varepsilon - u) : \nabla u \,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \otimes u_\varepsilon : (\nabla R_\varepsilon[u] - \eta_\varepsilon \nabla u) \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathcal{F}_\varepsilon(t)} \eta_\varepsilon (u_\varepsilon - u) \otimes (u_\varepsilon - u) : \nabla u \,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^T \int_{\mathcal{F}_\varepsilon(t)} \nabla\eta_\varepsilon \cdot u \otimes (u_\varepsilon - u) : \nabla u \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{47}$$

Before estimating the right-hand side of the above equality, let us recall from (18) and (19) the following bounds hold

$$\|R_\varepsilon[u] - \eta_\varepsilon u\|_{L^2_x} \le C\varepsilon^{3/2}\|u\|_{L^\infty_x}(1 + \|\nabla\eta_\varepsilon\|_{L^3_x}),$$
$$\|\nabla R_\varepsilon[u] - \eta_\varepsilon \nabla u\|_{L^{3/2}_x} \le C\varepsilon^2\|u\|_{W^{1,\infty}_x},$$

and

$$\begin{aligned}
\|\sqrt{\eta_\varepsilon}(u_\varepsilon - u)\|^2_{L^2(\mathcal{F}_\varepsilon(t))} \le \|u_\varepsilon\|^2_{L^2(\mathcal{F}_\varepsilon(t))} &- 2\int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot R_\varepsilon[u] \,\mathrm{d}x + \|R_\varepsilon[u]\|^2_{L^2(\mathcal{F}_\varepsilon(t))} \\
&- 2\int_{\mathcal{F}_\varepsilon(t)} u_\varepsilon \cdot (\eta_\varepsilon u - R_\varepsilon[u]) \,\mathrm{d}x + \int_{\mathcal{F}_\varepsilon(t)} u \cdot (\eta_\varepsilon u - R_\varepsilon[u]) \,\mathrm{d}x \\
&+ \int_{\mathcal{F}_\varepsilon(t)} R_\varepsilon[u] \cdot (\eta_\varepsilon u - R_\varepsilon[u]) \,\mathrm{d}x
\end{aligned}$$

$$\leq \|u_\varepsilon - R_\varepsilon[u]\|^2_{L^2(\mathcal{F}_\varepsilon(t))} + C(\|u_\varepsilon\|_{L^2_x} + \|u\|_{L^2_x} + \|R_\varepsilon[u]\|_{L^2_x})\|R_\varepsilon[u] - \eta_\varepsilon u\|_{L^2_x}$$
$$\leq \|u_\varepsilon - R_\varepsilon[u]\|^2_{L^2(\mathcal{F}_\varepsilon(t))} + C\varepsilon^{3/2}.$$

Using equality (47) and the fact that $\nabla\eta_\varepsilon$ has support contained in the ball $B_{2\varepsilon}(h_\varepsilon(t))$, we deduce

$$\begin{aligned}|J_2| \leq & \|u_\varepsilon\|^2_{L^2_t(L^6_x)}\|\nabla R_\varepsilon[u] - \eta_\varepsilon\nabla u\|_{L^\infty_t(L^{3/2}_x)} + \|\sqrt{\eta_\varepsilon}(u_\varepsilon - u)\|^2_{L^2_t(L^2(\mathcal{F}_\varepsilon(t)))}\|\nabla u\|_{L^\infty_t(L^\infty_x)} \\ & + \|1\|_{L^2(B_{2\varepsilon}(h_\varepsilon(t)))}\|\nabla\eta_\varepsilon\|_{L^3_x}\|u\|_{L^\infty_t(L^\infty_x)}(\|u_\varepsilon\|_{L^2_t(L^6_x)} + \|u\|_{L^2_t(L^6_x)})\|\nabla u\|_{L^2_t(L^\infty_x)} \\ \leq & C\varepsilon^2 + C\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon|u_\varepsilon - R_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t + C\varepsilon^{3/2}. \end{aligned} \tag{48}$$

After applying some Hölder inequality, the J_3 term is bounded as follows

$$|J_3| \leq \varepsilon^{3/2}\|u_\varepsilon\|_{L^2_t(L^6_x)}\|\nabla\eta_\varepsilon\|_{L^3_x}\|Du\|_{L^2_t(L^\infty_x)} \leq C\varepsilon^{3/2}. \tag{49}$$

We now tackle J_4

$$\begin{aligned}|J_4| &\leq \left|2\int_0^T\int_{\mathbb{R}^d} Du_\varepsilon : DR_\varepsilon[u] - \eta_\varepsilon Du_\varepsilon : Du\,\mathrm{d}x\mathrm{d}t\right| \\ &\leq \|Du_\varepsilon\|_{L^2_t(L^2_x)}\|DR_\varepsilon[u] - \eta_\varepsilon Du\|_{L^2_t(L^2_x)} \\ &\leq C\varepsilon^{3/2}, \end{aligned} \tag{50}$$

which follows from the estimate (19) that reads $\|DR_\varepsilon[u] - \eta_\varepsilon Du\|_{L^2_x} \leq C\varepsilon^{3/2}\|u\|_{W^{1,\infty}_x}$.

Finally we estimate the term J_5 as follows

$$\begin{aligned}|J_5| &\leq \left|\int_0^T\int_{\mathbb{R}^d} u_\varepsilon\cdot\nabla\eta_\varepsilon p\,\mathrm{d}x\mathrm{d}t\right| \\ &\leq \|u_\varepsilon\|_{L^2_t(L^6_x)}\|1\|_{L^2(B_{2\varepsilon}(h_\varepsilon(t)))}\|\nabla\eta_\varepsilon\|_{L^3_x}\|p\|_{L^2_t(L^\infty)} \\ &\leq C\varepsilon^{3/2}. \end{aligned} \tag{51}$$

If we collect estimates (46)–(48)–(49)–(50)–(51) and equality (45), we deduce

$$|Rest^1_\varepsilon| \leq C\int_0^T\int_{\mathbb{R}^d}\rho_\varepsilon|u_\varepsilon - R_\varepsilon[u]|^2\,\mathrm{d}x\mathrm{d}t + C(m_\varepsilon + \varepsilon^{3/2} + \varepsilon^{5/2}|\mathcal{S}^{in}_\varepsilon|^{-1/6}). \tag{52}$$

Let us recall that

$$|\widetilde{Rest_\varepsilon}| \leq |Rest^1_\varepsilon| + |Rest^2_\varepsilon|.$$

This together with estimates (52) and (36) implies the lemma.

Let us now explain how to reproduce the above bounds in dimension two. First of all, let us recall that $|B_\varepsilon(0)| \leq C\varepsilon^2$ and $\|\nabla\eta_\varepsilon\|_{L^p} \leq C\varepsilon^{(2-p)/2}$, in particular the L^p norm of $\nabla\eta_\varepsilon$ is uniformly bounded in ε only for $p \in [1, 2]$. Moreover, let us notice that the a priori bound $\|u_\varepsilon\|_{L^2_t(L^q_x)} \leq C_q$ holds for any $q \in [2, \infty)$ by the Sobolev embedding in dimension two, see Lemma 4. Using this information, we can estimate, for example,

$$|J_3| \leq \|u_\varepsilon\|_{L^2_t(L^q_x)} \|\nabla\eta_\varepsilon\|_{L^p_x} \|Du\|_{L^2_t(L^\infty_x)} \leq C\varepsilon^{\tilde{\delta}},$$

where we choose $p = 2/(\tilde{\delta} + 1)$ and $1/p + 1/q = 1$.

The bounds of all the other terms follow similarly. □

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A Bogovskiĭ Operator

This appendix is dedicated to recall some facts about the Bogovskiĭ operators. Let us recall that a Bogovskiĭ operator is a right inverse of the divergence on $\tilde{L}^p$ which is the space of L^p functions with integral zero. Due to the non-uniqueness of this operator, we choose B_1 to satisfy the following extra property.

Theorem 4 *There exists a Bogovskiĭ operator B_1 such that*

$$\mathcal{B}_1 : \tilde{L}^p(B_2(0) \setminus B_1(0); \mathbb{R}) \longrightarrow W^{1,p}_0(B_2(0) \setminus B_1(0); \mathbb{R}^d)$$

and it is linear and continuous for any $1 < p < +\infty$,

$$\operatorname{div}(\mathcal{B}_1[f]) = f \text{ for any } f \in \tilde{L}^p(B_2(0) \setminus B_1(0))$$

$$\text{and} \quad \|\mathcal{B}_1[f]\|_{L^\infty(B_2(0)\setminus B_1(0))} \le C\|f\|_{L^2(B_2(0)\setminus B_1(0))}.$$

Moreover, for any vector field $F \in L^p(B_2(0)\setminus B_1(0))$ *such that* $F\cdot n = 0$ *on* $\partial B_2(0)\cup \partial B_1(0)$*, it holds*

$$\|\mathcal{B}_1[\mathrm{div}\,(F)]\|_{L^p(B_2(0)\setminus B_1(0))} \le C\|F\|_{L^p(B_2(0)\setminus B_1(0))}.$$

We refer to subsection 3.3.1.2 of Novotny and Straskraba (2004) for a proof of the above theorem and for more details.

Let us recall that $\mathcal{B}_\varepsilon[f](x) = \varepsilon\mathcal{B}_1[f(\varepsilon y)](x/\varepsilon)$, in particular it satisfies the following uniform estimates.

Lemma 3 *Let* $1 < p < +\infty$*. The operator* $\mathcal{B}_\varepsilon$ *is a Bogovskiĭ operator in* $B_{2\varepsilon}(0)\setminus B_\varepsilon(0)$*. Moreover, there exists a constant* C *independent of* ε *such that*

$$\|\mathcal{B}_\varepsilon[f]\|_{W^{1,p}(B_{2\varepsilon}(0)\setminus B_1(0))} \le C\|f\|_{L^p(B_{2\varepsilon}(0)\setminus B_1(0))},$$

for any $f \in \tilde{L}^p(B_{2\varepsilon}(0)\setminus B_1(0))$*. And*

$$\|\mathcal{B}_\varepsilon[\mathrm{div}\,(F)]\|_{L^p(B_{2\varepsilon}(0)\setminus B_\varepsilon(0))} \le C\|F\|_{L^p(B_{2\varepsilon}(0)\setminus B_\varepsilon(0))},$$

for any vector field $F \in L^p(B_{2\varepsilon}(0)\setminus B_\varepsilon(0))$ *such that* $F\cdot n = 0$ *on* $\partial B_\varepsilon(0)\cup\partial B_\varepsilon(0)$.

The proof of the above result is consequence of the scaling.

B A Remark on Sobolev Embeddings

In this section, we show that Sobolev embeddings $W^{1,2}(\mathbb{R}^2)\subset L^p(\mathbb{R}^2)$ for $p\in[2,\infty)$ holds when we replace $W^{1,2}(\mathbb{R}^2)$ with $\dot{H}^1(\mathbb{R}^2)\cap L^2(\mathcal{F}_\varepsilon(t))$. Here $\dot{H}^1(\mathbb{R}^2)$ denotes the closure of $C_c^\infty(\mathbb{R}^2)$ respect to the norm $\|f\|_{\dot{H}^1} = \|\nabla f\|_{L^2(\mathbb{R}^2)}$.

Lemma 4 *Let* $u\in H^1(\mathbb{R}^2)$ *and* $p\in[2,\infty)$*, then we have*

$$\|u\|_{L^p(\mathbb{R}^2)} \le C(\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|u\|_{L^2(\mathcal{F}_\varepsilon(t))}).$$

Proof The estimate is well known if in the right-hand side we replace $\|u\|_{L^2(\mathcal{F}_\varepsilon(t))}$ by $\|u\|_{L^2(\mathbb{R}^2)}$. Let us show that

$$\|u\|_{L^2(\mathbb{R}^2)} \le C(\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|u\|_{L^2(\mathcal{F}_\varepsilon(t))}). \tag{53}$$

To see this, let us recall that $\mathcal{S}_\varepsilon(t)\subset B_\varepsilon(h_\varepsilon(t))$. By translation invariance of the norms, we can assume $h_\varepsilon(t)=0$. We introduce the space

$$X = \left\{ v\in H^1(B_2(0)) \text{ such that } \bar{v} = \frac{1}{|B_2(0)\setminus B_1(0)|}\int_{|B_2(0)\setminus B_1(0)|} v\,\mathrm{d}x = 0\right\}.$$

The Poincaré inequality in X implies the existence of a constant C_X such that

$$\|v\|_{L^2(B_2(0))} \le C_X \|\nabla v\|_{L^2(B_2(0))}.$$

For $u \in H^1(B_2(0))$, let us use the notation

$$\bar{u} = \frac{1}{|B_2(0) \setminus B_1(0)|} \int_{|B_2(0)\setminus B_1(0)|} u \,\mathrm{d}x.$$

Notice that $u - \bar{u} \in X$. We estimate

$$\|u\|_{L^2(B_2(0))} \le \|u - \bar{u}\|_{L^2(B_2(0))} + \|\bar{u}\|_{L^2(B_2(0))} \le C_X \|\nabla u\|_{L^2(B_2(0))} + C\|u\|_{L^2(B_2(0))\setminus B_1(0))},$$

which implies (53). □

References

Al Baba, H., Ghosh, A., Muha, B., Nečasová, Š: $L^p - L^q$-strong solution to fluid-rigid body interaction system with Navier slip boundary condition. J. Elliptic Parabol. Equ. **7**(2), 439–489 (2021)

Bravin, M., Nečasová, Š.: On the velocity of a small rigid body in a viscous incompressible fluid in dimension two and three (2022). arXiv:2208.12351

Bravin, M.: Energy equality and uniqueness of weak solutions of a "viscous incompressible fluid+ rigid body" system with Navier slip-with-friction conditions in a 2D bounded domain. J. Math. Fluid Mech. **21**(2), 1–31 (2019)

Bravin, M.: On the 2D "viscous incompressible fluid+ rigid body" system with Navier conditions and unbounded energy. Comptes Rendus. Mathématique **358**(3), 303–319 (2020)

Bravin, M., Nečasová, Š: On the vanishing rigid body problem in a viscous compressible fluid. J. Differ. Equ. **345**, 45–77 (2023)

Chemetov, N.V., Nečasová, Š, Muha, B.: Weak-strong uniqueness for fluid-rigid body interaction problem with slip boundary condition. J. Math. Phys. **60**, 13 (2019)

Conca, C., Martín, Jorge San, H., Tucsnak, M.: Motion of a rigid body in a viscous fluid. Comptes Rendus de l'Académie des Sciences-Series I-Mathematics **328**(6), 473–478 (1999)

Desjardins, B., Esteban, M.J.: Existence of weak solutions for the motion of rigid bodies in a viscous fluid. Arch. Ration. Mech. Anal. **146**(1), 59–71 (1999)

Feireisl, E., Roy, A., Zarnescu, A.: On the motion of a small rigid body in a viscous compressible fluid (2022). arXiv:2208.07933

Feireisl, E., Roy, A., Zarnescu, A.: On the motion of several small rigid bodies in a viscous incompressible fluid (2022). arXiv:2209.09284

Feireisl, E.: On the motion of rigid bodies in a viscous fluid. Appl. Math. **47**(6), 463–484 (2002)

Feireisl, E.: On the motion of rigid bodies in a viscous incompressible fluid. J. Evol. Equ. **3**, 419–441 (2003)

Ferriere, G., Hillairet, M.: Unbounded-energy solutions to the fluid+ disk system and long-time behavior for large initial data. Comptes Rendus. Mathématique **361**(G2), 453–485 (2023)

Geissert, M., Götze, K., Hieber, M.: L^p-theory for strong solutions to fluid-rigid body interaction in Newtonian and generalized Newtonian fluids. Trans. Am. Math. Soc. **365**(3), 1393–1439 (2013)

Gérard-Varet, D., Hillairet, M.: Existence of weak solutions up to collision for viscous fluid-solid systems with slip. Commun. Pure Appl. Math. **67**(12), 2022–2076 (2014)

Glass, O., Sueur, F.: Uniqueness results for weak solutions of two-dimensional fluid-solid systems. Arch. Ration. Mech. Anal. **218**(2), 907–944 (2015)

Glass, O., Sueur, F.: Dynamics of several rigid bodies in a two-dimensional ideal fluid and convergence to vortex systems (2019). arXiv:1910.03158

Glass, O., Lacave, C., Sueur, F.: On the motion of a small body immersed in a two-dimensional incompressible perfect fluid. Bull. Soc. Math. France **142**(3), 489–536 (2014a)

Glass, O., Munnier, A., Sueur, F.: Dynamics of a point vortex as limits of a shrinking solid in an irrotational fluid (2014b). arXiv:1402.5387

Glass, O., Lacave, C., Sueur, F.: On the motion of a small light body immersed in a two dimensional incompressible perfect fluid with vorticity. Commun. Math. Phys. **341**(3), 1015–1065 (2016)

Glass, O., Munnier, A., Sueur, F.: Point vortex dynamics as zero-radius limit of the motion of a rigid body in an irrotational fluid. Invent. Math. **214**(1), 171–287 (2018)

Glass, O., Lacave, C., Munnier, A., Sueur, F.: Dynamics of rigid bodies in a two dimensional incompressible perfect fluid. J. Differ. Equ. **267**(6), 3561–3577 (2019)

Grandmont, C., Maday, Y.: Existence for an unsteady fluid-structure interaction problem. ESAIM: Math. Model. Numer. Anal. **34**(3), 609–636 (2000)

Gunzburger, M.D., Lee, H.C., Seregin, G.A.: Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions. J. Math. Fluid Mech. **2**(3), 219–266 (2000)

He, J., Iftimie, D.: A small solid body with large density in a planar fluid is negligible. J. Dyn. Diff. Equat. **31**(3), 1671–1688 (2019)

He, J., Iftimie, D.: On the small rigid body limit in 3D incompressible flows. J. Lond. Math. Soc. **104**(2), 668–687 (2021)

Judakov, N.V.: The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid. Dinamika Splošn. Sredy **18**, 249–253 (1974)

Lacave, C., Takahashi, T.: Small moving rigid body into a viscous incompressible fluid. Arch. Ration. Mech. Anal. **223**(3), 1307–1335 (2017)

Maity, D., Tucsnak, M.: Lp-Lq maximal regularity for some operators associated with linearized incompressible fluid-rigid body problems. Mathe. Anal Fluid Mech. Select. Recent Results **710**, 175–201 (2018)

Muha, B., Nečasová, Š., Radošević, A. On the regularity of weak solutions to the fluid-rigid body interaction problem (2022). arXiv:2211.03080

Muha, B., Nečasová, Š, Radošević, A.: A uniqueness result for 3D incompressible fluid-rigid body interaction problem. J. Math. Fluid Mech. **23**(1), 1–39 (2021)

Novotny, A., Straskraba, I.: Introduction to the Mathematical Theory of Compressible Flow (Vol. 27). OUP Oxford (2004)

Planas, G., Sueur, F.: On the“viscous incompressible fluid+ rigid body” system with Navier conditions. Annales de l'Institut Henri Poincaré C **31**(1), 55–80 (2014)

Serre, D.: Chute libre d'un solide dans un fluide visqueux incompressible. Exist. Jpn J. Appl. Math. **4**(1), 99–110 (1987)

Takahashi, T.: Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain. Adv. Differer. Equ. **8**(12), 1499–1532 (2003)

Takahashi, T., Tucsnak, M.: Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid. J. Math. Fluid Mech. **6**(1), 53–77 (2004)